1/8/2017
Monday

General representation, $z = a + ib$.

Conjugate, $\bar{z} = a - ib$.

Real Part $\quad \text{Re } z = a = \dfrac{z+\bar{z}}{2}$

Imaginary Part $\quad \text{Im } z = b = \dfrac{z-\bar{z}}{2i}$

Modulus, $|z| = \sqrt{a^2+b^2} = \sqrt{(\text{Re } z)^2 + (\text{Im } z)^2}$

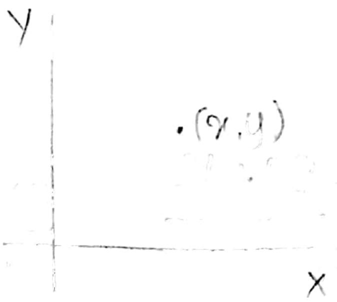
Polar form

Co-ordinates (r, θ)

$$z = re^{i\theta}$$
$$= r(\cos\theta + i\sin\theta)$$

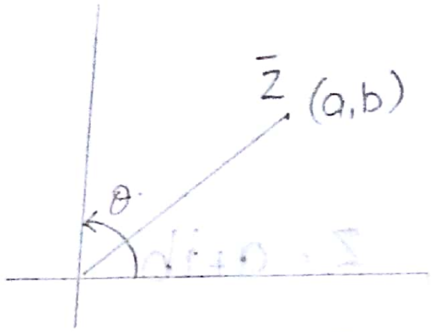
$\Rightarrow$ $x = r\cos\theta$
$y = r\sin\theta$
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Complex Plane / Argand Plane



$(x, y) \rightarrow x + iy$

Polar Representation



$r = |z|$

Addition & Substraction. & Multiplication & Division.

? Let $z_1 = -5+2i$ & $z_2 = 3-5i$. find z_1+z_2 & z_1-z_2

$\rightarrow$ $z_1 + z_2 = -2-3i$

$\rightarrow$ $z_1 - z_2 = -8+7i$

$\rightarrow$ $z_1 z_2 = (-5+2i)(3-5i)$

$= -15+25i+6i+10 = -5+31i$

$\rightarrow$ $\frac{z_1}{z_2} = \frac{-5+2i}{3-5i} \cdot \frac{3+5i}{3+5i}$

$= \frac{-15-25i+6i+-10}{9+25} = \frac{-25-19i}{34} = \frac{-25}{34} - \frac{19}{34}i$

? $z_1 = 2-3i$; $z_2 = -1-4i$

$\frac{z_1}{z_2} = \frac{2-3i}{-1-4i} \cdot \frac{-1+4i}{-1+4i} = \frac{-2+8i+3i+12}{1+16} = \frac{10+11i}{17}$

$= \frac{10}{17} + \frac{11}{17}i$

? $\frac{z_1}{z_2} = \frac{-10-i}{-2-3i} \quad \frac{-2+3i}{-2+3i}$

$= \frac{20-30i+2i+3}{4+9} = \frac{23-28i}{13} = \frac{23}{13} - \frac{28}{13}i$

Circle Representation in Complex Plane

> centre $(0,0)$; $x^2+y^2 = r^2$

$|z|^2 = r^2$

$|z| = r$

> centre $(a,0)$; $|z-a| = r$

$|x+iy-a| = r$

$|(x-a)+iy|^2 = r^2$

$(x-a)^2+y^2 = r^2$

$|z-a| = r$

$|x+iy-a| = r$

$|(x-a)+iy| = r$

$\sqrt{(x-a)^2+y^2} = r$

$(x-a)^2+y^2 = r^2$

$(x-h)^2+(y-k)^2 = r^2$

> centre $(0,a)$; $|z-ai| = r$

> centre $(-a,0)$; $|z+a| = r$

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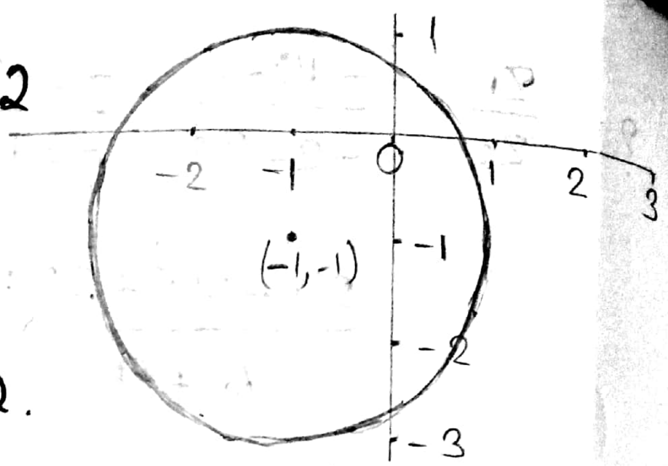
centre $(0,-a)$; $|z+ai| = r$

? Draw the circle $|z+(1+i)| = 2$.

Radius : 2.

center : $(-1,-1)$

$$|(x+iy)+(1+i)|=2$$



$$|(x+1)+i(y+1)|=2$$

$$\sqrt{(x+1)^2+(y+1)^2}=2.$$

$$(x+1)^2+(y+1)^2=4 \approx (x-h)^2+(y-k)^2=r^2.$$

center : $(-1,-1)$

radius : 2.

