MODULE IV

RESIDUE INTEGRATION

Singularities

The points where the function is not analytic is called singularity.

$f(z) = \frac{1}{z}$ here singularity at $z=0$.

A singularity $z = z_0$ of a function $f(z)$ is called the isolated singularity if there exist a neighbourhood a circle with centre (z_0) which contains no other singularity of $f(z)$

eg: $f(z) = z^2-1$ has isolated singularities at $z=1$ & $z=-1$

1) Removable Singularity

An isolated singularity z_0 is called removable singularity if principle part of $f(z)$ at $z=z_0$ in Laurent series expansion

$$f(z) = \frac{\sin z}{z}$$

$$= \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} \cdots\right]$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} \cdots$$

here the principle part has no terms. Hence $z=0$ is a removable singularity if $z=z_0$ is a removable singularity then $\boxed{\lim_{z\to z_0} f(z) = \text{finite value}}$

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2) Pole

An isolated singularity z_0 of $f(z)$ is said to be a pole if the principle part contains finite number of terms.

eg : $f(z) = \frac{e^z}{z} = \frac{1}{z}\left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots\right)$

$= \frac{1}{z} + \frac{1}{1!} + \frac{z}{2!} + \frac{z^3}{3!} + \cdots$

So here the principle part contains finite number of terms (1). Hence the singularity $z=0$ is called a pole.

Note :- If $z = z_0$ is a pole then.

$$\boxed{\lim_{z \to z_0} f(z) = \infty}$$

Note : The pole of order 1 it is simple pole
The pole of order 2 it is called double pole

3) Essential Singularity

An isolated singularities $z = z_0$ is called an essential singularity if the principle part of $f(z)$ at $z = z_0$ has infinite number of term.

eg : $f(z) = e^{1/z}$

$= 1 + \frac{(1/z)}{1!} + \frac{(1/z)^2}{2!} + \cdots$

$= 1 + \frac{1}{z\,1!} + \frac{1}{z^2\,2!} + \cdots$

$= \infty$

limit does not exist here

Note: If $z = z_0$ if an essential singularity.

$\lim_{z \to z_0} f(z) =$ not defined

Zero of $f(z)$

zero of an analytic function $f(z)$ in a domain at $z = z_0$ in D such that $f(z_0) = 0$.

$f(z)$ has zero of order n if $f'(z_0) = 0$, $f''(z_0) = 0$

$f^{n-1}(z_0) = 0$ at $f^{n}(z_0) \neq 0$

Note :-

The first order zero is called simple zero.

Let $f(z)$ be analytic at $z = z_0$ and have a zero of nth order at $z = z_0$ then $1/f(z)$ has a pole of order n

Residue

We have the laurent series expansion of $f(z)$ as

$$f(z) = \left[a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\right] + \left[a_{-1}(z-z_0)^{-1} + a_{-2}(z-z_0)^{-2} + \cdots\right]$$

The residue of $f(z)$ at $z = z_0$ is defined as the coefficient of $(z-z_0)^{-1}$ in the principle part of laurent series and is denoted as

$$\underset{z=z_0}{\text{Res}}\, f(z) = a_{-1}$$

we can also find a_{-1} as

$$a_{-1} = \frac{1}{2\pi i} \oint f(z)\, dz$$

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Calculation of Residues.

I $z = z_0$ is a simple pole

$$\operatorname{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z-z_0) f(z)$$

II $z = z_0$ is a pole of order 'm'

$$\operatorname{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z)\right]$$

III If $z = z_0$ is a singularity and if $f(z) = \frac{p(z)}{q(z)}$, $p(z_0) \neq 0$

$p(z_0) \neq 0$ & $q(z_0) = 0$.

$$\operatorname{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)}, \quad q'(z_0) \neq 0.$$

? Find the residue of $f(z) = \frac{2z}{z+4}$ at its pole.

A Singularity at $z = -4$ & order $= 1$

$$\operatorname{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z-z_0) f(z)$$

$$\operatorname{Res}_{z=-4} f(z) = \lim_{z \to -4} (z - -4)\left(\frac{2z}{z+4}\right)$$

$$= \lim_{z \to -4} z+4 \frac{(2z)}{z+4}$$

$$= \lim_{z \to -4} 2z = \underline{\underline{-8}}$$

? Evaluate residue of $f(z) = \frac{1}{z^3(z+4)}$ at its poles.

A Singularity at $z=0$ of order 3.
at $z=-4$ of order 1

At $z=0$

$$\underset{z=z_0}{\text{Res}}\, f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$$

$$\underset{z=0}{\text{Res}}\, f(z) = \frac{1}{2!} \lim_{z\to 0} \frac{d^2}{dz^2}\left[(z-0)^3 \cdot \frac{1}{z^3(z+4)}\right]$$

$$= \frac{1}{2!} \lim_{z\to 0} \frac{d^2}{dz^2}\left(\frac{1}{z+4}\right)$$

$\frac{1}{z+4}$

$\frac{d}{dz} = -1(z+4)^{-2}$

$\frac{d^2}{dz^2} = 2(z+4)^{-3}$

$$= \frac{1}{2!} \lim_{z\to 0} \frac{2}{(z+4)^3}$$

$$= \frac{1}{2} \cdot \frac{2}{4^3} = \underline{\underline{\frac{1}{64}}}$$

At $z=-4$

$$\underset{z=z_0}{\text{Res}}\, f(z) = \lim_{z\to z_0} (z-z_0) f(z)$$

$$= \lim_{z\to -4} (z+4) \frac{1}{z^3(z+4)}$$

$$= \lim_{z\to -4} \frac{1}{z^3} = \frac{1}{(-4)^3} = \underline{\underline{-\frac{1}{64}}}$$

? Find the residue $f(z) = \frac{e^z}{\cos \pi z}$ at the singularity which lies in $|z| = 1$

A. Singularity $\cos \pi z = 0$ when $z = 1/2, 3/2, 5/2 \ldots\ldots$

At $|z| = 1$ only $z = 1/2$ only lies.

$$\underset{z \to 1/2}{\text{Res}} f(z) = \lim_{z \to 1/2} (z - 1/2) \frac{e^z}{\cos \pi z}$$

