Module - VI

Eigen Values And Eigen Vectors

⇒ Let A be a square matrix of order n. Then $|A - \lambda I| = 0$ is called the characteristic equation of A.

⇒ The roots of the characteristic eqn is called the Eigen values or characteristic values or Latent roots.

⇒ The set of Eigen values is called spectrum of A.

⇒ The set of all solns to $Ax = \lambda x$ is called the Eigen space of A and is denoted by $E(\lambda)$.

⇒ The eigen values of A and A' are same (A' – transpose)

⇒ If λ is an Eigen value of A then λ^n is an eigen value of A^n

⇒ If λ is an Eigen value of A then $k\lambda$ is an Eigen value of kA and $\lambda - k$ is an Eigen value of $A - kI$

⇒ If λ is an Eigen value of a non-singular matrix then $1/\lambda$ is an eigen value of A^{-1} and $\frac{|A|}{\lambda}$ is the eigen value of adjoint of A

⇒ Eigen values of triangular matrices and diagonal matrices are its ~~diagonal~~ diagonal matrix

⇒ The sum of eigen values of a matrix is the sum of its diagonal elements

⇒ The product of Eigen values

⇒ The eigen values of a symmetric matrix are real and of a skew symmetric matrix are purely imaginary or zero.

? Find the Eigen values and Eigen vectors of the matrix

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

As $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = 0$.

$\lambda^2 -$ trace $A\lambda + |A| = 0$

$\lambda^3 - 2\lambda +$

$(1-\lambda)^2 - 4 = 0$

$1 - 2\lambda + \lambda^2 - 4 = 0$.

$\lambda^2 - 2\lambda - 3 = 0$.

$(\lambda-3)(\lambda+1) = 0$.

$\lambda = 3, -1$

To find eigen vectors.

Case I, $\lambda = 3$.

$[A - \lambda I] = \begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix}$

$= \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$ $R_1 \to R_1/_{-2}$.

$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$

which is in eshlon form.

$AX = \lambda X$

$\Rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 - x_2 = 0$ $\quad x_1 = f(x_2)$

Put $x_2 = a$ $\quad x_1 = a = x_2$

$x_1 = a$

Eigenvector $x_1 = \begin{bmatrix} a \\ a \end{bmatrix}$

$= a\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigenspace $E(\lambda) = E(3) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Case II $\lambda = -1$

$[A - \lambda I] = \begin{bmatrix} 1-^-1 & 2 \\ 2 & 1-^-1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$\sim = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ $R_1 \to R_1/2$.

$= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$

which is in eshloo form.

$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 + x_2 = 0$.

$x_2 = a$

$x_1 = -a$

Eigen $x_2 = \begin{bmatrix} -a \\ a \end{bmatrix}$

$= a\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Eigenspaces,

$E(\lambda_2) = E(-1) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

? Find Eigen vectors and Eigen values of matrix $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

: $[A - \lambda I] = 0$ $\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$

$= (2-\lambda)(3-\lambda)(2-\lambda) - 2\cancel{x}(2+\lambda) - 1(2-\lambda$

$+ 1(2-(3-\lambda)) - 2$

$= (2-\lambda)(\lambda^2 - 5\lambda + 4) - 2(1-\lambda) + 1(\lambda - 1)$.

$= 2\lambda^2 - 10\lambda + 8 - \lambda^3 + 5\lambda^2 - 4\lambda -$.

trace of A $= \cancel{7}2+3+2$

$\lambda^3 - (\text{trace of } A)\lambda^2 + (\text{sum of coefficent})$

$\lambda - |A| = 0$.

$\lambda^2 - (\text{trace of } A)\lambda +$

$|A| = 0$.

Sum of coeffcnt.

$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \cancel{2}(3\times 2) - (2) + 3(4-1) + 2(6-2)$

$= 4 + 3\times 1 + 4$.

$$\begin{matrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{matrix}$$

$|A| = \begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 2(4) + 2\times(1) + 1\times -1$

$= 8 + 2 - 1$

$= 5$

$\lambda = 5, 1, 1$

$\lambda = 5$

$[A - \lambda I] = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & [illegible] \end{bmatrix}$ $R_1 \leftrightarrow R_2$.

