Sri Lanka Institute of Information Technology

Faculty of Computing

Logic Control

Ms.Nilushi Dias

Year 01 Semester 01

Outline

Number Systems

Number Systems

- Mathematical notation/symbols for representing values (numbers).
- In different systems, different symbols are used.

Positional Number System

- The most common base used in everyday activities is 10 (Decimal System).
- Different bases are used in other situations.
- The base can be written as a subscript to the number for easy identification.
  - Example: 1265_{10} 010000001_2
- 4 types of positional number systems are discussed:
  - Decimal (Base = 10, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9).
  - Binary (Base = 2, 0, 1).
  - Octal (Base = 8, 0, 1, 2, 3, 4, 5, 6, 7, 8).
  - Hexadecimal (Base = 16, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F).

Positional Number System (cont'd)

- We are used to dealing with numbers in the decimal system.
- This is probably a result of having ten fingers.
- It is a positional number system.
  - Roman number system is not such a system.
- The position of the symbol denotes the magnitude.
- A positional number system uses a base (aka radix).
- A number system with base b is a system that uses distinct symbols for b digits.

Positional Number System (cont'd)

- To determine the quantity, it is necessary to multiply each digit by an integer power of the base and then form the sum of all weighted digits.

Example (724110):-

7		2		4		1
×		×		×		×
10^3		10^2		10^1		10^0
=		=		=		=
7000	+	200	+	40	+	1
			7241			

Binary Number System

- Mainly used in computers and computer-based devices.
- A computer contains electronic components that uses voltages.
- Therefore, numbers are represented in a computer with a base of 2.
- All other information is represented using a binary code as well.
    - Letters of the alphabet and punctuation marks
    - Microprocessor instruction
    - Graphics/Video
    - Sound

Number Base Conversion

- Should be able to convert a number in one base to another base.
- Examples will be discussed in converting,

$$Decimal \leftrightarrow Binary$$

$$Decimal \leftrightarrow Octal$$

$$Decimal \leftrightarrow Hexadecimal$$

$$Binary \leftrightarrow Octal$$

Repeated Division Method

- Can be easily used to convert a decimal number to another base.
  1. Divide the number successively by the base.
  2. After each division record the remainder.
  3. The result is read from the last remainder upwards.

Repeated Division Method (Example)

- Convert 123_{10} to a binary representation.
  - $123/2 \quad q = 61 \quad r = 1$
  - $61/2 \quad q = 30 \quad r = 1$
  - $30/2 \quad q = 15 \quad r = 0$
  - $15/2 \quad q = 7 \quad r = 1$
  - $7/2 \quad q = 4 \quad r = 1$
  - $4/2 \quad q = 2 \quad r = 0$
  - $2/2 \quad q = 1 \quad r = 0$
- $123_{10} = 0011011_2$

Repeated Subtraction Method

- Can be used to convert a decimal number to binary.
  1. Starting with the 1s place, write down all of the binary place values in order until you get to the first binary place value that is GREATER THAN the decimal number you are trying to convert.
  2. AMark out the largest place value (it just tells us how many place values we need).
  3. Subtract the largest place value from the decimal number. Place a "1" under that place value.

Repeated Subtraction Method

4. For the rest of the place values, try to subtract each one from the previous result.
    - If you can, place a “1” under that place value.
    - If you can’t, place a “0” under that place value.
5. Repeat Step 4 until all of the place values have been processed.

- Convert 123_{10} to binary using the repeated subtraction method

Other Base Conversions

- Binary/Octal/Hexadecimal to Decimal
  1. Take the left most none zero bit,
  2. Multiply by the base and add it to the bit on its right.
  3. Now take this result, multiply by the base it and add it to the next bit on the right,
  4. Continue in this way until the right-most bit has been added in.

The fundamental setup of positional number systems can be used as well.

Other Base Conversions

- Binary to Octal/Hexadecimal
  1. Form the bits into groups of three/four starting at the right and move leftwards.
  2. Replace each group of three bits with the corresponding octal/hexadecimal digit.

- Octal/Hexadecimal to Binary

The opposite of the above process is used

Conversion of Fractions

- Decimal Fractions to Binary Fractions

1. Begin with the decimal fraction and multiply by 2. The whole number part of the result is the first binary digit to the right of the point.
2. Disregard the whole number part of the previous result and multiply by 2 once again. The whole number part of this new result is the second binary digit to the right of the point.
3. Continue this process until we get a zero as our decimal part or until we recognize an infinite repeating pattern.

Conversion of Fractions (Example)

- Convert 0.625_{10} to binary.
- Convert 0.1_{10} to binary.
- Convert 1.625_{10} to binary.

Conversion of Fractions (cont'd.)

- Binary Fractions to Decimal Fractions

The fundamental setup of positional number systems used in converting binary integers to decimals can be used here.

- Represent 10.01101_2 as a decimal number

Summary

- Students should be able to,
  - Understand the numerical system.
  - Explain why computer designers chose to use the binary system for representing information in computers.
  - Explain different number systems.
  - Translate numbers between number system
- Understanding the pattern in each set of conversions will make it easier to remember the methods.

Boolean Algebra

Boolean Algebra

- A variable used in an algebraic formula so far, is assumed to take a set of numerical values.
- All variables in boolean equations can take only one of two possible values.
- Used symbols for the two values are 0 and 1.
- Rules first defined for logic by George Boole (1854), were adapted for the use in designing electronic circuits.
- The circuits in computers and other electronic devices have inputs, each of which is either a 0 or a 1.

Boolean Operators

- One major advantage in using these rules is to simplify an electronic circuit.
- Boolean algebra provides the operations and the rules for working with boolean variables.
- Three (3) boolean operators are discussed.
  - Complement
  - Boolean sum
  - Boolean product
- Ten (10) rules are also discussed (aka Boolean Identities).

Boolean Operators

- **Complement**
  - Defined as the opposite of the value that a boolean variable takes.
  - Denoted with a bar (E.g.: A).
  - $0 = 1$ and $1 = 0$.
- **Boolean Sum**
  - Defined as the output to be 1 if at least one variable is 1.
  - Denoted with the symbol $+$ or by OR.
  - $0 + 0 = 0$, $0 + 1 = 1$, $1 + 0 = 1$ and $1 + 1 = 1$.

Boolean Operators (cont'd.)

- **Boolean Product**
  - Defined as the output to be 0 if at least one variable is 0.
  - Denoted with the symbol ( ) or by AND.
  - $0 \quad 0 = 0$, $0 \quad 1 = 0$, $1 \quad 0 = 0$ and $1 \quad 1 = 1$.
  - When there is no danger of confusion, the symbol can be omitted.

- Order of boolean operators,
  1. Complement.
  2. Boolean products.
  3. Boolean sums.

Boolean Identities

1. Law of Double Complement
   - $\bar{\bar{A}} = A$
2. Idempotent Laws
   - $A + A = A$
   - $A \cdot A = A$
3. Identity Laws
   - $A + 0 = A$
   - $A \cdot 1 = A$
4. Domination/Null/Universal Bound Laws
   - $A + 1 = 1$
   - $A \cdot 0 = 0$

Boolean Identities (cont'd.)

5. Commutative Laws
   - $A + B = B + A$
   - $A \cdot B = B \cdot A$
6. Associative Laws
   - $A + (B + C) = (A + B) + C.$
   - $A \cdot (B \cdot C) = (A \cdot B) \cdot C$
7. Commutative Laws
   - $A \cdot (B + C) = (A \cdot B) + (A \cdot C)$
   - $A + B \cdot C = (A + B) \cdot (A + C)$

Boolean Identities (cont'd.)

