Week #1 Assignment

Deadline: 17th June, 11:59 PM

1. **Gram-Schmidt Procedure**
   Let
   $$|v_1\rangle = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix}, \quad |v_2\rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad |v_3\rangle = \begin{pmatrix} 0 \\ 1 \\ i \end{pmatrix}.$$
   (a) Use the Gram–Schmidt procedure to construct an orthonormal basis $\{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$ from $\{|v_1\rangle, |v_2\rangle, |v_3\rangle\}$.
   (b) Verify explicitly that $\langle e_i|e_j\rangle = \delta_{ij}$.

2. **Anti-Hermitian Operators**
   Prove that an anti-hermitian operator, defined as $A^\dagger = -A$ has imaginary non-degenerate(distinct) eigenvalues.

3. **Expectation Values for a General Qubit**
   Consider the pure qubit state
   $$|\psi\rangle = \cos\frac{\theta}{2}\,|0\rangle + e^{i\phi}\sin\frac{\theta}{2}\,|1\rangle.$$
   (a) Compute $\langle\sigma_x\rangle$, $\langle\sigma_y\rangle$, and $\langle\sigma_z\rangle$ in terms of θ and ϕ.
   (b) For $\theta = \pi/3$ and $\phi = \pi/4$, evaluate these expectation values.
   (c) Show that the Bloch vector $\big(\langle\sigma_x\rangle, \langle\sigma_y\rangle, \langle\sigma_z\rangle\big)$ has unit length.

4. **Generalized Uncertainty Principle**
   Let $\Delta A := A - \langle A\rangle$ and $|\alpha\rangle = \Delta A\,|\psi\rangle, |\beta\rangle = \Delta B\,|\psi\rangle$ for any general state $|\psi\rangle$. Using the Cauchy-Schwarz Inequality on $|\alpha\rangle$ and $|\beta\rangle$, prove the following:
   $$\langle\Delta A^2\rangle\langle\Delta B^2\rangle \geq \frac{1}{4}\,|\langle[A,B]\rangle|^2$$
   where $[.,.]$ is the commutator. This is the generalized uncertainty relation.
   (**Hint:** You'll be needing the result from Problem 2).

5. **Partial Trace and Mixedness**
   Let
   $$|\Psi\rangle_{AB} = \frac{1}{\sqrt{3}}\big(|00\rangle + |11\rangle + |22\rangle\big)$$
   be a pure state of two qutrits.

(a) Compute $\rho_{AB} = |\Psi\rangle \langle\Psi|$.

(b) Perform the partial trace over system B, $\rho_A = \mathrm{Tr}_B\, \rho_{AB}$.

(c) Determine whether ρ_A is pure or mixed by checking $\mathrm{Tr}\, \rho_A^2$.

6. **Bloch-Sphere Visualization**

   (a) Show that the set of all valid single-qubit density matrices $\rho = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma})$ corresponds to $|\vec{r}| \leq 1$.

   (b) For the family

   $$\rho(p) = p\ |0\rangle \langle 0| + (1-p)\,\frac{I}{2}, \quad 0 \leq p \leq 1,$$

   find the length $r(p)$ of its Bloch vector and sketch r versus p.

   (c) Interpret the special cases $p = 0$ and $p = 1$.