We cannot do this by I since 0/0 form.
So take type III

$$\underset{z = z_0}{\text{Res}} f(z) = \frac{p(z_0)}{q'(z_0)}$$

$p(z) = e^z$
$p(1/2) \neq 0$
$q(z) = \cos \pi z$
$q'(z) = -\pi \sin \pi z$
$q'(1/2) = -\pi \sin \pi/2$
$= -\pi$
$q(1/2) = 0$

$$\underset{z = 1/2}{\text{Res}} f(z) = \frac{e^{1/2}}{-\pi}$$

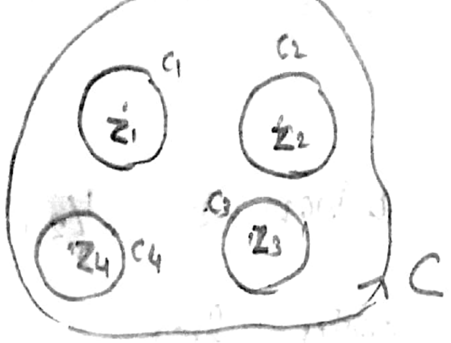
Cauchy's Residue Theorum

Let $f(z)$ be analytic inside a simple closed curve c and ~~insie~~ on c except for finit number of singularity $z_1, z_2, z_3 \ldots z_k$ inside c when the $\oint_c f(z)$ taken counterclockwise around c equals to $2\pi i$ time the sum of ~~redue~~ residues of $f(z)$ at $z_1, z_2, \ldots, z_k$

ie $\oint_c f(z)\,dz = 2\pi i \left[\underset{z=z_1}{\text{Res}} f(z) + \underset{z=z_2}{\text{Res}} f(z) + \cdots + \underset{z=z_k}{\text{Res}} f(z) \right]$

$= 2\pi i \sum^{k} f(z) = 2\pi i \sum \underset{z=z_i}{\text{Res}} f(z)$

Proof



Let c be a simple closed curve in a domain D. Let $z_1, z_2, \dots z_k$ be the finite no:of singularities lying inside c. Then let $c_1, c_2, c_3, \dots c_k$ be the circles that enclosing the points $z_1, z_2, \dots z_k$ respectively and the circles are disjoined with each other. Then by cauchy's theorum we have

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \dots + \oint_{C_k} f(z)\,dz$$

By the definition of residue we have

$$\oint_C f(z)\,dz = 2\pi i \operatorname*{Res}_{z=z_1} f(z) + 2\pi i \operatorname*{Res}_{z=z_2} f(z) + \dots + 2\pi i \operatorname*{Res}_{z=z_k} f(z)$$

since residue at $z=z_0$ is $\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{2\pi i}\oint_C f(z)\,dz$

$$\oint_C f(z)\,dz = 2\pi i \left[\operatorname*{Res}_{z=z_1} f(z) + \operatorname*{Res}_{z=z_2} f(z) + \dots + \operatorname*{Res}_{z=z_k} f(z)\right]$$

$$= 2\pi i \sum_{i=1}^{k} \operatorname*{Res}_{z=z_i} f(z).$$

? Evaluate $\oint_c \frac{2z-1}{z(z+1)(z-3)} dz$ where c is $|z| = 2$.

A. Singularity at $z=0 < 2$, $z=-1 < 2$ } lies inside, order = 1

$z = 3 > 2$ lies outside.

$\oint_c$ At $z=0$.

$$\underset{z=0}{\text{Res}}\, f(z) = \lim_{z \to z_0} (z-z_0)\, f(z)$$

$$= \lim_{z \to 0} (z-0) \cdot \frac{2z-1}{z(z+1)(z-3)}$$

$$= \frac{-1}{1(-3)} = \frac{1}{3}$$

At $z = -1$

$$\underset{z=-1}{\text{Res}}\, f(z) = \lim_{z \to z_0} (z-z_0)\, f(z)$$

$$= \lim_{z \to -1} (z-(-1)) \frac{2z-1}{z(z+1)(z-3)}$$

$$= \lim_{z \to -1} \frac{2z-1}{z(z-3)} = \frac{-2-1}{-1(-4)} = \frac{-3}{4}$$

By cauchy's theorum. write formula.

$$\oint_c \frac{2z-1}{z(z+1)(z+3)} dz = 2\pi i \left[\frac{1}{3} - \frac{3}{4}\right]$$

$$= 2\pi i \left(\frac{-5}{12}\right) = \frac{-5}{6}\pi i$$

? Evaluate $\oint_c \frac{dz}{z^2(z+1)(z-1)}$, $|z| = 3$

A. Singularity at $z=0$ — order = 3.

$z=-1$, $z=1$ } lies inside } order = 1

At $z = 0$.

$$\operatorname{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z)\right]$$

$$= \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2} \left(z^2 \frac{dz.}{z^2(z+1)(z-1)} \right)$$

$$= \frac{1}{2} \lim_{z \to 0} \frac{d^2}{dz^2} \frac{1}{(z+1)(z-1)} = \lim_{z \to 0} \frac{-1}{(z^2-1)^2} \cdot 2z$$

$(a+b)(a-b)$

$$= \frac{1}{2} \cdot 0 = \underline{\underline{0}}$$

$\cancel{\frac{1}{(z+1)(z-1)}}$

$\cancel{\frac{1}{z^2-1}}$

$\frac{d}{dz} = -1(z^2-1)^{-2} \cdot 2z$

$\frac{d^2}{dz^2} = -2 \left[(z^2-1)^{-2} \cdot 1 + z \cdot -2(z^2-1) \right]$

At 0

$\frac{d^2}{dz^2} = -2(1+) = \underline{\underline{-2}}$

At $z = -1$

$$\operatorname{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z-z_0) f(z)$$

$$= \lim_{z \to -1} (z+1) \cdot \frac{1}{z^2(z+1)(z-1)}$$

$$= \frac{1}{1(-2)} = \underline{\underline{-\frac{1}{2}}}$$

At $z = 1$

$$\operatorname{Res}_{z=1} f(z) = \lim_{z \to 1} (z-1) \frac{1}{z^2(z+1)(z-1)}$$

$$= \underline{\underline{\frac{1}{2}}}$$

By cauchy's theorum,

$$\oint_C \frac{dz}{z^2(z+1)(z-1)} = 2\pi i \left[0 - \frac{1}{2} + \frac{1}{2} \right]$$

$$= \underline{\underline{0}}$$

1) ? Evaluate $\frac{dz}{z^2+4}$ where c is , $|z-i| = 2$.

