An Introduction to Game Theory

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§1 Motivation

Before we answer why, we need to know what game theory actually encompasses. Game theory is the name given to the methodology of using mathematical tools to model and analyze situations of interactive decision making. These are situations involving several decision makers (called players) with different goals, in which the decision of each affects the outcome for all the decision makers. Game theory was developed to study strategic interactions where the outcome for each participant depends not only on their own decisions but also on the decisions of others. The motivation behind it is to understand how rational individuals or groups behave in situations of conflict, cooperation, or competition. By modelling these interactions mathematically, game theory helps predict possible outcomes and identify optimal strategies. Its applications span economics, political science, evolutionary biology, and computer science, offering insights into everything from market behaviour to international diplomacy. Let us start by analysing one of the most well-known games of all time, the game of chess.

§2 The game of chess

Before we begin our analysis, remember that although the theory introduced in this analysis is specifically related to chess, the same theory will be developed to apply later in this course. I also assume that most of you are familiar with the basic rules of chess, if you are not aware of the basic rules, you can refer to: https://www.chesshouse.com/pages/chess-rules. Let us now begin our analysis.

For the purposes of our analysis all we need to assume is that the game is finite, i.e., the number of possible turns is bounded (even if that bound is an astronomically large number). This does not apply, strictly speaking, to the game of chess, but since our lifetimes are finite, we can safely assume that every chess match is finite.
We will denote the set of all possible board positions in chess by X. A board position by definition includes the identity of each piece on the board, and the board square on which it is located.
A board position, however, does not provide full details on the sequence of moves that led to it: there may well be two or more sequences of moves leading to the same board position. We therefore need to distinguish between a "board position" and a "game situation," which is defined as follows.

Definition 2.1. A game situation (in the game of chess) is a finite sequence $(x_0, x_1, x_2, ..., x_K)$ of board positions in X satisfying:

1. x_0 is the opening board position
2. For each even integer k, $0 \leq k \leq K$, going from board position x_k to x_{k+1} can be accomplished by a single legal move on the part of White.
3. For each odd integer k, $0 \leq k \leq K$, going from board position x_k to x_{k+1} can be accomplished by a single legal move on the part of Black.

Definition 2.2. A strategy for White is a function s_W that associates every game situation $(x_0, x_1, x_2, ..., x_K) \in H$, where K is even, with a board position x_{K+1}, such that going from board position x_K to x_{K+1} can be accomplished by a single legal move on the part of White.
Analogously, a strategy for Black is a function s_B that associates every game situation $(x_0, x_1, x_2, ..., x_K) \in H$, where K is odd, with a board position x_{K+1} such that going from board position x_K to x_{K+1} can be accomplished by a single legal move on the part of Black.
An entire course of moves (from the opening move to the closing one) is termed a play of the game.

Definition 2.3. A strategy s_W is a winning strategy for White if for every strategy s_B of Black, the play of the game determined by the pair (s_W, s_B) ends in victory for White. A strategy s_W is a strategy guaranteeing at least a draw for White if for every strategy s_B of Black, the play of the game determined by the pair (s_W, s_B) ends in either a victory for White or a draw.

Now, although these definitions may seem obvious, it is essential that we set up common notation used throughout the course.
While it may seem obvious that in a game of chess, each game must end in either a win for White, a win for Black or a draw, this is not a trivial statement to prove. It is a direct result of one of the earliest theorems in game theory.We state this theorem in the following manner:

Theorem 2.4

In chess, one and only one of the following must be true:

1. White has a winning strategy.
2. Black has a winning strategy.
3. Each of the two players has a strategy guaranteeing at least a draw.

We start with several definitions that are needed for the proof of the theorem. The set of game situations can be depicted by a tree.Such a tree is called a game tree. . Each vertex of the game tree represents a possible game situation. Denote the set of vertices of the game tree by H.
The root vertex is the opening game situation x_0, and for each vertex x, the set of children vertices of x is the set of game situations that can be reached from x in one legal move. For example, in his opening move, White can move one of his pawns one or two squares forward, or one of his two knights. So White has 20 possible opening moves, which means that the root vertex of the tree has 20 children vertices. Every vertex that can be reached from x by a sequence of moves is called a descendant of x. Every leaf of

the tree corresponds to a terminal game situation, in which either White has won, Black has won, or a draw has been declared.

