Linear Vector Space

The idea is to generalise the notion of a vector from an "arrow" & redefine it as a member of a Linear Vector Space (LVS)

→ Advantage is that many more mathematical objects such as fns, matrices etc. start behaving as vectors. $\boxed{\text{Abstraction}}$

To define an LVS, we first define a field

Field : Any set of nos. which are closed under operations of addition, multiplication, subtraction & division comprises a field.

Let us now imagine a set of abstract objects denoted by the symbol $| \rangle$. Two kinds of operations are permitted with $| \rangle$.

→ We can add two $| \rangle$'s to get another $| \rangle$

→ We can multiply a $| \rangle$ with a no. coming from the field to get another $| \rangle$. i.e. $|c\rangle = \alpha \cdot |b\rangle$

Set of rules which define an LVS

$\boxed{A}$: a) If $|a\rangle, |b\rangle \in S$, then $|a\rangle + |b\rangle \in S$

b) If $|a\rangle \in S$ & α is a no. from the field, then $\alpha \cdot |a\rangle \in S$

c) There exists a null element indicated by $|0\rangle \in S$ such that $|a\rangle + |0\rangle = |a\rangle$

d) For any $|a\rangle \in S$, there exists another $|a'\rangle \in S$

such that $|a\rangle + |a'\rangle = |0\rangle$

$\boxed{B}$: For any $|a\rangle, |b\rangle$ & $|c\rangle$ in S in complex nos. α & β (from a field)

e) $|a\rangle + |b\rangle = |b\rangle + |a\rangle$ [Commutative law]

f) $(|a\rangle + |b\rangle) + |c\rangle = |a\rangle + (|b\rangle + |c\rangle)$ [Associative law of addition]

g) $1 \cdot |a\rangle = |a\rangle$

h) $\alpha \cdot (\beta \cdot |a\rangle) = (\alpha \cdot \beta) \cdot |a\rangle$ [Associative law of multiplication]

i) $(\alpha + \beta) \cdot |a\rangle = \alpha \cdot |a\rangle + \beta \cdot |a\rangle$ [Distributive law w.r.t. addition of complex nos.]

j) $\alpha \cdot (|a\rangle + |b\rangle) = \alpha \cdot |a\rangle + \alpha \cdot |b\rangle$ [Distributive law w.r.t. addition of $|\,\rangle$'s]

A set S of $|\,\rangle$'s whose elements satisfy all the properties A & B is called a Linear Vector Space. Elements of this set are called vectors.

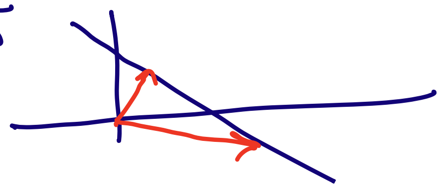
Pls. convince yourself that arrows on the plane (or in 3D) are vectors as per the above defn.

$\boxed{\text{What is the use of all this abstraction?}}$

Many more objects also behave as vectors in addition to the ones which have magnitude & direcn [the school defn of a vector]

Decide which is a LVS :-

1) The set of all arrows which end on the line $y = -3x+5$



2) Polynomials wit eal coefficients of degree 2.

$2x^2+5x \qquad -2x^2$

3) Polynomials with real coefficients up to degree 2

The vectors $|1\rangle, |2\rangle, |3\rangle \ldots |N\rangle$ are said to be linearly independent if

$$\sum_i a_i \cdot |i\rangle = |0\rangle \quad \text{implies all } a_i = 0$$

The max^m no. of linearly ind. (LI) vectors in an LVS is called the dimension of the space.

The set of vectors $|i\rangle$ that are LI & have the property that each vector $|a\rangle \in S$, can be expressed as a linear combination of the vectors $|i\rangle$ is called a basis of S.

