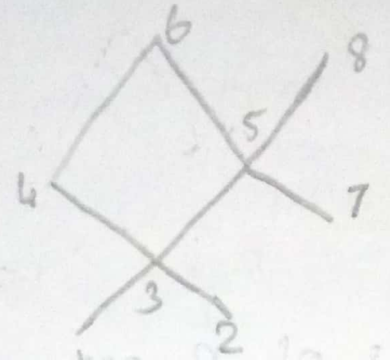


Upper bound is 5, 6.

Sup (B) = 5

Lower bound = 3

LATTICES

A lattice is a poset in which every pair has a supremum & infimum

Join

Consider a poset L under the order ≤ Let a, b ∈ L. Then supremum of a & b is called join of a & b and is denoted by $a \oplus b$ or $a \vee b$

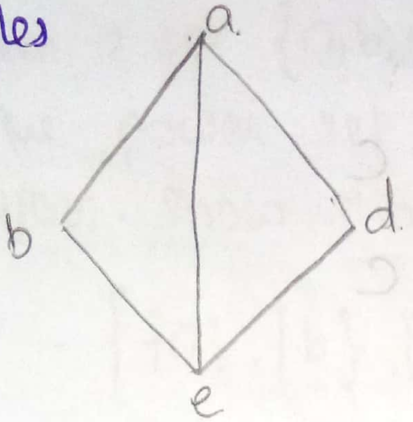
Meet

Consider the poset L under the order ≤. Let a, b ∈ L. The infimum of a, b is called the meet of a & b is denoted by $a * b$ or $a \wedge b$.

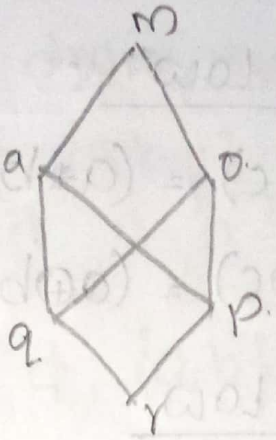
Remark:

The lattice is denoted by $(L, *, \oplus)$

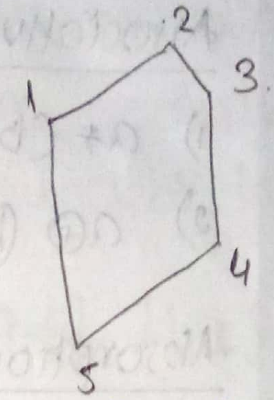
15/9/17 Examples



fig(1)



fig(3)



fig(2)

→ from figure (1), every pair {a,b}, {b,e}, {a,e}, {a,c}, {c,e} {e,d}, {a,d}, {a,e}. All have supremum & infimum. and hence figure(1) is a lattice.

→ In figure(2) every pair has infimum & supremum. So fig (2) is a lattice.

→ figure (3) is not a lattice if we are considering pairs {p,q} it has upper bounds. n, o & m, but no supremum violates the condition.

Some Properties of Lattice.

Idempotent law.

1) $a * a = a$

2) $a \oplus a = a$

where $a \in L$.

Commutative Law.

1) $a \oplus b = b \oplus a$

2) $a * b = b * a$

where $a, b \in L$.

Associative Law

1) $a * (b * c) = (a * b) * c$

2) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$

Absorption Law

1) $a * (a \oplus b) = a$

2) $a \oplus (a * b) = a.$

Distribution Law

1) $a * (b \oplus c) = (a * b) \oplus (a * c)$

2) $a \oplus (b * c) = (a \oplus b) * (a \oplus c)$

The Bounded Lattice

A littice L is called a bounded lattice if it has a greatest element and a least element from the above figures (1) & (2) are bounded. lattices.

It has no greatest element but have least element and hence not a bounded lines.

Remark:-

The greatest element is denoted by either 'I' or 1.

The least element is denoted by 'o'.

? Consider a set $\{a,b,c\}$ draw the Hasse diagram for the power set of $\{a,b,c\}$ with odering inclusion. Show that it is a bounded lattic.

A. $P = \{\{a\},\{b\},\{c\},\phi,\{a,b\}\{a,c\}\{b,c\}$

Complete Lattices

A lattice is said to be complete if every nonempty subset has a supremum & infimum.

Remark:-

Every finite lattices must be complete.

Ex: figure (1) & (2) drawn above are complete lattice.

Distributive Lattice.

The lattice is called distributive lattice if it satisfies the distributive law.

ie $a, b, c \in L$. then

1) $a * (b \oplus c) = (a * b) \oplus (a * c)$
2) $a \oplus (b * c) = (a \oplus b) * (a \oplus b)$

16/9/17.

