Sri Lanka Institute of Information Technology

Department of Information Systems Engineering

Discrete Mathematics

(IE2082)

Unit1 - Logical Form & Logical Equivalence

INTRODUCTION TO LOGIC

- Logic is the science of reasoning, proof, thinking, or inference. Logic allows us to analyze a piece of reasoning, and determine whether it is correct or not. To use the technical terms, we determine whether the reasoning is valid or invalid.
- One does not need to study logic in order to reason correctly. However, a little basic knowledge of logic is often helpful when constructing or analyzing an argument.

1. Logical Form

Definition

- A statement (or proposition) is a sentence that is true or false, but not both.
- The truth value of a statement is whether it is true or false.
- An argument is a sequence of statements aimed at demonstrating the truth of an assertion.
- The conclusion is the assertion at the end of an argument
- The preceding statements used for the conclusion are called premises.

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Which of the following are statements:

(i) The moon is round

Statement - either true or false, can't be both

(ii) It is raining outside

Not statement - depends upon where you are

(iii) $2 + 2 = 5$

Statement - always false

(iv) $x + y = 3$

Not statement - depends upon values of x & y

Definitions :

1. If p is a statement, the negation of p is "not p" or "It is not the case that p" and is denoted $\sim p$. It has opposite truth value from p: if p is true, $\sim p$ is false, and if p is false, $\sim p$ is true.

2. If p and q are statements, the **conjunction** of p and q is "p and q" denoted $p \wedge q$. If either p or q is false, or both are false, then $p \wedge q$ is false (so $p \wedge q$ is true only when p and q are both true).

3. If p and q are statements, the **disjunction** of p and q is "p or q" denoted $p \vee q$. If either p or q is true, or both are true, then $p \vee q$ is true (so $p \vee q$ is false only when p and q are both false).

Note that ;

1. p or q denoted $p \vee q$ is true means either p is true or q is true or both are true
2. p and q denoted $p \wedge q$ is true means both p and q are true. If ether p or q is false then $p \wedge q$ is false
3. p but q means p and q
4. neither p nor q means $\sim p$ and $\sim q$

That is $\sim (p \vee q)$ [negation of either p or q -—- proof later)

$\vee$, $\wedge$ and $\sim$ are called **connectives** as then can combine statements together

TRUTH TABLES

We now define three important Truth Tables

Let p and q are statements, then;

1. Truth table for ~ (negation)

p	~ p
T	F
F	T

Example: Let p : 2 + 2 = 3. Here p is false, then ~ p is true

TRUTH TABLES

Let p and q are statements, then;

2.Truth table for p Λq (**conjunction** of p and q)

Note that p Λq is true if both p and q are true

p	q	p Λq
T	T	T
T	F	F
F	T	F
F	F	F

Example

p:2 +2 = 4

q:Monkey is a bird

Then p Λq: 2+2=4 and Monkey is a bird is false

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TRUTH TABLES

Let p and q are statements, then;

3.Truth table for $p \vee q$ (**disjunction** of p and q)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Note that $p \vee q$ is false if both p and q Are false

Example

p:2 +2 = 4

q:Monkey is a bird

Then $p \vee q$: 2+2=4 or Monkey is a bird is TRUE

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Exercise

p and q are propositions

Then write down the truth table for;

a. $\sim(p \wedge q)$ c. $\sim(p \vee q)$ e. $\sim p \wedge \sim q$

b. $p \vee \sim q$ d. $\sim p \vee \sim q$

p	q	$\sim p$	$\sim q$	$(p \wedge q)$	$p \vee q$	$\sim(p \wedge q)$	$\sim(p \vee q)$	$p \vee \sim q$	$\sim p \vee \sim q$	$\sim p \wedge \sim q$
T	T	F	F	T	T	F	F	T	F	F
T	F	F	T	F	T	T	F	T	T	F
F	T	T	F	F	T	T	F	F	T	F
F	F	T	T	F	F	T	T	T	T	T

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If two logical statements or prepositions have the same truth value then those two statements are said to be **Logically Equivalent.**

Thus in the earlier truth table we get the same truth values for the prepositions ~(p ∧q) and ~ p ∨ ~q and therefore they are equivalent.

This is indicated by ~(p ∧q) ≡ ~ p ∨ ~q

To test whether two prepositions p and q are Logically Equivelant, construct a truth table for p and involving the same statement variables and check their truth values; if they are the same then they are logically equivalent.