8. De Morgan's Laws
   - $\overline{(A \cdot B)} = \overline{A} + \overline{B}$
   - $\overline{(A + B)} = \overline{A} \cdot \overline{B}$
9. Absorption Laws
   - $A \cdot (A + B) = A$
   - $A + A \cdot B = A$
10. Inverse Laws / Unit Zero Properties
    - $A + \overline{A} = 1$
    - $A \cdot \overline{A} = 0$

Examples

1. Find the values of the following expressions.
   i. $1 \cdot \overline{0}$
   ii. $1 + \overline{1}$
   iii. $\overline{(1 + 0)}$
2. Prove both variants of the absorption law using other boolean identities.
3. Simplify the following expressions.
   i. $A\overline{B}D + A\overline{BD}$
   ii. $(\overline{A} + B)(A + B)$
   iii. $M = \overline{WXYZ} + \overline{W}X\overline{YZ} + WX\overline{YZ} + W\overline{XYZ}$

Truth Tables

- To verify the above rules, a truth table can be used.
- It's also known as a Table of Combinations.
- It's a table displaying all possible values for the variables and the outcomes for a boolean expression.
- If there are n number of variables, there will be 2^n number of rows in the truth table.
- If the truth table for two boolean expressions shows the same outcomes for the same values for the variables, it can be concluded that the expressions are the same/equal.

Example

1. Use a table to express the values of each of these Boolean functions.
   i. $\overline{A}B$
   ii. $M = x\overline{y} + (\overline{xyz})$
   iii. $F(x, y, z) = \overline{y}(xz + \overline{xz})$

2. Using a truth table, show that,

$$x\overline{y} + y\overline{z} + \overline{x}z = \overline{x}y + \overline{y}z + x\overline{z}$$

Sum of Products (SoP)

- In some cases, the truth table might be known and we might want to know the expression that gives the truth table.
- This can be done by representing as a Sum of Products (SoP) of the variables and their complements.
- Steps:-
  1. Select the rows in the truth table that gives 1 as the outcome.
  2. Write how we can obtain 1 for the first selected row by using the product of the variables.
  3. Repeat step two for all selected rows and use the sum to combine all results

Example

Find the boolean expression for F from the given truth table

A	B	C	F
0	0	0	**1**
0	0	1	**0**
0	1	0	**1**
0	1	1	**1**
1	0	0	**0**
1	0	1	**0**
1	1	0	**1**
1	1	1	**1**

Product of Sums (PoS)

- Used for the same reason as a SoP.
- Product of Sums (PoS) has opposite steps of SoP.
- Steps:-
  1. Select the rows in the truth table that gives 0 as the outcome.
  2. Write how we can obtain 0 for the first selected row by using the sum of the variables.
  3. Repeat step two for all selected rows and use the product to combine all results.
- Conversion can be done between the two using De Morgan's rule.

Duality Principle

- In a boolean expression, if all the sums $(+)$ and products $(\ .\ )$ are exchanged as well as if 1's and 0's are exchanged, the resulting expression is the opposite of the initial expression.
- This property is observed between SoP and PoS.
- The duel of the complement of one form is equal to the expression in the other form.

Summary

- Students should be able to,
  - Understand the boolean expressions.
  - Learn laws and rules of boolean algebra.
  - Simplify boolean expressions using boolean identities.
  - Use Sum of Products (SoP) and Product of Sums (PoS) to find boolean expressions.
  - Understand similarities and differences between boolean variables as opposed to regular variables.

Computer Arithmetic

Introduction

Recap:

- Binary numbers are a number system with base 2.
- Information represented inside a computer takes binary values.
- Previous lecture dealt with the conversions between different number systems.
- This lecture deals with basic mathematical operations (such as addition, subtraction, multiplication and division) for binary numbers.

Binary Addition

- Addition in the decimal number system.
  - Add values rightmost position (least significant).
  - If this addition is grater than 10, 1 is carried to the 2nd position and added.
  - This process is carried for all the positions.
- Binary addition follows the same set of rules.
  - If the addition is greater that 2, 1 is carried to the 2nd next position.

Binary Addition

- Evaluate the following.
  - $101_2 + 101_2$
  - $00011010_2 + 00001100_2$
  - $10001 \ + \ 11101$
  - $1110 \ + \ 1111$
  - $101101 \ + \ 11001$
  - $10111 \ + \ 110101$
  - $1011001 \ + \ 111010$
  - $11011 \ + \ 1001010$

Binary Subtraction

- Similar to subtraction in the decimal number system.
- Inverse of addition.
- If the values cannot be subtracted, borrow from the next position.
- Subtraction table,
    - 0 - 0 = 0
    - 1 - 0 = 1
    - 1 - 1 = 0
    - 0 - 1 = 1 with a borrow of 1.

Examples

- Evaluate the following.
  - 10110 - 10010
  - 1011011 - 10010
  - 100010110 - 1111010
  - 1010110 - 101010
  - 101101 - 100111
  - 1000101 - 101100
  - 1110110 - 1010111
- Compare the above results by converting them to decimal numbers.

Multiplication and Division

- Similar to multiplication and division in the decimal number system.
- Rules of binary multiplication,
  - $0 \times 0 = 0$
  - $0 \times 1 = 0$
  - $1 \times 0 = 0$
  - $1 \times 1 = 1$
- Rules of binary division,
  - $0 \div 1 = 0$
  - $1 \div 1 = 1$

Examples

- Evaluate the following.
  - 1100×1010
  - 1111×101
  - 0011×11
  - 1100110×1000
  - $1000 \div 10$
  - $1010 \div 11$
  - $1111 \div 111$
- Compare the above results by converting them to decimal numbers.

Complementary Arithmetic

- Complements are used in digital computers for simplifying,
  - the subtraction operation
  - the logical manipulation.
- Two types of compliments for each base b
- system.
  - r's compliment
  - (r - 1)'s compliment
  - Example: For binary numbers, 2's complement and 1's complement.

Examples

- Get the 1's and 2's compliments of the following binary numbers.
  - 1100011
  - 0001111
  - 1010100
  - 1111011

Logic Gates

Logic Gates

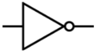
- A computer, or other electronic device, is made up of a number of circuits.
- The components in a logical circuit takes 0 and 1 as inputs.
- 1 is the state where there is a voltage on the input and 0 is the state where there is no voltage on the input.

Logic Gates

- Therefore, boolean algebra is used to model the circuitry in electronic devices.
- The absence of a voltage is usually denoted as 0 (zero) and the presence of a voltage is denoted by 1 (one).
- As mentioned earlier, concepts of boolean algebra can be used to simplify logical circuitry.
- There are a set of components that matches the boolean operators discussed earlier.
- These components are called **Logic Gates**.

Basic Logic Gates (NOT Gate

- **Complement**→ NOT Gate.
- Also known as *inverter* or *complementer*.
- Consists of a single input and a single output.
- As in the complement, the input gets inverted.
- Symbol:

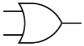


- Truth Table:

Input (A)	Output ($\bar{A}$)
0	1
1	0

Basic Logic Gates (OR Gate)

- Boolean Sum $\rightarrow$ OR Gate.
- Consists of two inputs and a single output.
- As in the sum, the output is 1 if at least one input is 1.
- Symbol:

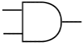


- Truth Table:

Input (A)	Input (B)	Output (A + B)
0	0	0
0	1	1
1	0	1
1	1	1

Basic Logic Gates (AND Gate)

- **Boolean Product** $\rightarrow$ AND Gate.
- Consists of two inputs and a single output.
- As in the product, the output is 0 if at least one input is 0.
- Symbol:

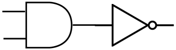


- Truth Table:

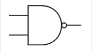
Input (A)	Input (B)	Output (A · B)
0	0	0
0	1	0
1	0	0
1	1	1

Derived Logic Gates (NAND Gate)

- Created by combining an AND gate with a NOT gate.