Linear Operator :- An operator (consumes a vector & produces another vector) A is called a linear operator if

$$A \cdot \{\alpha \cdot |a\rangle + \beta \cdot |b\rangle\}$$
$$= \alpha \cdot A \cdot |a\rangle + \beta \cdot A \cdot |b\rangle$$

α, β are from the field

In general linear operators do not commute

i.e. $A \cdot B \cdot |\rangle \neq B \cdot A \cdot |\rangle$

Statement w/o proof

Every linear operator in a finite dimensional LVS can be represented by a matrix provided a basis is chosen.

How to do this:

Consider the operator

$$T \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 3a_1 + a_2 \\ a_1 + a_3 \\ a_1 - a_3 \end{bmatrix}$$

Std. basis

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

T operates on col^n matrices with 3 entries.

Is T a linear operator?

$$T \left\{ \alpha \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \beta \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right\}$$

$$= T \cdot \begin{bmatrix} \alpha a_1 + \beta b_1 \\ \alpha a_2 + \beta b_2 \\ \alpha a_3 + \beta b_3 \end{bmatrix} = \begin{bmatrix} 3\alpha a_1 + 3\beta b_1 + \alpha a_2 + \beta b_2 \\ \alpha a_1 + \beta b_1 + \alpha a_3 + \beta b_3 \\ \alpha(a_1 - a_3) + \beta(b_1 - b_3) \end{bmatrix}$$

$$= \begin{bmatrix} \alpha(3a_1 + a_2) + \beta(3b_1 + b_2) \\ \alpha(a_1 + a_3) + \beta(b_1 + b_3) \\ \alpha(a_1 - a_3) + \beta(b_1 - b_3) \end{bmatrix}$$

$$= \alpha \begin{bmatrix} 3a_1 + a_2 \\ a_1 + a_3 \\ a_1 - a_3 \end{bmatrix} + \beta \begin{bmatrix} 3b_1 + b_2 \\ b_1 + b_3 \\ b_1 - b_3 \end{bmatrix}$$

$$= \alpha \cdot T \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \beta \cdot T \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\therefore T \cdot \left\{ \alpha \cdot \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \beta \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right\}$$

$\boxed{\#1}$ Find the matrix representation of T in the standard basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

order is imp

Ordered basis. Don't switch

Algo

Apply T on basis vectors:

$$T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Matrix rep. of T in the std. basis

$$\begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

#2 Consider the LVS of polynomials with real coeff. up to degree 2.

$$f(x) = b_1 + b_2 x + b_2 x^2$$

Consider the operator $\frac{d}{dx} \equiv A$

$\boxed{\text{Is this a Linear operator ?}}$ $\longrightarrow$ Check

Choose a set of basis vectors $\{1, x, x^2\}$

Find the matrix rep. of A in this basis

Algo : $A \cdot |1\rangle \equiv \frac{d}{dx} \cdot (1) = 0 = 0 + 0 \cdot x + 0 \cdot x^2$

$A \cdot |2\rangle \equiv \frac{d}{dx} (x) = 1 = 1 + 0 \cdot x + 0 \cdot x^2$

$A \cdot |3\rangle \equiv \frac{d}{dx} (x^2) = 2x = 0 + 2 \cdot x + 0 \cdot x^2$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Let's go a bit further.

Consider the effect of this operator on the vector $1+5x+3x^2$

$$\frac{d}{dx}(1+5x+3x^2) = 5+6x$$
$$= 5+6x+0 \cdot x^2$$

Take the matrix rep. of A & operate

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 0 \end{bmatrix}$$

Consider a 2nd linear operator $B \equiv x \cdot \frac{d}{dx}$ [verify]

Find the matrix rep. of B in the same basis

$$x\frac{d}{dx}|1\rangle = x\frac{d}{dx}(1) = 0 = 0 + 0 \cdot x + 0 \cdot x^2$$

$$x\frac{d}{dx}|2\rangle = x\frac{d}{dx}(x) = x = 0 + 1 \cdot x + 0 \cdot x^2$$

$$x\frac{d}{dx}|3\rangle = x\frac{d}{dx}(x^2) = 2x^2 = 0 + 0 \cdot x + 2 \cdot x^2$$