? $|z+i|=3$

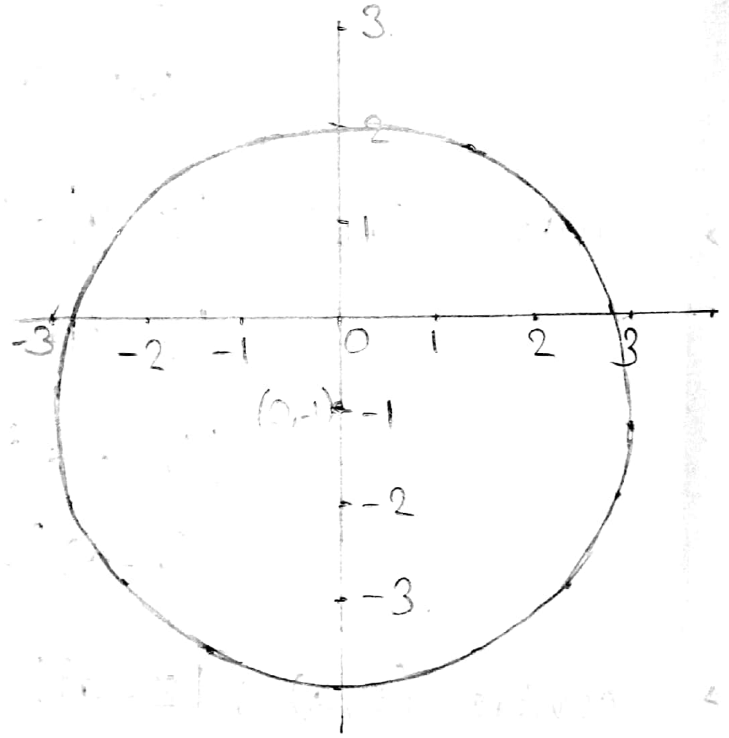
$$|x+iy+i|=3$$

$$|x+i(y+1)|=3$$

$$\sqrt{x^2+(y+1)^2}=3$$

centre : $(0,-1)$

radius : 3

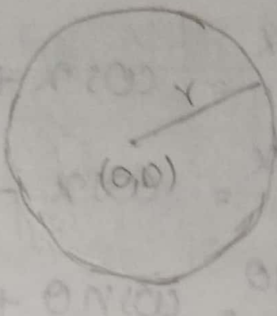


<u>Disk</u>

The equation of an open disk with centre (0,0) and radius 'r' is denoted as

$$|z|<r$$

Equation of a closed disk with centre (0,0) & radius, $|z| \leq r$.



<u>Note</u> :

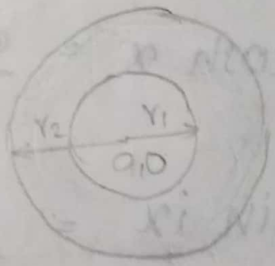
$|z-a| < \rho$

Open disk with centre (a,0) & radius ρ

It is also called ρ-neighbourhood.

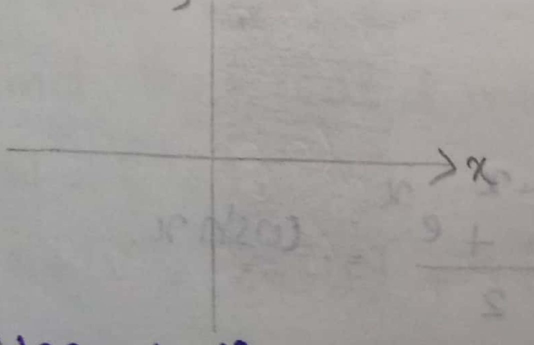
Annulus

$r_1 \leq |z-a| \leq r_2$

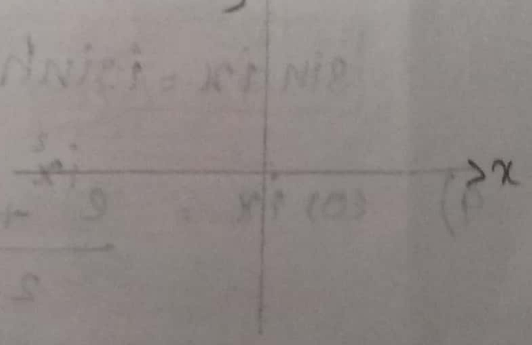


Half Planes

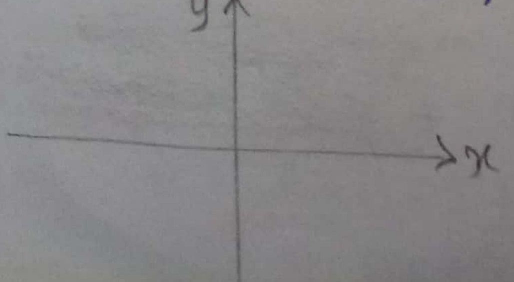
Right half plane ; $x>0$.



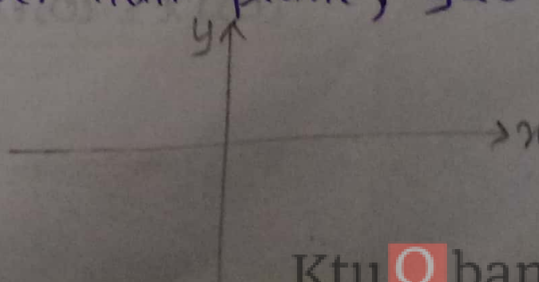
Left half plane ; $x<0$.



Upper half plane ; $y>0$



Lower half plane ; $y<0$.



3/8/2017
Thursday

1) $e^{ix} = \cos x + i\sin x$. (Euler's formula)

2) $e^{-ix} = \cos x - i\sin x$.

3) $e^{in\theta} = \cos n\theta + i\sin n\theta$ (De Moivre's formula)

4) $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

5) $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

6) $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$

7) $\cosh x = \dfrac{e^{x} + e^{-x}}{2}$

8) $\sin ix = \dfrac{e^{i^2x} - e^{-i^2x}}{2i} = \dfrac{e^{-x} - e^{x}}{2i} = -\dfrac{\sinh x}{i}$ $\quad \boxed{\frac{1}{i} = -i}$

$= i\sinh x$

$$\boxed{\sin ix = i\sinh x}$$

9) $\cos ix = \dfrac{e^{i^2x} + e^{-i^2x}}{2} = \dfrac{e^{-x} + e^{x}}{2} = \cosh x$.

$$\boxed{\cos ix = \cosh x}$$

FUNCTION OF A COMPLEX VARIABLE

Let $w = f(z)$ be a function of complex variable where to each value of complex variable z we assign another complex w.