- Symbol:

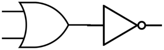


- Truth Table:

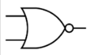
Input (A)	Input (B)	Output $(\overline{A \cdot B})$
0	0	1
0	1	1
1	0	1
1	1	0

Derived Logic Gates (NOR Gate)

- Created by combining an OR gate with a NOT gate.



- Symbol:



- Truth Table:

Input (A)	Input (B)	Output $(\overline{A+B})$
0	0	1
0	1	0
1	0	0
1	1	0

Derived Logic Gates (XOR Gate)

- Similar to the OR Gate.
- Consists of two inputs and a single output.
- The output is 1 if ONLY ONE input is 1.
- Symbol:

- Truth Table:

Input (A)	Input (B)	Output (A ⊕ B)
0	0	0
0	1	1
1	0	1
1	1	0

Summary

- Students should be able to,
  - Get an understanding about the need and usage of logic gates.
  - Understand basic logic gates and the connection with boolean operators.
  - The functions obtained by the logic gates.
  - Draw truth tables for the logic gates.
  - Draw circuit diagrams for the logic gates using the standard symbols.
  - Drawing circuit diagrams from boolean expressions and vice versa.

Thank You!

Sri Lanka Institute of Information Technology

Information Technology

Partial Fractions

Ms.Nilushi Dias

Year 01 Semester 01

Outline

Learning outcomes

- Factorize the denominator of an algebraic fraction into its prime factors
- Separate an algebraic fraction into its partial fractions
- Recognise the rules of partial fractions

Partial fractions

To reverse the process, namely, to separate an algebraic fraction into its partial fractions we proceed as follows. Consider the fraction:

$$\frac{8x-28}{x^2-6x+8}$$

Firstly, the denominator is factorized to give:

$$\frac{8x-28}{x^2-6x+8}=\frac{8x-28}{(x-2)(x-4)}$$

Partial fractions

Next, it is assumed that a partial fraction break down is possible in the form:

$$\frac{8x-28}{(x-2)(x-4)} = \frac{A}{x-2} + \frac{B}{x-4}$$

The assumption is validated by finding the values of A and B

Partial fractions

To find the values of A and B the two partial fractions are added to give:

$$\frac{8x-28}{(x-2)(x-4)} = \frac{A}{x-2} + \frac{B}{x-4}$$

$$= \frac{A(x-4)+B(x-2)}{(x-2)(x-4)}$$

Partial fractions

Since:

$$\frac{8x-28}{(x-2)(x-4)} = \frac{A(x-4)+B(x-2)}{(x-2)(x-4)}$$

And since the denominators are identical the numerators must be identical as well. That is:

$$8x-28 \equiv A(x-4)+B(x-2)$$

Partial fractions

Consider the identity:

$$8x - 28 \equiv A(x-4) + B(x-2)$$

$$\text{Let } x = 4 \text{ then } 32 - 28 \equiv A(0) + B(2) \text{ so } B = 2$$

$$\text{Let } x = 2 \text{ then } 16 - 28 \equiv A(-2) + B(0) \text{ so } A = 6$$

Therefore:

$$\frac{8x-28}{(x-2)(x-4)} = \frac{6}{x-2} + \frac{2}{x-4}$$

For this procedure to be successful the numerator of the original fraction must be of at least one degree less than the degree of the denominator. If this is not the case the original fraction must be reduced by division. For example:

$$\frac{x^2+3x-10}{x^2-2x-3}=\frac{x^2-2x-3+5x-7}{x^2-2x-3}$$

$$=1+\frac{5x-7}{x^2-2x-3}$$

$$=1+\frac{5x-7}{(x+1)(x-3)}=1+\frac{3}{x+1}+\frac{2}{x-3}$$

Denominators with quadratic factors

A similar procedure is applied if one of the factors in the denominator is a quadratic. For example:

$$\frac{15x^2-x+2}{(x-5)(3x^2+4x-2)}=\frac{A}{x-5}+\frac{Bx+C}{3x^2+4x-2}$$

This results in:

$$15x^2-x+2\equiv A(3x^2+4x-2)+(Bx+C)(x-5)$$

$$=(3A+B)x^2+(4A-5B+C)x-2A-5C$$

Denominators with quadratic factors

Equating coefficients of powers of x yields:

$$[x^2] \quad 15 = 3A + B$$

$$[x] \quad -1 = 4A - 5B + C$$

$$[CT] \quad 2 = -2A - 5C$$

Three equations in three unknowns with solution:

A = 4, B = 3 and C = -2 so that:

Denominators with quadratic factors

Substituting A = 4, B = 3 and C = 2

$$\frac{15x^2-x+2}{(x-5)(3x^2+4x-2)}=\frac{4}{x-5}+\frac{3x-2}{3x^2+4x-2}$$

Denominators with quadratic factors

Repeated factors in the denominator of the original fraction of the form:

$$(ax+b)^2$$

give partial fractions of the form:

$$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}$$

Denominators with quadratic factors

Similarly, repeated factors in the denominator of the original fraction of the form:

$$(ax+b)^3$$

give partial fractions of the form:

$$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}+\frac{C}{(ax+b)^3}$$

Thank You!

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Sri Lanka Institute of Information Technology

Information Technolog

Trigonometry

Ms.Nilushi Dias

Year 01 and Semester 01

Outline

Angles

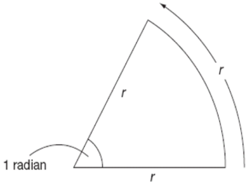
- Trigonometric identities
- Trigonometric formulas

Angles: Rotation

- When a straight line is rotated about a point, it sweeps out an angle that can be measured in degrees or radians.
- A straight line rotating through a full angle and returning to its starting point is said to have rotated through 360 degrees (360°).
- One degree $=$ 60 minutes and one minute $=$ 60 seconds.

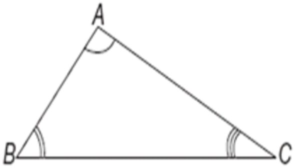
Angles: Radians

- When a straight line of length r is rotated about one end so that the other end describes an arc of length r, the line is said to have rotated through 1 radian (1 rad).



Angles: Triangles

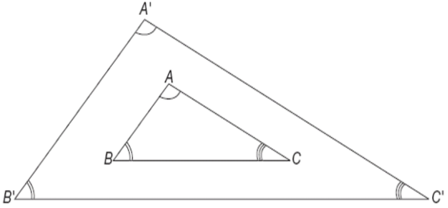
- All triangles possess shape and size.
- The shape of a triangle is governed by the **three angles** and the size by the **lengths of the three sides**.



Angles: Trigonometric Ratios

For similar triangles

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$$



so that:

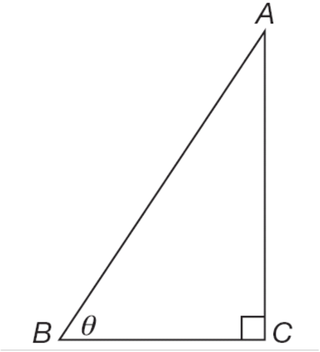
$$\frac{AB}{AC} = \frac{A'B'}{A'C'} \quad \text{and} \quad \frac{AB}{BC} = \frac{A'B'}{B'C'} \quad \text{and} \quad \frac{AC}{BC} = \frac{A'C'}{B'C'}$$

Angles: Trigonometric Ratios

sine of angle $\theta = \dfrac{AC}{AB}$ - denoted by $\sin\theta$

cosine of angle $\theta = \dfrac{AC}{BC}$ - denoted by $\cos\theta$

tangent of angle $\theta = \dfrac{AC}{BC}$ - denoted by $\tan\theta$



Angles: Reciprocal Ratios

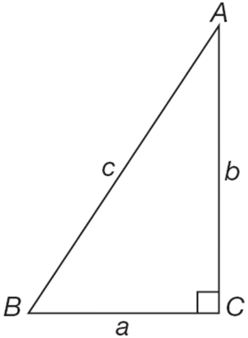
$$\textit{cosecant} \text{ of angle } \theta = \frac{1}{\sin\theta} \text{ - denoted by } \operatorname{cosec}\theta$$

$$\textit{secant} \text{ of angle } \theta = \frac{1}{\cos\theta} \text{ - denoted by } \sec\theta$$

$$\textit{cotangent} \text{ of angle } \theta = \frac{1}{\tan\theta} \text{ - denoted by } \cot\theta$$

