MODULE - 3

COMPLEX INTEGRATION

There are two types of complex integration

I Line integral in the complex plane

II Contoure integration

Complex definite integrals are ~~for~~ called complex line integral. And they are written as $\int_c f(z)dz$

Here c is a given terms or a portion of it.

And c is called path of integration.

H.W

1) (continue...)

$$\frac{i+1}{i-1} = \frac{-1-i}{-1-i}$$

$$\frac{-i+1-1-i}{2} = -i$$

$$\frac{(w-i)i}{(w+i)-i} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{w-i}{w+i} = \frac{(z-1)i}{z+1}$$

$$(w-i)(z+1) = (zi-i)(w+i)$$

$$wz+w-iz-i = izw-z-iw+1$$

$$w(z+1-iz+i) = -z+1+iz+i$$

$$w = \frac{z(i-1)+(1+i)}{z(1-i)+(1+i)}$$

3/10/2017.

Properties

1) $\int_C (kf(z))\,dz = k\int_C f(z)\,dz.$

2) $\int_{z_0}^{z_1} f(z)\,dz = -\int_{z_1}^{z_0} f(z)\,dz$

3) $C = C_1 + C_2 + C_3 + \cdots + C_n.$

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots + \int_{C_n} f(z)\,dz$$

Indefinite Integral

An indefinite integral is a function whose derivative equals a given analytic function in a region.

Line integral in a complex Plane.

I First Evaluation Method

The first evaluation method is applied only to analytic function.

Theorum:-

Let $f(z)$ be an analytic function in a simply connected domain D. then there exist an indefinite integral of $f(z)$ in the domain D.

[ie, there exist an analytic function $F(z)$ such that $\boxed{F'(z) = f(z)}$] .

And for all paths in D joining any two points z_0 & z_1 in D we have

$$\boxed{\int_{z_0}^{z_1} f(z)\, dz = F(z_1) - F(z_0)}$$

? Evaluate $\int_0^{1+i} z^2\, dz$

A. z^2 is analytic function

$$\therefore \int_0^{1+i} z^2\, dz = \left[\frac{z^3}{3}\right]_0^{1+i} = \frac{1}{3}(1+i)^3 = \frac{1}{3}(1+3i+-3-i)$$

$$= \frac{2i-2}{3} = \frac{2(i-1)}{3}$$

? Evaluate $\int_{-\pi i}^{\pi i} \cos z\, dz$

A. $\cos z$ is analytic

$$\therefore \int_{-\pi i}^{\pi i} \cos z\, dz = 2\int_0^{\pi i} \cos z\, dz = 2\left[\sin z\right]_0^{\pi i}$$

$$= 2\sin \pi i$$

$$= 2i \sinh \pi$$

? Evaluate $\int_{8+\pi i}^{8-3\pi i} e^{z/2}\, dz.$

A. $e^{z/2}$ is analytic function.

$$\int_{8+\pi i}^{8-3\pi i} e^{z/2}\, dz = \frac{2}{1}\left[e^{z/2}\right]_{8+\pi i}^{8-3\pi i}$$

e^{at}
log

$$= 2\left[e^{8-3\pi i/2} - e^{8+\pi i/2}\right]$$

$-\frac{3\pi i}{2} \div 2$

$$= 2\, e^{8/2} \cdot e^{-3\pi i/2} - e^{8/2} \cdot e^{\pi i/2}$$

$$= 2e^4\left[e^{-3\pi i/2} - e^{\pi i/2}\right]$$

$$\cancel{= 2\log e^4\left[\log e^{-3\pi i/2} - \log e^{\pi i/2}\right]}$$

$$\cancel{= 2\times 4\left[-\frac{3\pi i}{2} - \frac{\pi i}{2}\right]}$$

$$\cancel{= 8\left[-\frac{4\pi i}{2}\right] = -16\pi i}$$

$$= 2e^4\left[\cos -\frac{3\pi i}{2} - i\sin\frac{3\pi i}{2} - \cos\frac{\pi i}{2} - i\sin\frac{\pi i}{2}\right]$$

$$= 2e^4\left[0 - i\sin\frac{3\pi}{2} - 0 - i\sin\frac{\pi}{2}\right]$$

$$= 2e^4\left[-i(-1) - i(1)\right]$$

$$= 2e^4\left[i - i\right] = 0$$

4/10/17

? Evaluate $\int_{-i}^{i} \frac{dz}{z}$ where the domain is the complex plane ~~which~~ without zero and the -ve real values.

A. Excluding the point '0' & the -ve real axis values the given function is analytic. Hence we can apply Ist evaluation method.

$$\int_{-i}^{i} \frac{dz}{z} = 2\int_{0}^{i} \frac{dz}{z} = \cancel{2\left[\log z\right]_0^i}$$

only for even function
$f(-x) = f(x)$ → even
$f(-x) = -f(x)$ → odd

$$= \cancel{2(\log i - \log 0)}$$

$$= \cancel{2(\log i - 1)}$$

$$\int_{-i}^{i} \frac{dz}{z} = \left[\log z\right]_{-i}^{i} = \frac{\cancel{\log i}}{\cancel{\log -i}} \log i - \log(-i)$$

$$= \log\left(\frac{i}{-i}\right) = \cancel{\log(-1)} \; \cancel{\pi i}.$$

1) $\int_{-a}^{a}$ even functn $= 2\int_{0}^{a} f(x)\,dx$.

2) $\int_{-a}^{a}$ odd functn $= 0$.

$f(-x) = f(x)$ → even
$f(-x) = -f(x)$ → odd

? Evaluate the following integrals using I evalutn method.

$\int_C e^z dz$ where C is the path ~~for~~ is a line joining $\frac{\pi}{2i}$ to πi

A e^z is analytic function.

$$\int_{\frac{\pi}{2}i}^{\pi i} e^z dz = \left[e^z\right]_{\pi/2 i}^{\pi i}$$

$$= e^{\pi i} - e^{\pi/2\, i}$$

$$= \cos\pi + i\sin\pi - \left[\cos \pi/2 + i\sin\pi/2\right]$$

$$= 1 + i\times 0 - [0+i] = \underline{\underline{-1-i}}$$

? Evaluate $\int_C \cos 2z\, dz$ where c is a path joining $-\pi i$ to πi

$$\int_{-\pi i}^{\pi i} \cos 2z\, dz = \left[\frac{\sin 2z}{2}\right]_{-\pi i}^{\pi i}$$

$$= \frac{1}{2}\left[\sin 2\pi i - \sin 2(-\pi i)\right]$$

$$= \frac{1}{2}\left[\sin 2\pi i + \sin 2\pi i\right]$$

$$= \frac{1}{2}\cdot 2\sin 2\pi i$$

$$= \underline{\underline{\sin 2\pi i}}$$

2) Second Evaluation Method. (Use of parametric representation of a path)

This method is applied to any continous complex function

Theorum:-

Let 'c' be a piecewise smooth curve represented by $z = z(t)$ where $a \le t \le b$.