We have $z = x+iy$ & $w = u+iv$ where u & v are real & imaginary part of w respectively.

And u & v are functions of x & y.

$$\therefore \; w = u(x,y) + iv(x,y)$$

For Example:

Let $w = z^2$

$$u+iv = (x+iy)^2$$

$$u+iv = x^2 + 2ixy - y^2$$

Equating real & imaginary parts.

$$u = x^2 - y^2$$

$$v = 2xy$$

Q. Find the real & imaginary parts of

? $w = z^2 + 3z$

$$u+iv = (x+iy)^2 + 3(x+iy)$$

$$= x^2 + 2ixy - y^2 + 3x + 3iy$$

$$u = x^2 - y^2 + 3x$$

$$v = 2xy + 3y$$

$$= y(2x+3)$$

? $W = 2iz + 6\bar{z}$

$u+iv = 2i(x+iy) + 6(x-iy)$

$= 2ix - 2y + 6x - 6iy$

$u = 6x - 2y$

$v = 2x - 6y$

? $W = \sin z$

$u+iv = \sin(x+iy)$

$= \sin x \cos iy + \cos x \sin iy$

$= \sin x \cosh y + \cos x \, i \sinh y$

$u = \sin x \cosh y$

$v = \cos x \sinh y$

? $W = \cos \bar{z}$

$u+iv = \cos(x-iy)$

$= \cos x \cos iy + \sin x \sin iy$

$= \cos x \cosh y + \sin x \, i \sinh y$

$u = \cos x \cosh y$

$v = \sin x \sinh y$

4/8/2017.

? Find the values of real f and image f.

$$f(z) = \frac{1}{1+z} \quad \text{at} \quad z = (1+i)$$

$$u+iv = \frac{1}{1+(x+iy)} = \frac{1+x-iy}{(1+x)^2-(iy)^2} = \frac{1+x-iy}{(1+x)^2+y^2}.$$

$$u = \frac{1+x}{(1+x)^2+y^2} \qquad v = \frac{-y}{(1+x)^2+y^2}$$

from $z = (1+i) \Rightarrow x = 1 \quad y = 1$

$$u = \frac{1+1}{2^2+1} = \frac{2}{5} \quad ; \quad v = \frac{-1}{2^2+1} = \frac{-1}{5}$$

? $f(z) = \frac{z-1}{z+1}$ at $z = 2i$

$$u+iv = \frac{x+iy-1}{x+iy+1}$$

$$= \frac{x+iy-1}{x+1+iy} = \frac{x-1+iy}{x+1+iy} = \frac{x-1+iy}{x+1+iy}\frac{(x+1-iy)}{(x+1-iy)}$$

$$= \frac{x^2+x-xyi-x-1+iy+xyi+yi-(iy)^2}{(x+1)^2-(iy)^2}$$

$$= \frac{x^2+2yi+y^2-1}{(x+1)^2+y^2}.$$

$$u = \frac{x^2+y^2-1}{(x+1)^2+y^2} \qquad v = \frac{2y}{(x+1)^2+y^2}$$

when $z = 2i$ $\quad x = 0 \quad y = 2$

$$u = \frac{4-1}{1+4} = \frac{3}{5} \qquad v = \frac{4}{4+1} = \frac{4}{5}$$

? find values of real f & image f.

$f(z) = 5z^2 - 12z + 3 + 2i$ $\quad$ at $\quad z = 4 - 3i$

$u + vi = 5(x+iy)^2 - 12(x+iy) + 3 + 2i$

$= 5(x^2 + 2xyi - y^2) - 12x - 12yi + 3 + 2i$

$= 5x^2 - 5y^2 - 12x + 3 + i(10xy - 12y + 2)$

$u = 5x^2 - 5y^2 - 12x + 3$

$v = 10xy - 12y + 2.$

when $z = 4 - 3i$ $\quad x = 4 \quad y = -3.$

$u = 5x^2 - 5y^2 - 12x + 3.$

$= 5 \times 16 - 5 \times 9 - 12 \times 4 + 3.$

$= 80 - 45 - 48 + 3 = \underline{\underline{-10}}.$

$v = 10xy - 12y + 2$

$= 10 \times 4 \times -3 - 12 \times 3 + 2 = \underline{\underline{-82}}$

? $f(z) = |z|^2 \; Im\left(\frac{1}{z}\right)$ at $z = i$

$u + iv = \left(\sqrt{x^2+y^2}\right)^2 Img\left(\frac{1}{x+iy}\right)$

$= x^2 + y^2 \; Img\left(\frac{x-iy}{x^2+y^2}\right)$

$= x^2 + y^2 \times \frac{-y}{x^2+y^2}$

$= -y$

When $z = i$ $\quad x = 0$ $\quad y = 1$

$u = -1$

? Find the value of $f(z) = \frac{Re\, z}{1-|z|}$ at $z = 1-i$

$$u + iv = \frac{Re(x+iy)}{1-\sqrt{x^2+y^2}}$$

$$= \frac{x}{1-\sqrt{x^2+y^2}}$$

$$u = \frac{x}{1-\sqrt{x^2+y^2}}$$

when $z = 1-i$ $\quad x = 1$ $\quad y = -1$

$$u = \frac{1}{1-\sqrt{1+1}} = \frac{1}{1-\sqrt{2}}$$

? Find the value of $\frac{z^3}{(z-1)^3}$ at $-i$

5/8/2017

Limit of Function of a Complex Variable.