Mat. rep. of B in this basis $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\boxed{\text{Product of linear operators is itself a LO}}$

$$\frac{d}{dx}\left(x\frac{d}{dx}\right) = x\frac{d^2}{dx^2} + \frac{d}{dx} = A \cdot B$$

Let A.B operate on the vector $17 + 15x + 18x^2$

$$\left(x\frac{d^2}{dx^2} + \frac{d}{dx}\right)\{17 + 15x + 18x^2\}$$

$$= x\frac{d^2}{dx^2}(18x^2) + \frac{d}{dx}(15x) + 36x$$

$= 36x + 15 + 36x = 15 + 72x + 0 \cdot x^2$

Now check if the matrix rep. do the same job.

$$A \cdot B \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore A \cdot B \cdot \{17 + 15x + 18x^2\}$ in matrix notation is

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 17 \\ 15 \\ 18 \end{bmatrix} = \begin{bmatrix} 15 \\ 72 \\ 0 \end{bmatrix}$$

$\boxed{\text{Does } A \text{ \& } B \text{ commute ?}}$

$$x \frac{d}{dx} \cdot \left(\frac{d}{dx} \right) = x \frac{d^2}{dx^2}$$

$$\frac{d}{dx} \cdot \left(x \frac{d}{dx} \right) = \frac{d}{dx} + x \frac{d^2}{dx^2}$$

$\boxed{\text{Do the matrix representations commute ?}}$

Back to solving $A \cdot \vec{x} = \vec{b}$ iteratively

Eigenvectors & Eigenvalues

$$A \cdot \vec{v} = \lambda \cdot \vec{v}$$

For $\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ find $\vec{v}$ & λ

$\lambda = 3$ & $1 \longrightarrow \vec{e}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{e}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ← Determined up to an arbitrary multiplicative constant.

Earlier we saw that the norm of the iteration matrix R, decides convergence. However this was not a necessary condition

$$\| \vec{x}^{(k+1)} - \vec{x}^{(e)} \| \leq \| R \|^m \cdot \underbrace{\| \vec{x}^{(0)} - \vec{x}^{(e)} \|}_{\text{initial error}}$$

Let us take the some example for an arbitrary matrix A.

$$\vec{e}^{(0)} = \vec{x}^{(e)} - \vec{x}^{(0)}$$

Recall $\vec{x}^{(k+1)} = R \cdot \vec{x}^{(k)} + \vec{c}$

$[\vec{x}^{(e)} = R \cdot \vec{x}^{(e)} + \vec{c}]$

$\therefore \vec{e}^{(k+1)} = R \cdot \vec{e}^{(k)}$

$$\Rightarrow \vec{e}^{(k+1)} = R^2 \cdot \vec{e}^{(k-1)}$$

$$= R^{(k+1)} \cdot \vec{e}^{(0)}$$

Let the eigenvectors of the R-matrix be $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_n$ [assume they span the n-dimensional space]

$$\vec{e}^{(0)} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n$$

$$\vec{e}^{(1)} = R \cdot \vec{e}^{(0)}$$

$$= R \cdot [c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n]$$

$$= [c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2 + \dots + c_n \lambda_n \vec{v}_n]$$

$$\vec{e}^{(2)} = R \cdot \vec{e}^{(1)}$$

$$= c_1 \lambda_1^2 \vec{v}_1 + c_2 \lambda_2^2 \vec{v}_2 + \dots c_n \lambda_n^2 \vec{v}_n$$

$$\vec{e}^{(k+1)} = c_1 \lambda_1^{k+1} \vec{v}_1 + c_2 \lambda_2^{k+1} \vec{v}_2 + \dots c_n \lambda_n^{k+1} \vec{v}_n$$

It is thus clear that

$\|\vec{e}^{(k+1)}\| \to 0$ if $|\lambda_{max}| < 1$

Def^n: Spectral radius of a matrix B

$$\rho(B) = \max_i |\lambda_i|$$

Thus in terms of the spectral radius, we have the necessary & sufficient condition for convergence.

