SLIIT

Discover Your Future

IT2070 – Data Structures and Algorithms

Recursive Functions can be written in several ways
1)Recursive Definition
2)Graphical Representation
3)Recurrence Equation-if it is an equation we have to solve this.

Lecture 05

Introduction to Recursion

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Recursion –Example 1

This is the recursive Definition

Factorial

$n! = n * (n-1) * (n-2) * \ldots * 2 * 1$, and that $0! = 1$.

n could be any number within these conditions

A recursive definition is

$$(n)! = \begin{cases} n * (n-1)! & \text{if } n > 0 \\ 1 & \text{if } n = 0 \end{cases}$$

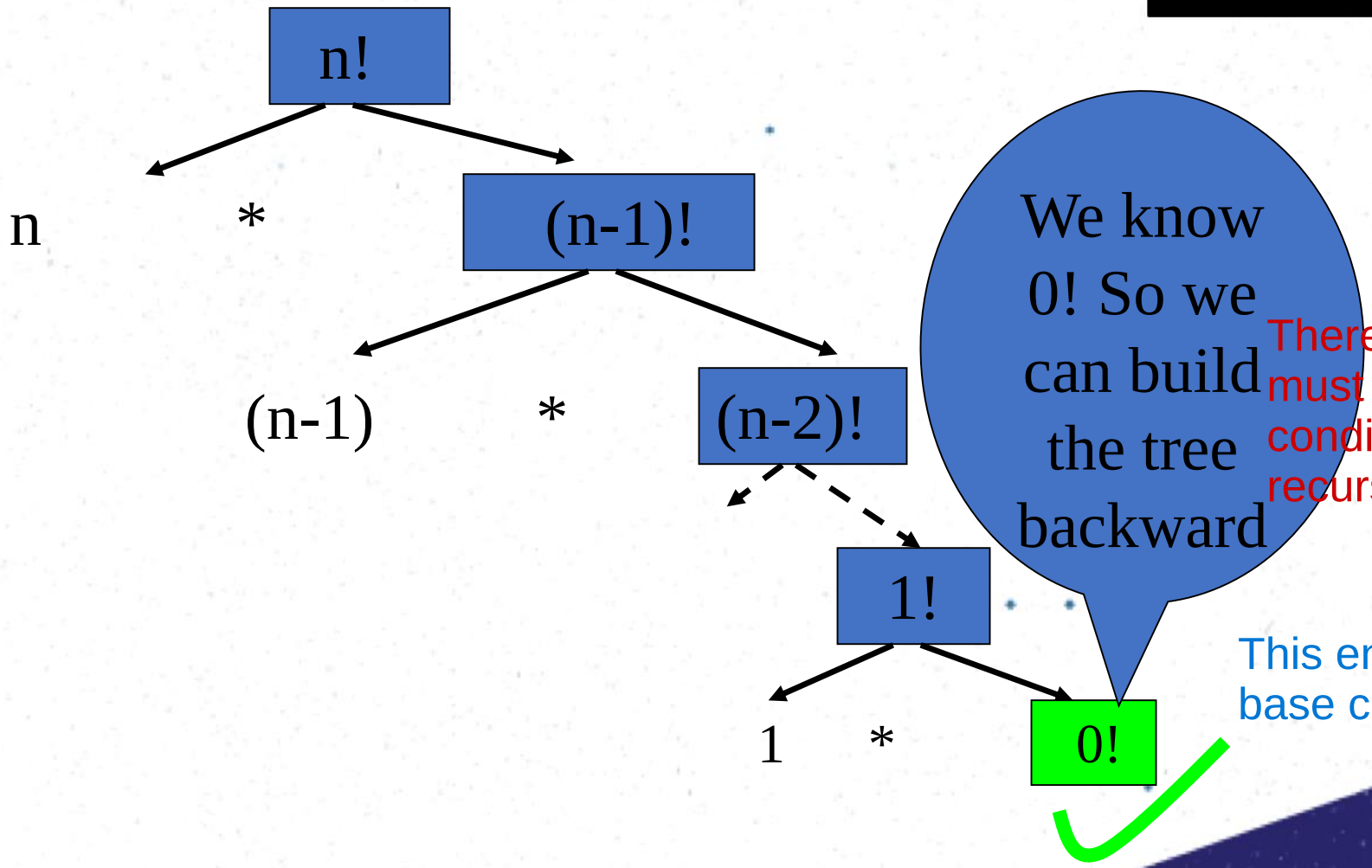
Recursive Condition

Base Condition

Base condition/Termination point/Initial Conditions

This is the graphical representation

Factorial -A graphical view



There for there must be a base condition in a recursive function

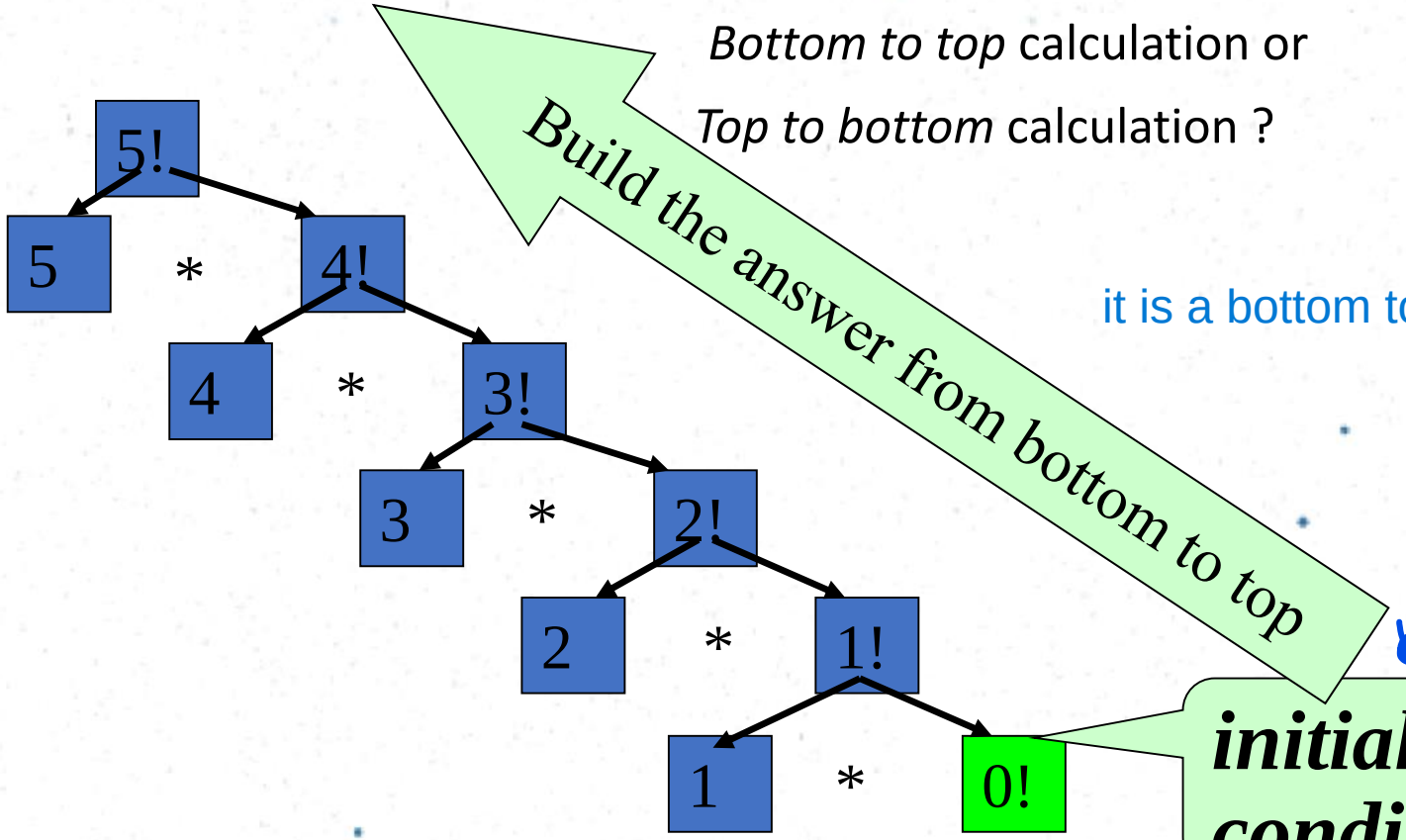
This end with the base condition

Exercise

- Draw the recursive tree for 5!
- How does it calculate 5! ? Is it:

Bottom to top calculation or

Top to bottom calculation ?



it is a bottom to top calculation

Factorial(contd.)

