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1. Prove that, the set of rational numbers Q, equipped with the two binary operations of addition and multiplication, forms a field.

Answer: We take the rational numbers $\mathbb{Q}$ to be the set of equivalence classes of ordered pairs (a, b) with $a, b \in \mathbb{Z}$ and $b \neq 0$, where $(a,b) \sim (a', b')$ iff $ab' = a'b$. We identify the class of (a, b) with the usual $\frac{a}{b}$.

Define addition and multiplication in the usual way:

$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$, $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$,

for, $b \neq 0$, $d \neq 0$. Below we show these operations make $\mathbb{Q}$ a field.

1 The operations are well-defined:

We must check that, if

$\frac{a}{b} = \frac{a'}{b'}$ and $\frac{c}{d} = \frac{c'}{d'}$ then

$$\frac{ad+bc}{bd} = \frac{a'd'+b'c'}{b'd'} \text{ and } \frac{ac}{bd} = \frac{a'c'}{b'd'}$$

From, $\frac{a}{b} = \frac{a'}{b'}$ and $\frac{c}{d} = \frac{c'}{d'}$ we have,

$$ab' = a'b \text{ and } cd' = c'd$$

Compute,

$$(ad+bc)b'd' = (a'b)(dd') + (c'd)(bb')$$

$= (ab')(dd') + (cd')(bb')$

and similarly expand the right-hand numerator times $bdb'd'$. Rearranging and using $ab' = a'b$, $cd' = c'd$ shows both cross-products are equal, therefore the sums (and similarly the products) represent the same equivalence class. So the addition and multiplication are well-defined.

2. $(Q, +)$ is an abelian Group:

Take any $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in Q$.

closure: $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ is a rational number since $bd \neq 0$.

Associativity: Follows from associativity of integer addition:

$$\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f}=\frac{ad+bc}{bd}+\frac{e}{f}$$

$$=\frac{(ad+bc)f+e(bd)}{(bd)f}$$

and a similar expansion for $\frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right)$; both give the same numerator by associativity of commutativity of integer operations.

Identity: $0=\frac{0}{1}$ satisfies

$$\frac{a}{b}+0=\frac{a}{b}$$

Inverse: additive inverse of $\frac{a}{b}$ is $\frac{-a}{b}$ because

$$\frac{a}{b} + \frac{-a}{b} = \frac{0}{b}$$

<u>Commutativity:</u> $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

$$= \frac{bc+ad}{db} = \frac{c}{d} + \frac{a}{b}.$$

Thus, $(\mathbb{Q}, +)$ is an abelian group.

<u>3. Multiplication on $\mathbb{Q}/\{0\}$ is an abelian group (except we first show ring axioms):</u>

<u>closure:</u> product $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ is rational since $bd \neq 0$.

<u>associativity and commutativity:</u> follow from associativity and commutativity of integer multiplication

$$\left(\frac{a}{b}\cdot\frac{c}{d}\right)\cdot\frac{e}{f} = \frac{ac}{bd}\cdot\frac{e}{f}$$

$$= \frac{a\,(ce)}{b\,(df)} = \frac{a}{b}\cdot\frac{ce}{df}$$

$$= \frac{a}{b}\cdot\left(\frac{c}{d}\cdot\frac{e}{f}\right)$$

Multiplicative identity:

$1 = \frac{1}{1}$ satisfies $\frac{a}{b}\cdot 1 = \frac{a}{b}$.

Distributivity: For addition and multiplication,

$$\frac{a}{b}\left(\frac{c}{d}+\frac{e}{f}\right) = \frac{a}{b}\cdot\frac{cf+ed}{df}$$

$$= \frac{a(cf+ed)}{bdf}$$

$$= \frac{acf+aed}{bdf}$$

$$= \frac{ac}{bd} + \frac{ae}{bf}$$

$= \frac{a}{b} \cdot \frac{c}{d} + \frac{a}{b} \cdot \frac{e}{f}$

using integer distributivity.

So, $\mathbb{Q}$ is a commutative ring with unity 1.

4. <u>Multiplicative inverses exist for nonzero rationals:</u>

Take a nonzero rational $\frac{a}{b}$ (so, $a \neq 0$, $b \neq 0$). Its multiplicative inverse is $\frac{b}{a}$ because $\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ba} = \frac{1}{1} = 1$

We also must check this inverse is well defined: if $\frac{a}{b} = \frac{a'}{b'}$ and $a \neq 0$, then $ab' = a'b$.

Multiplying both sides by $1/(aa')$

is informal but the correct check is: $\frac{b}{a} = \frac{b'}{a'}$ if and only if $ba' = b'a$; but from $ab' = a'b$ we get exactly $ba' = b'a$. So, inverses agree for different representatives. (Thus the operation of taking $\frac{a}{b} \to \frac{b}{a}$ is well defined on equivalence classes).

5. Nontriviality: $0 \neq 1$

clearly $\frac{0}{1} \neq \frac{1}{1}$ because if $0.1 = 1.1$ then $0 = 1$, contradicting the integer's properties. So, the field is not the zero ring.

putting the pieces together. Q with the usual addition and multiplication is a commutative ring with unity in which every nonzero element has a multiplicative inverse. Therefore Q is a field.