Let $f(z)$ be a continous function on 'c' then

$$\int_c f(z)\, dz = \int_a^b f(z(t)) \frac{dz}{dt} \cdot dt$$

$$= \int_a^b f(z(t))\, \dot{z}(t)\, dt$$

$z = x + iy$

$z(t) = x(t) + iy(t)$

$a \le t \le b$

$\frac{dz}{dt} = \dot{z}(t)$

where $\dot{z}(t) = \frac{dz}{dt}$

→ Steps for applying 2nd evaluation method.

Step 1: Represent the path 'c' in the form $z(t)$

Step 2: Calculate the derivative $\dot{z}(t) = \frac{dz}{dt}$

Step 3: Substitute $z(t)$ & z in $f(z)$ ie $x = x(t)$ & $y = y(t)$

Step 4: Integrate $f(z(t))$ into $\dot{z}(t)$ over 't' from 'a' to 'b'

? Evaluate $\int_c \frac{dz}{z}$ where c is the unit circle taken counterclockwise.

A. $\oint_C \frac{dz}{z}$

We have the curve represented by parameter

$z(t) = x(t) + i y(t)$

$= \cos t + i \sin t$

$= e^{it} \quad , \quad 0 \leq t \leq 2\pi$

for circle,
$x(t) = \cos t$
$y(t) = \sin t$
circle always $(0, 2\pi)$

$\frac{dz}{dt} = \dot{z}(t) = i e^{it}$

$\int_C \frac{dz}{z} = \int_0^{2\pi} \frac{1}{e^{it}} (i e^{it})\, dt$

$= \int_0^{2\pi} i\, dt = [it]_0^{2\pi} = \underline{\underline{2\pi i}}$

? Evaluate $\int_0^{1+2i} \operatorname{Re} z \, dz$ ~~where C is the st.~~
here the path is the line joining 0 & $1+2i$

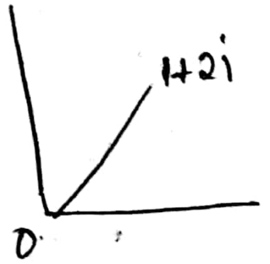
A. $z(t) = t + 2ti$ which is the path $0 \leq t \leq 1$

$\dot{z}(t) = 1 + 2i$

$= \int_0^1 t(1+2i)\,dt = \int_0^1 (t + 2ti)\,dt$

$= \left[\frac{t^2}{2} + t^2 i\right]_0^1 = \frac{1}{2} + i$

$(0,0)$ to $(1,2)$



$(0,0)$ to $(1,2)$

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1}$$

$$\frac{x-0}{1} = \frac{y}{2}$$

$$\underline{\underline{2x = y}}$$

$y = 2x.$

Parametric form of $y = 2x$.

Let $x = t$, $y = 2t$

$$z(t) = x(t) + iy(t)$$
$$= t + i2t$$

$$\frac{dz}{dt} = \dot{z}(t) = 1+2i$$

since real part 0 to 1

Range of t is $0 \le t \le 1$.

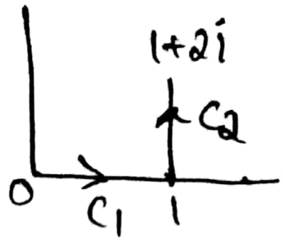
$$\int_0^{1+2i} \mathrm{Re}\, z\, dz = \int_0^1 \mathrm{Re}(z(t)) \frac{dz}{dt} \cdot dt$$

$$= \int_0^1 t(1+2i)\, dt$$

$$= 1+2i \left[\frac{t^2}{2}\right]_0^1$$

$$= \underline{\underline{\frac{1+2i}{2}}}$$

5/10/17 ? Evaluate $\int_c \operatorname{Re} z \, dz$ where c is a curve consisting of c_1 and c_2



A. Here $c = c_1 + c_2$

$$\therefore \int_c \operatorname{Re} z \, dz = \int_{c_1} \operatorname{Re} z \, dz + \int_{c_2} \operatorname{Re} z \, dz$$

Along c_1

$(0,0)$ & $(1,0)$ line joining these points.

$$\frac{x-0}{1} = \frac{y-0}{0}$$

$$y = 0$$

Parametric form, $y = 0$, $x(t) = t$ — in x axis, only varying variable is x.

$$z(t) = x(t) + iy(t)$$

$$= t + 0 = t \qquad \frac{dz}{dt} = 1$$

Range $0 \le t \le 1$.

$$\int_0^{1+2i} \operatorname{Re} z \, dz = \int_0^1 \operatorname{Re}(z(t)) \, \dot{z}(t) \, dt$$

$$= \int_0^1 t \cdot 1 \, dt = \left[\frac{t^2}{2}\right]_0^1 = \frac{1}{2}$$

Along c_2

$(1,0)$ to $(1,2)$ $\qquad \frac{x-1}{0} = \frac{y-0}{2}$

$$2(x-1) = 0$$

$$x - 1 = 0$$

$$\underline{x = 1}$$

Parametric form, $x(t) = 1$, $y(t) = t$

$$z(t) = x(t) + i\,y(t)$$

$$= 1 + it$$

$$\frac{dz}{dt} = i$$

Range ~~(or x)~~ $0 \le t \le 2$ since y varies from 0 to 2.

$$\int_0^{1+2i} \operatorname{Re} z \, dz = \int_0^2 \operatorname{Re} z \cdot \dot{z}(t)\, dt$$

$$= \int_0^2 1 \cdot i \, dt$$

$$= i\,[2-0] = \underline{2i}$$

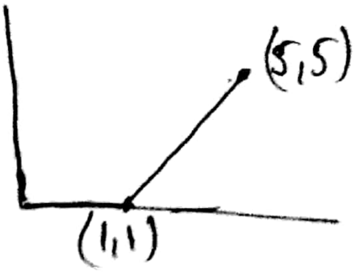
$$\int_C \operatorname{Re} z \, dz = \int_{C_1} \operatorname{Re} z \, dz + \int_{C_2} \operatorname{Re} z \, dz$$

$$= \underline{\frac{1}{2} + 2i}$$

1) ? Evaluate $\int_C \operatorname{Re} z \, dz$ where c is the shortest path from $1+i$ to $5+5i$

2) ? Evaluate $\oint_C \frac{z + \frac{1}{2}}{z}$ where c is the unit circle taken counterclockwise 0 to 2π