2) ? find residue

i) $f(z) = \frac{z-2}{z^2-z}$

ii) $f(z) = \frac{2z+3}{z^2+2z+5}$

iii) $f(z) = \cot \pi z$, $|z| = 1$

1) $\oint_c \frac{dz}{z^2+4}$

Singularity at $z^2 = -4$

$z = \pm 2i$

$z = 2i$ $|2i - i| = 1 < 2$. inside.

$z = -2i$ $|-2i - i| = 3 > 2$ outside

At $z = 2i$

$\operatorname{Res}_{z=z_0} f(z) = \lim_{z \to z_0} (z - z_0) f(z)$

$= \lim_{z \to 2i} (z-2i) \cancel{dz} \frac{1}{z^2+4}$

$= \lim_{z \to 2i} z - 2i \frac{1}{(z+2i)(z-2i)}$

$= \lim_{z \to 2i} \frac{1}{z+2i}$

$= \frac{1}{4i}$

By cauchy's theorum,

$$\oint_C \frac{dz}{z^2+4} = 2\pi i \frac{1}{4i} = \frac{\pi}{2}$$

? Evaluate $\oint_C \frac{\tan z}{z^2-1} dz$, $|z| = 3/2$

Singularity at $z^2=1$ $\qquad \cos z = 0$

$z = \pm 1$ $\qquad z = \pi/2, 3\pi/2, 5\pi/2 \ldots$

At $z=1$ $\quad |z| = 1 < 3/2$ } inside

$z=-1$ $\quad |z| = 1 < 3/2$

$z = \pi/2$. $|\frac{3.14}{2}| = 1.57 > 3/2$. outside

At $z=1$

$$\underset{z=1}{\text{Res}}\, f(z) = \lim_{z\to 1} (z-1) \frac{\tan z}{(z+1)(z-1)}$$

$$= \lim_{z\to 1} \frac{\tan z}{z+1} = \frac{\tan 1}{2}$$

At $z=-1$

$$\underset{z=-1}{\text{Res}}\, f(z) = \lim_{z\to -1} (z+1) \frac{\tan z}{(z+1)(z-1)}$$

$$= \lim_{z\to -1} \frac{\tan z}{z-1} = \frac{\tan(-1)}{-2} = \frac{\tan 1}{2}$$

By residue theorem,

$$\oint_C \frac{\tan z}{z^2-1} dz = 2\pi i \left(\frac{\tan 1}{2} + \frac{\tan 1}{2} \right)$$

$$= 2\pi i \tan 1$$

? $\oint_C \frac{z-23}{z^2-4z-5}\,dz$; $|z-2-i| = 3.2$

A. $\oint_C \frac{z-23}{(z-5)(z+1)}\,dz$

Singularity at $z=5$ & $z=-1$

At $z=5$, $|5-2-i| = |3-i| = \sqrt{9+1} = \sqrt{10} = 3.16$.

At $z=-1$ $|-1-2-i| = |-3-i| = \sqrt{9+1} = \sqrt{10} = 3.16$.

At $z=5$

$$\operatorname{Res}_{z=5} f(z) = \lim_{z\to 5} (z-5)\frac{z-23}{(z-5)(z+1)}$$

$$= \lim_{z\to 5} \frac{z-23}{z+1} = \frac{-18}{6} = -3$$

At $z=-1$

$$\operatorname{Res}_{z=-1} f(z) = \lim_{z\to -1} (z+1)\frac{z-23}{(z-5)(z+1)}$$

$$= \lim_{z\to -1} \frac{z-23}{z-5}$$

$$= \frac{-1-23}{-1-5} = \frac{-24}{-6} = \underline{\underline{4}}$$

$$\oint_C \frac{z-23}{z^2-4z-5}\,dz = 2\pi i\,(-3+4)$$

$$= \underline{\underline{2\pi i}}$$

? Evaluate $\oint_C e^{1/z}\,dz$ and c is $|z|=1$

A. Singularity at $z=0$ lies inside.

~~At z=0~~ it

Here $z=0$ is an essential singularity so the formula for calculation of residues cannot be applied. Hence by definition.

$$e^{1/z} = 1 + \frac{1}{z(1!)} + \frac{1}{z^2(2!)} + \cdots$$

$$\underset{z=0}{\text{Res}}\, f(z) = \text{coeff of } 1/z$$

$$= \underline{\underline{1}}.$$

$$\oint_C e^{1/z}\,dz = 2\pi i\,[1] = \underline{\underline{2\pi i}}$$

? Evaluate $\oint_C \frac{4-3z}{z^2-z}$, c is $|z|=2$.

A. Singularity at $z(z-1)=0$

ie $z=0$ & $z=1$ lies inside

At $z=0$

$$\underset{z=0}{\text{Res}}\, f(z) = \lim_{z\to 0} (z-0)\frac{4-3z}{z(z-1)}$$

$$= \lim_{z\to 0} \frac{4-3z}{z-1}$$

$$= \frac{4}{-1} = \underline{\underline{-4}}$$

At $z=1$

$$\underset{z=1}{\text{Res}}\, f(z) = \lim_{z\to 1} (z-1)\frac{4-3z}{z(z-1)}$$

$$= \lim_{z\to 1} \frac{4-3z}{z} = \frac{4-3}{1} = \underline{\underline{1}}$$

$$\oint_C \frac{4-3z}{z(z-1)}\,dz = 2\pi i\,(-4+1) = \underline{\underline{-6\pi i}}$$

? Find residue at $|z|=1$

i) $f(z) = \dfrac{z-2}{z^2-z} = \dfrac{z-2}{z(z-1)}$

Singularity at $z=0$ & $z=1$ lies inside

At $z=0$

$$\underset{z=0}{\text{Res}}\, f(z) = \lim_{z\to 0}(z-0)\frac{z-2}{z(z-1)} = \lim_{z\to 0}\frac{z-2}{z-1} = \frac{-2}{-1} = \underline{\underline{2}}$$

At $z=1$

$$\underset{z=1}{\text{Res}}\, f(z) = \lim_{z\to 1}(z-1)\frac{z-2}{z(z-1)} = \lim_{z\to 1}\frac{z-2}{z} = \frac{1-2}{1} = \underline{\underline{-1}}$$

ii) $f(z) = \dfrac{2z+3}{z^2+2z+5}$

Singularity at $z=-1-2i$ & $z=-1+2i$

$$\left.\begin{aligned} z=-1-2i, \quad |z| = \sqrt{1+4} = \sqrt{5} > 1 \\ z=-1+2i, \quad |z| = \sqrt{1+4} = \sqrt{5} > 1 \end{aligned}\right\} \text{lies outside}$$

iii) $f(z) = \cot \pi z.$

Singularity at $\sin \pi z \ldots$

$z = 0, 1, 2, \ldots$

$z=0 \quad |z|=0<1$ lies inside.