Given a vertex $x \in H$, we may consider the subtree beginning at x, which is by definition the tree whose root is x that is obtained by removing all vertices that are not descendants of x. This subtree of the game tree, which we will denote by $\Gamma(x)$, corresponds to a game that is called the subgame beginning at x. We will denote by n_x the number of vertices in $\Gamma(x)$. The game $\Gamma(x_0)$ is by definition the game that starts with the opening situation of the game, and is therefore the standard chess game. If y is a child vertex of x, then $\Gamma(y)$ is a subtree of $\Gamma(x)$ that does not contain x. In particular, $n_x > n_y$. Moreover, $n_x = 1$ if and only if x is a terminal situation of the game, i.e., the players cannot implement any moves at this subgame. In such a case, the strategy of a player is denoted by Φ.

Now that we have setup the notation, we proceed to the proof, which if you guess it already, inducts on the tree, a very popular proof technique to be used with trees.

Proof. The proof proceeds by induction on n_x, the number of vertices in the subgame $\Gamma(x)$. Suppose x is a vertex such that $n_x = 1$. As noted above, that means that x is a terminal vertex. If the White King has been removed from the board, Black has won, in which case Φ is a winning strategy for Black. If the Black King has been removed from the board, White has won, in which case Φ is a winning strategy for White. Alternatively, if both Kings are on the board at the end of play, the game has ended in a draw, in which case Φ is a strategy guaranteeing a draw for both Black and White.

Next, suppose that x is a vertex such that $n_x > 1$. Assume by induction that at all vertices y satisfying $n_y < n_x$, one and only one of the alternatives (1), (2), or (3) is true in the subgame $\Gamma(y)$.

Suppose, without loss of generality, that White has the first move in $\Gamma(x)$. Any board position y that can be reached from x satisfies $n_y < n_x$, and so the inductive assumption is true in the corresponding subgame $\Gamma(y)$. Denote by $C(x)$ the collection of vertices that can be reached from x in one of White's moves.

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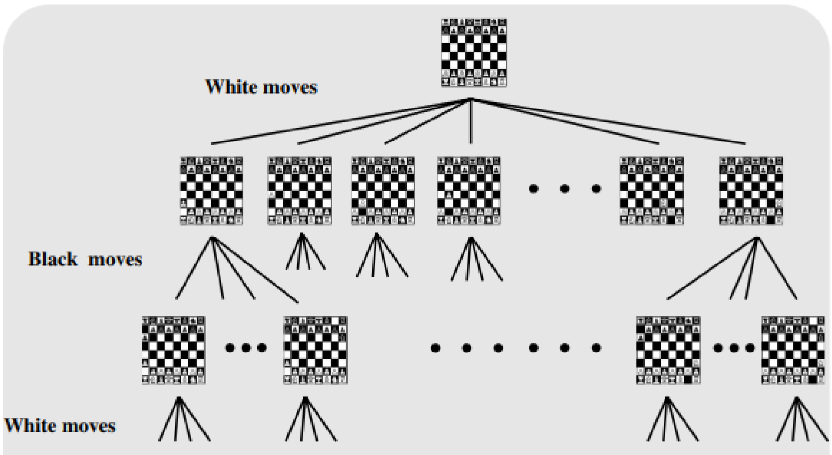

1. If there is a vertex $y_0 \in C(x)$ such that White has a winning strategy in $\Gamma(y_0)$, then alternative (i) is true in $\Gamma(x)$: the winning strategy for White in $\Gamma(x)$ is to choose as his first move the move leading to vertex y_0, and to follow the winning strategy in $\Gamma(y_0)$ at all subsequent moves.

2. If Black has a winning strategy in $\Gamma(y)$ for every vertex $y \in C(x)$, then alternative (ii) is true in $\Gamma(x)$: Black can win by ascertaining what the vertex y is after White's

first move, and following his winning strategy in $\Gamma(y)$ at all subsequent moves.

3. Otherwise:
   (1) does not hold, i.e., White has no winning strategy in $\Gamma(y)$ for any $y \in C(x)$. Because the induction hypothesis holds for every vertex $y \in C(x)$, either Black has a winning strategy in $\Gamma(y)$, or both players have a strategy guaranteeing at least a draw in $\Gamma(y)$.