1) $\int_C \text{Re } z \, dz$



$(1,1)$ to $(5,5)$

$$\frac{x-1}{5-1} = \frac{y-1}{5-1}$$

$$\frac{x-1}{4} = \frac{y-1}{4}$$

$$4x-4 = 4y-4$$

$$x-1 = y-1$$

$$x-y = -1+1$$

$$x-y = 0$$

$$x = y.$$

Parametric form $x(t) = t$ & $y(t) = t$

$$z(t) = x(t) + iy(t)$$

$$= t + it$$

$$\frac{dz}{dt} = \dot{z}(t) = 1+i$$

Range $1 \leq t \leq 5$

$$\int_C \text{Re } z \, dz = \int_1^5 t(1+i) \, dt$$

$$= \frac{1+i}{2} \left[t^2\right]_1^5 = \frac{1+i}{2} \cdot 24 = \underline{\underline{12(1+i)}}$$

2) $\int_C z + \frac{1}{z}$ where c is the unit circle taken counter clockwise

$0 \leq t \leq 2\pi$

$z(t) = r\cos t + r\sin t\, i$

$= \cos t + \sin t\, i$

$= e^{it}$

$\frac{dz}{dt} = \dot{z}(t) = e^{it} \cdot i$

$\oint z + \frac{1}{z}\, dz = \int_0^{2\pi} \left(e^{it} + \frac{1}{e^{it}}\right) ie^{it}\, dt$

$= \int_0^{2\pi} (ie^{2t} + i)\, dt$

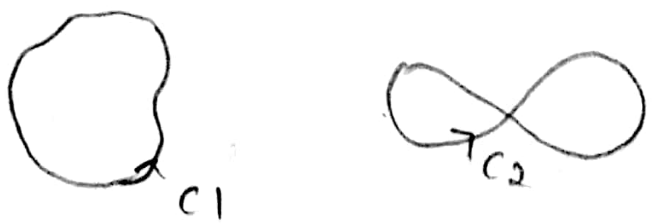
$= \left[\frac{e^{2t}}{2} + t\right]_0^{2\pi} = \frac{e^{4\pi}}{2} + 2\pi i - \frac{1}{2} + 0.$

$= \frac{e^{4\pi} - 1}{2} + 2\pi i$

7/10/17

Simple Closed path

A curve (or path) which is not intersecting or touching itself ~~or~~ it is ~~A~~ is called simple closed path.



C_1 is simple closed path. C_2 not simple closed path.

Simply connected Domain.

A domain D in the complex plane is said to be simple connected domain if every simple closed path c in D encloses only

(1) (2) (3)

(1) simply connected (2) & (3) are multiply connected.

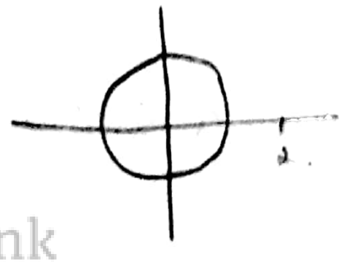
Cauchy's Integral Theorum.

If $f(z)$ is analytic in a simply connected domain, D, then for every simple closed path c in D.

$$\boxed{\oint_C f(z)\,dz = 0}$$

? Evaluate $\oint_C \frac{dz}{z-2}$ where c is $|z| = 1$

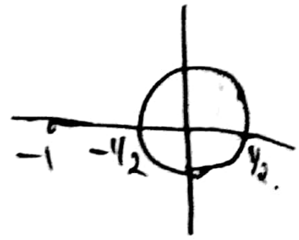
A. The given function has a singularity (the point where the function is not analytic) at the point $z = 2$ lies outside the unit circle $|z| = 1$



∴ The given function is analytic inside $|z| = 1$.
Hence by cauchy's theorum,
$\oint_C \frac{dz}{z-2} = 0$ //

? Evaluate $\oint_C \frac{e^z}{z+1} dz$, $|z| = 1/2$

A. The function has singularity at $z = -1$, which lies outside the circle



∴ The given function is analytic inside $|z| = 1/2$.

∴ By cauchy's theorum.

$$\oint_C \frac{e^z}{z+1} dz = 0 //$$

$|z| = 1/2$
origin point
$z = -1$
$|-1| = 1 > 1/2$
∴ outside

? Evaluate $\oint_C \frac{z^2+5}{z-3} dz$, $|z| = 1$

A. The function has singularity at $z=3$. which lies outside the circle.

∴ The given function is analytic inside $|z| = 1$

∴ By cauchy's theorum.

$$\oint_C \frac{z^2+5}{z-3} dz = 0 //$$

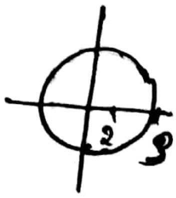
$|z| = 1$
$|3| = 3 > 1$

9/10/19 Cauchy's Integral Formula,

Let $f(z)$ be analytic in a simply connected domain 'D'. Then for any point 'z_0' in 'D' and for any simple closed path 'c' in 'D' that encloses z_0

$$\boxed{\oint_c \frac{f(z)\,dz}{z-z_0} = 2\pi i\, f(z_0)}$$

? Evaluate $\oint_c \frac{dz}{z-2}$, where c is $|z|=3$



A. Here the singularity $z=2$ lies inside $|z|=3$ so we have to apply cauchy's integral formula.

$$\oint_c \frac{f(z)\,dz}{z-z_0} = 2\pi i\, f(z_0)$$

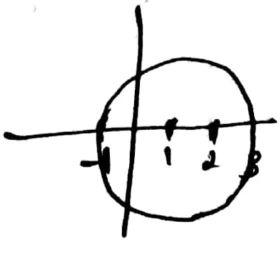
$$\oint_c \frac{dz}{z-2} = 2\pi i\, f(2) \quad \text{where} \quad f(z)=1, \text{analytic}, \; f(2)=1$$

$$= \underline{\underline{2\pi i}}$$

? Evaluate $\oint_c \frac{(z+2)\,dz}{z-2}$, where c is ~~|z|~~ $|z-1|=2$

A. Here the singularity $z=2$ ~~lies~~. ~~lies~~

$|z-1|=2$ centre $(1,0)$ radius $=2$



$|z-1| = |2-1| = 1 < 2$

Here singularity $z=2$ lies inside $|z-1|=2$.