$z=1 \quad |z|=1$

At $z=0$

$$\operatorname*{Res}_{z=0} f(z) = \cancel{\lim_{z\to 0}} \cancel{(z-0)} \cancel{\cos\pi z} \frac{\cos 0}{\pi}$$

$$= \frac{1}{\pi}$$

$q(z) = \sin\pi z$

$q'(z) = \pi\cos\pi z$

$q'(0) = \pi$

$q'(1) = \pi\cos\pi$

$= -\pi$

At $z=1$

$$\operatorname*{Res}_{z=1} f(z) = \frac{\cos\pi}{-\pi} = \frac{-1}{-\pi} = \frac{1}{\pi}$$

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? Evaluate $\oint_C \frac{dz}{(z^2+4)^2}$, $|z-i|=2$

Singularity $z^2=-4$

At $z=2i$, $|z-i| = |2i-i| = |i| = 1 < 2$ lies inside

At $z=-2i$, $|z-i| = |-2i-i| = |-3i| = 9 > 2$ lies outside

At $z=2i$

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}} \left((z-z_0)^m \cancel{dz} f(z)\right)$$

$$= \frac{1}{1!} \lim_{z\to 2i} \frac{d}{dz}\left[(z-2i)^2 \frac{dz}{((z+2i)(z-2i))^2}\right]$$

$$= \lim_{z\to 2i} \frac{d}{dz}\left(\frac{1}{(z+2i)^2}\right)$$

$$= \lim_{z \to 2i} \left[-2 \cdot (z+2i)^{-3}\right]$$

$$= -2 \frac{1}{(4i)^3} = \frac{-2}{-64i} = \cancel{\frac{+}{64i}} \frac{1}{32i}$$

$$\oint_C \frac{dz}{(z^2+4)^2} = 2\pi i \frac{1}{32i} = \frac{\pi}{16}$$

Q. Evaluate $\oint_C \frac{e^z}{\cos \pi z} dz$, $|z| = 1$

A. Singularity $\cos \pi z$

$$z = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$$

$$\left.\begin{array}{l} z = 1/2 \Rightarrow |z| < 1 \\ z = -1/2 \Rightarrow |z| < 1 \end{array}\right\} \text{inside}$$

At $z = 1/2$

$$\operatorname*{Res}_{z=1/2} f(z) = \frac{e^z}{-\pi} = \frac{e^{1/2}}{-\pi}$$

$q(z) = \cos \pi z$

$q'(z) = -\pi \sin \pi z$

$q'(1/2) = -\pi \sin \pi/2$

$= -\pi$

At $z = -1/2$

$$\operatorname*{Res}_{z=-1/2} f(z) = \frac{e^z}{\pi} = \frac{e^{-1/2}}{\pi}$$

$q'(-1/2) = \pi$

$$\oint_C \frac{e^z dz}{\cos \pi z} = \cancel{2\pi i -\pi + 1}$$

$$= 2\pi i \left(\frac{+e^{1/2}}{\pi} - \frac{e^{1/2}}{\pi}\right) = \cancel{0}$$

$$\cancel{= 2\pi i \frac{e^{1/2}}{-\pi}} = \frac{2\pi i}{-\pi}\left(e^{1/2} - e^{-1/2}\right)$$

$$= -2i\left[2\sinh(1/2)\right]$$

? Evaluate $\oint_C \left[\frac{z e^{\pi z}}{z^4-16} + z e^{\pi/z}\right] dz \quad 9x^2+y^2=9$

$z^4-16=0 \Rightarrow (z^2+4)(z^2-4)=0$

$\Rightarrow z^2+4=0 \Rightarrow z=\pm 2i$

$\Rightarrow z^2-4=0 \Rightarrow z=\pm 2.$

Singularity at $z=\pm 2i, \pm 2, 0.$ $z=0$ (essential)

$\oint_C \frac{z e^{\pi z}}{z^4-16} dz + \oint_C z e^{\pi/z} dz$

$I = I_1 + I_2$

At $z=2i$	$9x^2+y^2=4.$	}	inside
$z=-2i$	$9x^2+y^2=4$		
$z=2$	$9x^2+y^2=36$	}	outside
$z=-2$	$9x^2+y^2=36.$		
$z=0$	$9x^2+y^2=0$		inside

At $z=2i$

$$\operatorname*{Res}_{z=2i} f(z) = \lim_{z=2i} (z-2i) \frac{z e^{\pi z}}{(z+2i)(z-2i)(z^2-4)}$$

$$= \lim_{z\to 2i} \frac{z e^{\pi z}}{(z+2i)(z^2-4)}$$

$$= \frac{2i e^{2\pi i}}{(4i)(-4-4)} = \frac{e^{2\pi i}}{-16} = \frac{\cos 2\pi + i\sin 2\pi}{-16}$$

$$= \frac{1+0}{-16} = \frac{1}{-16}$$

At $z = -2i$

$$\operatorname*{Res}_{z=-2i} f(z) = \lim_{z\to -2i} (z+2i) \frac{z e^{\pi z}}{(z+2i)(z-2i)(z^2-4)}$$

$$= \lim_{z\to -2i} \frac{z e^{\pi z}}{(z-2i)(z^2-4)}$$

$$= \frac{-2i e^{-2\pi i}}{(-4i)(-4-4)} = \frac{e^{-2\pi i}}{-16} = \frac{\cos(-2\pi)+i\sin(-2\pi)}{-16} = -\frac{1}{16}$$

$$I_1 = 2\pi i \left[\cancel{\frac{e^{2\pi i}}{-16} + \frac{e^{-2\pi i}}{-16}}\right] = 2\pi i\left(\frac{-1}{16} - \frac{1}{16}\right)$$

$$= \cancel{\frac{2\pi i}{-16}\left(e^{2\pi i} + e^{-2\pi i}\right)} = 2\pi i\left(\frac{-1}{8}\right) = -\frac{\pi i}{4}$$

$$= \cancel{-\frac{\pi i}{8}\left[2\cosh(2\pi i)\right]} = \cancel{-\frac{\pi i}{4}\cosh(2\pi i)}$$