- The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.

$$\boxed{a^2 + b^2 = c^2}$$



Angles: Special Triangles

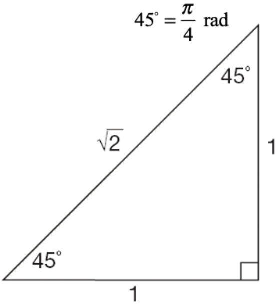
Right-angled isosceles

Angles measured in degrees:

$$\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}} \quad \text{and} \quad \tan 45^\circ = 1$$

Angles measured in radians:

$$\sin \pi/4 = \cos \pi/4 = \frac{1}{\sqrt{2}} \quad \text{and} \quad \tan \pi/4 = 1$$



Angles: Special Triangles

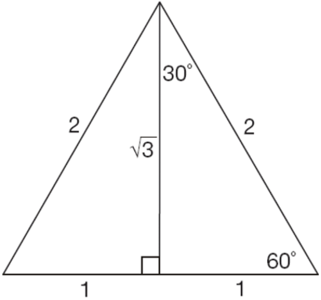
Half equilateral

Angles measured in degrees:

$$\sin 30^\circ = \cos 60^\circ = \frac{1}{2}$$

$$\sin 60^\circ = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\tan 60^\circ = \frac{1}{\tan 30^\circ} = \sqrt{3}$$



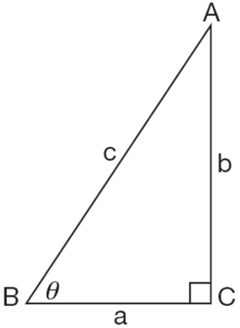
Trigonometric Identities: The fundamental identity

- The fundamental trigonometric identity is derived from Pythagoras' theorem.

$$a^2 + b^2 = c^2 \quad \text{so} \quad \frac{a^2}{c^2} + \frac{b^2}{c^2} = 1$$

that is:

$$\boxed{\cos^2\theta + \sin^2\theta = 1}$$



Trigonometric Identities: Two more identities

- Dividing the fundamental identity by $\cos^2\theta$

$$\cos^2\theta + \sin^2\theta = 1 \quad \text{so that} \quad \frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

that is:

$$\boxed{1 + \tan^2\theta = \sec^2\theta}$$

Trigonometric Identities: Two more identities

- Dividing the fundamental identity by $\sin^2\theta$

$$\cos^2\theta + \sin^2\theta = 1 \quad \text{so that} \quad \frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

that is:

$$\boxed{\cot^2\theta + 1 = \operatorname{cosec}^2\theta}$$

Trigonometric Formulas

- Sums and differences of angles

$$\cos(\theta+\phi)=\cos\theta\cos\phi-\sin\theta\sin\phi$$
$$\cos(\theta-\phi)=\cos\theta\cos\phi+\sin\theta\sin\phi$$

$$\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi$$
$$\sin(\theta-\phi)=\sin\theta\cos\phi-\cos\theta\sin\phi$$

$$\tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$$
$$\tan(\theta-\phi)=\frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}$$

Trigonometric Formulas

- Double angles

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\cos 2\theta = 1 - 2\sin^2\theta$$
$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Trigonometric Formulas

- Sums and differences of ratios

$$\sin\theta + \sin\phi = 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}$$
$$\sin\theta - \sin\phi = 2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}$$

$$\cos\theta + \cos\phi = 2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}$$
$$\cos\theta - \cos\phi = -2\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}$$

Trigonometric Formulas

- Products of ratios

$$2\sin\theta\cos\phi = \sin(\theta+\phi) + \sin(\theta-\phi)$$

$$2\cos\theta\cos\phi = \cos(\theta+\phi) + \cos(\theta-\phi)$$

$$2\sin\theta\sin\phi = \cos(\theta-\phi) - \cos(\theta+\phi)$$

Thank You!

Sri Lanka Institute of Information Technology

Faculty of Computing

Differentiation

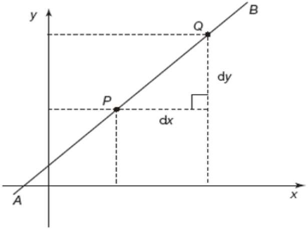
Ms.Nilushi Dias

Year 01 and Semester 01

The Gradient of a Straight-Line Graph

- The gradient of the sloping straight line in the figure is defined as

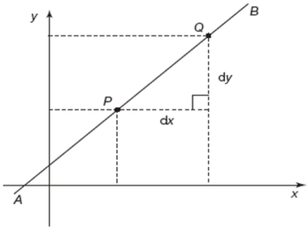
$$\frac{\text{the vertical distance the line rises and falls between the two points P and Q}}{\text{the horizontal distance between P and Q}}$$



The gradient of a straight-line graph

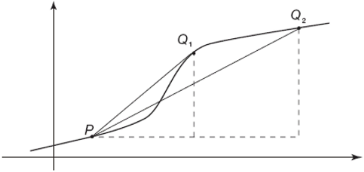
- The gradient of the sloping straight line in the figure is given as:

$$\frac{dy}{dx} \text{ and its value is denoted by the symbol } m$$



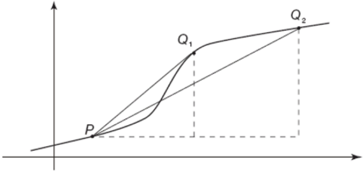
The Gradient of a Curve at a Given Point

- If we take two points P and Q on a curve and calculate, as we did for the straight line, the ratio of the vertical distance the curve rises or falls and the horizontal distance between P and Q the result will depend on the points chosen:



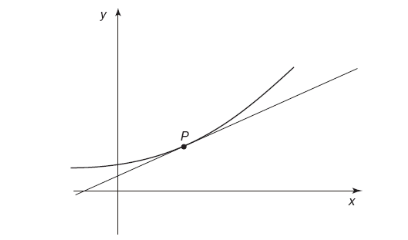
The gradient of a curve at a given point

- The result depends on the points chosen because the gradient of the curve varies along its length. Because of this the gradient of a curve is not defined between two points as in the case of a straight line but at a single point.



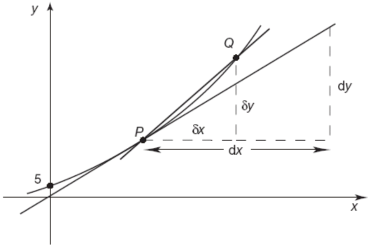
The gradient of a curve at a given point

- The gradient of a curve at a point P is defined to be the gradient of the tangent at that point:



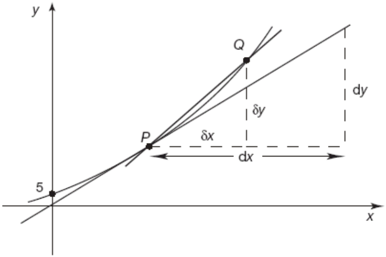
Algebraic determination of the gradient of a curve

The gradient of the chord PQ is $\dfrac{\delta y}{\delta x}$ and the gradient of the tangent at P is $\dfrac{dy}{dx}$



Algebraic determination of the gradient of a curve

- As Q moves to P so the chord rotates. When Q reaches P the chord is coincident with the tangent.
- For example, consider the graph of $y = 2x^2 + 5$



