24-677
Spring 2021
Final Exam
5/13/21
Time Limit: 180 Minutes

Name (Print): ________________________

This exam contains 5 pages (including this cover page) and 4 problems.

You may use your calculator on this exam.

Please read the following rules carefully and sign here to certify your compliance:
A digital signature is acceptable, or you can sign your name to your paper submission.

Signature: ________________________

You are required to show your work on each problem on this exam. The following rules apply:

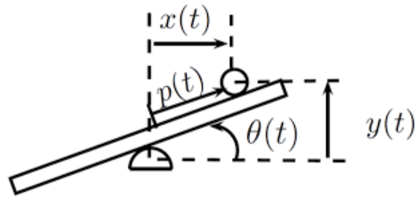
- **Individual effort:** This is not a homework, where collaboration is encouraged. All of the work you submit must be your own, and **you are not permitted to discuss this exam with your fellow students until Friday 5/14**. **As discussed on day 1, cheating on an exam will result in course failure.**
- **Internet usage:** This is an open book / notes exam, but **you are permitted to use the internet only to access the course Canvas materials** and submit your exam to Gradescope. The Lockdown browser will be used to enforce compliance.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- **Manage your time**. To maximize partial credit, you should clearly set up each problem before getting bogged down in algebra and calculations for any one problem.
- You can write your answers in the Exam itself, or on your own paper. If the latter, be sure that the problems are clearly identified for the graders.

1. (40 points) The figure below shows the classic ball and beam control system, where the position of a ball on a beam is controlled by manipulating the beam angle. The nonlinear system dynamics are given by

$$\frac{7}{5}\ddot{p} + g\sin\theta - p\dot{\theta}^2 = 0$$

$$\left(mp^2 + \frac{1}{12}ML^2\right)\ddot{\theta} + 2mp\dot{p}\dot{\theta} + mgp\cos\theta = \tau,$$

where m is the mass of the ball, M is the mass of the beam, and the equations are valid for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. We are interested in controlling the position of the ball along the beam $p(t)$.



(a) (10 points) Using the state vector $\mathbf{x} = \begin{bmatrix} p \\ \dot{p} \\ \theta \\ \dot{\theta} \end{bmatrix}$, write the state equation in the form $\dot{x} = f(x, u)$. Find the equilibrium point(s) for the system and linearize the system about the equilibrium point(s).

(b) (10 points) Using values $g = 9.81$, $m = 1$, $M = 10$, and $L = 0.5$, assess the stability of the system about the origin. Is it Lyapunov stable? Asymptotically stable? BIBO stable? HINT: You should have a sparse system - take advantage of that in your computation!

(c) (10 points) Consider the linearization about the origin. Is the system controllable via the torque input τ?

(d) (10 points) Consider the linearization about the origin. Is the system observable by measuring the location of the ball $p(t)$ alone?

2. (40 points) Consider the model of a simple oscillator given by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -100 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

(a) (10 points) Find the eigenvalues and eigenvectors for the system.

(b) (10 points) This system with complex eigenvalues can be placed into real modal canonical form by applying a similarity transformation $M = \begin{bmatrix} u & v \end{bmatrix}$ where u is the vector of real eigenvector components for one of the two eigenvalues and v is the vector of imaginary eigenvector components for same eigenvalue. Transform the given system into real modal canonical form.

(c) (10 points) Discretize the original system using a time step of $T = 0.01$ and the ZOH method. Give your answer as a discrete time state equation.

(d) (10 points) Solving the state equation from Part c forward three steps results in

$$x_3 = \begin{bmatrix} 0.955 & 0.03 \\ -2.96 & 0.995 \end{bmatrix} x_0 + \begin{bmatrix} 2.5 \times 10^{-5} \\ 0.97 \times 10^{-3} \end{bmatrix} u_0 + \begin{bmatrix} 1.5 \times 10^{-5} \\ 0.99 \times 10^{-3} \end{bmatrix} u_1 + \begin{bmatrix} 5 \times 10^{-6} \\ 0.001 \end{bmatrix} u_2.$$

Find the optimal control sequence that minimizes $\|\mathbf{u}\|_2^2$ while driving the system from state $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to state $x_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

3. (20 points) A standard model from the voltage - speed relationship for a permanent magnet brushed DC motor is

$$G(s) = \frac{K}{\tau s + 1}.$$

In this problem you will design an optimal output feedback controller that achieves strong reference tracking while being immune to encoder noise. Assume that $K = 10$ and $\tau = 0.1$.

(a) (10 points) Assume that the motor driver has a voltage noise of $W = 0.01\ V^2$ and the encoder has a noise of $V = 0.001$ rad^2. Using a forward difference model of the plant dynamics with $T = 0.001$, find the Kalman gain L <u>assuming an infinite time horizon</u>. What is the expected steady-state variance of the state estimate?

(b) (10 points) Using the same forward difference plant dynamics, find the optimal control sequence $\mathbf{u}$ that best follows the reference $r_1 = 2,\ r_2 = 5$ starting from $x_0 = 5$ with respect to the cost function

$$J = 10(r_2 - y_2)^2 + \sum_{k=0}^{1} 10(r_k - y_k)^2 + 2u_k^2.$$

4. (10 points) Consider the system

$$x_{k+1} = \begin{bmatrix} 1 & 1 \\ 10 & 1 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_k,$$

$$y_k = \begin{bmatrix} 0 & 1 \end{bmatrix} x_k.$$

Your partner designed a deadbeat controller and observer for this system, which resulted in $K = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $L = \begin{bmatrix} 1.1 \\ 2 \end{bmatrix}$. Starting from initial state $x_0 = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$ find

- The state of the open loop system at $k = 10$. Assume that $u_k = 0 \ \forall k$.
- The state of the closed loop system at $k = 4$. Assume that the observer states are initialized to zero.