$$\rho(R) = \rho(A_1^{-1} \cdot A_2) < 1$$

Note that $\|R\| < 1$ is only a sufficient condition for convergence but not a necessary

W/o Proof: The iteration $\vec{x}^{(k+1)} = R \cdot \vec{x}^{(k)} + \vec{c}$ converges for any initial guess vector $\vec{x}^{(0)}$ if and only if $\rho(R) < 1$

↳ necessary & sufficient condition

$\boxed{\text{Not for exam}}$

↳ If the spectral radius $\rho(T) < 1$, then

$$(I-T)^{-1} \stackrel{?}{=} \frac{I}{I-T} = I + T + T^2 + T^3 + \dots$$

↳ Reminds you of what ?

Digression : $\|A\|_2 = \left[\lambda_{max}(A^TA)\right]^{1/2}$ → max. magnitude eigenvalue

$$= \left[\rho(A^TA)\right]^{1/2}$$

Note that $\rho(A) \neq \|A\|_2$

Prove that $\rho(B) \leq \|B\|$ for any norm

↳ Homework

Example :

$$2x + y + z = 4$$
$$x + 2y + z = 4$$
$$x + y + 2z = 4$$

For the GS method,

$$R = -(L+D)^{-1} \cdot U$$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R = -\begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= -\frac{1}{8}\begin{bmatrix} 4 & 0 & 0 \\ -2 & 4 & 0 \\ -1 & -2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1/2 & -1/2 \\ 0 & 1/4 & -1/4 \\ 0 & 1/8 & +3/8 \end{bmatrix}$$

$$\|R\|_1 = \frac{4}{8} + \frac{2}{8} + \frac{3}{8} = \frac{9}{8} > 1$$

Will GS converge?

Try GS with the initial guess

$$\vec{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{x}^{(1)} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}, \quad \vec{x}^{(2)} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$

$$\vec{x}^{(e)} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Does it converge?

Now try computing $\rho_{GS}(R)$. For the complex eigenvalues $\left[\lambda = a + ib, \ |\lambda| = \sqrt{a^2 + b^2}\right]$. You may encounter complex eigenvalues for the R-matrix for this example.

We saw earlier that the decomposition of the error vector into eigenvectors of the R-matrix

$$\vec{e}^{(k+1)} = c_1 \lambda_1^{k+1} \vec{v}_1 + c_2 \lambda_2^{k+1} \vec{v}_2 + \dots c_n \lambda_n^{k+1} \vec{v}_n$$

then the iteration converges.

If $\rho(R) = \max_i |\lambda_i(R)| < 1$,

$\boxed{\text{What determines the rate of convergence when } \rho(R) < 1 \text{ ?}}$

Compare two R matrices below

$$\begin{bmatrix} 0.6 & 0.5 \\ 0.6 & 0.5 \end{bmatrix} \quad \& \quad \begin{bmatrix} 0.6 & 1.1 \\ 0 & 0.5 \end{bmatrix}$$

$\lambda_{max} = 1.1$ $\qquad$ $\lambda_{max} = 0.6$

Which will converge?

A 3rd algo for solving $A\vec{x} = \vec{b}$ iteratively

$$A\vec{x} = \vec{b}$$

Recall Gauss-Siedel is the decomposition $(A = L + D + U)$

$$A \cdot \vec{x} = \vec{b}$$

$$\Rightarrow (L+D)\vec{x} = -U \cdot \vec{x} + \vec{b}$$

$$\Rightarrow \vec{x} = -(L+D)^{-1} \cdot U \cdot \vec{x} + \vec{b} \longrightarrow \text{Gauss-Siedel}$$

In relaxation methods, we do the following

$$x_i^{(k)} = (1-\omega)\, x_i^{(k-1)} + \omega\, x_{i,\,G.S}^{(k)}$$

$\boxed{\omega : \text{free parameter}}$

$$\Rightarrow x_i^{(k)} = (1-\omega)\, x_i^{(k-1)} + \omega \left[\frac{b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{N} a_{ij} x_j^{(k-1)}}{a_{ii}} \right]$$