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If a logical statement or preposition has the same truth value T for every combination of primitive statements , p and q, then that statement is said to be a **Tautology** (denoted by t) and if it has the same truth value F then it is called a **Contradiction** (denoted by c)

Example Tautology (t) Contradiction (c)

p	q	(p ∧q)	~(p ∧q)	(p ∧q)∨ ~(p ∧q)	(p ∧q) ∧ ~(p ∧q)
T	T	T	F	T	F
T	F	F	T	T	F
F	T	F	T	T	F
F	F	F	T	T	F

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Exercise
What is the Truth value of the following compound proposition?

"Either potatoes grows on trees and Margaret Thacher is the first British Female PM, or it is not true that crow is a bird"

Let p - potatoes grows on trees
q - Margaret Thacher is the first British Female PM,
r – Crow is a bird

The proposition reads as (p and q) or (not r)

That is $(p \wedge q) \vee \sim r \equiv$ (F and T) or (not T)

(F) or F $\equiv$ F

So the complete proposition is F

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Theorems on Logical Equivalence

Commutative law	$(p \wedge q) \equiv (q \wedge p)$	$p \vee q \equiv q \vee p$
Associative law	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive law	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
Identity law	$(p \wedge t) \equiv p$	$p \vee c \equiv p$
Negation law	$p \vee \sim p \equiv t$	$p \wedge \sim p) \equiv c$
Double negative law	$\sim(\sim p) = p$	
Idempotent law	$p \wedge p \equiv p$	$p \vee p \equiv p$
De Morgan's law	$\sim(p \wedge q) \equiv (\sim p \vee \sim q)$	$\sim(p \vee q) \equiv (\sim p \wedge \sim q)$
Universal bound laws	$p \vee t \equiv t$	$(p \wedge c) \equiv c$
Absorption law	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Negation of t and c	$\sim t \equiv c$	$\sim c \equiv t$

Theorems on Logical Equivalence

Verify the following logical equivalences

a. $\sim(\sim p \land q) \land (p \lor q) \equiv p$

b. $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$

Conditional Prepositions

Many statements, particularly in Mathematics, are of the form " If p then q". Such statements are called *conditional statements* and are denote by $p \rightarrow q$

It is equivalent to:

i. p only if q
ii. p is sufficient for q
iii. q is necessary for p
iv. p implies q

The conditional $p \rightarrow q$ is true except in the case that p is true and q is false

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Example

Consider the statement" If ABC is an equilateral triangle, then the three angles are equal"

Let p : ABC is an equilateral triangle

And q : three angles are equal

Then the conditional statement is $p \rightarrow q$

It is equivalent to:

i. p only if q - *ABC is an equilateral triangle only if 3 angles are equal*

ii. p is sufficient for q - *ABC is an equilateral triangle is sufficient to state the 3 angles are equal*

iii. q is necessary for p - *It is necessary that three angles are equal for ABC to be an equilateral triangle*

iv. p implies q - *ABC is an equilateral triangle, implies that the three angles are equal*

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The truth table for $p \rightarrow q$ is given below

p	Q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Note:
The statement gives you the state of q only if p is true. But it does not imply anything when p is false. But the statement is true.

Exercise

Decide whether each of the following is true

$p \rightarrow (p \wedge q)$ is a tautology

$p \rightarrow (p \vee q)$ is a tautology

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The truth table for $p \rightarrow (p \wedge q)$ and $p \rightarrow (p \vee q)$ is given below

p	Q	$(p \wedge q)$	$(p \vee q)$	$p \rightarrow (p \wedge q)$	$p \rightarrow (p \vee q)$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	T	T	T	T	T
T	F	F	T	F	T	T
F	T	F	T	T	T	T
F	F	F	F	T	T	T

$p \rightarrow (p \wedge q)$ is not a tautology

$p \rightarrow (p \vee q)$ is a tautology and is correct

$(p \wedge q) \rightarrow (p \vee q)$ is a tautology and is correct

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Show that

$p \rightarrow (p \vee q)$, $(p \wedge q) \rightarrow (p \vee q)$ are tautologies where as

$(p \wedge q) \wedge \sim p$ is a contradiction

p	q	$\sim p$	$(p \wedge q)$	$(p \vee q)$	$p \rightarrow (p \wedge q)$	$p \rightarrow (p \vee q)$	$(p \wedge q) \wedge \sim p$	$(p \wedge q) \rightarrow (p \vee q)$
T	T	F	T	T	T	T	F	T
T	F	F	F	T	F	T	F	T
F	T	T	F	T	T	T	F	T
F	F	T	F	F	T	T	F	T

Different statements of a conditional statement

The truth table for $p \to q$ is given below

p	Q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

The conditional statement of the truth table is;

" If p is true then q is true"

Then the;