At $z = 0$

$$z e^{\pi/z} = z\left[1 + \frac{\pi}{z\,1!} + \frac{\pi^2}{z^2\,2!} + \frac{\pi^3}{z^3\,3!} + \cdots\right]$$

$$= z + \frac{\pi}{1!} + \frac{\pi^2}{z\,2!} + \frac{\pi^3}{z^3\,3!} + \cdots$$

$$I_2 = \oint_C z e^{\pi/z} = \text{coef of } 1/z \times 2\pi i$$

$$= \frac{\pi^2}{2!} = \frac{\pi^2}{2} 2\pi i = \pi^3 i$$

$$\oint \left[\frac{z e^{\pi z}}{z^4 - \ldots} + z e^{\pi/z}\right] dz = -\frac{\pi i}{4}\cancel{\cosh(2\pi i)} + \pi^3 i$$

1/11/17

Evaluation of Real Definite Integrals Using Residue Integration

Type I : Integrals of the form $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$

Method

Type I is along a unit circle $|z|=1$

$$z = e^{i\theta} \Rightarrow dz = e^{i\theta}\, i\, d\theta$$

$$d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$$

$$d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+1/z}{2}$$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-1/z}{2i}$$

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = \oint_C F\left(\frac{z+1/z}{2}, \frac{z-1/z}{2i}\right)\frac{dz}{iz}$$

where c is $|z|=1$

Then to evaluate type I

Apply Cauchy's residue theorem in which we find out the residues at singularities which lies inside. $|z|=1$

? Evaluate $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$

A. It is of Type I

$$z = e^{i\theta}$$

$$d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{z+1/z}{2}$$

0 to $2\pi \Rightarrow c$ is $|z| = 1$

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \oint_c \frac{\frac{dz}{iz}}{2+\frac{(z+1/z)}{2}}$$

$$= \oint_c \frac{\frac{dz}{iz}}{2+\frac{(z^2+1)}{2z}}$$

$$= \oint_c \frac{\frac{dz}{i\cancel{z}}}{\frac{4z+z^2+1}{\cancel{z}}}$$

$$= \oint_c \frac{2}{i}\left[\frac{dz}{z^2+4z+1}\right]$$

Singularity when $z^2+4z+1 = 0$.

$$z = -2+\sqrt{3}$$

$$z = -2-\sqrt{3}$$

The singularity $z = -2+\sqrt{3}$ lies inside $|z| = 1$

$$\operatorname*{Res}_{z=-2+\sqrt{3}} f(z) = \lim_{z \to -2+\sqrt{3}} (z-(-2+\sqrt{3})) \cdot \frac{2}{i} \frac{1}{z^2+4z+1}$$

$$= \lim_{z \to -2+\sqrt{3}} \cancel{(z-(-2+\sqrt{3}))} \frac{2}{i} \frac{1}{\cancel{(z-(-2+\sqrt{3}))}(z-(-2-\sqrt{3}))}$$

$$= \lim_{z \to -2+\sqrt{3}} \frac{2}{i} \frac{1}{(z-(-2-\sqrt{3}))}$$

$$= \frac{2}{i} \frac{1}{(-2+\sqrt{3})-(-2-\sqrt{3})}$$

$$= \frac{2}{i} \frac{1}{-2+\sqrt{3}+2+\sqrt{3}} = \frac{2}{i} \frac{1}{2\sqrt{3}} = \frac{1}{i\sqrt{3}}$$

By residue theorem.

$$\oint_0^{2\pi} \frac{d\theta}{2+\cos\theta} = 2\pi i \left[\operatorname*{Res}_{z=-2+\sqrt{3}} f(z) \right]$$

$$= 2\pi i \left(\frac{1}{i\sqrt{3}} \right)$$

$$= \frac{2\pi}{\sqrt{3}}$$

2/11/17 ? Evaluate $\int_0^{\pi} \frac{d\theta}{(5-3\cos\theta)^2}$

A. This is not of Type I, make it to 0 to 2π

$z = e^{i\theta}$

If $f(2\pi-\theta) = f\theta$ then $\int_0^{2\pi} f(\theta)\, d\theta = 2\int_0^{\pi} f(\theta)\, d\theta$

or $\int_0^{\pi} f(\theta)\, d\theta = \frac{1}{2}\int_0^{2\pi} f(\theta)\, d\theta$

$$f(\theta) = \frac{1}{(5-3(\cos\theta))^2}$$

$$f(2\pi-\theta) = \frac{1}{[5-3\cos(2\pi-\theta)]^2}$$

$$= \frac{1}{(5-3\cos\theta)^2} = f(\theta)$$

$$\therefore \int_0^{\pi} \frac{1}{(5-3\cos\theta)^2} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{(5-3\cos\theta)^2}$$

Now it is of Type I.

$$z = e^{i\theta}$$

$$d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{z+1/z}{2}$$

$$\int_0^{2\pi} \frac{d\theta}{(5-3\cos\theta)^2} = \frac{\frac{dz}{iz}}{\left(5-3\left(\frac{z+1/z}{2}\right)\right)^2}$$

$$= \frac{dz/iz}{\left(5-3\,\frac{z^2+1}{2z}\right)^2}$$

$$= \frac{dz/iz}{\left(\frac{10z-3(z^2+1)}{2z}\right)^2}$$

$$= \frac{\frac{dz}{iz}}{\left(\frac{10z-3z^2-3}{2z}\right)^2} = \frac{\frac{dz}{iz}}{\left(\frac{-(-10z+3z^2+3)}{2z}\right)^2}$$

$$= \frac{\frac{dz}{iz}}{\frac{(3z^2-10z+3)^2}{4z^2}}$$

$$= \frac{4z\,dz}{i(3z^2-10z+3)^2}$$

$$\frac{1}{2}\oint_0^{2\pi} \frac{d\theta}{(5-3\cos\theta)^2} = \frac{1}{2i}\oint_C \frac{4z\,dz}{(3z^2-10z+3)^2}$$

Singularity when $3z^2-10z+3=0$.