Let $w = f(z)$ be a function of complex variable z. The function $f(z)$ is said to have a limit l as z approaches to a point z_0 as is written as.

$$\lim_{z \to z_0} f(z) = l.$$

ie, when the distance b/w z & z_0 coming closer & closer then the distance b/w $f(z)$ and l coming closer and closer.

Mathematically we can write if $\forall$ positive real number $\varepsilon > 0$ we can find another positive real number δ such that $\forall$ $z \neq z_0$,

$$|f(z) - l| = \varepsilon \text{ whenever } |z - z_0| < \delta$$

Continuity of f(z)

i) $f(z)$ is defined at z_0.

ii) $\lim_{z \to z_0}$ exist.

iii) $\lim_{z \to z_0} f(z) = f(z_0)$

Differentiabily of f(z) at z_0.

The derivative of a function $f(z)$ at z_0 is defined as $f'(z_0) = \lim_{z \to z_0} \dfrac{f(z) - f(z_0)}{z - z_0}$.

In general we can write derivative as.

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Note:-

for a function of real variables we have only two directions along which $x \to x_0$ (ie left limit & right limit at x_0).

for function of complex variable we have infinite ways of approaches are possible. if along any two different path we get different values we can conclude that limit does not exist.

? Show that the function $f(z) = z^2$ is differentiable for all z. And hence find the derivative of $f(z)$

A. $f(z) = z^2$

$f(z+\Delta z) = (z+\Delta z)^2$

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$\lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{2z\Delta z + (\Delta z)^2}{\Delta z}$$

$$\lim_{\Delta z \to 0} \frac{\Delta z(2z+\Delta z)}{\Delta z} = \lim_{\Delta z \to 0} (2z+\Delta z) = \underline{\underline{2z}}$$

? Proove that $\bar{z}$ is not differentiable.

$f(z) = \bar{z}$

$f(z+\Delta z) + \bar{z} + \overline{\Delta z}$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

$z = x + iy$

$\Delta z = \Delta x + i\Delta y$

case 1: $\Delta z \to 0$ along x axis.

$y = 0 \Rightarrow \Delta y = 0$

$\Delta z \to 0 \Rightarrow \Delta x \to 0$

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta x \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = \underline{\underline{1}}$$

case 2: $\Delta z \to 0$ along y axis.

$x = 0 \Rightarrow \Delta x = 0$

$\Delta z \to 0 \Rightarrow \Delta y \to 0$

$$\lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{\Delta y \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} =$$

$$= \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = \underline{\underline{-1}}$$

$\therefore$ $\bar{z}$ not differentiable.

18/2017.

? Show that $\lim\limits_{z\to 0} \frac{xy}{x^2+y^2}$ does not exist.

A.

$$\lim_{z\to 0} \frac{xy}{x^2+y^2}$$

To prove that limit is not existing we can take a general path such as $y=mx$ which is a straight line.

$$z\to 0 \Rightarrow \boxed{y=mx}$$

$$z\to 0 \Rightarrow x+iy \to 0$$

$$\Rightarrow x+imx \to 0.$$

$$\Rightarrow x(1+im) \to 0.$$

$$\Rightarrow x\to 0.$$

$$\lim_{z\to 0} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{xmx}{x^2+(mx)^2}$$

$$= \lim_{x\to 0} \frac{mx^2}{x^2+m^2x^2} = \lim_{x\to 0} \frac{m}{1+m^2}$$

$$= \underline{\underline{\frac{m}{1+m^2}}}.$$

ie limit is not unique. Hence limit does not exist.

? Check whether $f(z)$ is continuous at $z=0$ when $f(0)=0$. for $f(z) = |z|^2 . \operatorname{Im}(1/z)$

A. $f(z) = |z|^2 \operatorname{Im}(1/z)$

$$u+iv = \left(\sqrt{x^2+y^2}\right)^2 \cdot \operatorname{Img}\left(\frac{1}{x+iy}\right)$$

$$= x^2+y^2 \operatorname{Img}\left(\frac{x-iy}{x^2+y^2}\right)$$

$$= x^2+y^2 \, \frac{-y}{x^2+y^2} = \underline{\underline{-y}}.$$

* Along straight line

$z \to 0 \quad y = mx.$

$z \to 0 \Rightarrow x+iy \to 0$

$\Rightarrow x+imx \to 0$

$\Rightarrow x(1+im) \to 0$

$\Rightarrow x \to 0$

$$\lim_{z\to 0} -y = \lim_{x\to 0} -mx = \underline{\underline{0}}.$$

* Along x axis. $\quad y=0.$

$\cancel{y} \to 0 \qquad z \to 0 \Rightarrow x+iy \to 0$

$\boxed{x \to 0}$.

$$\lim_{\cancel{x}\to 0} -y = \underline{\underline{0}}$$

* Along y axis $\quad x=0$

$y \to 0. \qquad z \to 0 \Rightarrow x+iy \to 0$

$\Rightarrow y\, \cancel{imx} \to 0.$

$$\lim_{y\to 0} -y = \lim_{x\to 0} -mx = \underline{\underline{0}}.$$

∴ It is continous.

Q. Show that $f(z) = \frac{(x+y)^2}{x^2+y^2}$ is discontinous at the origin given that $f(0)=0$.

A. To prove limit does not exist and it is discontinous.

Along straight line

$y = mx$

$z \to 0 \Rightarrow x+iy \to 0$

$x + imx \to 0$

$x(1+im) \to 0$

$x \to 0$

$$\lim_{z\to 0} \frac{(x+y)^2}{x^2+y^2} = \lim_{x\to 0} \frac{(x+mx)^2}{x^2+(mx)^2}$$

$$= \lim_{x\to 0} \frac{x^2+2x^2m+m^2x^2}{x^2+m^2x^2}$$

$$= \lim_{x\to 0} \frac{1+2m+m^2}{1+m^2} = \frac{1+2m+m^2}{1+m^2}$$

$$= \frac{(1+m)^2}{1+m^2}$$

ie, limit does not exist $\therefore$ It is discontinous.