At Q: $y + \delta y = 2(x+\delta x)^2 + 5$

So $= 2x^2 + 4x.\delta x + 2[\delta x]^2 + 5$

As $\delta y = 4x.\delta x + 2[\delta x]^2$ and $\dfrac{\delta y}{\delta x} = 4x + 2.\delta x$

$\delta x \to 0$ so $\dfrac{\delta y}{\delta x} \to$ the gradient of the tangent at $P = \dfrac{dy}{dx}$

Therefore

$$\boxed{\frac{dy}{dx} = 4x}$$

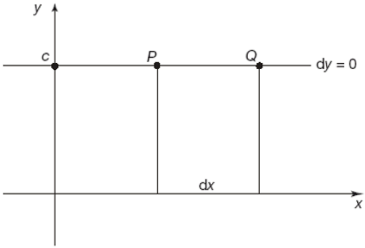
called *the derivative of y with respect to x.*

Derivatives of powers of x

- Two straight lines
- Two curves

(a) $y = c$ (constant)

$dy = 0$ therefore $\boxed{\dfrac{dy}{dx} = 0}$



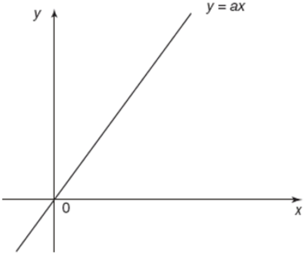
Derivatives of powers of x

Two straight lines

(b) $y = ax$

$$y + dy = a(x + dx)$$

$$dy = a.dx \quad \text{therefore} \quad \boxed{\frac{dy}{dx} = a}$$



Derivatives of powers of x

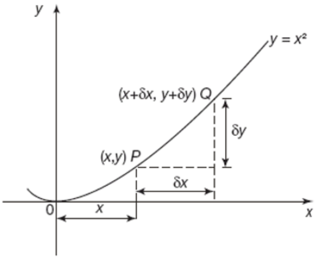
Two curves

(a) $y = x^2$

so $y + \delta y = (x + \delta x)^2$

$\delta y = 2x.\delta x + [\delta x]^2$ therefore $\dfrac{\delta y}{\delta x} = 2x + \delta x$

therefore $\boxed{\dfrac{dy}{dx} = 2x}$



Derivatives of powers of x

Two curves

(b) $y = x^3$

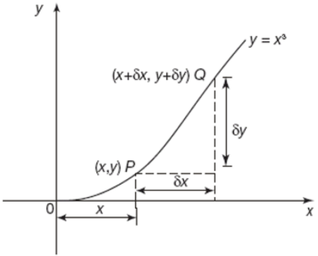
$$y + \delta y = (x + \delta x)^3$$

so

$$\delta y = 3x^2.\delta x + 3x.[\delta x]^2 + [\delta x]^3$$

therefore $\dfrac{\delta y}{\delta x} = 3x^2 + 3x.\delta x + [\delta x]^2$

therefore $\boxed{\dfrac{dy}{dx} = 3x^2}$



Derivatives of powers of x

- A clear pattern is emerging:

y	$\frac{dy}{dx}$
c	0
x	1
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$
x^5	$5x^4$

$$\boxed{\text{If } y = x^n \text{ then } \frac{dy}{dx} = nx^{n-1}}$$

Differentiation of polynomials

- To differentiate a polynomial, we differentiate each term in turn:

$$\text{If } y = x^4 + 5x^3 - 4x^2 + 7x - 2$$

$$\text{then } \frac{dy}{dx} = 4x^3 + 5 \times 3x^2 - 4 \times 2x + 7 \times 1 - 0$$

$$\text{Therefore } \frac{dy}{dx} = 4x^3 + 15x^2 - 8x + 7$$

Derivatives – an alternative notation

- The double statement:

$$\text{If } y = x^4 + 5x^3 - 4x^2 + 7x - 2$$

$$\text{then } \frac{dy}{dx} = 4x^3 + 5 \times 3x^2 - 4 \times 2x + 7 \times 1 - 0$$

- can be written as

$$\frac{d}{dx}\left(x^4 + 5x^3 - 4x^2 + 7x - 2\right) = 4x^3 + 15x^2 - 8x + 7$$

Derivatives – an alternative notation

- The derivative of the derivative of y is called the second derivative of y and is written as:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

So, if:

$$y = x^4 + 5x^3 - 4x^2 + 7x - 2$$

$$\frac{dy}{dx} = 4x^3 + 15x^2 - 8x + 7$$

then

$$\frac{d^2y}{dx^2} = 12x^2 + 30x - 8$$

Standard derivatives and rules

Standard derivatives

- The table of standard derivatives can be extended to include trigonometric and the exponential functions:

$$\frac{d}{dx}\left(\sin x\right)=\cos x$$

$$\frac{d}{dx}\left(\cos x\right)=-\sin x$$

$$\frac{d}{dx}\left(e^{x}\right)=e^{x}$$

Differentiation of products of functions

- Given the product of functions of x:

$$y = uv$$

then:

$$\boxed{\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$$

- This is called the product rule.

Differentiation of a quotient of two functions

- Given the quotient of functions of x:

$$y = \frac{u}{v}$$

then:

$$\boxed{\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}}$$

- This is called the quotient rule.

Functions of a function

Differentiation of a function of a function

- To differentiate a function of a function we employ the chain rule.
- If y is a function of u which is itself a function of x so that:

$$y(x) = y(u[x])$$

Then:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

- This is called the chain rule.

- Many functions of a function can be differentiated at sight by a slight modification to the list of standard derivatives:

F is a function of x			
y	$\frac{dy}{dx}$	y	$\frac{dy}{dx}$
F^n	$nF^{n-1}.\frac{dF}{dx}$	$\cos F$	$-\sin F.\frac{dF}{dx}$
$a.F^n$	$a.nF^{n-1}.\frac{dF}{dx}$	$\tan F$	$\sec^2 F.\frac{dF}{dx}$
$\sin F$	$\cos F.\frac{dF}{dx}$	e^F	$e^F.\frac{dF}{dx}$

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Sri Lanka Institute of Information Technology

Faculty of Computing

Integration

Ms.Nilushi Dias

Year 01 and Semester 01

Outline

- Integration is the reverse process of differentiation. For example:

$$\frac{d}{dx}(x^4) = 4x^3$$

- The integral of $4x^3$ is then written as:

$$\int 4x^3 . dx$$

- Its value is, however:

$$\int 4x^3 . dx = x^4 + C \quad \text{where } C \text{ is called the } \textit{constant of integration}$$

Standard integrals

- Just as with derivatives we can construct a table of standard integrals:

$f(x)$	$\int f(x).\mathrm{d}x$
x^n	$\frac{x^{n+1}}{n+1} + C$
1	$x + C$
a	$ax + C$
$\sin x$	$-\cos x + C$
$\cos x$	$\sin x + C$
$\sec^2 x$	$\tan x + C$
e^x	$e^x + C$
a^x	$a^x / \ln a + C$
$\frac{1}{x}$	$\ln x + C$

Integration of polynomial expressions

- Just as polynomials are differentiated term by term so they are integrated, also term by term. For example:

$$\int \left(4x^3 + 5x^2 - 2x + 7\right).dx = x^4 + \frac{5x^3}{3} - x^2 + 7x + C$$

Functions of a linear function of x

- To integrate

$$\int (ax+b)^n .dx$$

- we change the variable by letting u = ax + b so that du = a.dx. Substituting into the integral yield:

$$\int (ax+b)^n .dx = \int u^n .du$$
$$= \frac{u^{n+1}}{n+1} + C$$
$$= \frac{(ax+b)^{n+1}}{n+1} + C$$

Integration by partial fractions

To integrate $\int \frac{7x+8}{2x^2+11x+5}dx$ we note that $\frac{7x+8}{2x^2+11x+5} = \frac{3}{x+5} + \frac{1}{2x+1}$

so that:

$$\int \frac{7x+8}{2x^2+11x+5}dx = \int \frac{3}{x+5}dx + \int \frac{1}{2x+1}dx$$