$f(z)=z+2$

$f(2)=4$

$$\oint \frac{(z+2)}{z-2}\,dz = 2\pi i \cdot f(2)$$

$$= \underline{\underline{8\pi i}}$$

? $\oint_c \frac{e^{2z}}{\pi z - i}$, $|z| = 1$

$\pi z - i = 0$
$\pi z = i$
$z = i/\pi$

A. ~~$\pi z - i = 0$~~
~~$\pi z = i$~~ Not of the form of formula.
~~$z = i/\pi$~~

~~$|z| = 1$~~
i
~~divide the with π.~~

$$\oint_c \frac{e^{2z}}{\pi(z - i/\pi)}$$

The point of singularity $z = i/\pi$

$|z| = 1$

$|i/\pi| = 1$

$\frac{1}{\pi}|i| = 1$ $\qquad |i| = 1$

$\frac{1}{3.14} < 1$

$$\oint_c \frac{e^{2z}}{\pi(z - i/\pi)} = \frac{1}{\pi} 2\pi i \, f(i/\pi)$$

$$= \underline{\underline{2ie^{2i/\pi}}}$$

$f(z) = e^{2z}$

$f(i/\pi) = e^{2i/\pi}$

'OR'

$$\oint_c \frac{e^{2z}/\pi}{z - i/\pi} = 2\pi i \, f(i/\pi)$$

$$= \underline{\underline{2i \, e^{2i/\pi}}}$$

$f(z) = e^{2z}/\pi$

$f(i/\pi) = \frac{e^{2i/\pi}}{\pi}$

? Evaluate $\oint_C \frac{3z^2+z}{z^2-1}$, $|z-1| = 1$.

A. ~~Here~~ ~~singularity~~ at ~~z^2-1~~ ~~$z = \pm 1$~~

$$\oint_C \frac{3z^2+z}{(z+1)(z-1)}$$

Resolving into partial fraction.

$$\frac{1}{(z+1)(z-1)} = \frac{A}{z+1} + \frac{B}{z-1}$$

$$= \frac{A(z-1) + B(z+1)}{(z+1)(z-1)}$$

~~$Az - A + Bz + B = 1$~~

$$A(z-1) + B(z+1) = 1$$

put $z = 1$

$2B = 1$

$B = 1/2$

put $z = -1$

$-2A = 1$

$A = -1/2$

$$\frac{1}{(z+1)(z-1)} = \frac{-1/2}{z+1} + \frac{+1/2}{z-1}$$

$$\oint_C \frac{3z^2+z}{(z+1)(z-1)} = \underbrace{\oint_C \frac{(3z^2+z)(-1/2)}{z+1}dz}_{I} + \underbrace{\oint_C \frac{(3z^2+z)\,1/2}{z-1}dz}_{II}$$

$$I = \oint_C \frac{(3z^2+z)(-1/2)}{z+1}\, dz$$

At $z=-1$

$|z-1| = 1$

$|-2| = 2 > 1$ outside

$$\oint_C \frac{(3z^2+z)(-1/2)}{z+1} = 0$$

At $z=1$

$|z-1| = 1$

$|1-1| = 0 < 1$

inside

$$II = \oint_C \frac{(3z^2+z)(1/2)}{z-1}\, dz$$

$f(z) = (3z^2+z)\frac{1}{2}$

$= 2\pi i \cdot f(1)$

$f(1) = \frac{1}{2}(3+1) = 2$

$= 4\pi i$

$$\therefore \oint_C \frac{3z^2+z}{z^2-1} = 4\pi i$$

10/10/19

Another Method

$$\oint_C \frac{3z^2+z}{z^2-1}\, dz, \quad |z-1| = 1$$

$$\oint \frac{3z^2+z}{(z+1)(z-1)}$$

$z=1 \Rightarrow$ inside

$z=-1 \Rightarrow$ outside

$$\oint \frac{\frac{3z^2+z}{z+1}}{z-1}\, dz = 2\pi i f(1)$$

$= 4\pi i$

$f(z) = \frac{3z^2+z}{z+1}$

$f(1) = \frac{3+1}{2} = 2$

? Evaluate the following

1) $\oint_C \frac{z^2 - z + 1}{z-1}\, dz$, $|z| = 1$. $2\pi i$

2) $\oint_C \frac{\sin z}{z + 4iz}\, dz$, $|z - 4 - 2i| = 6.5$

0.

3) $\oint \frac{\tan z}{z - i}\, dz$ where c is the boundary of the triangle with vertices $(0, -1+2i, 1+2i)$ 

$-2\pi \tanh 1$

singularity, if $\frac{\sin z}{\cos z}$ ...

4) $\oint_C \frac{dz}{z^2+4}$ where c is $4x^2 + (y-2)^2 = 4$

4) A. $\oint_C \frac{dz}{z^2+4}$

$\oint_C \frac{dz}{(z+2i)(z-2i)}$

Singularity points are $z = 2i$ & $z = -2i$

put in $4x^2 + (y-2)^2 = 4$

$z = 2i \Rightarrow x + iy = 2i$

$x = 0,\ y = 2.$

$4x^2 + (y-2)^2 = 0 + 0 = 0 < 4$

$\Rightarrow z = 2i$ is inside

$z = -2i \Rightarrow x = 0 \quad y = -2$

$4x^2 + (y-2)^2 = 0 + (-4)^2 = \underline{16} > 4 \Rightarrow z = -2i$ is outside

$\oint_C \frac{\frac{1}{z+2i}}{z - 2i}\, dz = 2\pi i f(2i)$

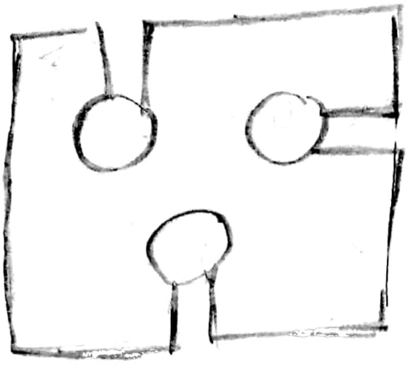
$= 2\pi i \cdot \frac{1}{4i}$

$= \pi/2$ //

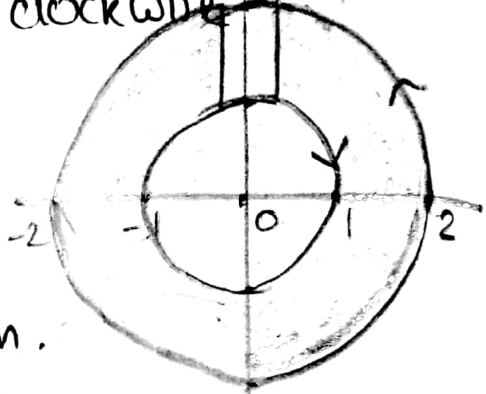
$f(z) = \frac{1}{z+2i}$

$f(2i) = \frac{1}{4i}$

Cauchy's Theorum for Multiply Connected Region



? Evaluate $\oint_c \frac{e^z}{z} dz$ where c consists of $|z|=2$ counterclockwise and $|z|=1$ clockwise