9/8/17

Q. $f(z) = \begin{cases} \frac{\operatorname{Im} z}{|z|}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

check whether $f(z)$ is continous at $z=0$.

since check whether check 2/3 cases.

A. $f(z) = \frac{\operatorname{Im} z}{|z|} = \frac{y}{\sqrt{x^2+y^2}}$

* Along straight line.

$y = mx$.

$z \to 0 \Rightarrow x + iy \to 0$

$x + imx \to 0$.

$x(1+im) \to 0$.

$x \to 0$.

$$\lim_{z\to 0} f(z) = \lim_{x\to 0} \frac{y}{\sqrt{x^2+y^2}} = \lim_{x\to 0} \frac{mx}{\sqrt{x^2+m^2x^2}}$$

$$= \lim_{x\to 0} \frac{m}{\sqrt{1+m^2}} = \frac{m}{\sqrt{1+m^2}}.$$

* Along x-axis.

$y = 0$.

$z \to 0 \Rightarrow x + iy \to 0$.

$x \to 0$.

$$\lim_{z\to 0} f(z) = \lim_{x\to 0} \frac{y}{\sqrt{x^2+y^2}} = \underline{\underline{0}}$$

* Along y-axis.

$x = 0$.

$z \to 0 \Rightarrow y \to 0$.

$$\lim_{z\to 0} f(z) = \lim_{y\to 0} \frac{y}{\sqrt{x^2+y^2}} = \lim_{y\to 0} \frac{y}{y} = 1.$$

entire all complex plane

Analytic Functions

Let $w = f(z)$ be a function of complex variable

- A function $f(z)$ is said to be analytic in a domain D if if $f(z)$ is defined and differentiable at every points of D.
- The function $f(z)$ is said to be analytic at a point $z = z_0$ if $f(z)$ is analytic in a neighbourhood of z_0.
- Analytic functions are also called Regular functions or Holomorphic function.
- In a function $f(z)$ is analytic at every points in the complex plane then it is called entire function.
- Polynomials, exponential functions, trignometric function etc are ($\sin z$, $\cos z$) are always analytic.

NB

- Necessary & sufficient conditions for $f(z)$ to be analytic

Let $f(z) = u(x,y) + iv(x,y)$ be defined and continous in some neighbourhood of a point z and differentiable at z itself. (It is analytic) then,

i) The first order partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \quad \text{exist.}$$

ii) The partial derivatives satisfies Cauchy-Riemann Equation (CR equation)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence if f(z) is analytic, the partial derivatives will satisfies the CR equations at every point of the domain.

? Check whether $f(z) = z^2$ is analytic.

$$f(z) = z^2$$

$$u + iv = (x+iy)^2$$

$$= x^2 - y^2 + i2xy$$

$$u(x,y) = x^2 - y^2$$

$$v(x,y) = 2xy$$

$$\frac{\partial u}{\partial x} = 2x \quad ; \quad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial v}{\partial x} = 2y \quad ; \quad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Here the partial derivaties exists and it satisfies CR equation.

$\Rightarrow$ f(z) is analytic.

? Check whether $f(z) = \bar{z}$ is analytic

$$f(z) = \bar{z}$$

$$u + iv = \overline{x + iy}$$

$$= x - iy$$

$$u(x,y) = x$$

$$v(x,y) = -y$$

$$\frac{\partial u}{\partial x} = 1 \quad ; \quad \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial v}{\partial x} = 0 \quad ; \quad \frac{\partial v}{\partial y} = -1$$

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

It does not satisfy CR equation.

$\therefore$ $f(z)$ is not analytic.

? $W = \sin z$. Check whether it is analytic.

$$u + iv = \sin(x + iy)$$

$$= \sin x \cos iy + \cos x \sin iy$$

$$= \sin x \cosh y + \cos x \, i \sinh y$$

$$u(x,y) = \sin x \cosh y$$

$$v(x,y) = \cos x \sinh y$$

$\frac{\partial u}{\partial x} = \cosh y \cos x$ $\quad\quad$ $\frac{\partial u}{\partial y} = \sin x . \sinh y.$

$\frac{\partial v}{\partial x} = \sinh y \cdot -\sin x$ $\quad\quad$ $\frac{\partial v}{\partial y} = \cos x \cosh y.$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\boxed{\begin{array}{l} \frac{d}{dx}(\sinh x) = \cosh x \\ \frac{d}{dx}(\cosh x) = \sinh x. \end{array}}$

Here all partial derivatives exists and it satisfies CR equation.

$\Rightarrow$ The function is analytic.

? $f(z) = e^z$

$u + iv = e^{x+iy}$

$= e^x . e^{iy}$

$= e^x . (\cos y + i\sin y)$

$= e^x \cos y + e^x i \sin y$

$u(x,y) = e^x \cos y$

$v(x,y) = e^x \sin y$

$\frac{\partial u}{\partial x} = \cos y e^x$ $\quad\quad$ $\frac{\partial u}{\partial y} = e^x . -\sin y$

$\frac{\partial v}{\partial x} = e^x \sin y$ $\quad\quad$ $\frac{\partial v}{\partial y} = e^x \cos y.$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$

$\therefore f(z)$ is analytic.

? Check whether $f(z) = \log z$ which is analytic or not.

$f(z) = \log z$.