$z = 3, \frac{1}{3}$

At $z = 3$ $|z| = 3 > 1$ lies outside. $3z^2 - 10z + 3$

At $z = 1/3$ $|z| = 1/3 < 1$ lies inside $\cancel{(z-3)(z-1/3)}$ is wrong since we assume it as 0

NOTE:

Let $az^2 + bz + c = 0$ has roots at $z = \alpha$ & $z = \beta$

then $\boxed{az^2 + bz + c = a(z-\alpha)(z-\beta)}$

This is done whenever coefficient of $z^2 \neq 1$

$$f(z) = \frac{4z}{(3z^2 - 10z + 3)^2}$$

$3(z-3)(z-1/3)$

$(z-3)(3z-1)$

$$= \frac{4z}{(3(z-3)(z-1/3))^2}$$

At $z = 1/3$

$$\operatorname{Res}_{z=1/3} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z)\right] \frac{1}{2i}$$

$$= \frac{1}{1!} \lim_{z \to 1/3} \frac{d}{dz} \cancel{(z-1/3)^2} \frac{1}{2i} \frac{4z}{(3(z-3)\cancel{(z-1/3)})^2}$$

$$= \lim_{z \to 1/3} \frac{d}{dz} \frac{1}{2i} \frac{4z}{9(z-3)^2}$$

$$= \lim_{z \to 1/3} \frac{1}{2i} \frac{4}{9} \frac{(z-3)^2 \cdot 1 - z \cdot 2(z-3)}{(z-3)^4}$$

$$= \lim_{z \to 1/3} \frac{1}{2i} \frac{4}{9} \left[\frac{(z-3) - 2z}{(z-3)^3}\right]$$

$$= \frac{2}{9i}\left[\frac{1/3 - 3 - 2/3}{(1/3 - 3)^3}\right]$$

$$= \frac{2}{9i}\left[\frac{-3-\frac{1}{3}}{\left(-\frac{8}{3}\right)^3}\right]$$

$$= \frac{2}{9i}\left[\frac{-\frac{10}{3}}{-\frac{512}{27}}\right]$$

$$= \frac{2}{9i}\cdot\frac{45}{256} = \frac{5}{128i}$$

$$\int_0^{2\pi} \frac{d\theta}{(5-3\cos\theta)^2} = 2\pi i$$

$$= 2\pi i \times \frac{5}{128i}$$

$9\pi/2$

$$= \frac{5\pi}{64}$$

~~When z=3~~

Res

Q. Evaluate $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5-4\cos\theta}\, d\theta$

A. This is of Type I.

$$z = e^{i\theta}$$

$$d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{z+1/z}{2} \quad ; \quad \cancel{\cos\theta}\ \sin\theta = \frac{z-1/z}{2i}$$

$$\int_0^{2\pi} \frac{\sin^2\theta}{5-4\cos\theta}\, d\theta = \oint_C \frac{\left(\frac{z-1/z}{2i}\right)^2}{5-4\left(\frac{z+1/z}{2}\right)} \cdot \frac{dz}{iz}$$

$$= \oint_C \frac{\left(\frac{z^2-1}{z\,2i}\right)^2}{\frac{5-2(z^2+1)}{z}} \frac{dz}{iz}$$

$$= \oint_C \frac{\left(\frac{z^2-1}{z\,2i}\right)^2}{5z-2(z^2+1)} \frac{dz}{i}$$

$$= \frac{(z^2-1)^2}{-4z^2} \cdot \frac{dz}{i} \cdot \frac{1}{5z-2z^2-2}$$

$$= \oint_C \frac{(z^2-1)^2\, dz}{i4z^2(2z^2-5z+2)}$$

Singularity at $z=0$ order $=2$. lies inside.

$$2z^2-5z+2=0$$

$$z=\frac{5\pm\sqrt{25-16}}{4}=\frac{5\pm3}{4}=\frac{8}{4},\frac{2}{4}=2,\frac{1}{2}$$

$z=2$. lies outside

$z=1/2$. lies inside

At $z=0$

$$\underset{z=0}{\text{Res}}\, f(z)=\frac{1}{1!}\lim_{z\to0}(z-0)^2\frac{d}{dz}\frac{(z^2-1)^2}{i4z^2(2z^2-5z+2)}$$

$$=\lim_{z\to0}\frac{d}{dz}\frac{(z^2-1)^2}{i4(2z^2-5z+2)}=\lim_{z\to0}\frac{1}{4i}\frac{(z-2)(z-1/2)2(z^2-1)2z-(z^2-1)^2(4z-5)}{(2z^2-5z+2)^2}$$

$$=\frac{1}{4i}\frac{0+5}{4}=\frac{5}{16i}$$

At $z=1/2$

$$\underset{z=1/2}{\text{Res}}\, f(z)=\lim_{z\to1/2}(z-1/2)\frac{(z^2-1)^2}{i4z^2\cdot2(z-2)(z-1/2)}$$

$$=\lim_{z\to1/2}\frac{(z^2-1)^2}{i8z^2(z-2)}$$

$$=\frac{(1/4-1)^2}{i\cdot8\cdot\frac{1}{4}\cdot\left(\frac{1}{2}-2\right)}=\cancel{\frac{x}{2i\left(\frac{-3}{2}\right)}}=\cancel{\frac{-x}{3i}}$$

$$=\frac{1}{8i}\frac{9/16}{-3/8}=\frac{1}{8i}\frac{-3}{2}=\frac{-3}{16i}$$

$$\int_0^{2\pi} \frac{\sin^2\theta}{5-4\cos\theta}\, d\theta = 2\pi i \left(\frac{5}{16i} - \frac{3}{16i}\right)$$

$$= 2\pi i \left(\frac{+2}{16i}\right) = \frac{+4\pi}{16} \ldots$$

$$= \frac{\pi}{4}$$

$\frac{1}{8} - \frac{1}{3}$

$\frac{3-8}{24}$

$\frac{-5}{24}$

4/11/17

Q. $\int_0^{\pi} \frac{d\theta}{a+b\cos\theta}, \quad a>|b|$

A. This is not of Type 1

$$\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} \qquad \left(f(2\pi-\theta) = f(\theta)\right)$$

$$\Rightarrow \frac{1}{a+b(2\pi-\theta)} = \frac{1}{a+b\cos\theta}$$

Now this of Type 1

$$z = e^{i\theta}$$

$$d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{z+1/z}{2} \qquad \theta \to 0 \text{ to } 2\pi \Rightarrow C = |z| = 1$$