1. Converse statement is $q \to p$
   That is "If q is true then p is true"

2. Inverse statement is $\sim p \to \sim q$
   That is "If p is false then q is false"

3. The Contra-positive statement is $\sim q \to \sim p$
   That is "If q is false then p is false"

Example: What are the Converse, Inverse and Contra-positive statements of the statement " If Sunil can cross the lake then Sunil can swim to the island"

Let If Sunil can cross the lake ------- > p

Sunil can swim to the island --------- > q

The conditional statement is $p \to q$

1. Converse statement is $q \to p$
   " If Sunil can swim to the island then Sunil can cross the lake"

2. Inverse statement is $\sim p \to \sim q$
   " If Sunil cannot cross the lake then Sunil cannot swim to the island"
3. The Contra-positive statement is $\sim q \to \sim p$
   " If Sunil cannot swim to the island then Sunil cannot cross the lake"

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Different statements of a conditional statement (in symbolical form) in the truth table

p	q	~p	~q	p → q	q → p	~p → ~ q	~q → ~ p
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Note that:

1. Converse statement q → p is equivalent to the Inverse statement ~ p → ~ q

3. The Conditional statement is equivalent to the Contra-positive statement ~ q → ~ p

Ways To State An Implication

a. if p then q
b. p is sufficient for q
c. q if p
d. q when p
e. a necessary condition for p is q
f. a sufficient condition for q is p
g. q unless ~ p
h. p implies q
i. p only if q
j. q whenever p
k. q follows from p
l. If ~q then ~p

Exercise

Let p and q be the propositions: p - It is below freezing
q – It is snowing

Write the following propositions using p and q an logical connectives.

a. It is below freezing and snowing
b. It is below freezing but not snowing
c. It is not below freezing and it is not snowing
d. It is either snowing or below freezing (or both)
e. If it is below freezing, it also snowing.
f. It is either below freezing or it is snowing, but it is not snowing if it is below freezing.
g. That is is below freezing is necessary for it to be snowing

Exercise

State the converse, inverse and the contra-positive statements of the following conditional statements.

a. If it snows today, I will ski tomorrow

b. I come to the class whenever there is going to be a quiz.

c. A positive integer is a prime only if it has no divisors other than one and itself.

Bi-conditional statement

Consider $p \longrightarrow q$, This means that if p is true then q is true. In other words if p happens the q always happens.

Similarly $q \longrightarrow p$ means if q happens then p always happens.

Now consider $(p \longrightarrow q) \wedge (q \longrightarrow p)$ means if p happens then q always happens and if q happens then p always happens. This is denoted in mathematics by p iff q. It is denoted by $p \longleftrightarrow q$

TRUTH Table for p iff q

p	q	$P \longleftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Quantifiers

We need quantifiers to formally express the meaning of the words "all" and "some". The two most important quantifiers are:

Universal quantifier, "For all". Symbol: $\forall$

Existential quantifier, "There exists". Symbol: $\exists$

$\forall x\, P(x)$ asserts that $P(x)$ is true for every x in the domain.

$\exists x\, P(x)$ asserts that $P(x)$ is true for some x in the domain

Examples.

(i) For every x, $x^2 > 0$

(ii) For all A and B, $A \cap B = B \cap A$.

If p(x) is a predicate we write $(\forall x)p(x)$ to denote the statement "For all x, p(x) is true". Similarly, $(\forall x)(\forall y)p(x, y)$ denotes "For all x and all y, p(x, y)" is true.

Quantifiers

Examples.

(i) Let p(x) denote $x^2 > 0$. Then $(\forall x)p(x)$ denotes "For every x, $x^2 > 0$" or x, "For each x, $x^2 > 0$".

(i) Let p(A, B) denote $A \cap B = B \cap A$. Then $(\forall A)(\forall B)p(A, B)$ denotes "For all A and B, $A \cap B = B \cap A$".

But if necessary we may specify the domain, U say, if it is not clear: "For every real number x, $x^2 > 0$" may be denoted by $(\forall x \in R)p(x)$ instead of $(\forall x)p(x)$.

If p(x) is a PREDICATE then $(\forall x)p(x)$ is a STATEMENT.
$(\forall x)p(x)$ is true if p(x) is true for every $x \in U$, whereas while $(\forall x)p(x)$ is false if p(x) is false for at least one $x \in U$.

Quantifiers

Similar remarks apply to $(\forall x)(\forall y)p(x, y)$, etc.

Examples.

(i) Let p(x) denote $x^2 > 0$ where U = R. Then $(\forall x)p(x)$ is true.