$\begin{bmatrix} 1 & -2 & 1 \\ 0 & -4 & 4 \\ 0 & 4 & -4 \end{bmatrix}$ $R_2 \to R_2 + 3R_1$, $R_3 \to R_3 - R_1$

$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 4 & -4 \end{bmatrix}$ $R_2 \to R_2/-4$

$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 - 4R_2$

$Ax = \lambda X$

$\Rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$x_1 - 2x_2 + x_3 = 0$.

$x_2 - x_3 = 0$.

Put $x_3 = a$.

$x_2 = a$.

$x_1 = a$

$X_1 = \begin{bmatrix} a \\ a \\ a \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$E(5) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Case ii $\lambda = 1$

$[A - \lambda I] = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \to R_2 - R_1, \quad R_3 \to R_3 - R_1$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_2 + x_3 = 0$.

Put $x_2 = a \quad x_3 = b$.

$x_1 = -2a - b$.

$$X_2 = \begin{bmatrix} -2a-b \\ a \\ b \end{bmatrix} = \begin{bmatrix} -2a \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} -b \\ 0 \\ b \end{bmatrix} = a\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$X_2 = a\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad X_3 = b\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$E(1) = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{array}{r|rrrrr} 2 & 1 & 1 & -21 & -45 & (1 \\ & & 2 & 6 & -30 & \\ \hline & 1 & 3 & -15 & -75 & \end{array}$$

? Find the Eigen Values and Eigen vectors of $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

A:

$$A - \lambda I \quad \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} = 0. \quad \text{trace} \quad A = 6.$$

$= \cancel{(1-\lambda)(2-\lambda)(3-\lambda) - 0 \times (3-\lambda)(2) + -1(2) - 2(2-\lambda)}$

$= \cancel{(1-\lambda)}$

sum of coefficient

$$\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix} = (2\times3) - (1\times2) + 2(1\times3) - (-1\times2) + 3(1\times2) - (0\times1)$$

$= 6 - 2 + 2\times5 + 3\times2$

$= 4 + 10 + 2$

$= 11$

$$|A| = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix} = 1(4) - 0(1) + -1(2-4)$$

$= 4 + 2 = 6$

$\lambda^3 - 6\lambda^2 + 11A - 6 = 0$.

$1 - 6 + 11 - 6 = 0$.

$\lambda = 1 \quad \lambda =$

$$\begin{array}{r|rrrr} 1 & 1 & -6 & 11 & -6 \\ & & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 2)(\lambda - 3)\ \lambda = 2, 3, 1$

$\lambda = 2$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ 2 & 2 & 1 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + R_1 \\ R_3 \to R_3 + 2R_2 . \end{matrix}$$

Find the Eigen values and vectors of matrix $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

here the given matrix is a upper triangular matrix.

$\therefore$ its eigen values are its diagonal elts $\lambda = 2\,2\,2$.

The characteristic eqn is $(\lambda-2)^3 = 0$

$\lambda = 2$.

$$[A - I\lambda] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

echlon form.

$x_2 = 0$.

$x_3 = 0$

$x_1 = a$.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$0x_1 + x_2 + 0x_3 = 0$

$0x_1 + 0x_2 + 0x_3 = 0$.

eigen vector is $\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$

$x_1 = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$E(2) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Diagonalise the matrix $a = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

$A^T = A$

$\therefore$ A is symmetric

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix} = 0.$$

$$(2-\lambda)((2-\lambda)(2-\lambda)-0) - 0 + 1(0-(2-\lambda)) = 0.$$

$$(2-\lambda)^3 - (2-\lambda) = 0.$$

$$(2-\lambda^2)\left[(2-\lambda)^2 - 1\right] = 0$$

$$(2-\lambda)\left[4 - 4\lambda + \lambda^2 - 1\right] = 0$$

$$(2-\lambda)\left[\lambda^2 - 4\lambda + 3\right] = 0.$$

$$(2-\lambda)(\lambda-3)(\lambda-1) = 0.$$

$$\lambda = \underline{\underline{1, 2, 3}}.$$

$\underline{\lambda = 1}$

$$[A-\lambda I] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_1$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 + x_3 = 0$$

$$x_2 = 0.$$

$\underline{\lambda = 2}$

$$[A-\lambda I] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$x_3 = 0$

$x_1 = 0. \quad x_2 = a.$

$$x_2 = \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} = a \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$E(2) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\sqrt{0^2 + 1^2 + 0}$