These partial fractions are 'functions of a linear function of x'

based on the standard integral $\int \frac{1}{x}dx$ and so

$$\int \frac{7x+8}{2x^2+11x+5}dx = 3\ln(x+5) + \frac{1}{2}\ln(2x+1) + C$$

Areas under curves

- Area A, bounded by the curve y = f(x), the x-axis and the ordinates x = a and x = b, is given by:

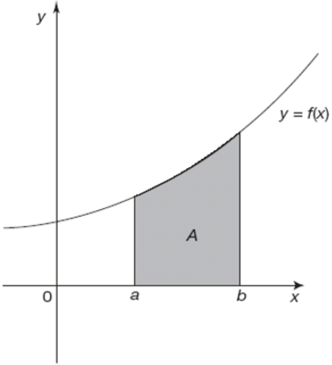
$$A = \int_{x=a}^{b} f(x)dx$$

$$= [F(x)]_{x=a}^{b}$$

$$= F(b) - F(a)$$

Where,

$$F'(x) = f(x)$$



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Sri Lanka Institute of Information Technology

Faculty of Computing

Complex Numbers

Ms.Nilushi Dias

Year 01 and Semester 01

Outline

- The numerals were devised to enable written calculations and records of quantities and measurements. When a grouping of symbols such as $\sqrt{-1}$ occurs to which there is no corresponding quantity we ask ourselves why such a grouping occurs and can we make anything of it?
- In response we carry on manipulating with it to see if anything worthwhile comes to light.
- We call $\sqrt{-1}$ an **imaginary** number to distinguish it from those numbers to which we can associate quantity which we call **real** numbers.

The symbol j

Quadratic equations

- The solutions to the quadratic equation:

$$x^2 - 1 = 0$$

are:

$$x = +1 \text{ and } x = -1$$

- The solutions to the quadratic equation:

$$x^2 + 1 = 0$$

are:

$$x = +\sqrt{-1} \text{ and } x = -\sqrt{-1}$$

We avoid the clumsy notation by defining $j = \sqrt{-1}$

Powers of j

- Positive integer powers
- Negative integer powers

Powers of j

Positive integer powers

Because: $j = \sqrt{-1}$

so:

$$j^2 = -1$$

$$j^3 = j^2 j = -j$$

$$j^4 = \left(j^2\right)^2 = (-1)^2 = 1$$

$$j^5 = j^4 j = j$$

Powers of j

Negative integer powers

Because: $j^2 = -1$ so $j = -\dfrac{1}{j} = -j^{-1}$

and so:

$$j^{-1} = -j$$

$$j^{-2} = \left(j^2\right)^{-1} = (-1)^{-1} = -1$$

$$j^{-3} = \left(j^{-2}\right)\left(j^{-1}\right) = (-1)(-j) = j$$

$$j^{-4} = \left(j^{-2}\right)^2 = (-1)^2 = 1$$

- A complex number is a mixture of a real number and an imaginary number. The symbol z is used to denote a complex number.
- In the complex number $z = 3 + j5$:
  - the number 3 is called the real part of z and denoted by Re(z)
  - the number 5 is called the imaginary part of z, denoted by Im(z)

- The real parts and the imaginary parts are added (subtracted) separately:
- and so:

$$
\begin{aligned}
(4+j5)+(3-j2)& \\
&=4+j5+3-j2 \\
&=4+3+j5-j2 \\
&=7+j3
\end{aligned}
$$

Complex numbers

Multiplication

- Complex numbers are multiplied just like any other binomial product:
- and so:

$$
\begin{aligned}
&(4+j5)\times(3-j2) \\
&\quad = 4(3-j2)+j5(3-j2) \\
&\quad = 12-j8+j15-j^2 10 \\
&\quad = 12-j8+j15+10 \quad \text{because } j^2=-1 \\
&\quad = 22+j7
\end{aligned}
$$

- The complex conjugate of a complex number is obtained by switching the sign of the imaginary part. So that:

$$(5 + j8) \text{ and } (5 - j8)$$

- Are complex conjugates of each other.
- The product of a complex number and its complex conjugate is entirely real:

$$\begin{aligned}
&(a + jb)\times(a - jb) \\
&\qquad = a(a - jb) + jb(a - jb) \\
&\qquad = a^2 - jba + jba - j^2b^2 \\
&\qquad = a^2 + b^2
\end{aligned}$$

- To divide two complex numbers both numerator and denominator are multiplied by the complex conjugate of the denominator:

$$
\begin{aligned}
\frac{7-j4}{4+j3} &= \frac{(7-j4)}{(4+j3)} \times \frac{(4-j3)}{(4-j3)} \\
&= \frac{(7-j4)\times(4-j3)}{(4+j3)\times(4-j3)} \\
&= \frac{(16-j37)}{(16+9)} \\
&= \frac{16}{25} - j\frac{37}{25}
\end{aligned}
$$

- If two complex numbers are equal then their respective real parts are equal and their respective imaginary parts are equal.

$$\text{If } a + jb = c + jd \text{ then } a = c \text{ and } b = d$$

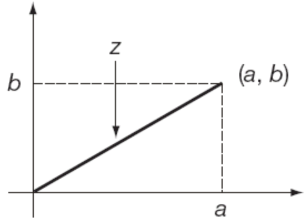
Graphical representation of a complex number

- Argand diagram
- Graphical addition of a complex number

Graphical representation of a complex number

Argand diagram

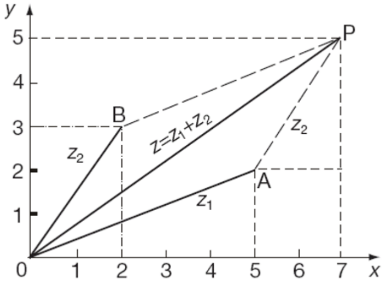
- The complex number $z = a + jb$ can be represented by the line joining the origin to the point (a, b) set against Cartesian axes.



- This is called the **Argand** diagram and the plane of points is called the complex plane.

- Complex numbers add (subtract) according to the parallelogram rule:

$$(5 + j2) + (2 + j3) = 7 + j5$$



Polar form of a complex number

A complex number can be expressed in polar coordinates r and θ

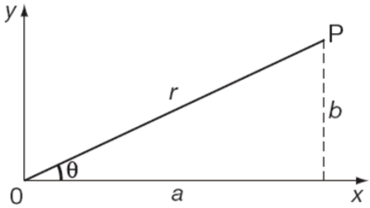
$$\begin{aligned} z &= a + jb \\ &= r(\cos\theta + j\sin\theta) \end{aligned}$$

where:

$$a = r\cos\theta, \quad b = r\sin\theta$$

and:

$$r^2 = a^2 + b^2$$



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Exponential form of a complex number

- Therefore:

$$\boxed{z = r\left(\cos\theta + j\sin\theta\right) = re^{j\theta}}$$

Polar-form calculations

Notation

- The polar form of a complex number is readily obtained from the Argand diagram of the number in Cartesian form.