$u + iv = \log(x+iy)$

$z = re^{i\theta}$

$u + iv = \log(re^{i\theta})$

$= \log r + \log(e^{i\theta})$

$= \log r + i\theta$

$= \log\sqrt{x^2+y^2} + i\tan^{-1}(y/x)$

$r = |z| = \sqrt{x^2+y^2}$
$\theta = \tan^{-1}(y/x)$

$u(x,y) = \log\sqrt{x^2+y^2}$

$v(x,y) = \tan^{-1}(y/x)$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2\sqrt{x^2+y^2}} \, 2x. = \frac{2x}{2(\sqrt{x^2+y^2})^2} = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y = \frac{y}{x^2+y^2}$$

x^{-1}
$-1x^{-2}$.

$$\frac{\partial v}{\partial x} = \frac{-1}{1+(y/x)^2} \frac{y}{x^2} = \frac{-y}{x^2+y^2}.$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+(y/x)^2} \cdot \cancel{2y}\frac{1}{x} = \frac{\cancel{2y}}{\cancel{1+(y/x)^2}}.$$

$$= \frac{x^2}{x^2+y^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}.$$

Here $\log z$ is analytic except the origin $(0,0)$

? Proove that an analytic function is constant if it's real part is constant.

A. Given that $f(z)$ is analytic

$$f(z) = u(x,y) + i v(x,y)$$

where $u = \text{const}$

Let $u = C$

Differentiating partially

$$\frac{\partial u}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = 0.$$

Since analytic, by CR equation.

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 0.$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0.$$

Since $\frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x}$ is zero.,

$\Rightarrow v = \text{const}.$

$\therefore f(x) = u + iv$ is constant.

? Prove that an analytic function $f(z)$ is constant if it's imaginary part is constant.

A. Given that $f(z)$ is analytic.

$$f(z) = u(x,y) + iv(x,y).$$

where $v = \text{const}$.

$v = c$

Differentiate partially.

$\frac{\partial v}{\partial x} = 0$ $\frac{\partial v}{\partial y} = 0$.

Since it is analytic., it satisfies, CR equation,

$\therefore \Rightarrow \frac{\partial u}{\partial x} = 0$ & $\frac{\partial u}{\partial y} = 0$

Since $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y} = 0$.

u is a constant.

$f(z) = u + iv$ is a constant.

? An analytic function $f(z)$ is constant if its modulus is constant.

A. $f(z) = u(x,y) + iv(x,y)$.

Given that $f(z)$ is analytic

$f(z) = u + iv$.

Also given that $|f(z)| = \text{const}$.

ie, $\sqrt{u^2+v^2} = \text{const}$.

$\Rightarrow u^2 + v^2 = k$ — (1)

Diff (1) w.r.t x

$$\frac{\partial u}{\partial x} \cdot 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0.$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \quad \text{—— (2)}$$

Diff (1) w.r.t y

$$2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0.$$

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0 \quad \text{—— (3)}$$

According to CR equation.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \; ; \; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$(3) \Rightarrow -u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x} = 0. \quad \text{—— (4)}$$

Multiply (2) with u.

$$u^2\frac{\partial u}{\partial x} + uv\frac{\partial v}{\partial x} = 0 \quad \text{—— (5)}$$

(4) × v.

$$-uv\frac{\partial u}{\partial x} + v^2 \cdot \frac{\partial v}{\partial x} = 0 \quad \text{—— (6)}$$

(5) + (6)

$$u^2\frac{\partial u}{\partial x} + uv\frac{\partial v}{\partial x} \mp uv\frac{\partial u}{\partial x} + v^2\frac{\partial u}{\partial x} = 0.$$

$$u^2\frac{\partial u}{\partial x} + v^2\frac{\partial u}{\partial x} = 0.$$

$$k\frac{\partial u}{\partial x} = 0 \qquad \frac{\partial u}{\partial x} = 0. \qquad u = \text{const}$$

$(2) \Rightarrow \quad u \cdot \frac{\partial v}{\partial y} + -v \frac{\partial u}{\partial y} = 0 \quad —(7)$

$(7) \times u \quad u^2 \frac{\partial v}{\partial y} - uv \frac{\partial u}{\partial y} = 0. \quad —(8)$

$(3) \times v \quad uv \frac{\partial u}{\partial y} + v^2 \frac{\partial v}{\partial y} = 0. \quad —(9)$

$(8) + (9)$

$$u^2 \frac{\partial v}{\partial y} - uv \frac{\partial u}{\partial y} + uv \frac{\partial u}{\partial y} + v^2 \frac{\partial v}{\partial y} = 0.$$

$$u^2 \frac{\partial v}{\partial y} + v^2 \frac{\partial v}{\partial y} = 0.$$

$$u^2 + v^2 \left(\frac{\partial v}{\partial y} \right) = 0.$$

$$k \frac{\partial v}{\partial y} = 0.$$

$$\frac{\partial v}{\partial y} = 0.$$

v is const.

$\Rightarrow f(z)$ is constant since u & v are constant.

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? An analytic function $f(z)$ is constant if its argument is constant.

A. $f(z) = u + iv$ is analytic

$\text{Arg} f(z) = \text{const}.$

$\Rightarrow \tan^{-1}\left(\frac{v}{u}\right) = \text{const}.$

$$\tan^{-1}\left(\frac{v}{u}\right) = c$$

$$\frac{v}{u} = \tan c$$

$$\frac{v}{u} = k \quad \text{——— (1).}$$

$$v = ku.$$

v-utank =0.

$$\frac{\partial v}{\partial x} = k\frac{\partial u}{\partial x} \quad \text{—— (2)} \quad \frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial y} = k\frac{\partial u}{\partial y} \quad \text{—— (3).} \quad \frac{\partial v}{\partial y}.$$

Since f(z) is an analytic function, it obeys CR equation.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(2) ÷ (3).

$$\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}$$

$$\frac{\partial u}{\partial x} \times \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} \times \frac{\partial v}{\partial y}$$

$$= -\frac{\partial u}{\partial y} \times \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} \times \frac{\partial u}{\partial x}.$$

$$\left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 = \cancel{0}\cdot k.$$

$$u = \cancel{0}k$$

$$u + iv = k$$

$$\frac{\partial v}{\partial x} = k\frac{\partial v}{\partial y} = -k^2\frac{\partial v}{\partial x}.$$

$$\frac{\partial u}{\partial x}(1+k^2) = 0$$

v = const

$$\frac{\partial v}{\partial x} = k\frac{\partial u}{\partial x}$$

$$\frac{\partial v}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0$$

$\Rightarrow u$ = const.