$\boxed{\text{Set } \omega = 1 \text{ to recover G.S.}}$

$$\Rightarrow a_{ii}\, x_i^{(k)} = (1-\omega)\, a_{ii}\, x_i^{(k-1)} + \omega \left[\quad \right]$$

In matrix notation

$$D\vec{x}^{(k)} = (1-\omega)\, D\vec{x}^{(k-1)} + \omega \left[\vec{b} - L\vec{x}^{(k)} - U\vec{x}^{(k-1)} \right]$$

$$\Rightarrow (D + \omega L)\cdot\vec{x}^{(k)} = \left[(1-\omega)D - \omega U\right]\cdot\vec{x}^{(k-1)} + \omega\vec{b}$$

$$\Rightarrow \vec{x}^{(k)} = (D + \omega L)^{-1}\cdot\left[(1-\omega)D - \omega U\right]\cdot\vec{x}^{(k-1)} + \omega (D+\omega L)^{-1}\cdot\vec{b}$$

The matrix decomp. of A for relaxation methods is

$$A_1 = \frac{D}{\omega} + L \quad \& \quad A_2 = \left(\frac{1-\omega}{\omega}\right) D - U$$

$$\boxed{A_1 - A_2 = L + D + U}$$

$\therefore\ A\vec{x} = \vec{b}$

$$\Rightarrow (A_1 - A_2)\cdot\vec{x} = \vec{b}$$

$$\Rightarrow \left[\left(\frac{D}{\omega} + L\right) - \left\{\left(\frac{1-\omega}{\omega}\right) D - U\right\}\right]\vec{x} = \vec{b}$$

$$\Rightarrow \left[(D + \omega L) - \left\{(1-\omega)D - \omega U\right\}\right]\vec{x} = \omega\vec{b}$$

$$\Rightarrow (D + \omega L)\vec{x}^{(k)} = \left[(1-\omega)D - \omega U\right]\vec{x}^{(k-1)} + \omega\vec{b}$$

$$\Rightarrow \vec{x}^{(k)} = (D + \omega L)^{-1}\left[(1-\omega)D - \omega U\right]\vec{x}^{(k-1)} + \omega (D+\omega L)^{-1}\vec{b}$$

✓

The idea is to choose "ω" optimally such that the iteration converges fastest. Note that ω can be $\gtrless 1$ implying over-relaxation OR under-relaxation.

e.g.
$$9x_1 + x_2 + x_3 = 10$$
$$2x_1 + 10x_2 + 3x_3 = 19$$
$$3x_1 + 4x_2 + 11x_3 = 0$$

$$\vec{X}^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \omega = 1.2 \text{ (choose)}$$

↳ over-relaxation

$$x^{(1)}_{1,G.S.} = \frac{10 - x_2^{(0)} - x_3^{(0)}}{9} = 10/9$$

$$x_1^{(1)} = (1 - 1.2)\underbrace{x_1^{(0)}}_{0} + 1.2\, x_{1,G.S.}$$

$$= 1.2 \times \frac{10}{9} = 1.332$$

$$x^{(1)}_{2,G.S.} = \frac{19 - 2x_1^{(1)} - 3x_3^{(0)}}{10}$$

$$= \frac{19 - 2 \times 1.332 - 3(0)}{10} = 1.6336$$

$$x_2^{(1)} = (1 - 1.2)\, x_2^{(0)} + 1.2\, x_{2,G.S.}$$

$$= 1.2 \times 1.6336 = 1.96$$

$$x^{(1)}_{3,G.S.} = -\frac{3x_1^{(1)} - 4x_2^{(2)}}{11} = \frac{-3\times 1.332 - 4\times 1.96}{11}$$

$$=$$

$$x_3^{(1)} = (1-1.2)\,x_3^{(0)} + 1.2\,x_{3,G.S}$$

$$= 1.2\times \ldots$$

$$\vec{x}^{(1)} = \begin{bmatrix} 1.332 \\ 1.96 \\ -1.1836 \end{bmatrix} \qquad \text{Exact soln} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