Recursive Definition

$$(n)! = \{n * (n-1)! \quad \text{if } n > 0\}$$
$$\{1 \quad \text{if } n = 0\}$$

PsuedoCode

```
int factorial(int n) {
    if (n == 0)
        return 1;
    else
        return (n * factorial(n-1));
}
```

Compare

Recursion

What is recursion?

A function that calls itself directly or indirectly to solve a smaller version of its task until a final call which does not require a self-call is a ***recursive*** function.

Recurrence equation

There are two types of Recursion equations

- Mathematical function that defines the running time of recursive functions.

we are going to come up with an equation to solve recursive functions

- This describes the overall running time on a problem of size *n* in terms of the running time on smaller inputs.

check example 1

Lets divide these as type 1 equation and tyoe2 equation

Ex:

$T(N) = T(N-1) + b$

$T(N) = T(N/2) + c$

Recursive functions are complicated and we might get equations like this dependending on different scenarios

for the recursion

$T(n) = T(n-1) + K$

rest of values (ex: a/b)

Recurrence - Example1

T(n)

Find the Running time of the following function

```
int factorial(int n) {
    if (n == 0)              //A
        return 1;            //B
    else
        return (n * factorial(n -1));    //C
}
```

statement A (a μs)

statement B (b μs)

for operation C μs for this T(n-1) μs

the time we give for factorial n is T(n)

∴ the time we give for (n-1) is T(n-1)

Statement A takes time a → for the conditional evaluation

Statement B takes time b → for the return assignment

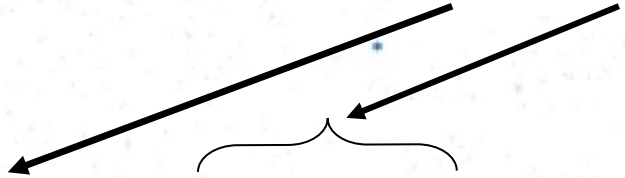
Statement C takes time:

c → for the operations(multiplication & return)

T(n-1) → to determine (n-1)!

Recurrence - Example1 (Contd.)

T(n) = Time to execute A & C



$$T(n) = a + c + T(n-1)$$

- This method is called iteration method (or repeated substitution)

$$T(n) = T(n/2) + 2$$

Exercise

This is called Repeated substitution method and Both equations can be solved using this

- Solve the recurrence

  T(n) = T(n/2) + 2

  You are given that

  n = 16 and

  T(1) = 1

this method is called repeated substitution method

$$T(16) = T(8) + 2$$
$$T(8) = T(4) + 2$$
$$T(4) = T(2) + 2$$
$$T(2) = T(1) + 2$$
$$T(1) = 1 \text{ (given)}$$

$$T(16) + T(8) + T(4) + T(2) + T(1) = T(8) + 2 + T(4) + 2 + T(2) + 2 + T(1) + 2 + 1$$

$$T(16) = 2 + 2 + 2 + 2 + 1$$
$$= 9$$

Finding a solution to a recurrence.

- Other methods
  - Recursion tree.
  - Master Theorem.

repeated substitution method (iteration method)