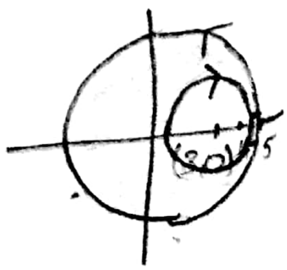


A. Here using cut we can make a multiply connected region into a simply connected domain.
Now the singularity $z=0$ lies outside the simply connected domain.
Hence by Cauchy's theorum,

$$\oint_c \frac{e^z}{z} dz = 0$$

? Evaluate $\oint_c \frac{e^{-z}\sin z}{z-4} dz$ where c consist of $|z|=5$ counter clockwise & $|z-3|=3/2$ clockwise

A. Here using cut we can make a multiply connected region into a simple connected domain.



Singularity at $z=4$ lies outside.
Hence $\oint_c \frac{e^z}{z} dz = 0$

Cauchy's Integral Formula for Derivatives of Analytic Functions.

If $f(z)$ is analytic in a simply connected domain D. It has derivatives of all orders in D which are also analytic in D. Then the values of derivatives at a point z_0 are given by

$$f'(z_0) = \frac{1!}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^2}$$

$$f''(z_0) = \frac{2!}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^3}$$

$$f'''(z_0) = \frac{3!}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^4}$$

$$\boxed{f^n(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}}$$

? Evaluate $\oint_C \frac{\cos z}{z^2}\,dz$ where C is $|z|=1$

A. Singularity at $z=0$

$|z|=1$

$0<1 \Rightarrow z=0$ lies inside.

$$f^n(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

Comparing $n+1=2$

$n=1$

$$f'(z_0) = \frac{1!}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^2}$$

$$= \frac{1!}{2\pi i} \oint_C \frac{\cos z \, dz}{z^2}$$

$z_0 = 0$
$f(z) = \cos z$
$f'(z) = -\sin z$
$f'(z_0) = f'(0)$
$= \sin 0 = 0$

$$\oint_C \frac{\cos z}{z^2} dz = 2\pi i f'(z_0)$$

$$= 0$$

? Evaluate $\oint_C \frac{e^{2z}}{(z+1)^4} dz$ where C is $|z| = 2$

A. Singularity at $z = -1$

$|z| = 1 < 2$ $\quad z = -1$ lies inside.

$$f^{3}(z_0) = \frac{3!}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^4} dz$$

$$= \frac{3!}{2\pi i} \oint_C \frac{e^{2z}}{(z+1)^4}$$

$z_0 = -1$
$f(z) = e^{2z}$
$f'(z) = 2e^{2z}$
~~$f'(z_0) = f'(-1)$~~
~~$2e^{-2}$~~
$f''(z) = 4e^{2z}$
$f'''(z) = 8e^{2z}$
$f'''(-1) = 8e^{-2}$

$$\frac{e^{2z}}{(z+1)^4} = \frac{f^{(3)}(z_0) \cdot 2\pi i}{3!}$$

$$= \frac{8e^{-2} \cdot 2\pi i}{3 \times 2 \times 1} = \frac{8\pi i e^{-2}}{3}$$

? Evaluate $\oint_C \frac{\sin^2 z \, dz}{(z-\pi/6)^3}$, $|z|=1$.

$f'(z) = \sin 2z$
$f''(z) = \cos 2z(2)$
$f''(\pi/6) = 2\cos 2\frac{\pi}{6}$
$= 2(1/2) = 1$

A. Singularity point $z = \pi/6$.

$|\pi/6| = 1$

$|3.14/6| < 1 \quad \therefore z = \pi/6$ lies inside.

$$f''(z_0) = \frac{2!}{2\pi i} \oint_C \frac{\sin^2 z}{(z-\pi/6)^3} dz$$

~~$f(z) = \sin^2 z$~~
~~$f'(z) = 2\sin z.\cos z$~~
~~$f''(z) = 2[\sin z.\sin z + \cos z.\cos z]$~~
~~$= 2[\sin^2 z + \cos^2 z]$~~
~~$f''(\pi/6) = 2\left(\frac{1}{4}+\frac{1}{4}\right)$~~
~~$= 2\frac{2}{8} = \frac{1}{2}$~~

$$\oint_C \frac{\sin^2 z}{(z-\pi/6)^3} dz = \frac{f''(z_0)\, 2\pi i}{2!}$$

$$= \frac{1 . 2\pi i}{2} = \frac{2\pi i}{2} = 2\pi i$$

? Evaluate $\oint_C \frac{\sinh 2z}{(z-1/2)^4}$, $|z|=1$

A. Singularity at $z = 1/2$.

$|z| = |1/2| < 1 \quad z = 1/2$ lies inside.

$$f'''(z_0) = \frac{3!}{2\pi i} \oint_C \frac{\sinh 2z}{(z-1/2)^4} dz$$

$f(z) = \sinh 2z$
$f'(z) = 2\cosh 2z$
$f''(z) = 4\sinh 2z$
$f'''(z) = 8\cosh 2z$
$f'''(1/2) = 8\cosh(2 \cdot 1/2)$
$= 8\cosh 1$

$$\oint_C \frac{\sinh 2z}{(z-1/2)^4} dz = f'''(z_0) \frac{2\pi i}{3!}$$

$$= \frac{8\cosh 1 . 2\pi i}{6}$$

$$= \frac{8}{3}\pi i \cosh 1$$

? Evaluate $\oint_c \frac{\tan \pi z}{z^2} dz$ where c is the ellipse $16x^2+y^2=1$

$\oint_c \frac{\sin \pi z}{(\cos \pi z) z^2} dz.$

$\frac{\sin \pi z}{\cos \pi z} \neq 0$

A. Singularity at $z=0$ & $\cos \pi z=0$

$\cancel{16x^2+y^2}$

at $z=0$ & $z=(2n+1)/2$.

At $z=0$,

$16x^2+y^2 = \underline{0} < 1$ $z=0$ lies inside.