$\sqrt{1} = 1$

$\underline{\lambda = 3}$

$$[A-\lambda I] = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \quad R_1 \to R_1 \times -1$$

$$\begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_1$$

$\sqrt{1^2 + 1^2}$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \to R_2 \times -1.$$

$\sqrt{2}$

$1^2 + 0$

$\sqrt{2}$

$x_1 - x_3 = 0.$

$x_2 = 0.$

Put $x_3 = a.$

$x_1 = a.$

$$x_3 = \begin{bmatrix} a \\ 0 \\ a \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \; E(3) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Normalized Eigen vector.

$$X_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

modal matrix $P = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$

here

$X_1 \cdot X_2 = 0$.

$X_2 \cdot X_3 = 0$

$X_1 \cdot X_3 = 0$

$D = P^T A P$

$= P^T A P$.

$$= \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

? Diagonalise the matrix $A = \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}$

$$[A\lambda - I] = \begin{bmatrix} 1-\lambda & -3 & 3 \\ 0 & -5-\lambda & 6 \\ 0 & -3 & 4-\lambda \end{bmatrix}$$

trace of A = 0.

sum of coefficient = $\begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}$

$= (-5\times4) - (-3\times6) + 4 + -5$.

$= -20 - -18 + 4 + -5$

$= -2 + 4 + -5 = -3$

$|A| \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 6 & -3 & 4 \end{bmatrix} = 1(-20 - -18) - -3(0) + 3)$

$= -2$

$\lambda^3 - 3\lambda + 2 = 0$.

$$\begin{array}{c|cccc} & ~~1~~ & ~~-3~~ & 2 & \\ 1 & & ~~1~~ & -2 & \\ \hline & ~~1~~ & ~~-2~~ & 0 & \end{array}$$

~~$\lambda^2 - 2\lambda = 0$~~

$$\begin{array}{c|cccc} & 1 & 0 & -3 & 2 \\ 1 & & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & 0 \end{array}$$

$\lambda^2 + \lambda - 2 = 0$.

$\lambda = 1, 1, -2$.

$\lambda = 1$

$$[A - \lambda I] = \begin{bmatrix} 0 & -3 & 3 \\ 0 & -6 & 6 \\ 6 & -3 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & -1 \\ 0 & -6 & 6 \\ 0 & -3 & 3 \end{bmatrix} \quad R_1 \to R_1/-3.$$

$$\sim \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \to R_2 + 6R_1,\ R_3 \to R_3 + 3R_1$$

$$= \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$n - r$

$3 - 1 = 2$

$\cancel{x_2 - x_3 = 0}$

$\cancel{x_1 - x_2 + x_3 = 0}$

$\cancel{x_2 - 2x_3 = 0}$

$\cancel{\text{Put } x_3 = a}$

$\cancel{x_2 = 2a}$

$\cancel{x_1 = x_2 - x_3}$

$\cancel{= 2a - a}$

$= a$

$$\cancel{X_1 = \begin{bmatrix} a \\ 2a \\ a \end{bmatrix} = a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}}$$

$x_2 - x_3 = 0$

$x_2 = x_3 = 0$

$$x_1 = b = \begin{bmatrix} b \\ a \\ a \end{bmatrix}$$

$$= \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ a \\ a \end{bmatrix}$$

$$\Rightarrow b \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda = 1$

$$X_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$\lambda = -2$

$$(A - \lambda I) = 0 \quad \begin{bmatrix} 3 & -3 & 3 \\ 0 & -3 & 6 \\ 0 & -3 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & -3 & 6 \\ 0 & -3 & 6 \end{bmatrix} \quad R_1 \to R_1/3 \quad \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & -3 & 6 \end{bmatrix} \quad R_2 \to R_2/-3,\ R_3 \to R_3 + 3R_2$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - x_2 + x_3 = 0.$

$x_2 - 2x_3 = 0$

Put $x_3 = a$.