recursive tree

Master theorem

The recursion-tree method

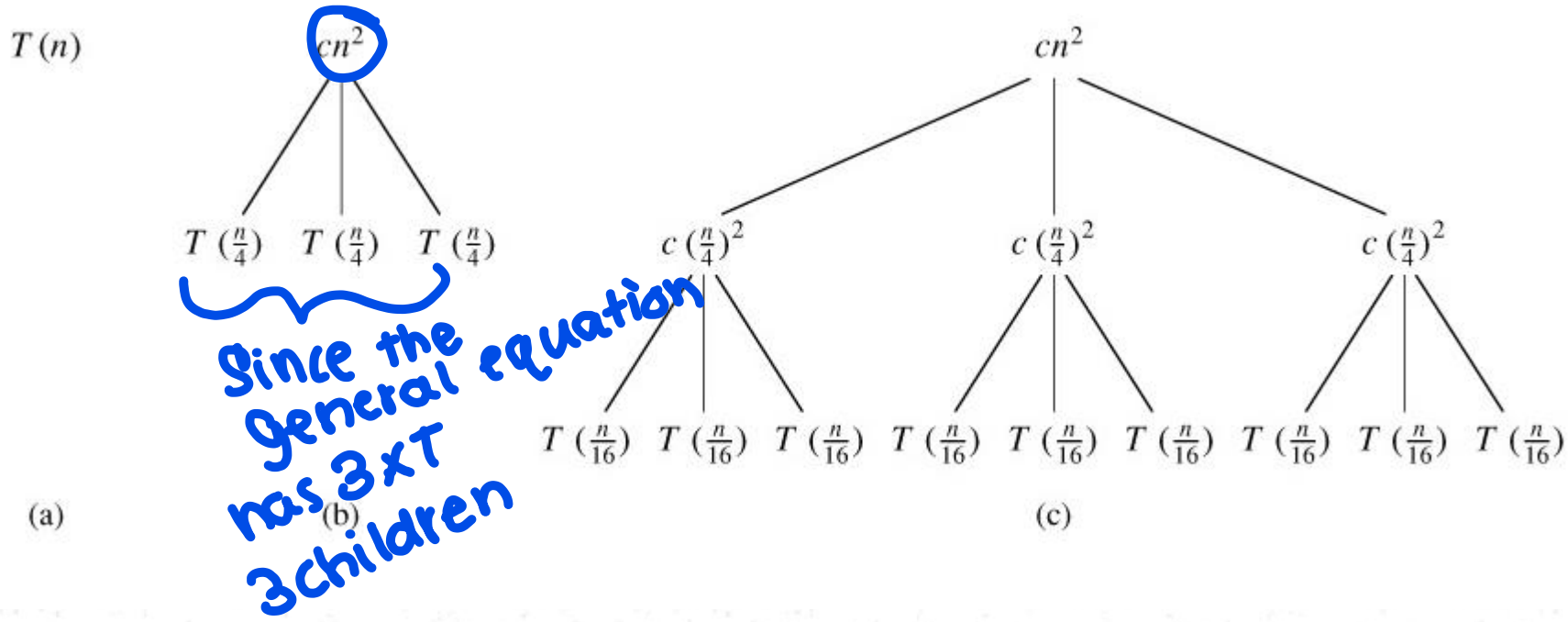
This method isnt discuss a lot

- Although the substitution method can provide a succinct proof that a solution to a recurrence is correct, it is sometimes difficult to come up with a good guess. Drawing out a recursion tree,is a straightforward way to devise a good guess. In a **recursion tree**, each node represents the cost of a single subproblem somewhere in the set of recursive function invocations.
- A recursion tree is best used to generate a good guess, which is then verified by the substitution method.