Given:

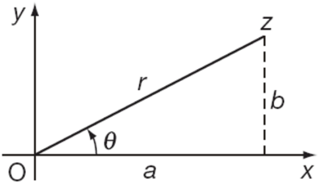
$$z = a + jb$$

then:

$$r^2 = a^2 + b^2 \quad \text{so} \quad r = \sqrt{a^2 + b^2}$$

and

$$\tan\theta = \frac{b}{a} \quad \text{so} \quad \theta = \tan^{-1}\frac{b}{a}$$



- The length r is called the **modulus** of the complex number and the angle θ is called the **argument** of the complex number

Polar-form calculations

Multiplication

- When two complex numbers, written in polar form, are multiplied the product is given as a complex number whose modulus is the product of the two moduli and whose argument is the sum of the two arguments.

$$\text{If } z_1 = r_1(\cos\theta_1 + j\sin\theta_1) \text{ and } z_2 = r_2(\cos\theta_2 + j\sin\theta_2)$$

$$\text{then } z_1 z_2 = r_1 r_2(\cos[\theta_1+\theta_2] + j\sin[\theta_1+\theta_2])$$

- When two complex numbers, written in polar form, are divided the quotient is given as a complex number whose modulus is the quotient of the two moduli and whose argument is the difference of the two arguments.

$$\text{If } z_1 = r_1(\cos\theta_1 + j\sin\theta_1) \text{ and } z_2 = r_2(\cos\theta_2 + j\sin\theta_2)$$

$$\text{then } \frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos[\theta_1 - \theta_2] + j\sin[\theta_1 - \theta_2]\right)$$

Roots of a complex number

- De Moivre's theorem
- nth roots

- If a complex number is raised to the power n the result is a complex number whose modulus is the original modulus raised to the power n and whose argument is the original argument multiplied by n.

$$\text{If } z = r(\cos\theta + j\sin\theta)$$

$$\text{then } z^n = \left[r(\cos\theta + j\sin\theta)\right]^n = r^n(\cos n\theta + j\sin n\theta)$$

Thank You!

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Faculty of Computing

Matrices

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FACULTY OF COMPUTING

Outline

Matrices – definitions

- A matrix is a set of real or complex numbers (called elements) arranged in rows and columns to form a rectangular array.
- A matrix having m rows and n columns is called an m $\times$ n matrix.
- For example:

$$\begin{bmatrix} 2 & 0 & 1 \\ 3 & 0 & 0 \\ 5 & 1 & 1 \end{bmatrix}$$

is a 2 $\times$ 3 matrix.

Matrices – definitions

- Row matrix
  - A row matrix consists of a single row. For example:

$$\begin{bmatrix} 4 & 3 & 7 & 2 \end{bmatrix}$$

- Column matrix
  - A column matrix consists of a single column. For example:

$$\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix}$$

Matrices – definitions

- Double suffix notation
  - Each element of a matrix has its own address denoted by double suffices, the first indicating the row and the second indicating the column. For example, the elements of 3×4 matrix can be written as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

Matrix notation

- Where there is no ambiguity a matrix can be represented by a single general element in brackets or by a capital letter in bold type.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

can be denoted by (a_{ij}) or by $\mathbf{A}$

Equal matrices

- Two matrices are equal if corresponding elements throughout are equal.

$$\mathbf{A} = \mathbf{B} \text{ that is } (a_{ij}) = (b_{ij}) \text{ if } a_{ij} = b_{ij} \text{ for all values of i and j}$$

Addition and subtraction of matrices

- Two matrices are added (or subtracted) by adding (or subtracting) corresponding elements. For example:

$$\begin{aligned}\begin{pmatrix}4 & 2 & 3\\ 5 & 7 & 6\end{pmatrix}+\begin{pmatrix}1 & 8 & 9\\ 3 & 5 & 4\end{pmatrix}&=\begin{pmatrix}4+1 & 2+8 & 3+9\\ 5+3 & 7+5 & 6+4\end{pmatrix}\\ &=\begin{pmatrix}5 & 10 & 12\\ 8 & 12 & 10\end{pmatrix}\end{aligned}$$

Multiplication of matrices

Scalar multiplication

- To multiply a matrix by a single number (a scalar), each individual element of the matrix is multiplied by that number. For example:

$$4\times\begin{pmatrix}3 & 2 & 5\\ 6 & 1 & 7\end{pmatrix}=\begin{pmatrix}12 & 8 & 20\\ 24 & 4 & 28\end{pmatrix}$$

That is:

$$k\left(a_{ij}\right)=\left(ka_{ij}\right)$$

- Two matrices can only be multiplied when the number of columns in the first matrix equals the number of rows in the second matrix.
- The *ij*th element of the product matrix is obtained by multiplying each element in the *i*th row of the first matrix by the corresponding element in the *j*th column of the second matrix and the element products added.
- For example:

Multiplication of matrices

Multiplication of two matrices

$$\text{If } \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix}$$

$$\text{then } \mathbf{A.B} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} . \begin{pmatrix} b_{11} \\ b_{21} \\ b_{23} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} \end{pmatrix}$$

If $\mathbf{A}=\left(a_{ij}\right)$ is an $n\times m$ matrix and

$\mathbf{B}=\left(b_{ij}\right)$ is an $m\times q$ matrix then

$\mathbf{C}=\mathbf{A.B}=\left(c_{ij}\right)$ is an $n\times q$ matrix where

$$c_{ij}=\sum_{k=1}^{m} a_{ik}b_{kj}$$

Transpose of a matrix

- If a new matrix is formed by interchanging rows and columns the new matrix is called the transpose of the original matrix. For example, if:

$$\mathbf{A} = \begin{pmatrix} 4 & 6 \\ 7 & 9 \\ 2 & 5 \end{pmatrix} \text{ then } \mathbf{A}^{\mathbf{T}} = \begin{pmatrix} 4 & 7 & 2 \\ 6 & 9 & 5 \end{pmatrix}$$

Special matrices

Square matrix

- A square matrix is of order m $\times$ m.
- A square matrix is symmetric if $a_{ij} = a_{ji}$. For example:

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 8 & 9 \\ 5 & 9 & 4 \end{pmatrix}$$

- A square matrix is skew-symmetric if $a_{ij} = a_{ji}$. For example:

$$\begin{pmatrix} 0 & 2 & 5 \\ -2 & 0 & 9 \\ -5 & -9 & 0 \end{pmatrix}$$

Special matrices

Diagonal matrix

- A diagonal matrix is a square matrix with all elements zero except those on the leading diagonal. For example:

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

- A unit matrix is a diagonal matrix with all elements on the leading diagonal being equal to unity. For example:

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- The product of matrix A and the unit matrix is the matrix A:

$$\mathbf{A.I} = \mathbf{A}$$

- A null matrix is one whose elements are all zero.

$$\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- Notice that

$$\mathbf{A.0} = \mathbf{0}$$

- But that if $A.B = 0$ we cannot deduce that $A = 0$ or $B = 0$

Determinant of a square matrix

Singular matrix

- Every square matrix has its associated determinant. For example, the determinant of

$$\begin{pmatrix} 5 & 2 & 1 \\ 0 & 6 & 3 \\ 8 & 4 & 7 \end{pmatrix} \text{ is } \begin{vmatrix} 5 & 2 & 1 \\ 0 & 6 & 3 \\ 8 & 4 & 7 \end{vmatrix} = 150$$

- The determinant of a matrix is equal to the determinant of its transpose.
- A matrix whose determinant is zero is called a singular matrix.

- Each element a_{ij} of a square matrix has a minor which is the value of the determinant obtained from the matrix after eliminating the ith row and jth column to which the element is common.
- The cofactor of element a_{ij} is then given as the minor of a_{ij} multiplied by

$$(-1)^{i+j}$$

- Let square matrix C be constructed from the square matrix A where the elements of C are the respective cofactors of the elements of A so that if:

$$\mathbf{A}=\left(a_{ij}\right) \text{ and } A_{ij} \text{ is the cofactor of } a_{ij} \text{ then } \mathbf{C}=\left(A_{ij}\right)$$

- Then the transpose of C is called the adjoint of A, denoted by adjA.

Determinants

The symbol:

$$a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$ (evaluated by cross multiplication as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$)

- Is called a second-order determinant; second-order because it has two rows and two columns.