   (2) does not hold, i.e., there is a vertex $y_0 \in C(x)$ such that Black does not have a winning strategy in $\Gamma(y_0)$. But because (1) does not hold, White also does not have a winning strategy in $\Gamma(y_0)$. Therefore, by the induction hypothesis applied to $\Gamma(y_0)$, both players have a strategy guaranteeing at least a draw in $\Gamma(y_0)$. As we now show, in this case, in $\Gamma(x)$ both players have a strategy guaranteeing at least a draw. White can guarantee at least a draw by choosing a move leading to vertex y_0, and from there by following the strategy that guarantees at least a draw in $\Gamma(y_0)$. Black can guarantee at least a draw by ascertaining what the board position y is after White's first move, and at all subsequent moves in $\Gamma(y)$ either by following a winning strategy or following a strategy that guarantees at least a draw in that subgame.This completes the proof of the theorem.

Now while this result was proved in the context of the game of chess, we need to recognise what specific properties of the game we used to arrive at this result, so as to conclude similar results for games with those properties in general. We realise we used 3 such characteristics of the game :

1. The game is finite.
2. The strategies of the players determine the play of the game. In other words, there is no element of chance in the game; neither dice nor card draws are involved.
3. Each player, at each turn, knows the moves that were made at all previous stages of the game.

Therefore although this result might seem true for all games in general, if any game fails to satisfy any of these properties, it does not follow the theorem.
The game of chess is part of a huge class of games, known as extensive form games. In game theory, an extensive-form game represents a game as a tree, detailing the sequential moves of players, their available actions, and the outcomes of those actions. This representation captures the full sequential play of the game, including what each player knows at each stage and the payoffs for each outcome. On the other hand, we have another class games, which we now focus on, called normal form or strategic form games , which ignores such dynamic aspects of the game.

§3 The Prisoner's Dilemma

The Prisoner's dilemma is a very simple game to understand, but the optimal strategy is not what most people assume it to be. While the formal proof to why the best strategy is optimal will not be included in this week, we simply describe the game now. It appears in literature in the form of the following story :

1. If you confess and your friend refuses to confess, you will be released from custody and receive immunity as a state's witness.

2. If you refuse to confess and your friend confesses, you will receive the maximum penalty for your crime (ten years of incarceration).

3. If both of you sit tight and refuse to confess, we will make use of evidence that you have committed tax evasion to ensure that both of you are sentenced to a year in prison.

4. If both of you confess, it will count in your favour and we will reduce each of your prison terms to six years.

This situation defines a two-player strategic-form game in which each player has two strategies: D, which stands for Defection, betraying your fellow criminal by confessing, and C, which stands for Cooperation, cooperating with your fellow criminal and not confessing to the crime.
This is a table demonstrating the different possible outcomes for the game:

	D	C
D	$6, 6$	$0, 10$
C	$10, 0$	$1, 1$

Now what makes the prisoner's dilemma interesting is that C, C seems to be the obvious optimal strategy since it results in the minimum total years in prison for the 2 players, however the answer and the explanation are not that trivial. In fact the answer, which will not be revealed now, is counter intuitive, until explained, which requires concepts of domination and stability, which will be covered soon.

§4 The idea of rationality

Throughout this course, when it comes to strategic form or normal games, we inherently assume certain things relating to the player's rationality, and for the sake o completeness, we now state these assumptions formally.We must understand what rationality means.In its mildest form, rationality implies that every player is motivated by maximizing his own payoff. In a stricter sense, it implies that every player always maximizes his utility, thus being able to perfectly calculate the probabilistic result of every action. We now state some assumptions that we hold to be true :

1. All the players in the game are rational.

2. The fact that all players are rational is common knowledge among the players.

The reason why the second assumption is important is something we will not explore now, however to better understand what it means we define what common knowledge means in the context of a game :

Definition 4.1. A fact is common knowledge among the players of a game if for any finite chain of players $i_1, i_2, ..., i_k$ the following holds: player i_1 knows that player i_2 knows that player i_3 knows . . . that player i_k knows the fact.

Although assumption 2 implies assumption 1, we have stated both to show that the first assumption is not enough for players to come with optimal strategies. These assumptions will be crucial as we seek to eliminate certain strategies on the basis of the rationality of the players.

§5 Further reading

1. To get a better understanding of utility functions, which will be useful for understanding concepts next week, read chapter 2 of Game Theory, by Maschler,Solan and Zamir.

2. If you're interested in knowing more about extensive form games, and how they are represented, refer to chapter 3 of the book above.