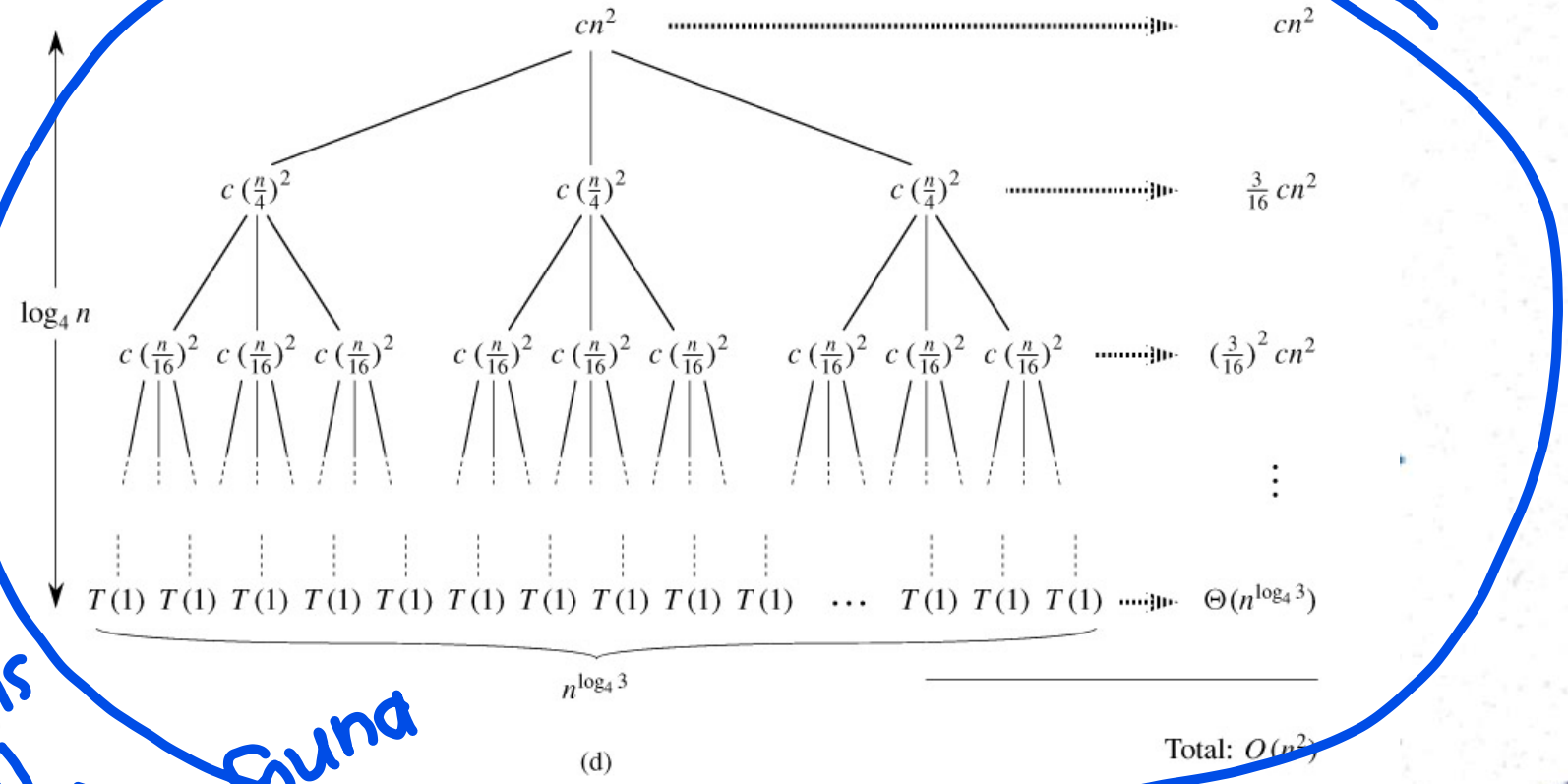
it is a good guess about the recurtion time

Recursion tree for $T(n) = 3T(n/4) + cn^2$

root



Recursion tree for $T(n) = 3T(n/4) + cn^2$



You have to further extend like this until T(1) is found

The Master Method

- The Master method applies to recurrences of the form

$$T(n) = a\,T(n/b) + f(n)$$

when the equation is given and you want to use master method it should be in this format (equation)

where $a \geq 1$ and $b > 1$, and $f(n)$ is an asymptotically positive function.

we just need to apply this (no need to prove)

The recurrence describes the running time of an algorithm that divides a problem of size n into a subproblems, each of size n/b, where a and b are positive constants. The a subproblems are solved recursively, each in time T (n/b). The cost of dividing the problem and combining the results of the subproblems is described by the function f (n).

The master theorem

- The master method depends on the following theorem.
- Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n),$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ can be bounded asymptotically as follows.

The master theorem

Compare $n^{\log_b a}$ vs. $f(n)$:

Case 1: $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$.
($f(n)$ is polynomially smaller than $n^{\log_b a}$.)
Solution: $T(n) = \Theta(n^{\log_b a})$.

Case 2: $f(n) = \Theta(n^{\log_b a} \lg^k n)$, where $k \geq 0$.

($f(n)$ is within a polylog factor of $n^{\log_b a}$, but not smaller.)
Solution: $T(n) = \Theta(n^{\log_b a} \lg^{k+1} n)$.
(Intuitively: cost is $n^{\log_b a} \lg^k n$ at each level, and there are $\Theta(\lg n)$ levels.)
Simple case: $k = 0 \Rightarrow f(n) = \Theta(n^{\log_b a}) \Rightarrow T(n) = \Theta(n^{\log_b a} \lg n)$.

