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MODULE

LINEAR ALGEBRA

Linear Systems of Equations

Consider the linear system of equations.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2.$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

$$\underset{m\times n}{A}\ \underset{n\times 1}{X} = \underset{m\times 1}{B}$$

'A' known as coefficient matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$ column matrix of unknowns.

$$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$ column matrix of constants.

Elementary Row Operations.

1) $R_i \leftrightarrow R_j$ (Interchange of rows)
2) $R_i \rightarrow kR_i$ $(k \neq 0)$
3) $R_i \rightarrow R_i + kR_j$

Echelon form (Row equivalent Echelon form)

To reduce a matrix to Echelon form apply only row transformation.

Step 1: In each row first non-zero element should be 1.

Step 2: All the elements below that 1 should be 0.

Step 3: Then proceed through next non-zero row and proceed as usual & finally we get a form called Echelon form in which all the non-zero elements are towards the right

? Reduce $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 15 & 3 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 15 & 3 \end{bmatrix} \quad R_2 \to R_2 - 4R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 15 & 3 \end{bmatrix} \quad R_2 \to -\frac{1}{3} R_2.$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -27 \end{bmatrix} \quad R_3 \to R_3 - 15R_2$$

? Reduce $A = \begin{bmatrix} 0 & 8 & 6 \\ -2 & 4 & -6 \\ 1 & 1 & -1 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ -2 & 4 & -6 \\ 0 & 8 & 6 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 6 & -8 \\ 0 & 8 & 6 \end{bmatrix} \quad R_2 \to R_2 + 2R_1$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -4/3 \\ 0 & 8 & 6 \end{bmatrix} \quad R_2 \to \frac{R_2}{6}$$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -4/3 \\ 0 & 0 & 50/3 \end{bmatrix} \quad R_3 \to R_3 - 8R_2$$

This is of Echelon form

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? Reduce the matrix $A = \begin{bmatrix} 2 & 3 & 1 & -11 \\ 5 & -2 & 5 & -4 \\ 1 & -1 & 3 & -3 \\ 3 & 4 & -7 & 2 \end{bmatrix}$

A.

$$\sim \begin{bmatrix} 1 & -1 & 3 & -3 \\ 5 & -2 & 5 & -4 \\ 2 & 3 & 1 & -11 \\ 3 & 4 & -7 & 2 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & -1 & 3 & -3 \\ 0 & 3 & -10 & 11 \\ 0 & 5 & -5 & -5 \\ 0 & 7 & -16 & 11 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 3 & -3 \\ 0 & 5 & -5 & -5 \\ 0 & 3 & -10 & 11 \\ 0 & 7 & -16 & 11 \end{bmatrix} \quad R_2 \leftrightarrow R_3.$$

$$\sim \begin{bmatrix} 1 & -1 & 3 & -3 \\ 0 & 1 & -1 & -1 \\ 0 & 3 & -10 & 11 \\ 0 & 7 & -16 & 11 \end{bmatrix} \quad R_2 \to \frac{1}{5} R_2.$$

$$\sim \begin{bmatrix} 1 & -1 & 3 & -3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -7 & 14 \\ 0 & 0 & -9 & 18 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 - 3R_2 \\ R_4 \to R_4 - 7R_2. \end{array}$$

$$\sim \begin{bmatrix} 1 & -1 & 3 & -3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -9 & 18 \end{bmatrix} \quad R_3 \to -\frac{1}{7} R_3$$

$$\sim \begin{bmatrix} 1 & -1 & 3 & -3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \to R_4 + 9R_3$$

 This is now in Echelon form.

Rank of a Matrix

If a matrix is in Echelon form, then the rank is given by no:of non zero rows in Echelon form.

In previous question, Rank A = 3

Gauss Elimination Method for Solving a Linear System of Equation

Consider a linear system of equations.