$x_2 = 2a$.

$$2a - a = a \quad X_1 \begin{bmatrix} a \\ 2a \\ a \end{bmatrix} \quad a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

(Normal)

Modal Matrix.

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$D = P^{-1} AP$

$$P^{-1} = \frac{\text{adj}(P)}{|P|}$$

Sym.

$P^T A P$

$P^T AP$

$$\begin{array}{cccccc} 1 & 1 & 0 & 1 & 1 & 0 \\ 2 & 0 & 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 2 & 0 & 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \end{array}$$

$$= \frac{\begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & 1 & -2 \end{bmatrix}}{-1}$$

$$= \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & -1 & 2 \end{bmatrix}.$$

$$D = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Examine whether the matrix $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$ is diagonalisable.

$$A^T = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 2 & 2 \\ 2 & 1 & 2 \end{bmatrix}, \quad A^T \neq A \text{ asymmetric}$$

$$\boxed{A\lambda - I} = \begin{bmatrix} 1-\lambda & 2 & 2 \\ 0 & 2-\lambda & 1 \\ -1 & 2 & 2-\lambda \end{bmatrix}$$

Trace of $A = 5$.

Sum of coefficient $= \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{bmatrix}$

$$\cancel{= (2\times 2) - 2(2\times -2) + (2+0)}$$

$$\cancel{= 1(4-2) - 2(2-2) + 2(2-0)}$$

$$\cancel{= 2 - 8 + 4}$$

$$= (4-2) + (2 - -2) + 2$$

$$= 2 + 4 + 2 = 8$$

$$2 + 2$$

$-|A|$

$$|A| = \begin{vmatrix} 1 & 2 & 2 \\ 0 & 2 & 1 \\ -1 & 2 & 2 \end{vmatrix} = 1\times(4-2) - 2(-1) + 2(-2)$$

$$\cancel{= 2 + 2 - 4}$$

$$4 \quad 2 - 2 + 4$$

$$= 4$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0.$$

$(\lambda-1)=0 \quad , \lambda=1.$

$$\begin{array}{c|cccc} & 1 & -5 & 8 & -4 \\ 1 & & 1 & -4 & 4 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

$\lambda^2 - 4\lambda + 4 = 0.$

$(\lambda-2)(\lambda-2)(\lambda-1).$

$\lambda = \cancel{0}\,2, 2, 1.$

$\underline{\lambda=1}$

$$\begin{bmatrix} 1-1 & 2 & 2 \\ 0 & 2-1 & 1 \\ -1 & 2 & 2-1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ -1 & 2 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} \cancel{0 \quad 1 \quad 1} \\ \cancel{0 \quad 1 \quad 1} \\ \cancel{-1 \quad 2 \quad 1} \end{bmatrix} \quad R_1/2$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \quad R_1 \leftrightarrow -R_3$$

$$\begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & \end{bmatrix} \qquad R_3 \to R_3 + R_1$$

$\underline{\lambda=2}$

$$[A-\lambda I] = \begin{bmatrix} -1 & 2 & 2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{bmatrix} \quad R_1 \to R_1 \times -1$$

$$\begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \quad R_3 \to R_3 + R_1$$

$$\begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + 2R_2.$$

$$\begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Q: If 2 is an Eigen value of $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ without using its characteristic eqn, find the other Eigen values also find the eigen values of $A^3, A^T, A^{-1}, 5A, A-3I$, adj A

A: Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values

Let $\lambda_1 = 2$

we know that $\lambda_1 + \lambda_2 + \lambda_3 = \text{trace of } A$

$$\lambda_1 + \lambda_2 + \lambda_3 = 11$$

$$2 + \lambda_2 + \lambda_3 = 11 \Rightarrow \lambda_2 + \lambda_3 = 11 - 2 = 9 \quad — (1)$$

$$\lambda_1 \times \lambda_2 \times \lambda_3 = |A|$$

$$|A| = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix}$$

$$= 3(15-1) - -1(-3-1) + 1(1-5)$$

$$= 3 \times 14 + 1(-4) + 1(-4)$$

$$= 3$$

Vector space:

A known empty set V with two operations called addition and scalar multiplication denoted by '+' and '$\cdot$' respectively is a vector space over a field F, if it satisfies the following condition.