$$\frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \frac{1}{2}\oint_C \frac{\frac{dz}{iz}}{a+b\left(\frac{z+1/z}{2}\right)}$$

$$= \frac{1}{2}\oint_C \frac{\frac{dz}{iz}}{a+b\frac{z^2+b}{2z}}$$

$$= \frac{1}{2}\oint_C \frac{\frac{dz}{iz}}{\frac{2az+bz^2+b}{2z}}$$

$$= \frac{1}{2}\oint_C \frac{\frac{dz}{i}}{\frac{2az+bz^2+b}{2}}$$

$$= \frac{1}{2}\cdot\frac{2}{i}\oint_C \frac{dz}{bz^2+2az+b}$$

$bz^2+2az+b$

$$z = \frac{-2a \pm \sqrt{4a^2-4b^2}}{2b}$$

$$= \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

Singularity at $z = \dfrac{-a + \sqrt{a^2-b^2}}{b}$

$$z = \frac{-a - \sqrt{a^2-b^2}}{b}$$

$a > |b|$ put $a = 2$, $b = 1$

$2 > |b|$, $z = \dfrac{-2+\sqrt{3}}{1}$ lies inside $z = \dfrac{-2-\sqrt{3}}{1}$ lies outside

At $z = \dfrac{-a+\sqrt{a^2-b^2}}{b}$

$$\operatorname*{Res}_{z=\frac{-a+\sqrt{a^2-b^2}}{b}} f(z) = \lim_{z \to \frac{-a+\sqrt{a^2-b^2}}{b}} \left(z - \left(\frac{-a+\sqrt{a^2-b^2}}{2}\right)\right)$$

$$\times \frac{1}{i}\frac{dz}{(bz^2+2az+b)}$$

$$= \lim_{z \to \frac{-a+\sqrt{a^2-b^2}}{b}} \left(z - \left(\cancel{\frac{-a+\sqrt{a^2-b^2}}{b}} \right) \right) \cdot \frac{1}{i} \frac{1}{b\left(z - \left(\frac{-a+\sqrt{a^2-b^2}}{b}\right)\right)}$$

$$z - \left(\frac{-a-\sqrt{a^2-b^2}}{b}\right)$$

$$= \lim_{z \to \frac{-a+\sqrt{a^2-b^2}}{b}} \frac{1}{ib\left[z - \left(\frac{-a-\sqrt{a^2-b^2}}{b}\right)\right]}$$

$$= \frac{1}{ib} \frac{1}{\frac{-a+\sqrt{a^2-b^2}}{b} - \left(\frac{-a-\sqrt{a^2-b^2}}{b}\right)}$$

$$= \frac{1}{ib} \frac{1}{\frac{2\sqrt{a^2-b^2}}{b}}$$

$$= \frac{1}{2i\sqrt{a^2-b^2}}$$

$$\int_0^{\pi} \frac{d\theta}{a+b\cos\theta} = 2\pi i \frac{1}{2i\sqrt{a^2-b^2}}$$

$$= \frac{\pi}{\sqrt{a^2-b^2}}$$

? Evaluate $\int_0^{2\pi} \frac{d\theta}{1-2a\sin\theta+a^2}$, $0 < a < 1$

A. This is of Type 1

$$z = e^{i\theta}$$

$$d\theta = \frac{dz}{iz}$$

$$\cancel{\cos\theta} = z$$

$$\sin\theta = \frac{z - 1/z}{2i}$$

$$\int_0^{2\pi} \frac{d\theta}{1 - 2a\sin\theta + a^2} = \oint_C \frac{dz/iz}{1 - 2a\left(\frac{z - 1/z}{2i}\right) + a^2}$$

$$= \oint_C \frac{dz/iz}{1 - 2a\left(\frac{z^2 - 1}{2zi}\right) + a^2}$$

$$= \oint_C \frac{dz/iz}{1 - \left(\frac{az^2 - a}{zi}\right) + a^2}$$

$$= \oint_C \frac{dz/iz}{\frac{zi - az^2 + a + zia^2}{zi}}$$

$$= \oint_C \frac{dz}{zi - az^2 + a + zia^2}$$

Singularity when $-az^2 + z(a^2+1)i + a = 0$

$$a = \frac{-ia^2 - i \pm \sqrt{(a^2+1)^2 i^2 - 4 \cdot -a \cdot a}}{-2a}$$

$$= \frac{-ia^2 - i \pm \sqrt{-(a^4+2a^2+1)+4a^2}}{-2a}$$

$$= \frac{-ia^2 - i \pm \sqrt{-a^4 - 2a^2 - 1 + 4a^2}}{-2a}$$

$$= \frac{-i(a^2+1) \pm \sqrt{-a^4 + 2a^2 - 1}}{-2a}$$

$$= \frac{-i(a^2+1) \pm \sqrt{-(a^4 - 2a^2 + 1)}}{-2a}$$

$$= \frac{-i(a^2+1) \pm \sqrt{-(a^2-1)^2}}{-2a}$$

$$= \frac{-i(a^2+1) \pm i(a^2-1)}{-2a}$$

$$= \frac{-i(a^2+1) + i(a^2-1)}{-2a}, \quad \frac{-i(a^2+1) - i(a^2-1)}{-2a}$$

$$= \frac{-ia^2 - i + ia^2 - i}{-2a}, \quad \frac{-ia^2 - i - ia^2 + i}{-2a}$$

$$= \frac{-2i}{-2a}, \quad \frac{-2ia^2}{-2a}$$

$$= \frac{i}{a}, \quad ia.$$

Type II

Integral of the form $\int_{-\infty}^{\infty} f(x)\,dx$

To evaluate Type II integral replace the terms in x by z.
We take the closed contour curve as a semicircle lie in the upper half plane (excluding singularities on the real axis) with radius R. And we assume that R is very large so that it encloses all the singularities of the given function.

Take the singularities which lie in the upper half planes and find residues.
Finally apply Cauchy's residue theorum to evaluate Type II.