$AX = B$ where A is coefficient matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

To solve the system

1) First form Augmented matrix $[A:B]$

which is formed by writing the matrix B to the right side of matrix A.

2) Reduce $[A:B]$ to echelon form

3) Check consistancy

I. i) If Rank $[A:B]$ = Rank A then the system is consistant.

ii) If Rank $[A:B] \neq$ Rank A then the system is inconsistant.

II If the system is consistant,

Rank $[A.B]$ = Rank $A = r$.

Let no:of unknowns $= n$

a) If $r = n$, then the system has a unique solution.

b) If $r < n$, then the system has infinite number of solutions.

III If Rank $[A:B] \neq$ Rank A.

ie, system has no solutions.

4) From the echelon matrix write the equations in the form $AX = B$

5) Apply Back substitution.

? Solve the system of equation $8y+6z=-4$
$-2x+4y-6z=18$, $x+y-z=2$.

A.

1) $A = \begin{bmatrix} 0 & 8 & 6 \\ -2 & 4 & -6 \\ 1 & 1 & -1 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $B = \begin{bmatrix} -4 \\ 18 \\ 2 \end{bmatrix}$

2) $A:B = \left[\begin{array}{ccc|c} 0 & 8 & 6 & -4 \\ -2 & 4 & -6 & 18 \\ 1 & 1 & -1 & 2 \end{array}\right]$

$\sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ -2 & 4 & -6 & 18 \\ 0 & 8 & 6 & -4 \end{bmatrix}$ $R_1 \leftrightarrow R_3$

$\sim \begin{bmatrix} 1 & +1 & -1 & 2 \\ 0 & 6 & -8 & 22 \\ 0 & 8 & 6 & -4 \end{bmatrix}$ $R_2 \to R_2 + 2R_1$

$\sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -8/6 & 22/6 \\ 0 & 8 & 6 & -4 \end{bmatrix}$ $R_2 \to \frac{1}{6} R_2$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -8/6 & 22/6 \\ 0 & 0 & 100/6 & -200/6 \end{bmatrix} R_3 \to R_3 - 8R_2$$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & -8/6 & 22/6 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad R_3 \to \frac{6}{100} R_3$$

3) Rank $A = 3 =$ Rank $[A.B]$

$\rightarrow$ It is consistant

$r = 3$, $n = 3$

$r = n \longrightarrow$ unique solution.

4) from Echelon matrix

$$AX = B \Rightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -8/6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 22/6 \\ -2 \end{bmatrix}$$

$$x + y - z = 2 \quad \text{——— (1)}$$

$$y - \frac{8}{6}z = \frac{22}{6} \quad \text{——— (2)}$$

$$z = -2. \quad \text{——— (3)}$$

5) Applying Back substitution, solution is.

(1) $\Rightarrow z = -2$ (2) $\Rightarrow y = 1$ (3) $\Rightarrow x = -1$

1. Solve $4y - 2z = 2$, $6x - 2y + z = 29$, $4x + 8y - 4z = 24$

2. Solve $x + 2y - z = 3$, $3x - y + 2z = 1$, $2x - 2y + 3z = 2$, $x - y + z = -1$

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = X \quad B = \begin{bmatrix} 3 \\ 1 \\ 2 \\ -1 \end{bmatrix}$$

$$A:B = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \\ 2 & -2 & 3 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -5/7 & 8/7 \\ 0 & -6 & 5 & -4 \\ 0 & -3 & 2 & -4 \end{bmatrix} \quad R_2 \to -\frac{1}{7} R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -5/7 & 8/7 \\ 0 & 0 & 5/7 & 20/7 \\ 0 & 0 & -1/7 & -4/7 \end{bmatrix} \quad \begin{array}{l} R_3 \to R_3 + 6R_2 \\ R_4 \to R_4 + 3R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -5/7 & 8/7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & -1/7 & -4/7 \end{bmatrix} \quad R_3 \to \frac{7}{5} R_3$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -5/7 & 8/7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \to R_4 + \frac{1}{7} R_3$$