1) If $u, v \in V$ then $u+v \in V$ (closure property under addition)
2) let $u, v, w \in V$ for any vectors $u+(v+w) = (u+v)+w$ (associative ppty)
3) There is a vector $0 \in V$, for any $u \in V$ $0+u = u+0 = u$ (existence of additive identity)
4) For every $u \in V$, there is $-u \in V$ s.t $u+-u = -u+u = 0$ (existence of additive inverse).
5) For any $u, v \in V$ $u+v = v+u$ (Commutative ppty).
6) For any $u \in V$ and $c \in F$ $c.u \in V$ (closure ppty under scalar multiplication)
7) for any $u, v \in V$ and $c \in F$ $c\cdot(u+v) = c\cdot u + c\cdot v$.
8) for any $u \in V$ and $a, b \in F$ $(ab)\cdot u = a\cdot(bu)$
9) for any $u \in V$ and ~~a,b~~ $\in F$ $a, b \in F$ $(a+b)\cdot u = a\cdot u + b\cdot u$
10) For any $u \in V$ ~~then u+v∈v~~ $\exists\, 1 \in F \cdot$ s.t $1\cdot u = u$.

? Check whether the set of real numbers with equal addition and multiplication is a vector square or not

i) For any $u, v \in \mathbb{R}$ $u+v \in \mathbb{R}$ (closure ppty satisfied)
ii) For $u, v, w \in \mathbb{R}$ $(u+v)+w = u+(v+w)$ (associative ppty satisfie
iii) for $u \in \mathbb{R}$ $\exists\, 0 \in \mathbb{R}$ s.t $u+0 = 0+u = u$.
iv) for $u \in \mathbb{R}$, $-u \in \mathbb{R}$ s.t $u+-u = -u+u = 0$.
v) for $u, v \in \mathbb{R}$, $a+v = v+u$.
vi) for $u \in \mathbb{R}$, $c \in F$ $c.u \in \mathbb{R}$
vii) for any $u \in V$ and $c \in F$ $c\cdot(u+v) = c\cdot u + c\cdot v$

$(a+b)\cdot u = a\cdot u + b\cdot u$
$(ab)\,u = a\,(bu)$
$a\,(u+v) = a\cdot u + a\cdot v$
$u \in \mathbb{R}$ $1\cdot u = u$

! eg: set of polynomials together with zero polynomial is a vector space.

set of all m/n matrixes with matrix addition and multiplication is a vector space.

? A two dimensional Eucledian plane is a vector space.

add Eucledian plane

i) $u, v \in \mathbb{R}^2$ $u+v \in \mathbb{R}^2$
ii) $u, v, w \in \mathbb{R}^2$ $(u+v+w) = u+(v+w)$.
iii) $u \in \mathbb{R}^2$ $0 = (0,0) \in \mathbb{R}^2$ $u+0 = 0+u = u$.
iv) $u \in \mathbb{R}^2$ $-u \in \mathbb{R}^2$. $u+-u = -u+u$
v) $u, v \in \mathbb{R}^2$ $u+v = v+u$.
vi) $u \in \mathbb{R}^2$ $c \in F$ $c.u \in \mathbb{R}^2$.
vii) $a, b \in \mathbb{R}$ $u \in \mathbb{R}^2$ $(a+b)\cdot u = a\cdot u + b\cdot u$
viii) $a, b \in F$ $u \in \mathbb{R}^2$ $(ab)u = a(bu)$.
ix) $a \in F$ $u, v \in \mathbb{R}^2$ $a(u+v) = a\cdot u + a\cdot v$.
x) $1 \in F$, $u \in \mathbb{R}^2$ $1\cdot u = u$