Note :-

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

which is called the Cauchy Principle value of the integral and is written as pr.v $\int_{-\infty}^{\infty} f(x)\,dx$

? Evaluate $\int_{-\infty}^{\infty} \frac{dx}{x^2+4}$

A. Consider $\oint_C \frac{dz}{z^2+4}$

Singularity at $z = +2i, -2i$

$z = 2i$ lies in the upper half plane

$$\operatorname{Res}_{z=2i} f(z) = \lim_{z\to 2i} (z-2i) \frac{1}{z^2+4}$$

$$= \lim_{z\to 2i} (z-2i) \frac{1}{(z+2i)(z-2i)}$$

$$= \lim_{z\to 2i} \frac{1}{z+2i} = \underline{\underline{\frac{1}{4i}}}$$

$$\oint_C \frac{dz}{z^2+4} = 2\pi i \, \frac{1}{4i}$$

$$= \underline{\underline{\frac{\pi}{2}}}$$

? Evaluate $\int_0^\infty \frac{x^2 dx}{(x^2+9)(x^2+4)^2}$

A. Note: $\int_{-a}^{a} f(x) = 2\int_0^a f(x)\,dx$ if $f(x)$ is even $f(-x) = f(x)$

$\int_{-a}^{a} f(x) = 0$ if $f(x)$ is odd. $f(-x) = -f(x)$

This function is even. So.

$$\therefore \int_0^{\infty} \frac{x^2\,dx}{(x^2+9)(x^2+4)^2} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+9)(x^2+4)^2}$$

Now this of type II.

~~Singularity at~~

Replacing

Consider $\oint_C \frac{1}{2}\frac{z^2\,dz}{(z^2+9)(z^2+4)^2}$

Singularities at $(z^2+9)(z^2+4)^2 = 0$.

$$z = +3i, -3i \quad \text{order} = 1$$

$$z = +2i, -2i \quad \text{order} = 2$$

$z = +3i$ & $z = 2i$ lies in the upper half plane.

At $z = 3i$

$$\operatorname*{Res}_{z=3i} f(z) = \lim_{z\to 3i} \cancel{(z-3i)}\,\frac{1}{2}\,\frac{z^2}{(z+3i)\cancel{(z-3i)}(z^2+4)^2}$$

$$= \lim_{z\to 3i} \frac{z^2}{2(z+3i)(z^2+4)^2}$$

$$= \frac{-9}{2(6i)(-9+4)^2} = \frac{-9}{(12i)(25)} = \frac{-3}{100i}$$

At $z = 2i$

$$\underset{z=2i}{\text{Res}}\, f(z) = \frac{1}{1!} \lim_{z \to 2i} \frac{d}{dz} \cancel{(z-2i)^2} \cdot \frac{z^2}{2(z^2+9)((z+2i)\cancel{(z-2i)})^2}$$

$$= \lim_{z \to 2i} \frac{d}{dz} \frac{z^2}{2(z^2+9)(z+2i)^2}$$

$$= \lim_{z \to 2i} \frac{1}{2} \frac{(z^2+9)(z+2i)^2 \cdot 2z - z^2 \frac{d}{dz}\left[(z^2+9)(z+2i)^2\right]}{(z^2+9)^2(z+2i)^4}$$

$$= \lim_{z \to 2i} \frac{1}{2} \frac{(z^2+9)(z+2i)^2 \cdot 2z - z^2\left[(z^2+9)\, 2(z+2i) + (z+2i)^2 2z\right]}{(z^2+9)^2(z+2i)^4}$$

$$= \lim_{z \to 2i} \frac{1}{2} \frac{(z^2+9)(z+2i)^2 2z - z^2(z^2+9)2(z+2i) + z^2(z+2i)^2(2z)}{(z^2+9)^2(z+2i)^4}$$

$$= \frac{1}{2} \frac{((-4+9)(4i)^2 4i) - (-4(-4+9)2(4i)) + (-4(4i)^2(4i))}{(-4+9)^2(4i)^4}$$

$$= \frac{1}{2}\left[\frac{-320i + 4[40i - 64i]}{25 \times 256}\right]$$

$$= \frac{1}{2}\left(\frac{-320i - 96i}{25 \times 256}\right) = \frac{-13i}{400}$$

$$\therefore \oint \frac{z^2\,dz}{(z^2+9)(z^2+16)^2} = 2\pi i\left(\frac{-3}{100i} - \frac{13i}{400}\right)$$

$$= \cancel{2\pi i\left(\frac{-1200+1300}{100i\times 400}\right)}$$

$$= \cancel{2\pi\left(\frac{-12+13}{4}\right) = \frac{\pi}{2}}$$

$$= 2\pi i\left[\frac{-3}{100i} + \frac{13}{400i}\right]$$

$$= \frac{\pi}{200}$$

Type III

Integrals of the form $\int_{-\infty}^{\infty} f(x) \sin mx\, dx$ or $\int_{-\infty}^{\infty} f(x) \cos mx\, dx$.

To evaluate type III consider the integral $\oint_C e^{imz} f(z)\, dz$. Find out the singularities which lie in the upper half plane. Then find residues at those singularities. Finally apply Cauchy's residue theorum. Then take the real part from the final integral if the given integral is $\int_{-\infty}^{\infty} f(x) \cos mx\, dx$

And take the imaginary part if the given integral is $\int_{-\infty}^{\infty} f(x) \sin mx$..

Here the closed curve is the semicircle in the upper half plane.

? Evaluate $\int_{-\infty}^{\infty} \frac{\cos x \, dx}{x^2+1}$

A. Here $m=1$

Consider $\oint_C \frac{e^{iz}}{z^2+1} dz$

Singularites at $z = i, -i$

$z=i$ lies in the upper half plane.

$$\underset{z=i}{\text{Res}}\, f(z) = \lim_{z \to i} (z-i) \frac{e^{iz}}{(z+i)(z-i)}$$

$$= \lim_{z\to i} \frac{e^{iz}}{z+i}$$

$$= \frac{e^{i^2}}{i+i} = \frac{e^{-1}}{2i}$$

Applying Cauchy's residue theorum

$$\oint_C \frac{e^{iz}}{z^2+1} dz = 2\pi i \frac{e^{-1}}{2i} = \pi e^{-1}$$

$$\int_{-\infty}^{\infty} \frac{\cos x \, dx}{1+x^2} = \text{Real part of } \oint_C \frac{e^{iz}}{z^2+1} dz$$

$$= \underline{\underline{\pi e^{-1}}}$$

? Evaluate $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2} dx$ where $a > 0$ $\quad \pi e^{-a}$

? Evaluate $\int_{0}^{\infty} \frac{x^2 dx}{(x^2+1)(x^2+9)}$