This in Echelon form

Rank $[A:B] = 3$

Rank $[A] = 3$ (look A of Echelon form)

$\Longrightarrow$ consistant

$r = 3$, $n = 3$ $\Longrightarrow$ unique solution

from echelon,

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -5/7 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 8/7 \\ 4 \end{bmatrix}$$

$$x + 2y - z = -3 \quad \text{——— (1)}$$

$$y - \frac{5}{7}z = \frac{8}{7} \quad \text{——— (2)}$$

$$z = 4 \quad \text{——— (3)}$$

$$(2) \Rightarrow y = \frac{8}{7} + \frac{20}{7} = \frac{28}{7} = 4$$

$$(1) \Rightarrow \cancel{x = -1}$$
$$x = 3 - 2 \cdot 4 + 4$$
$$= 3 - 8 + 4 = \underline{\underline{-1}}$$

Solution is

$$x = -1 \; ; \; y = 4 \; ; \; z = 4$$

? $4y - 2z = 2$; $6x - 2y + z = 29$; $4x + 8y - 4z = 2$

$$A = \begin{bmatrix} 0 & 4 & -2 \\ 6 & -2 & 1 \\ 4 & 8 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 2 \\ 29 \\ 24 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A:B = \begin{bmatrix} 0 & 4 & -2 & 2 \\ 6 & -2 & 1 & 29 \\ 4 & 8 & -4 & 24 \end{bmatrix}$$

$$\sim \begin{bmatrix} 4 & 8 & -4 & 24 \\ 6 & -2 & 1 & 29 \\ 0 & 4 & -2 & 2 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 6 \\ 6 & -2 & 1 & 29 \\ 0 & 4 & -2 & 2 \end{bmatrix} \quad R_1 \to \frac{1}{4} R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 6 \\ 0 & -14 & 7 & -7 \\ 0 & 4 & -2 & 2 \end{bmatrix} \quad R_2 \to R_2 - 6R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 6 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 4 & -2 & 2 \end{bmatrix} \quad R_2 \to -\frac{1}{14} R_2$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 6 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - 4R_2$$

This is in Echelon form.

Rank $[A:B] = 2$

Rank $[A] = 2$.

$\Rightarrow$ consistant.

$r = 2$, $n = 3$. $\Rightarrow$ unique solution.

~~from Echelon~~

$r < n \Rightarrow$ infinite solution.

To

To find the infinite solution assign arbitrary values to (n-r) unknown

Here, $n-r = 3-2 = 1$

Let $\boxed{z = a}$

from Echelon

$$AX = B$$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 1/2 \\ 0 \end{bmatrix}$$

$$x + 2y - z = 6 \quad \text{—— (1)}$$

$$y - \frac{1}{2}z = 1/2 \quad \text{—— (2)}$$

We have $z = a$

$$(2) \Rightarrow y - \frac{1}{2}a = \frac{1}{2}$$

$$y = \frac{1}{2} + \frac{a}{2} = \frac{(a+1)}{2}$$

$$(1) \Rightarrow x + 2y - z = 6$$

$$x + 2\frac{(a+1)}{2} - a = 6$$

$$x = 6 - a - 1 + a$$

$$= \underline{\underline{5}}$$

(i) $\Rightarrow$ $x+2y+$

$\therefore$ Solution is

$$x = 5 \;;\; y = \frac{a+1}{2} \;;\; z = a$$

? Solve the augomented matrix

$$\begin{array}{c} \hphantom{[}\overbrace{\hphantom{0 \quad 1 \quad 1 \quad -2}}^{A}\ \overbrace{\hphantom{0}}^{B} \\ \begin{bmatrix} 0 & 1 & 1 & -2 & 0 \\ 2 & -3 & -3 & 6 & 2 \\ 4 & 1 & 1 & -2 & 4 \end{bmatrix} \end{array}$$