? Check whether this ~~u+v=uv~~ set V of all +ve real no. with the operations $u+v = uv$ and $c.u = u^c$ is a vector space or not

i) $u, v \in \mathbb{R}^+$ $u+v = uv \in \mathbb{R}^+$
ii) $u, v, w \in \mathbb{R}^+$ $(u+v)+w = (uv)+w = uvw = u(vw)$
$= u+(vw) = u+(v+w)$.
iii) $u \in \mathbb{R}^+$ $1 \in \mathbb{R}^+$ $u+1 = 1+u = u\cdot 1 = u$

iv) $u \in \mathbb{R}^+$ $\frac{1}{u} \in \mathbb{R}^+$ $u + \frac{1}{u} = \frac{1}{u} + u = u \cdot \frac{1}{u} = 1$.

v) $u \in \mathbb{R}^+$ $c \in F$ $c \cdot u = u^c \in \mathbb{R}^+$

vi) $u \in \mathbb{R}^+$ $a, b \in F$ $(a+b) \cdot u = u^{a+b} = u^a \cdot u^b = u^a + u^b = a \cdot u + b \cdot u$

vii) $u \in \mathbb{R}^+$ $a, b \in F$ $(ab) \cdot u = u^{ab} = (u^b)^a = a(u^b) = a \cdot (bu)$.

ix) $u, v \in \mathbb{R}^+$ $a \in F$ $a(u+v) = (u+v)^a = u^a v^a = u^a + v^a = a \cdot u + a \cdot v$

x) $u \in \mathbb{R}^2$ $1 \in F$ $1 \cdot u = u^1 = u$.

Linear Independence and dependence

Linear combination

Let V be a vector space over a field F and let $F_1, F_2, \dots F_n$ element in V. Then any vector x in V is a linear combination of $x_1, x_2, x_3 \dots x_n$. If there exist scalars $a_1, a_2 \dots a_n$ s.t.

$x = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$

Let V be a vector space over a field F the vectors is said to be linearly dependend.

If $a_1 x_1 + a_2 x_2 + \dots + a_n x_n = 0 \Rightarrow a_1 = a_2 = \dots a_n = 0$.

the vector is said to be linearly independent

Spanning Set:

Let H be a subset of a vectorspace over the field of scalars F the subset H(v) is said to be a span in set of the vectorspace V. If each vector in V can be expressed as a linear combination of elts of H.

Basis and dimension of V.S

A subset H of a V.S 'V' is called the basis of the V.S, if it is both linearly independent and spanning set of the vector space.

The no: of elts in the basis is called dimension of V.S.

Q Express the vector $v = (1,-2,5)$ as a linear combination of the vectors $v_1 = (1,1,1)$, $v_2 = (1,2,3)$, $v_3 = (2,-1,1)$.

$v = v_1 a_1 + v_2 a_2 + v_3 a_3$.

$(1,-2,5) = a_1(1,1,1) + a_2(1,2,3) + a_3(2,-1,1)$

$= (a_1, a_1, a_1) + (a_2, 2a_2, 3a_2) + (2a_3, -a_3, a_3)$

$= (a_1 + a_2 + 2a_3), a_1 + 2a_2 - a_3, a_1 + 3a_2 + a_3)$.

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & -1 & 4 \end{bmatrix} \quad \begin{matrix} R_2 - R_2 - R_1 \\ R_3 \to R_3 - R_1. \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 5 & 10 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad R_3 \to R_3/5$$

which is in echlon-form.

by back substitn

$a_1 + a_2 + 2a_3 = 1$

$0a_1 + a_2 - 3a_3 = -3$

$0a_1 + 0a_2 + a_3 = 2.$

$a_3 = 2$

$a_2 = -3 + 3a_3 = -3+6 = 3.$

$a_1 = 1 - a_2 - 2a_3$

$= 1 - 3 - 2\times 2$

$= -6.$

$a_1 + a_2 + 2a_3 = 1$

$a_1 + 2a_2 - a_3 = -2$

$a_1 + 3a_2 + a_3 = 5.$

$\therefore (1,-2,5) = -6(1,1,1) + 3(1,2,3) + 2(2,-1,1)$

? $v = (2,-5,3)$ cannot be expressed as a linear combination of the vectors $(1,-3,2)$, $(2,-4,-1)$, $(1,-5,7)$