GLB	a	b	c	d	e
a	a	b	e	d	e
b	b	b.	e	e	e
c	e	e.	c	e.	e
d	d	e	e	d	e
e	e	e	e	e	e

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{a, b, c, d, e}

{a, b, c} {a, b, d} {b, c, e}
{a, c, d} {a, b, e} {b, d, e}
{a, d, e} {b, c, d} {c, d, e}

	b⊕c	a*(b⊕c)	a*b	a*c	(a*b)⊕(a*c)
{b, c, d}	c⊕d a				e

First to check the distributive lattice we have to first check whether it is a lattic and secondly whether every subset of three element satisfies distributive law.

Checking for lattice

*GLB	a	b	c	d	e
a	a	b	c	d	e
b	b	b	e	e	e
c	c	e	c	e	e
d	d	e	e	d	e
e	e	e	e	e	e

⊕LUB	a	b	c	d	e
a	a	a	a	a	a
b	a	b	a	a	b
c	a	a	c	a	c
d	a	a	a	d	d
e	a	b	c	d	e

From the tables it is clear that every pair has a supremum & infimum & hence a lattice

Now, according to distributive law.

$\forall\ a, b, c \in L$.

$$a * (b \oplus c) = (a * b) \oplus (a * c)$$

$$a \oplus (b * c) = (a \oplus b) * (a \oplus c)$$

Consider the subsets.

$\{a,b,c\}$ $\{a,b,d\}$ $\{a,c,d\}$ $\{e,d,c\}$ $\{e,b,d\}$ $\{e,c,d\}$

$\{b,c,d\}$ $\{e,d,a\}$ $\{e,c,a\}$ $\{e,d,a\}$

Consider the $\{b,c,d\}$ start with disjoint sets first.

We have to show that.

$$b * (c \oplus d) = (b * c) \oplus (b * d)$$

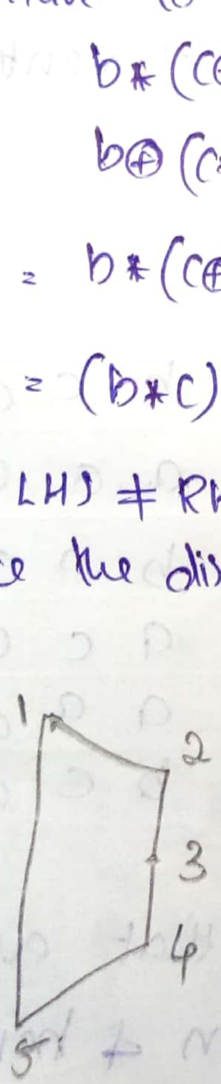
$$b \oplus (c * d) = (b \oplus c) * (b \oplus d)$$

$LHS = b * (c \oplus d) = b * a = b$

$RHS = (b * c) \oplus (b * d) = e \oplus e = e.$

$\therefore LHS \neq RHS$

Hence the distributive law fails.



26/9/17

Complemented Lattice

An element 'a' ∈ L were L is a lattice, is said to have a complement b then $a * b = 0$ and $a \oplus b = I$ (one) ie, Infimum(a,b) = least element & sup(a,b) = greatest element.

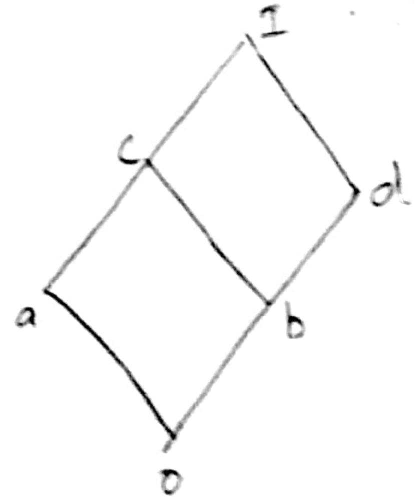
A complemented lattice is a bounded lattice with every element has atleast one complement.

Remark :

1) The complement of least element (zero) is the greatest element (one)
Also the complement of greatest element (one) is the least element (zero)

2) The complement is always symmetric functions ie. if 'a' is a complement of 'b' then, 'b' is the complement of 'a' also.

? Determine whether the given Hasse diagram is a complemented lattice.



A. This is a lattice since every pair has infimum & supremum

This is a bounded lattice since it has greatest & least element.

$a * b = 0$

$a \oplus b = c \neq 1$

b not complement of a

~~a*b~~

The complement of a is d

since $a * d = 0$ & $a \oplus d = 1$

The complement of d is a (by symmetry)

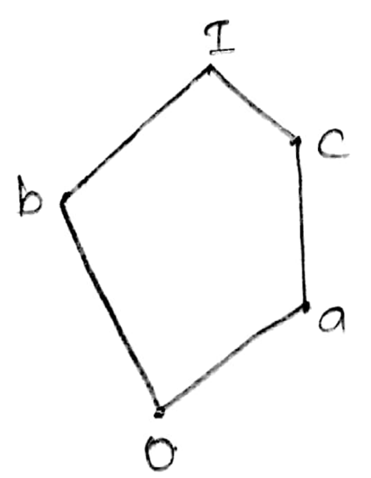
$c * d = 0$ $\qquad c * a = a$ $\qquad c * d \neq 0$

$c \oplus b = c$

$\Rightarrow$ c has no complement.