At $z=(2n+1)/2$ $z=1/2, 3/2, \ldots$

$\cancel{\frac{16 (2n+1)^2}{4}} =$

$z=1/2$ $x=1/2 \Rightarrow 16x^2+y^2.$

$16 \cdot \frac{1}{4} = 4 > 1$ lies outside

$$f'(z_0) = \frac{1!}{2\pi i} \oint_c \frac{f(z) dz}{z-z_0} dz.$$

$$= \frac{1!}{2\pi i} \oint_c \frac{\tan \pi z}{z^2} dz$$

$f(z) = \tan \pi z$

$f'(z) = \sec^2 \pi z$

$f'(0) = \underline{\pi}$

$$\oint_c \frac{\tan \pi z}{z^2} dz = f'(z_0) 2\pi i$$

$$= \pi \, 2\pi i$$

$$= \underline{2\pi^2 i}$$

Q Evaluate $\oint_C \dfrac{e^z\,dz}{(z+1)^2(z+2)}$ where C is $|z|=3$

A Here singularities are $z=-1$ & $z=-2$.
Both points lies inside of $|z|=3$
Resolving into partial fraction.

$$\frac{1}{(z+1)^2(z+2)} = \frac{A}{(z+1)} + \frac{B}{(z+1)^2} + \frac{C}{z+2}$$

$$1 = A(z+1)(z+2) + B(z+2) + C(z+1)^2$$

put $z=-2$

~~A(-1)(~~ $C = 1$

put $z=-1$

$B = 1$

put $z=0$

$2A + 2 + 1 = 1$

$2A = 1-3 = -2$.

$A = -1$

$$\frac{1}{(z+1)^2(z+2)} = \frac{-1}{z+1} + \frac{1}{(z+1)^2} + \frac{1}{z+2}$$

$$\oint_C \frac{e^z dz}{(z+1)^2(z+2)} = \underbrace{\oint_C \frac{-e^z dz}{z+1}}_{\text{I}} + \underbrace{\oint_C \frac{e^z}{(z+1)^2}dz}_{\text{II}} + \underbrace{\oint_C \frac{e^z}{z+2}dz}_{\text{III}}$$

I. $\oint_C \dfrac{-e^z dz}{z+1} = 2\pi i f(z_0)$ $\quad f(z) = -e^z$

$= -2\pi i e^{-1}$ $\quad f(-1) = -e^{-1}$

II. $\oint_C \dfrac{e^z}{(z+1)^2} dz = \dfrac{1!}{2\pi i} \oint \dfrac{e^z}{(z+1)^2} = f'(z_0)$ $\quad f(z) = e^z$, $f'(z) = e^z$, $f'(-1) = e^{-1}$

$= 2\pi i f'(z_0)$

$= \underline{\underline{2\pi i e^{-1}}}$

III. $\oint_C \dfrac{e^z}{z+2} dz = 2\pi i f(z_0)$ $\quad f(z) = e^z$, $f(-2) = e^{-2}$

$= 2\pi i e^{-2}$

$\therefore \oint_C \dfrac{e^z dz}{(z+1)^2(z+2)} = -2\pi i e^{-1} + 2\pi i e^{-1} + 2\pi i e^{-2}$

$= 2\pi i(-e^{-1} + e^{-1} + e^{-2})$

$= \underline{\underline{2\pi i e^{-2}}}$

Power Series

The general form of power series is given by $\sum_{n=0}^{\infty} a_n (z-z_0)^n$ where z_0 is called centre of the power series

Taylor Series

Let $f(z)$ be analytic in a domain D and let $z = z_0$ be any point in D then there exist the taylor series with centre z_0 that represents. $f(z)$. This is valid in the largest open disk $|z-z_0| < R$. Then the Taylor series is given by.

$$f(z) = f(z_0) + \frac{(z-z_0)}{1!} f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \cdots + \frac{(z-z_0)^n}{n!} f^n(z_0) + \cdots$$

When the centre $z_0 = 0$ then the representation of $f(z)$ as a power series is called Maclaurin series

? Represent $f(z) = \sin z$, $z = \pi/2$ as Taylor series.

$f(z) = \sin z$

$f(\pi/2) = \sin \pi/2 = 1$

$f'(z) = \cos z \Rightarrow f'(\pi/2) = \cos \pi/2 = 0.$

$f''(z) = -\sin z \Rightarrow f''(\pi/2) = -\sin \pi/2 = -1$

$f'''(z) = -\cos z = \underline{\underline{0}}$

$f^{iv}(z) = +\sin z = \underline{\underline{1}}$

$$f(z) = f(z_0) + \frac{(z-z_0)}{1!} f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \cdots$$

$$\sin z = 1 + \frac{(z-\pi/2)}{1!} \cdot 0 + \frac{(z-\pi/2)^2}{2!} \cdot -1 + 0 + \frac{(z-\pi/2)^4}{4!} \cdot 1 + \cdots$$

$$= 1 - \frac{(z-\pi/2)^2}{2!} + \frac{(z-\pi/2)^4}{4!} - \cdots$$

? Represent $f(z) = \cos z$ as a maclaurin series.

$$f(z) = f(z_0) + \frac{(z-z_0)}{1!} f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \cdots$$

$f(z) = \cos z \quad f(0) = 1$

$f'(z) = -\sin z \quad f'(0) = 0$

$f''(z) = -\cos z \quad f''(0) = -1$

$f'''(z) = \sin z \quad f'''(0) = 0$

$f^{iv}(z) = \cos z \quad f^{iv}(0) = 1$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots\dots$$

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? Expand $f(z) = ze^{2z}$ as a Taylor series about $z = -1$

A. $f(z) = ze^{2z}$ $\qquad f(-1) = -e^{-2}$

$f'(z) = 2ze^{2z} + e^{2z} \cdot 1$ $\qquad f'(-1) = -2e^{-2} + e^{-2} = -e^{-2}$

$f''(z) = 4ze^{2z} + \cancel{2}e^{2z} \cdot 1 + 2e^{2z}$ $\qquad f''(-1) = -4e^{-2} + 2e^{-2} + 2e^{-2} = 0$

$f'''(z) = 8 \cdot ze^{2z} + 4e^{2z} + 4e^{2z} + 4e^{2z}$ $\qquad f'''(-1) = -8e^{-2z} + 12 \cdot e^{-2} = 4e^{-2z}$

$$f(z) = f(z_0) + \frac{(z-z_0)}{1!} f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \frac{(z-z_0)^3}{3!} f'''(z_0) + \dots$$

$$f(z) = -e^{-2} + \frac{(z+1)}{1!}(-e^{-2}) + \cancel{\frac{(z+1)^2}{2!}(-e^{-2})} + \frac{(z+1)^3}{3!} 4e^{-2z}$$

$$= -e^{-2}\left[1 + \frac{(z+1)}{1!} + \cancel{\frac{(z+1)^2}{2!}} - \frac{4(z+1)^3}{3!}\right]$$