Laplace's Equation

If $f(z) = u+iv$ is analytic in a domain D, then u & v satisfy Laplace equation given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

and $$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Proof

Let $f(z) = w$
$= u+iv$.

CR equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ —(1) & $-\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$ —(2).

Diff (1) partially w.r.t x,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad —(3).$$

Diff (2) partially w.r.t y,

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} \quad —(4).$$

(3) + (4)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Since the

Since the function is continous we can say that $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$.

Diff (1) w.r.t y;

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \quad —(5)$$

Diff (2) w.r.t y,

$$\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial y^2} \quad —(6)$$

(5) + (6)

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0,$$

$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ since the function is continous.

Harmonic function.

Solutions of Laplace equation, having second order partial derivatives are called harmonic function.

The real & imaginary parts of an analytic function are always harmonic.

If $f(z) = u+iv$ is analytic then u & v are harmonic conjugates to each other.

? Verify that $u = x^2 - y^2 - y$ is harmonic

A. $u(x,y) = x^2 - y^2 - y$ we have to show that whether u satisfies laplace's equation.

$$\frac{\partial u}{\partial x} = 2x$$

$$\frac{\partial^2 u}{\partial x^2} = 2.$$

$$\frac{\partial u}{\partial y} = -2y - 1$$

$$\frac{\partial^2 u}{\partial y^2} = -2.$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = 2 - 2 = 0$$

It obey's Laplaces equation.

$\Rightarrow$ u is harmonic.

? Check whether $v = (2x-1)y$ is harmonic

A. $v = (2x-1)y = 2xy - y$

$$\frac{\partial v}{\partial x} = 2y \qquad \frac{\partial^2 v}{\partial x^2} = 0$$

$$\frac{\partial v}{\partial y} = 2x - 1 \qquad \frac{\partial^2 v}{\partial y^2} = 0.$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\Rightarrow$ v is harmonic

? $v = -2xy$

$$\frac{\partial v}{\partial x} = -2y \qquad \frac{\partial^2 v}{\partial x^2} = 0.$$

$$\frac{\partial v}{\partial y} = -2x \qquad \frac{\partial^2 v}{\partial y^2} = 0.$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$\Rightarrow$ v is harmonic

? $u = e^{-x} \sin 2y.$

$$\frac{\partial u}{\partial x} = -e^{-x} \sin 2y \qquad \frac{\partial^2 u}{\partial x^2} = e^{-x} \sin 2y$$

$$\frac{\partial u}{\partial y} = 2e^{-x} \cos 2y \qquad \frac{\partial^2 u}{\partial y^2} = -4e^{-x} \sin 2y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = +e^{-x} \sin 2y - 4e^{-x} \sin 2y.$$

$$= -3e^{-x} \sin 2y$$

$\Rightarrow u$ is not harmonic

? Find the value of 'a' so that $u = e^{-\pi x} \cos ay$ is harmonic.

A. $u = e^{-\pi x} \cos ay$.

$$\frac{\partial u}{\partial x} = e^{-\pi x} \cdot -\pi \cos ay$$

$$\frac{\partial^2 u}{\partial x^2} = \pi^2 e^{-\pi x} \cos ay$$

$$\frac{\partial u}{\partial y} = a e^{-\pi x} - \sin ay$$

$$\frac{\partial^2 u}{\partial y^2} = -a^2 e^{-\pi x} \cos ay$$

Since u is harmonic

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\pi^2 e^{-\pi x} \cos ay = a^2 e^{-\pi x} \cos ay$$

$$\pi^2 = a^2$$

$$\underline{\underline{a = \pi}}$$

Note :-

If $f(z) = u + iv$ is analytic then,

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}}$$

? Find the derivative of $f(z) = \log z$ where $f(z)$ is analytic everywhere except at zero.

A. $f(z) = \log z$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2} \qquad \frac{\partial v}{\partial x} = \frac{-y}{x^2+y^2}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{x-iy}{x^2+y^2}.$$

$$= \frac{x-iy}{(x+iy)(x-iy)} = \frac{1}{x+iy} = \frac{1}{z}$$

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? Show that $v = (2x-1)y$ is harmonic. hence find the harmonic conjugate of v such that $f(z) = u+iv$ is analytic

A. $v = (2x-1)y$

$$\frac{\partial v}{\partial x} = 2y \quad \text{—— (1)}$$

$$\frac{\partial v}{\partial y} = 2x-1 \quad \text{—— (2)}$$

By CR equation.

$(1) \Rightarrow -\frac{\partial u}{\partial y} = 2y$ —— (3)

$(2) \Rightarrow \frac{\partial u}{\partial x} = 2x-1$ —— (4)

$(3) \Rightarrow \frac{\partial u}{\partial y} = -2y$

$\int$ing. both sides w.r.t 'y'

$u = -y^2 + h(x)$ —— (5) since here $x \neq$ cons

(5) differentiate w.r.t 'x'

$\frac{\partial u}{\partial x} = h'(x)$ —— (6)

Equating (4) & (6)

$h'(x) = 2x-1$

$\int$ing $h(x) = x^2 - x + C$

$\therefore$ Harmonic conjugate, $u = -y^2 + x^2 - x + C$

$f(z) = u + iv$

$= (-y^2 + x^2 - x + C) + i((2x-1)y)$

$= -y^2 + x^2 - x + C + i2xy - iy$

$= (x^2 - y^2 + i2xy) - (x+iy) + C$

$f(z) = z^2 - z + C$

? Find the harmonic conjugate if $u = 6xy$ such that $f(z) = u+iv$ is analytic.