Another example :-

$$\begin{aligned} 3x_1 - x_2 + x_3 &= -1 \\ -x_1 + 3x_2 - x_3 &= 7 \\ x_1 - x_2 + 3x_3 &= -7 \end{aligned}$$

$$\omega = 1.25$$

$$\vec{x}^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{x}^{(1)} = \begin{bmatrix} -0.41667 \\ 2.7431 \\ -1.6001 \end{bmatrix}, \quad \vec{x}^{(2)} = \begin{bmatrix} 1.4972 \\ 2.1880 \\ -2.2288 \end{bmatrix}$$

$$\vec{x}^{(3)} = \begin{bmatrix} 0.9428 \\ 2.0007 \\ -1.9723 \end{bmatrix} \longrightarrow \vec{x}^{(e)} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

Positive definite matrix :- A matrix is (+ve) definite if $\vec{x}^T . A . \vec{x} > 0$ for all $\vec{x} \neq \vec{0}$ symmetric matrix.

A above need not be a symmetric matrix.

But even when A is not symmetric, only the symmetric part of A contributes ~~to~~ the above quadratic form. So we will refer to A as being symmetric $\left[A = \underbrace{\frac{1}{2}(A+A^T)}_{\text{symm. part of } A} + \frac{1}{2}(A-A^T) \right]$

Prove that all the eigenvalues of a symmetric (+ve) definite matrix > 0

Positive definiteness is an useful attribute because for $A.\vec{x} = \vec{b}$, if A is positive definite and $0 < \omega < 2$, then the relaxation method converges $\rightarrow$ $\boxed{\text{Ostrowski – Reich theorem}}$

We had earlier checked SOR on the system

$$9x_1 + x_2 + x_3 = 10$$
$$2x_1 + 10x_2 + 3x_3 = 19$$
$$3x_1 + 4x_2 + 11x_3 = 0$$

$$A = \begin{bmatrix} 9 & 1 & 1 \\ 2 & 10 & 3 \\ 3 & 4 & 11 \end{bmatrix}$$

$$\frac{1}{2}(A + A^T) = \frac{1}{2}\begin{bmatrix} 9 & 1 & 1 \\ 2 & 10 & 3 \\ 3 & 4 & 11 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 9 & 2 & 3 \\ 1 & 10 & 4 \\ 1 & 3 & 11 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 18 & 3 & 4 \\ 3 & 20 & 7 \\ 4 & 7 & 22 \end{bmatrix} = \underbrace{\begin{bmatrix} 9 & 3/2 & 2 \\ 3/2 & 10 & 7/2 \\ 2 & 7/2 & 11 \end{bmatrix}}_{}$$

↳ symmetric part of A

Check for +ve definiteness [check eigenvalues]

$\lambda_1 = 15.062$

$\lambda_2 = 8$ & $\lambda_3 = 6.93798$ all > 0

Not for exam

The Ostrowski-Reich theorem follows from a more general theorem Householder-John theorem which says for $A\cdot\vec{x} = \vec{b}$ with the splitting $A = A_1 - A_2$ and $\rho(A_1^{-1}\cdot A_2) < 1$ iff A is (+ve) definite, $A_1^* + A_2$ is +ve definite

$$A = \begin{bmatrix} 4 & 5 & 9 \\ 7 & 1 & 6 \\ 5 & 2 & 9 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Find the spectral radius of the iteration matrix for Gauss-Siedel

$\rho_{GS}(A_1^{-1}\cdot A_2) = 5.9$ [GS will not converge]

Multiply both sides of $A\cdot\vec{x} = \vec{b}$ by A^T

$$A^T\cdot A\cdot\vec{x} = A^T\cdot\vec{b}$$

and check if G.S. converges ?

Start with $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ in both cases

What is $\vec{x}^{(e)} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$

$\rightarrow$ Calculate $\frac{1}{2}(A + A^T)$ & its eigenvalues

$\rightarrow$ Calculate $Q = A^T . T$ & its eigenvalues

Evaluate +ve definiteness ?