A.
$$A = \begin{bmatrix} 0 & 1 & 1 & -2 \\ 2 & -3 & -3 & 6 \\ 4 & 1 & 1 & -2 \end{bmatrix} \quad X = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix}$$

$$A:B = \begin{bmatrix} 0 & 1 & 1 & -2 & 0 \\ 2 & -3 & -3 & 6 & 2 \\ 4 & 1 & 1 & -2 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & 1 & -2 & 4 \\ 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 1/4 & 1/4 & -1/2 & 1 \\ 2 & -3 & -3 & 6 & 2 \\ 0 & 1 & 1 & -2 & 0 \end{bmatrix} \quad R_1 \rightarrow \frac{1}{4} R_1$$

$$\sim \begin{bmatrix} 1 & 1/4 & 1/4 & -1/2 & 1 \\ 0 & -7/2 & -7/2 & 7 & 0 \\ 0 & 1 & 1 & -2 & 0 \end{bmatrix} \quad R_2 \to R_2 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 1/4 & 1/4 & -1/2 & 1 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & -2 & 0 \end{bmatrix} \quad R_2 \to -\frac{2}{7} R_2$$

$$\sim \begin{bmatrix} 1 & 1/4 & 1/4 & -1/2 & 1 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

This is in Echelon form

Rank $[A:B] = 2$

Rank $[A] = 2$.

$\Rightarrow$ consistant.

$r = 2 \qquad n = 4$

$r < n \Rightarrow$ infinite no:of solutions

$n - r = 4 - 2 = \underline{\underline{2}}$

Let $z = a \qquad y = b$

From Echelon form

$$AX = B$$

$$\begin{bmatrix} 1 & 1/4 & 1/4 & -1/2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$w + \frac{1}{4}x + \frac{1}{4}y - \frac{1}{2}z = 1 \quad \text{——— (1)}$$

$$x + y - 2z = 0 \quad \text{——— (2)}$$

Let $y = b$, $z = a$

$$(1) \Rightarrow x + b - 2a = 0$$

$$x = 2a - b$$

$$(2) \Rightarrow w + \frac{1}{4}(2a-b) + \frac{1}{4}b - \frac{1}{2}a = 1$$

$$w = 1 + \frac{a}{2} - \frac{b}{4} - \frac{(2a-b)}{4}$$

$$= 1 + \frac{a}{2} - \frac{b}{4} - \frac{a}{2} + \frac{b}{4}$$

$$w = 1$$

$\therefore$ Solution is

$$w = 1 \; ; \; x = 2a - b \; ; \; y = b \; ; \; z = a$$

? $x+2y=3$, $2x+4y=7$

$AX=B$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$$

$A:B$ $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$

This is in Echelon form

Rank $[A:B] = 2$

Rank $[A] = 1$

Rank $[A:B] \neq$ Rank $[A]$

$\Rightarrow$ system is inconsistant & has no solutions.

? Find the values of λ & μ so that

$x+2y+3z=4$; $x+3y+4z=5$; $x+3y+\lambda z=\mu$

1) Unique solution
2) Infinite solution
3) No solution

A. $A:B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 1 & 3 & \lambda & \mu \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & (\lambda-3) & (\mu-4) \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & (\lambda-4) & (\mu-5) \end{bmatrix} \quad R_3 \to R_3 - R_2$$

1) Unique Solution

Rank $(A:B)$ = Rank A = no:of unknowns = 3

$\lambda \neq 4$ & μ can be any real value

2) Infinite Solution

Rank $[A:B]$ = Rank A < no:of unknowns

$\lambda = 4$ & $\mu = 5$

$\lambda = 4$ Rank $A = 2$

$\mu = 5$ Rank $A:B = 2$.