A. $u = 6xy$

$$\frac{\partial u}{\partial x} = 6y \quad \text{—(1)} \qquad \frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial u}{\partial y} = 6x \quad \text{—(2)} \qquad \frac{\partial^2 u}{\partial y^2} = 0.$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\Rightarrow u = 6xy$ is harmonic

Applying CR equation in (1) & (2)

$$\frac{\partial v}{\partial y} = 6y \quad \text{—(3)}$$

$$-\frac{\partial v}{\partial x} = 6x. \quad \text{—(4)}$$

$$(3) \Rightarrow \frac{\partial v}{\partial y} = 6y$$

Sing $v = 3y^2 + h(x)$ —— (5)

Diff. (5) w.r.t 'x'

$$\frac{\partial u}{\partial x} = h'(x) \quad \text{—— (6)}$$

Equating (4) & (6)

$$h'(x) = -6x \quad \text{—(7)}$$

Sing. $h(x) = -3x^2 + c$

$v = 3y^2 - 3x^2 + c$

Harmonic conjugate, $v = 3y^2 - 3x^2 + c$ $\quad \frac{1}{i} = -i$ $\quad \frac{1}{-i} = i$

$f(z) = u + iv$

$= 6xy + i3y^2 - i3x^2 + ic$

$= -3i\ (x^2 - y^2 + \frac{1}{-i} 2xy) + ic$

$= -3i\ (x^2 - y^2 + i2xy) + ic$

$= -3i\ (x+iy)^2 + ic$

$= i[-3z^2 + c]$

? Find the value of 'a' so that $u = \cosh ax \cos y$ is harmonic hence find its harmonic conjugate.

A. $u = \cosh ax \cos y$

$\frac{\partial u}{\partial x} = a\cos y \sinh ax$ ——(1)

$\frac{\partial u}{\partial y} = \cosh ax \cdot {}^{-}\sin y$ ——(2)

~~Applying~~ CR ~~equation to (1) & (2).~~

$\frac{\partial v}{\partial y} = \cancel{a\cos y \sinh}$

Ktu Q bank

$$\frac{\partial^2 u}{\partial x^2} = a^2 \cos y \cosh ax$$

$$\frac{\partial^2 u}{\partial y^2} = -\cosh ax . \cos y$$

Since it is harmonic,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$a^2 \cos y \cosh ax = -\cosh ax \cos y.$$

$$a^2 = +1$$

$$\underline{\underline{a = \pm 1}}$$

Applying CR equation to (1) & (2)

$$\frac{\partial v}{\partial y} = \cos y \sinh x. \quad \text{--- (3)}$$

$$-\frac{\partial v}{\partial x} = -\cosh x . \sin y \quad \text{--- (4)}$$

(3) $\frac{\partial v}{\partial y} = \cos y \sinh x$

Sing $v = \sinh x \sin y + h(x)$ — (5)

Diff (5) w.r.t 'x'

$$\frac{\partial v}{\partial x} = \cosh x \sin y + h'(x). \quad \text{--- (6)}$$

Equating (6) & (4)

$$\cosh x \sin y = \cosh x \sin y + h'(x).$$

$$h'(x) = \cosh x \sin y - \cosh x \sin y.$$

$$h'(x) = 0.$$

$$h(x) = 0 + c$$

$$v = \sinh x \sin y + c$$

Harmonic conjugate, $v = \sinh x \sin y + c$

$$f(z) = u + iv$$

$$= \cosh x \cos y + i(\sinh x \sin y + c)$$

$$= \cosh x \cos y + i \sinh x \sin y + ic$$

$$= \cos ix \cos y + \sin ix \sin y + ic$$

$$= \cos(ix \mp y) + ic$$

$$= \cos i(x+iy) + ic$$

$$= \cos iz + ic$$

$$= \cosh z + ic$$

? Find the harmonic conjugate of $u = x^2 - y^2 - y$ $f(z) = u + iv$ is analytic

A. $u = x^2 - y^2 - y$

$$\frac{\partial u}{\partial x} = 2x. \quad \text{—— (1)}$$

$$\frac{\partial u}{\partial y} = -2y - 1 \quad \text{—— (2)}$$

By CR equation.

$$\frac{\partial v}{\partial y} = 2x \quad \text{—— (3)}$$

$$\frac{\partial v}{\partial x} = 2y+1 \quad \text{—— (4)}$$

$$(3) \Rightarrow \frac{\partial v}{\partial y} = 2x.$$

$$\int\text{ing} \quad v = 2xy + h(x) \quad \text{—— (5)}$$

Diff. (5) w.r.t 'x'

$$\frac{\partial v}{\partial x} = 2y + h'(x) \quad \text{—— (6)}$$

$\int$ing. $\cancel{v = 2xy + h(x)}$

Equating (6) & (4)

$$2y+1 = 2y + h'(x)$$

$$h'(x) = 1.$$

$$\int\text{ing} \quad h(x) = x$$

$$v = 2xy + x + c$$

Harmonic conjugate $v = 2xy + x$

$$f(z) = u + iv$$

$$= x^2 - y^2 - y + i(2xy + x + c)$$

$$= x^2 - y^2 + i2xy - y + ix + ic$$

$$= (x+iy)^2 + i^2 y + ix + ic$$

$$= z^2 + i(x+iy) + ic = \underline{\underline{z^2 + iz + ic}}$$