Since c has no complement it is not a complemented lattice.

? Check whether the given hasse diagram is a complemented lattice.



i) Checking for a lattice

This is a lattice.

ii) Bounded lattice

This is a bounded lattice since it has greatest & lowest element.

iii) Checking for complemented

$a * b = 0$ $\quad$ $b * c = a$

$a \oplus b \neq I = c$ $\quad$ $b \oplus c = I$.

It is not a complemented lattice.

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Sub Lattice

Let $(L, *, \oplus)$ be a lattice and $S \subseteq L$, then $(S, *, \oplus)$ is a sublattice if and only if S is closed under the operations of $*$ and $\oplus$ of L

ie, A non-empty set S of the lattice L is said to be a sublattice if S itself is a lattice w.r.t the operations of L

Remark

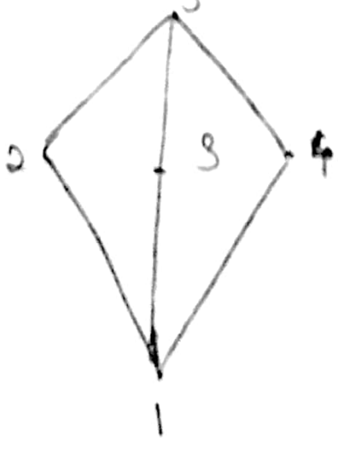
For finding sublattice we have to check.

i) for the poset & draw Hasse diagram

ii) Existance of supremum & infimum ∀ pair in the subset.

iii) The supremum & infimum element of each pair should belong to the subset under consideration

Q. Consider the lattice $L = \{1, 2, 3, 4, 5\}$ given by the Hasse diagram. Determine all the possible sublattices with three or more elements.



A. The sub lattices are $\{1,4,5\}$ $S=4$ $I=1$ $S=5$ $I=1$

This is a sublattice since every pair $(1,4),(1,5),(4,5)$ have a supremum & infimum. & these values belongs to set $1,4,5$.

The other sublattices are $\{1,2,5\}$, $\{1,3,5\}$, $\{1,2,3,5\}$, $\{1,3,4,5\}$, $\{1,2,4,5\}$, $\{1,2,3,4,5\}$

Remark.

$\{2,3,4\}$ is not a sublattice since for the pair $(2,3)$ supremum is 5 & infimum is 1 but they donot belong to $\{2,3,4\}$.

Lattice Homomorphism

Let $(L,*,\oplus)$ & ~~$(S,*,\oplus)$~~ $(S,\wedge,\vee)$ be two lattices then a mapping $g: L \to S$ is said to be a lattice homomorphism if it satisfies. for any $a,b \in L$

1) $g(a*b) = g(a) \wedge g(b)$
2) $g(a\oplus b) = g(a) \vee g(b)$

22/9/2017.

Boolean Algebra

A boolean algebra is a complemented distributed lattice and is denoted by $(B, *, \oplus, ', 0, 1)$

Since B is a lattice it satisfies all the properties of the lattice ie,

for any $a, b, c \in B$.

i) Idempotent law

$$a \oplus a = a$$
$$a * a = a.$$

ii) Commutative law

$$a * b = b * a$$
$$a \oplus b = b \oplus a$$

iii) Associative law

$$(a*b) * c = a * (b*c)$$
$$(a \oplus b) \oplus c = a \oplus (b \oplus c)$$

iv) Absorption law

$$a \oplus (a*b) = a.$$
$$a * (a \oplus b) = a$$

Since B is distributive, it satisfy the condition Distributive law.

$$a * (b \oplus c) = (a*b) \oplus (a*c)$$
$$a \oplus (b*c) = (a \oplus b) * (a \oplus c)$$

Since B is bounded, it satisfies the properties,

i) $0 \leq a \leq I$

ii) $a \oplus 0 = a$ & $a * 0 = 0$.

iii) $a \oplus I = I$ & $a * I = a$

Since B is complemented, it satisfies

i) $a \oplus a' = I$, $a * a' = 0$

ii) $I' = 0$, $0' = I$

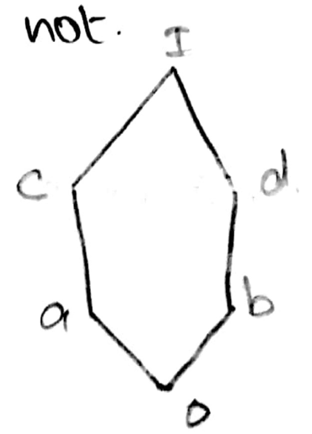
iii) $(a \oplus b)' = a' * b'$, $(a * b)' = a' \oplus b'$

Remark:

> If a finite lattice L does not contain 2^n elements for some positive integer n then L cannot be boolean algebra.