? Find the maclaurin series of $f(z) = \cos(2z^2)$

$f(z) = \cos(2z^2)$

$f'(z) = -\sin(2z^2) \cdot 4z \qquad 16z^2$

$f''(z) = -[\sin(2z^2) \cdot 4 + 4z \cdot \cos(2z^2) \cdot 4z]$

$f'''(z) = -[\cos(2z^2) \cdot 16z + \overset{16\times 2}{\cancel{4z \cdot -\sin(2z^2) \cdot 4z + 4\cos(2z^2) \cdot 4}}$

$+ 16[z^2 \cdot -\sin(2z^2) \cdot 4z + \cos(2z^2) \cdot 2z]]$

$f^{iv}(z) = -(\cos(2z^2) \cdot 16 + 16 \cdot z - \sin(2z^2) \cdot 4z) + 16[z^2 \cdot \cos(2z^2) \cdot 4z + \sin(2z^2) \cdot 2z$

$+ \cos(2z^2) \cdot 2 + 2z \cdot -\sin(2z^2) \cdot 4z]$

$f(0) = 1$

$f'(0) = 0$

$f''(0) = 0$

$f'''(0) = 0$

$f^{iv}(0) = -16 + 32 = \underline{\underline{16}}$

? Expand $f(z) = \log(1+z)$ in the region $|z|<1$

A. ~~f(z) =~~ $|z|<1$

$|z-z_0|<R$

$\therefore z_0 = 0 \quad , R=1$

$f(z) = \log(1+z)$

$f'(z) = \frac{1}{1+z} = (1+z)^{-1}$

$f''(z) = -1(1+z)^{-2}$

$f'''(z) = 2(1+z)^{-3}$

$f(0) = \log 1 = 0$

$f'(0) = 1^{-1} = 1$

$f''(0) = -1(1)^{-2} = -1$

$f'''(0) = \frac{2}{1^3} = 2.$

$$f(z) = f(z_0) + \frac{(z-z_0)}{1!} f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \frac{(z-z_0)^3}{3!} f'''(z_0) + \dots$$

$$f(z) = \frac{z}{1!} - \frac{z}{2!} + \frac{2z^3}{3!} + \dots$$

Important Special Taylor Series

① $e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots \dots \quad , |z|<\infty$

② $\sin z = \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \pm \dots \quad , |z|<\infty$

③ $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots \quad , |z|<\infty$

④ $\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots$

(5) $\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$

(6) $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} + \cdots$

(7) $\frac{1}{1-z} = (1-z)^{-1} = 1 + z + z^2 + \quad , \quad |z| < 1$

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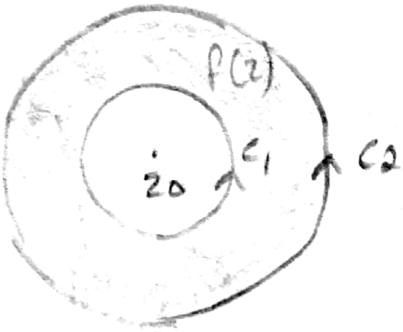
Laurent's Series



Let $f(z)$ be analytic in a domain containing two concentric circles C_1 & C_2 with centre z_0 and the annulus b/w them. Then $f(z)$ can be represented by Laurent's series

$$f(z) = \left[a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\right] + \left[a_{-1}(z-z_0)^{-1} + a_{-2}(z-z_0)^{-2} + \cdots\right]$$

$$= \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} a_{-n}(z-z_0)^{-n}$$

where $a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$

$$a_{-n} = \frac{1}{2\pi i} \oint_C \frac{f(z)\,dz}{(z-z_0)^{-n+1}}$$

Note :-

→ The Laurent's series can also be represented as $f(z) = \left[a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\right] + \left[b_1(z-z_0)^{-1} + b_2(z-z_0)^{-2} + \cdots\right]$

where $b_n = a_{-n}$

→ The Laurent's series consists of positive & negative powers of $(z-z_0)$.

The series containing positive powers of $(z-z_0)$ is called Taylor part.

The series containing negative powers of $(z-z_0)$ is called principle part / singular part.

? Expand $f(z) = \cancel{(1-z)(}\ \frac{1}{z(z-1)}$ as Laurent's series about a point $z=1$

A. $f(z) = \frac{1}{z(z-1)}$

$$1 = \frac{A}{z} + \frac{B}{z-1}$$

$1 = A(z-1) + Bz$

put $z=0$

$-A = 1$

$\underline{A = -1}$

put $z=1$

$\underline{B = 1}$

$$f(z) = \frac{-1}{z} + \frac{1}{z-1}$$

$$= \frac{-1}{z-1+1} + \frac{1}{z-1}$$

$$= \frac{-1}{1+(z-1)} + \frac{1}{z-1}$$

$$= -\left[1+(z-1)\right]^{-1} + \frac{1}{z-1}$$

$$= -\left[1-(z-1)+(z-1)^2-(z-1)^3+\dots\right] + \frac{1}{z-1}$$

? Expand $f(z) = \dfrac{z}{(z-1)(2-z)}$ as a Laurent's series in the regions i) $|z|<1$ ii) $1<|z|<2$ iii) $|z|>2$ iv) $|z-1|>1$ v) $0<|z-2|<1$

A. $f(z) = \dfrac{z}{(z-1)(2-z)} = \dfrac{A}{z-1} + \dfrac{B}{2-z}$

$$z = A(2-z) + B(z-1)$$

put $z=2$

$\underline{B=2}$

put $z=1$

$\underline{A=1}$

$$f(z) = \frac{z}{(z-1)(z-2)} = \frac{1}{z-1} + \frac{2}{2-z}$$

$$\frac{1}{1-x} = (1-x)^{-1} = 1+x+x^2+\dots$$

$$\frac{1}{1+x} = (1+x)^{-1} = 1-x+x^2-x^3+\dots$$

i) $|z|<1$

$$f(z) = \frac{1}{z-1} + \frac{2}{2-z}$$

$$= \frac{1}{-(1-z)} + \frac{2}{2(1-z/2)}$$

Make it as 1- form. / 1+ form.
Binary expation is valid only when 1- thing ie if $1-z$; then $|z|<1$. For the binomial expation this condition satisfies.

In 1st term $\frac{1}{-(1-z)}$ clearly $|z|<1$.

In 2nd term $\frac{2}{2(1-z/2)}$ $\frac{|z|}{2}<1$ {if $|z|<1$ its 1/2 will always be <1

In this term if we take $-z$ outside $\frac{2}{-z(1-\frac{2}{z})}$ $\left|\frac{2}{z}\right|<1; |z|>2$
So we cannot do that.

$$f(z) = \frac{1}{-(1-z)} + \frac{2}{2(1-z/2} = -[1+z+z^2+z^3+\dots] + \left[1+\frac{z}{2}+\frac{z^2}{4}+\dots\right]$$

ii) $1<|z|<2$.