- A third-order determinant has three rows and three columns. Each element of the determinant has an associated *minor* – a second order determinant obtained by eliminating the row and column to which it is common. For example:

$$\text{the minor of } a_1 \text{ is } \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} \text{ obtained thus } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

- To expand a third-order determinant about the first row we multiply each element of the row by its minor and add and subtract the products as follows:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Determinants of third order

Evaluation of a determinant about any row or column

- To expand a determinant about any row or column we multiply each element of the row or column by its minor and add and subtract the products according to the pattern:

$$\begin{array}{ccccccc} + & - & + & - & \cdots & \cdots & \cdots \\ - & + & - & + & \cdots & \cdots & \cdots \\ + & - & + & - & \cdots & \cdots & \cdots \\ - & + & - & + & \cdots & \cdots & \cdots \end{array}$$

Inverse of a square matrix

- If each element of the adjoint of a square matrix A is divided by the determinant of A then the resulting matrix is called the inverse of A, denoted by A^-1.

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}(\mathrm{adj}\mathbf{A})$$

- Note: if det A $=$ 0 then the inverse does not exist

Inverse of a square matrix

Product of a square matrix and its inverse

- The product of a square matrix and its inverse is the unit matrix:

$$\mathbf{A.A^{-1} = A^{-1}.A = I}$$

Solution of a set of linear equations

- The set of n simultaneous linear equations in n unknowns

$$\begin{array}{ccccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \ldots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \ldots & + & a_{2n}x_n & = & b_2 \\
\vdots & & \vdots & & \vdots & & & & \vdots & & \vdots \\
a_{n1}x_1 & + & a_{n2}x_2 & + & a_{n3}x_3 & + & \ldots & + & a_{nn}x_n & = & b_n
\end{array}$$

- can be written in matrix form as:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \quad \text{that is} \quad \mathbf{A.x} = \mathbf{b}$$

Solution of a set of linear equations

- Since:

$$\mathbf{A.x = b} \quad \text{then}$$

$$\mathbf{A^{-1}.Ax = A^{-1}.b} \quad \text{that is}$$

$$\mathbf{I.x = A^{-1}.b} \quad \text{and} \quad \mathbf{I.x = x}$$

- The solution is then:

$$\boxed{\mathbf{x = A^{-1}.b}}$$

- Given:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

- Create the augmented matrix B, where:

$$\mathbf{B} = \left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & b_n \end{array}\right)$$

$$\mathbf{B} = \left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & b_n \end{array} \right)$$

- Eliminate the elements other than a11 from the first column by subtracting a_{21}/a_{11} times the first row from the second row, a_{31}/a_{11} times the first row from the third row, etc. This gives a new matrix of the form:

$$\left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ 0 & c_{22} & c_{23} & \cdots & c_{2n} & d_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & c_{n2} & c_{n3} & \cdots & c_{nn} & d_n \end{array}\right)$$

- This process is repeated to eliminate the c_{i2} from the third and subsequent rows until a matrix of the following form is arrived at:

$$\left(\begin{array}{ccccc|c} a_{11} & \cdots & a_{1,n-2} & a_{1,n-1} & a_{1n} & b_1 \\ 0 & \cdots & p_{n-3,n-2} & p_{n-2,n-1} & p_{n-2,n} & q_2 \\ 0 & \cdots & 0 & p_{n-1,n-1} & p_{n-1,n} & \vdots \\ 0 & \cdots & 0 & 0 & p_{nn} & q_n \end{array}\right)$$

Solution of a set of linear equations

Gaussian elimination method for solving a set of linear equations

$$\left(\begin{array}{ccccc|c} a_{11} & \cdots & a_{1,n-2} & a_{1,n-1} & a_{1n} & b_1 \\ 0 & \cdots & p_{n-3,n-2} & p_{n-2,n-1} & p_{n-2,n} & q_2 \\ 0 & \cdots & 0 & p_{n-1,n-1} & p_{n-1,n} & \vdots \\ 0 & \cdots & 0 & 0 & p_{nn} & q_n \end{array}\right)$$

- From this augmented matrix we revert to the product:

$$\begin{pmatrix} a_{11} & \cdots & a_{1,n-2} & a_{1,n-1} & a_{1n} \\ 0 & \cdots & p_{n-3,n-2} & p_{n-2,n-1} & p_{n-2,n} \\ 0 & \cdots & 0 & p_{n-1,n-1} & p_{n-1,n} \\ 0 & \cdots & 0 & 0 & p_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$$

Solution of a set of linear equations

Gaussian elimination method for solving a set of linear equations

$$\begin{pmatrix} a_{11} & \cdots & a_{1,n-2} & a_{1,n-1} & a_{1n} \\ 0 & \cdots & p_{n-3,n-2} & p_{n-2,n-1} & p_{n-2,n} \\ 0 & \cdots & 0 & p_{n-1,n-1} & p_{n-1,n} \\ 0 & \cdots & 0 & 0 & p_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$$

- From this product the solution is derived by working backwards from the bottom starting with:

$$p_{nn} x_n = q_n \quad \text{so} \quad x_n = \frac{q_n}{p_{nn}}$$

Eigenvalues and eigenvectors

- Expressed as a set of separate equations:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

- That is:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= \lambda x_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= \lambda x_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots + a_{nn}x_n &= \lambda x_n \end{aligned}$$

- These can be rewritten as:

$$\begin{pmatrix} a_{11}-\lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn}-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

- That is:

$$\left(\mathbf{A}-\lambda\mathbf{I}\right).\mathbf{x}=\mathbf{0}$$

- Which means that, for non-trivial solutions:

$$\left|\mathbf{A}-\lambda\mathbf{I}\right|=0$$

Eigenvalues and eigenvectors
Eigenvalues

To find the eigenvalues of: $\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$

solve the characteristic equation: $\begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = 0$

That is: $(\lambda - 1)(\lambda - 5) = 0$

This gives eigenvalues $\boxed{\lambda_1 = 1;\ \lambda_2 = 5}$

To find the eigenvectors of $\mathbf{A}=\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$ solve the equation $\mathbf{A.x}=\lambda\mathbf{x}$

For the eigenvalues $\lambda = 1$ and $\lambda = 5$

For $\lambda=1$

$$\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=1\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and so } x_2=-3x_1 \text{ giving eigenvector } \begin{pmatrix} k \\ -3k \end{pmatrix}$$

For $\lambda=5$

$$\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}=5\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ and so } x_2=x_1 \text{ giving eigenvector } \begin{pmatrix} k \\ k \end{pmatrix}$$

Diagonalization

- A matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.
- In fact, $A = PDP^{-}1$ with D a diagonal matrix, if and only if the columns of P and n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

Diagonalization

- **Example:** Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

- That is, find an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}$$

Solution

- Step 1. **Find the eigenvalues of A.**
- Here, the characteristic equation turns out to involve a cubic polynomial that can be factored:

Diagonalization

$$0 = \det(A - \lambda I) = -\lambda^3 - 3\lambda^2 + 4$$
$$= -(\lambda - 1)(\lambda + 2)^2$$

- The eigenvalues are $\lambda = 1$ and $\lambda = -2$.
- Step 2. **Find three linearly independent eigenvectors of A.**
- Three vectors are needed because A is a matrix.
- This is a critical step.
- If it fails, then Theorem 5 says that A cannot be diagonalized.

Diagonalization

- Basis for $\lambda = 1 : \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

- Basis for $\lambda = -2 : \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

- You can check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set.

Diagonalization

- Step 3. **Construct P from the vectors in step 2.**
- The order of the vectors is unimportant.
- Using the order chosen in step 2, form

$$P = \begin{bmatrix} \mathrm{v}_1 & \mathrm{v}_2 & \mathrm{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

- Step 4. **Construct D from the corresponding eigenvalues.**
- In this step, it is essential that the order of the eigenvalues matches the order chosen for the columns of P.

Diagonalization

- Use the eigenvalue $\lambda = -2$ twice, once for each of the eigenvectors corresponding to $\lambda = -2$:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

- To avoid computing P^{-1} , simply verify that $AD = PD$.
- Compute

$$AP = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} = PD$$

Thank You!