Now $r = 2 < n = 3$

3) No solution

$\lambda = 4$, $\mu \neq 5$ Rank $A \neq$ Rank $(A:B)$

Homogenous & Non-Homogenous System of Equations.

$$AX = B$$

If $B \neq 0 \rightarrow$ Nonhomogenous

If $B = 0 \rightarrow$ Homogenous

Solution of Homogenous

$$AX = 0$$

$$[A:B] = \begin{array}{c} \quad A \qquad\quad B \\ \left[\begin{array}{cccc|c} - & - & \cdots & \cdot & 0 \\ - & - & \cdots & \cdot & 0 \\ - & - & \cdots & \cdot & 0 \end{array}\right] \end{array}$$

Rank $(A:B)$ = Rank $(A:0)$ = Rank A

A homogenous system is always consistant

The solution $x = 0$ is always a solution for homogenous system and this is called Trivial solution. For existance of non-trivial solution we should have $r < n$

? Solve $x + 2y + 3z = 0$; $2x + y + 3z = 0$; $3x + 2y + z = 0$.

A.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & +3 \\ 3 & 2 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since $B=0$, it is enough to reduce A.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & +3 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -9 \\ 0 & -4 & -8 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -4 & -8 \end{bmatrix} \quad R_2 \to -\frac{1}{3} R_2$$

$$R_3 \to R_3 + 4R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix}$$

Rank A = 3 = no: of unknowns

There exists only trivial solution $(r = n)$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$x+2y+3z=0 \quad \text{---} (1)$$

$$y+z=0 \quad \text{---} (2)$$

$$-4z=0 \quad \text{---} (3)$$

$z=0$

$y=0$

$x=0$.

Note:-

If the system has non-trivial solution proceed as the case of infinite solution in which assign arbitrary values to $(n-r)$ unknowns.

? Solve $x+3y-2z=0$; $2x-y+4z=0$; $x-11y+14z=0$

$$A=\begin{bmatrix}1&3&-2\\2&-1&4\\1&-11&14\end{bmatrix} \quad X=\begin{bmatrix}x\\y\\z\end{bmatrix} \quad B=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Reduce A.

$$\sim \begin{bmatrix}1&3&-2\\0&-7&8\\0&-14&16\end{bmatrix} \qquad \begin{matrix}R_2 \to R_2-2R_1\\ R_3 \to R_3-R_1\end{matrix}$$

$$\sim \begin{bmatrix}1&3&-2\\0&1&-8/7\\0&-14&16\end{bmatrix} \qquad R_2 \to -\frac{1}{7}R_2.$$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -8/7 \\ 0 & -16 & 16 \end{bmatrix} \quad R_3 \to R_3 + 16R_2$$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -8/7 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + 16R_2$$

Now this is in Echelon form.

Rank $A = 2$.

No: of unknowns $= 3$

$r = 2$, $n = 3$

$r < n$

Here $n - r = 1$

In Echelon form

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & -8/7 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + 3y - 2z = 0$ — (1)

$y - \frac{8}{7}z = 0$ — (2)

Let $z = a$

$(2) \Rightarrow \quad y - \frac{8a}{7} = 0$

$$y = \frac{8a}{7}.$$

$(1) \Rightarrow \quad x + 3 \cdot \frac{8a}{7} - 2a = 0.$

$$x = 2a - \frac{24a}{7} = \frac{-10a}{7}$$

$\therefore$ Solution is

$$x = \frac{-10a}{7} \qquad y = \frac{8a}{7} \qquad z = a.$$

Vector Space

$V \rightarrow$ set of vectors

$F \rightarrow$ set of scalars

$+$ (vector addition)

$\cdot$ (scalar multiplication)

Vector Addition

$a, b \in V$

If $(a+b) \in V$ then vector addition is defined

Properties

i) $a+b = b+a$ (commutative)

ii) $a+(b+c) = (a+b)+c$ (associative)

iii) $a+0 = a$ (additive Identity)

iv) $a+(-a) = 0$ (Additive Inverse)

Scalar Multiplication

$a \in V$, $k \in F$

$\Rightarrow ka \in V$ (scalar product)

$\Rightarrow$ scalar multiplication is defined.