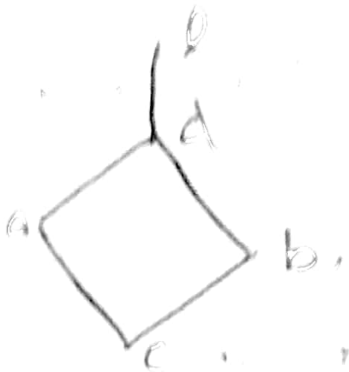
> For a distributed lattice, the complements are unique.

? Determine whether the following are boolean algebra or not.



A. This is not a boolean algebra since the number of elements = 6. cannot be written in the form 2^n where n is +ve integer.

?



This is not a boolean algebra since the no: of elements 5 cannot written in form of 2^n.

?

A. Since This a lattice, we can check whether it is complemented & distributed lattice.
This is a bounded lattice since it has least & greatest element.

$a * b = b \neq 0$

$a * 0 = 0$

$a \oplus 0 = a \neq I$

Here $0' = I$, $I' = 0$, but we cannot find complements for a & b Hence it is not a complemented lattice.

Sub Algebra

Consider the boolean algebra $(B, *, \oplus, ', 0, I)$ and A is said to be a sub algebra of B, then it should satisfy the conditions.

i) $A \subseteq B$

ii) A itself should be a boolean algebra.

Remark

Sub algebra can be also called as sub boolean algebra.

Direct Product

Let $(B_1, *_1, \oplus_1 ', 0_1, I_1)$ & $(B_2, *_2, \oplus_2, '', 0_2, I_2)$ be two different boolean algebra. The direct product two boolean algebras is defined to be a boolean algebra denoted by $(B_1 \times B_2, *_3, \oplus_3, ''', 0_3, I_3)$ and is defined by. For any $(a_1, b_1), (a_2, b_2) \in B_1 \times B_2$ it satisfies the condition.

i) $(a_1, b_1) *_3 (a_2, b_2) = (a_1 *_1 a_2, b_1 *_2 b_2)$

ii) $(a_1, b_1) \oplus_3 (a_2, b_2) = (a_1 \oplus_1 a_2, b_1 \oplus_2 b_2)$

iii) $(a_1, b_1)''' = (a_1', b_1'')$

iv) $0_3 = (0_1, 0_2)$ & $I_3 = (I_1, I_2)$

Boolean Homomorphism

Let $(B, *, \oplus, ', 0, I)$ & $(P, \wedge, \vee, -, \alpha, \beta)$ be two boolean algebras. A mapping $f : B \to P$ is called a boolean homomorphism if all the operations of boolean algebra are reserved. ie, for any $a, b \in B$ $f(a*b) = f(a) \wedge f(b)$.

$f(a \oplus b) = f(a) \vee f(b)$

$f(a') = \overline{f(a)}$

$f(0) = \alpha$

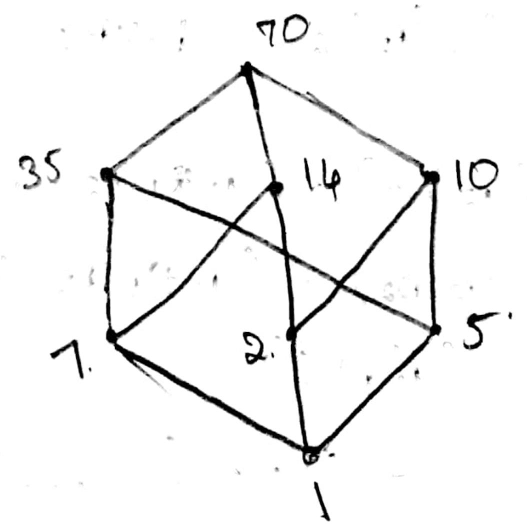
$f(I) = \beta$

D_n $n>0$
↓
collection of all divisors

? Consider the boolean algebra D_{70} whose Hasse diagram is given. List out the sub algebra of D_{70}.

A. $D_{70} = \{1, 2, 5, 7, 10, 14, 35, 70\}$

The Hasse diagram is



Here set of 3 doesn't work (usually take subset of 3 ele) Do definition of boolean algebra in this set

The subalgebra is A = {1, 7, 10, 70}

second algebra. {1, 2, 35, 70}