$1<|z|$ & $|z|<2$

$\frac{1}{|z|}<1$ $\qquad \frac{|z|}{2}<1$ $\qquad$ {find what is <1}

$$f(z) = \frac{1}{z-1} + \frac{2}{2-z} = \frac{1}{z(1-1/z)} + \frac{2}{2(1-z/2)}$$

$\frac{1}{|z|}<1$ $\qquad \frac{|z|}{2}<1$

$$= \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right] + \left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right]$$

$$= \left[\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots\right] + \left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right]$$

iii) $|z| > 2$

$1 > \frac{2}{|z|}$ $\qquad \frac{2}{|z|} < 1$

$\frac{2}{|z|} < 1$

$\frac{1}{|z|}$ is $\frac{1}{2}$ of $\frac{2}{|z|}$ $\therefore \frac{1}{|z|} < 1$

$$f(z) = \frac{1}{z-1} + \frac{2}{2-z} = \frac{1}{z(1-1/z)} + \frac{2}{-z(1-2/z)}$$

$$= \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right] + \frac{-2}{z}\left(1+\frac{2}{z}+\frac{2^2}{z^2}+\cdots\right]$$

iv) $|z-1| > 1$

$1 > \frac{1}{|z-1|}$

$\frac{1}{|z-1|} < 1$

$$f(z) = \frac{1}{z-1} + \frac{2}{2-z} = \frac{1}{z-1} + \frac{2}{-(z-2)}$$

$$= \frac{1}{z-1} + \frac{2}{-(z-1-1)} = \frac{1}{z-1} + \frac{2}{(z-1)\left(1-\frac{1}{(z-1)}\right)}$$

$$= \frac{1}{z-1} - \frac{2}{z-1}\left(1-\frac{1}{z-1}\right)^{-1}$$

$$= \frac{1}{z-1} - \frac{2}{z-1}\left[1+\frac{1}{z-1}+\frac{1}{(z-1)^2}+\frac{1}{(z-1)^3}+\cdots\right]$$

v) $0 < |z-2| < 1$

$0 < |z-2|$

$|z-2| < 1$

$$f(z) = \frac{1}{z-1} + \frac{2}{2-z}$$

$$= \frac{1}{z-1-1+1} + \frac{2}{-(z-2)} = \frac{1}{(z-2)+1} - 2\,\frac{1}{z-2}$$

$$= \frac{1}{(z-2)\left(1+\frac{1}{z-2}\right)} - \frac{2}{z-2}$$

$$= \frac{1}{z-2}\left[1-\frac{1}{z-1}+\frac{1}{(z-2)^2}+\frac{1}{(z-3)^3}+\cdots\right] - \frac{2}{z-2} \;//$$

21/10/2017 Module 3 (conti...)

? Expand $f(z) = \frac{\cos z}{z^4}$ as a Laurent series where $0 < |z| < R$.

A. $f(z) = \frac{1}{z^4}\left[1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots\right]$

$= \frac{1}{z^4} - \frac{1}{z^2(2!)} + \frac{1}{4!} - \frac{z^2}{6!} + \cdots$

? Expand $f(z) = z-3 \sin \frac{1}{z+2}$ about $z = -2$

A. $f(z) = (z-3) \sin \frac{1}{z+2}$

$= (z-3) \sin (z+2)^{-1}$

$= (z-3+2-2) \sin (z+2)^{-1}$

$= [(z+2)-5] \sin \frac{1}{z+2}$

$= (z+2) \sin \frac{1}{z+2} - 5 \sin \frac{1}{z+2}$

$= (z+2)\left[\frac{1}{z+2} - \frac{1}{(z+2)^3}\frac{1}{3!} + \frac{1}{(z+2)^5}\cdot\frac{1}{5!} - \cdots\right]$

$- 5\left[\frac{1}{z+2} - \frac{1}{(z+2)^3 3!} + \cdots\right]$

? Expand $f(z) = \frac{z^2-1}{(z+2)(z+3)}$ as a Laurent series

1) $|z| < 3$ 2) $2 < |z| < 3$

A. Since numerator & denominator same degree so reduce degree of numerator.

$$f(z) = \frac{z^2-1}{z^2+5z+6}$$

$$\begin{array}{r} 1 \ (Q) \\ (D)\ z^2+5z+6 \overline{) z^2-1} \\ z^2+5z+6 \\ \hline -5z-7 \ (R) \end{array}$$

$$= Q + \frac{R}{D}$$

$$= 1 - \frac{(5z+7)}{z^2+5z+6}$$

$$= 1 - \frac{(5z+7)}{(z+2)(z+3)} \quad \text{—— (1)}$$

$$\frac{5z+7}{(z+2)(z+3)} = \frac{A}{z+2} + \frac{B}{z+3}$$

$$5z+7 = A(z+3) + B(z+2)$$

put $z = -2$.

$-10+7 = A$

$\underline{\underline{A = -3}}$

put $z = -3$.

$-15+7 = -B$.

$\underline{\underline{B = 8}}$

$$\frac{5z+7}{(z+2)(z+3)} = \frac{-3}{z+2} + \frac{8}{z+3}$$

i) $|z| < 3$

$\frac{|z|}{3} < 1$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1 + \frac{3}{2(z/2+1)} - \frac{8}{3(1+z/3)}$$

$$= 1 + \frac{3}{2}\left(\frac{z}{2}+1\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left[1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right]$$

2) $2 < |z| < 3 \Rightarrow 2 < |z| \text{ \& } |z| < 3 \Rightarrow \frac{2}{|z|} < 1 \text{ \& } \frac{|z|}{3} < 1$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(\frac{z}{3}+1\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(\frac{z}{3}+1\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \frac{4}{z^2} - \frac{8}{z^3} + \frac{16}{z^4} - \cdots\right) - \frac{8}{3}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} \cdots\right)$$

$$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \frac{4}{z^2} - \frac{8}{z^3} + \cdots\right) - \frac{8}{3}\left(1 - \frac{z}{3} + \frac{z^2}{4} - \frac{z^3}{9} + \cdots\right)$$

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? Find the laurent's series.

$$f(z) = \frac{z+4}{(z+3)(z-1)^2} \qquad a)\ 0 < |z-1| < 4 \qquad b)\ |z-1| > 4$$

? $f(z) = \dfrac{7z-2}{z(z+1)(z-2)}$; $1 < |z+1| < 3$

? $f(z) = \dfrac{1}{z^3+3z+2}$; $1 < |z| < 2$

? $f(z) = \dfrac{2z^2+9z+5}{z^3+z^2-8z+2}$ as a Taylor series about $z=6$.