Properties

i) $1.a = a$

ii) $k(k'a) = (kk')a$

iii) $(k+k')a = ka + k'a$

iv) $k(a+b) = ka + kb$

If we satisfies all the above properties, then we forms a vector space over the field F

F deals with the set of real numbers.

Row ~~column~~ ~~matrix~~ vector $\rightarrow 1 \times n$

Column ~~matrix~~ vector $\rightarrow n \times 1$

The elements of R^2 are 2 row vectors.

$R^2 \rightarrow [a_1\ a_2]$

$R^3 \rightarrow [a_1\ a_2\ a_3]$

$R^4 \rightarrow [a_1\ a_2\ a_3\ a_4]$

$R^n \rightarrow [a_1, a_2 \ \ldots \ a_n]$

? $V = \{(a,b,c) \in R^3,\ a+2b+c=10\}$

Check whether V is a vector space.

A. Prove + & · is defined.

Let $u, v \in V$

$u = (a_1, b_1, c_1)$ such that $a_1 + 2b_1 + c_1 = 10$

$v = (a_2, b_2, c_2)$ such that $a_2 + 2b_2 + c_2 = 10$

$u + v = (a_1, b_1, c_1) + (a_2, b_2, c_2)$

$= (a_1 + a_2, b_1 + b_2, c_1 + c_2) \in R^3$

Consider

$$(a_1+a_2)+2(b_1+b_2)+c_1+c_2$$

$$a_1+2b_1+c_1+a_2+2b_2+c_2$$

$$10+10 = \underline{20} \neq 10.$$

$\Rightarrow$ Sum is not defined

$\therefore u+v \notin V$

Vector addition is not defined

Hence V doesnot form a vector space

Subspace

V is a vector space

Let W be a non-empty subset of V

For subspace, the following two conditions are satisfie

i) $u, v \in W \Rightarrow u+v \in W$

ii) $u \in W$, k any scalar $\Rightarrow ku \in W$.

Linear Independence of Vectors

Let $v_1, v_2 \ldots v_n$ be a set of vectors

then let $c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0$.

This is called linear combinational of vectors.

While solving

- If $c_1 = 0, c_2 = 0, \ldots c_n = 0$ ie ie, all scalars are zero.

$\Rightarrow$ It is linearly independent

- If the scalars are not all zero. (atleast one non-ze

$\Rightarrow$ linearly dependent.

* To check whether, a set of vectors are linearly independent.

Step 1: Write the given vectors as rows and form a matrix

Step 2: Reduce the matrix Echelon form

- If the rank of the matrix = no:of vectors then the set of vectors are linearly independent
- If rank < no:of vectors. then vectors are linearly dependent.

? Check whether the following vectors are linearly independent.

$[1 \ 2 \ 3 \ 4], [3 \ 4 \ 5 \ 6], [0 \ 1 \ 3 \ 4]$

A.
$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 0 & 1 & 3 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & -4 & -6 \\ 0 & 1 & 3 & 4 \end{bmatrix} \quad R_2 \rightarrow R_2 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 4 \end{bmatrix} \quad R_2 \rightarrow -\frac{1}{2} R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad R_3 \rightarrow R_3 - R_2$$

Rank = 3 = No: of vectors

$\Rightarrow$ linearly independent.