Case 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$ and $f(n)$ satisfies the regularity condition $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n.
($f(n)$ is polynomially greater than $n^{\log_b a}$.)
Solution: $T(n) = \Theta(f(n))$.
(Intuitively: cost is dominated by root.)

The master theorem

$$T(n) = \begin{cases} \Theta\left(n^{\log_b a}\right) & f(n) = O\left(n^{\log_b a - \varepsilon}\right) \rightarrow f(n) < n^{\log_b a} \\ \Theta\left(n^{\log_b a} \lg n\right) & f(n) = \Theta\left(n^{\log_b a}\right) \rightarrow f(n) = n^{\log_b a} \\ \Theta\left(f(n)\right) & f(n) = \Omega\left(n^{\log_b a + \varepsilon}\right) \rightarrow f(n) > n^{\log_b a} \\ & \text{if } af(n/b) \le cf(n) \text{ for } c < 1 \text{ and large } n \end{cases}$$

when we compare if our $f(n) < n^{\log_b a}$ our answer is this

when we compare

what you have to do here is, find a ✓ b ✓ $\log_b a$ ✓ $n^{\log_b a}$ compare with f(n)

Master Theorem – Case 1 example

Give tight asymptotic bound for

$$T(n) = 9T(n/3) + n$$

Solution:

a=9, b=3, and $f(n) = n$.

$$n^{\log_b a} = n^{\log_3 9} = n^2$$

$$f(n) = O(n^{\log_3 9 - \varepsilon}) \text{ for } \varepsilon = 1 \text{ or } f(n) < n^{\log_3 9} \rightarrow \text{case1}$$

$$\therefore T(n) = \Theta(n^2)$$

assume you are given an equation like this asking to find runing time

∴ if choosing master theorem equation should be according to

$T(n) = aT(n/b) + f(n)$

Since it is

a = 9 ✓

b = 3 ✓

f(n) = n ✓

$\log_b a = \log_3 9 = 2$

{ $n^{\log_b a} = n^2$, $f(n) = n$ } compare

since $n^2 > n$, $n^{\log_b a} > f(n)$

∴ answer ↑

$\Theta(n^2)$ = n^2 = $\Theta(n^{\log_b a})$

Master Theorem – Case 2 example

Give tight asymptotic bound for

$$T(n) = T(2n/3) + 1$$

$T(n) = aT(n/b) + f(n)$

Solution:

$a=1$, $b=3/2$, and $f(n) = 1$.

$a = 1$

$b = 3/2$

$f(n) = 1$

$$n^{\log_b a} = n^{\log_{3/2} 1} = n^0 = 1$$

$\log_b a = \log_{3/2} 1 = 0$

$$f(n) = \Theta(n^{\log_b a}) \text{ or } f(n) = n^{\log_b a} \rightarrow \text{case 2}$$

$n^{\log_b a} = n^0 = 1$

$f(n) = 1$

$f(n) = n^{\log_b a}$

$\Theta(1 \log n)$

$\Theta(n^{\log_b a} \log n)$

$$\therefore T(n) = \Theta(\log n)$$

Master Theorem – Case 3 example

- Give tight asymptotic bound for
- $T(n) = 3T(n/4) + n \log n$
- **Solution:**
- a=3, b=4, and $f(n) = n \log n$

$$n^{\log_b a} = n^{\log_4 3} = O(n^{0.793})$$

$$f(n) = \Omega(n^{\log_4 3 + \varepsilon}), \text{for } \varepsilon \approx 0.2 \text{ or } f(n) > n^{\log_4 3} \rightarrow \text{case } 3$$

$$Note: n \lg n \geq c.n^{\log_4 3}.n^{0.2}$$

Exercises.

- Use the master method to give tight asymptotic bounds for the following recurrences.
  1. $T(n) = 4T(n/2) + n$.
  2. $T(n) = 4T(n/2) + n2$. n^2
  3. $T(n) = 4T(n/2) + n3$. n^3

don't try

- Use the master method to show that the solution to the binary-search recurrence $T(n) = T(n/2) + \Theta(1)$ is $T(n) = \Theta(\lg n)$.