(ii) The statement "For every integer x, $x^2 > 5$" is false. e.g. x = 1. Here U = Z but there is at least one $x \in Z$ for which $x^2 > 5$ is true, eg. X=3

(iii) Let p(x, y) denote "If $x > y$ then $x^2 > y^2$", where U = R. Then $(\forall x)(\forall y)p(x, y)$ is false.

Take, for example, x = 1 and y = −2.
Then p(x, y) becomes "If $1 > -2$ then $1 > 4$". Here $1 > -2$ is T But $1 > 4$ is F . From the truth table for → we see that "If $1 > -2$ then $1 > 4$" is F . Hence $(\forall x)(\forall y)p(x, y)$ is F . [*Check this*]

Quantifiers

(iv) "For all x and all y, if $x > y$ then $2x > 2y$" is T.

The symbol ∀ is called the ***universal quantifier*** : it has the meaning "for all", "for every" or "for each".

We now also study ∃, the ***existential quantifier*** : it has the meaning "there is (at least one)", "there exists" or "for some".
Examples.
(i) Let p(x) denote $x^2 > 5$, where U = R. Then (∃x)p(x) denotes "There exists a real number x such that $x^2 > 5$". This can also be expressed as "$x^2 > 5$ for some real number x".

If p(x) is a PREDICATE then (∃x)p(x) is a STATEMENT. (∃x)p(x) is true if p(x) is true for at least one $x \in U$, whereas (∃x)p(x) is false if p(x) is false for all $x \in U$.

Quantifiers

Some standard sets : R – Real Numbers,
Z – Integers
Q – Rational Numbers
N – Natural Nos.

Examples.

(i) Let U = R. The statement $(\exists x)x^2 > 5$ is T because $x^2 > 5$ is T for at least one value of x, e.g. x = 3.

(ii) Let p(x) denote $x^2 < 0$, where U = R. Then (∃x)p(x) is F because p(x) is F for all $x \in U$.

(iii) $(\exists x)(\exists y)(x + y)^2 = x^2 + y^2$ (where U = R) is T: take x = 0, y = 0 for example.

Statements may involve both ∀ and ∃.

Quantifiers

Example. Consider the following statements.

(i) Everyone likes all of Beethoven's symphonies.

(ii) Everyone likes at least one of Beethoven's symphonies.

(iii) There is one Beethoven's symphony which everyone likes.

(iv) There is someone who likes all of Beethoven's symphonies.

(v) Every Beethoven's symphony is liked by someone.

(vi) There is someone who likes at least one of Beethoven's symphonies.

Quantifiers

If we let $p(x, y)$ denote the predicate "x likes y" where x belongs to the universal set of all University students and y belongs to the universal set of all Beethoven's symphonies then the statements become:

(i) $(\forall x)(\forall y)p(x, y)$
(ii) $(\forall x)(\exists y)p(x, y)$
(iii) $(\exists y)(\forall x)p(x, y)$
(iv) $(\exists x)(\forall y)p(x, y)$
(v) $(\forall y)(\exists x)p(x, y)$
(vi) $(\exists x)(\exists y)p(x, y)$

All have different meanings: in particular, $(\forall x)(\exists y)$ is not the same as $(\exists y)(\forall x)$.

Quantifiers

Example.

Consider the statements

(i) $(\forall x)(\exists y)x < y$ and
(ii) $(\exists y)(\forall x)x < y$ where $U = R$.

Statement (i) is true but statement (ii) is false.

Note that (i) states that whatever number x we choose we can find a number y which is greater than x (e.g. $y = x + 1$).
But (ii) states that there is a number y which is simultaneously greater than every number x: this is impossible because, with $x = y$, $x < y$ does not hold.

Quantifiers

Negation of quantifiers: Negation of $\forall x$ is $\exists x$ and the negation of $\exists x$ becomes $\forall x$.

That is $\sim\forall x \equiv \exists x$ and $\sim\exists x \equiv \forall x$.

Consider the following prepositions.

preposition	meaning
$\forall x\ p(x)$	All true
$\exists x\ p(x)$	At least one true
$\sim\exists x\ p(x)$	None true
$\forall x \sim p(x)$	All false
$\exists x \sim p(x)$	At least one false
$\sim\exists x \sim p(x)$	None false
$\sim\forall x\ p(x)$	Not all true
$\sim\forall x \sim p(x)$	Not all false

Quantifiers

Consider a statement of the form;

$\forall x \varepsilon D$, if $p(x)$ then $q(x)$

contra positive is the statement,

$\forall x \varepsilon D$, if $\sim q(x)$ then $\sim p(x)$

converse is the statement,

$\forall x \varepsilon D$, if $q(x)$ then $p(x)$

Its inverse is the statement,

$\forall x \varepsilon D$, if $\sim p(x)$ then $q(x)$