? $[3\ 0\ 2\ 2]$, $[-6, 42\ 24\ 54]$ $[21\ -21\ 0\ -15]$

A. $A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 0 & 2/3 & 2/3 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix} \quad R_1 \rightarrow \frac{1}{3} R_1$$

$$\sim \begin{bmatrix} 1 & 0 & 2/3 & 2/3 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix} \quad \begin{matrix} R_2 \rightarrow R_2 + 6R_1 \\ R_3 \rightarrow R_3 - 21R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2/3 & 2/3 \\ 0 & \cancel{42}1 & 2/3 & 29/21 \\ 0 & -21 & -14 & -29 \end{bmatrix} \qquad R_2 \to \frac{R_2 + 2R_3}{\cancel{-}42}$$

$$\sim \begin{bmatrix} 1 & 0 & 2/3 & 2/3 \\ 0 & 1 & 2/3 & 29/21 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_3 \to R_3 + 21R_2$$

Rank $A = 2$

no :of vectors $= 3$.

$r < n$

$\Rightarrow$ linearly dependent

Note:-

The rank r of a matrix A equals the maximum no:of linearly independent rows.

Span (Generating)

Let $v_1, v_2, \ldots v_n \in V$

$v = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n \quad \forall v \in V$

then we can say that $v_1, v_2 \ldots v_n$ will span V

Basis

A linearly independent set in V consisting of maximum possible number of vectors in V is called a basis for V.

ie, any largest possible set of linearly independe vectors in V forms a basis for V

The number of vectors in the basis is called Dimension of V

Note:-

The vectorspace R^n

$$\text{dimen}(R^n) = n$$

To find the basis write the given vectors as rows and form the matrix and reduce to echelon. The non-zero rows will form the basis

? Find the basis of the matrix $\begin{bmatrix} 0 & 0 & 5 \\ 3 & 5 & 0 \\ 5 & 0 & 0 \end{bmatrix}$

A. $A = \begin{bmatrix} 0 & 0 & 5 \\ 3 & 5 & 0 \\ 5 & 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 5 & 0 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ $R_1 \leftrightarrow R_3$

$\sim \begin{bmatrix} 1 & 0 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ $R_1 \rightarrow \frac{1}{5} R_1$

$\cancel{\begin{matrix} 1 & 0 & 0 \\ 0 & 5 & 0 \end{matrix}}$ ~~$R_2 \to R_2 - 3R_1$~~

This is in echelon form since below a non-zero there is zero.

Basis $= \{(1,0,0), (0,5,0), (0,0,5)\}$

Dimension $= 3$

Row space

It is the span of row vectors is called row space. And span of column ~~matrix~~ vectors is called ~~row~~ column space.

Basis of Row space & Column space

Basis of Row space.

To find the basis of row space

Step 1: Form a matrix having the given vectors as rows

Step 2: Reduce the matrix to Echelon

Step 3: The non zero rows will form the basis of row space.

Basis of Column space

To find the basis of column space

Step 1: Form a matrix A having given vectors as row.

Step 2: Find A^T (transpose of A)

Step 3: Reduce A^T to echelon form.

Step 4: The non-zero rows forms basis of rowspace of A^T

Step 5 : Take the transpose of basis elements to get the basis of columnspace of A.

? Find the basis of rowspace & columnspace of

$$A = \begin{bmatrix} 5 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

A. Rowspace Basis

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 5 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad R_1 \leftrightarrow R_3.$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -5 & -4 & -5 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad R_3 \rightarrow R_3 - 5R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -5 & -4 & -5 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3.$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -5 & -4 & -5 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad R_4 \to R_4 + R_3$$

$$\text{Basis} = \{[1\ 1\ 1\ 1],\ [0, -5\ -4\ -5],\ [0\ 0\ -1\ 0]\}$$

Dimension of rowspace = 3

<u>Columnspace Basis</u>

$$A^T = \begin{bmatrix} 5 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 5 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 5 & -4 & -5 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad R_3 \to R_3 - 5R_1$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 5 & -4 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 5 & -4 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \to R_4 - R_3$$

Basis of Rowspace of A^T

$$= \{ [1\ -1\ 1\ 1], [0\ 5\ -4\ -5], [0\ 0\ 1\ 0] \}$$

$\therefore$ Basis of Columnspace of A.

$$= \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -4 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Dimension of columnspace = 3.