$v = a_1v_1 + a_2v_2 + a_3v_3$

$(2,-5,3) = a_1(1,-3,2) + a_2(2,4,-1) + a_3(1,-5,7)$

$= (a_1, -3a_1, 2a_1) + (2a_2, 4a_2, -a_2) +$

$(a_3, -5a_3, 7a_3).$

$= (a_1 + 2a_2 + a_3), \; -3a_1 + 4a_2 + -5a_3,$

$2a_1 + -a_2 + 7a_3$

$a_1 + 2a_2 + a_3 = 2.$

$-3a_1 + 4a_2 - 5a_3 = -5$

$2a_1 - a_2 + 7a_3 = 3$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ -3 & 4 & -5 & -5 \\ 2 & -1 & 7 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 2 & -2 & 1 \\ 0 & -5 & 5 & -1 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 + 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & 1/2 \\ 0 & -5 & 5 & -1 \end{bmatrix} \quad R_2 \to R_2/2$$

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 9 \end{bmatrix} \quad R_3 \to R_3 + 5R_2$$

which is in Echlon form.

$a_1 + 2a_2 + a_3 = 2.$

$a_2 - a_3 = 1/2$

$a_2 = 1/2 + a_3.$

Show that the vectors $(1,1,1)$, $(1,2,3)$ & $(2,1,1)$ are linearly independent

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix} = 1(2+3) - 1(1+1) + 2(3-2)$$

$= 5 - 2 + 2.$

$= 5 \neq 0$ then values are linearly independent.

prove that the vectors (1,2,1), (2,1,4) and (4,5,6) are lineary dependent and find a relation connecting them.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 4 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 2 \\ 0 & -3 & 2 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{matrix} \quad \begin{matrix} x_2' = x_2 - 2x_1 \\ x_3' = x_3 - 4x_1 . \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -2/3 \\ 0 & -3 & 2 \end{bmatrix} \quad R_2 \to R_2/-3 \quad x_2'' = x_2'/-3 .$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + 3R_2 \quad \begin{matrix} x_3'' = x_3' + 3x_2' \\ x_3'' = 0 \end{matrix}$$

$\therefore$ the given set of vectors is linearly dependent.

$x_3' + 3x_2'' = 0.$

$x_3 - 4x_1 + 3(x_2'/-3) = 0.$

$x_3 - 4x_1 - (x_2') = 0$

$x_3 - 4x_1 - (x_2 - 2x_1) = 0$

$x_3 - 4x_1 - x_2 + 2x_1 = 0$

$x_3 - 2x_1 - x_2 = 0$

Prove that the vectors (2,3,0), (1,2,0). (1,3,0) are linearly dependent

? Prove that the vectors (1,-1,1), (0,1,2) and (3,0,-1) form a basis for R^3

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & -4 \end{bmatrix} \quad R_3 \to R_3 - 3R_1$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -10 \end{bmatrix} \quad R_3 \to R_3 - 3R_2 . \quad \begin{matrix} -4-3\times 2 \\ -4-6 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad R_3 \to R_3/-10$$

here rank = 3 . or dimension = 3.

we know that any linearly independent vectors of R^n form a basis for R^n, hence the three independent vector (1,-1,1), (0,1,2) and (3,0,-1) form a basis for R_3.

? Find the basis of the vectors (1,1,1), (1,2,3), (2,-1,1)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix}$$ Row reduced echlon form

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -1 \end{bmatrix}$$ $R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$-1 - 2 \times 1$

$-1 - 2$

$2 - 2 \times 1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$ $R_3 \to R_3 + 3R_2$

$-1 + 3 \times 2$

$-1 + 6$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$ $R_3 \to R_3/5$.

Rank = 3.

basis: $\{(1,1,1), (0,1,2), (0,0,1)\}$