1.3.7 Example: quantum teleportation

We will now apply the techniques of the last few pages to understand something non-trivial, surprising, and a lot of fun – quantum teleportation! Quantum teleportation is a technique for moving quantum states around, even in the absence of a quantum communications channel linking the sender of the quantum state to the recipient.

Here's how quantum teleportation works. Alice and Bob met long ago but now live far apart. While together they generated an EPR pair, each taking one qubit of the EPR pair when they separated. Many years later, Bob is in hiding, and Alice's mission, should she choose to accept it, is to deliver a qubit $|\psi\rangle$ to Bob. She does not know the state of the qubit, and moreover can only send *classical* information to Bob. Should Alice accept the mission?

Intuitively, things look pretty bad for Alice. She doesn't know the state $|\psi\rangle$ of the qubit she has to send to Bob, and the laws of quantum mechanics prevent her from determining the state when she only has a single copy of $|\psi\rangle$ in her possession. What's worse, even if she did know the state $|\psi\rangle$, describing it precisely takes an infinite amount of classical information since $|\psi\rangle$ takes values in a *continuous* space. So even if she did know $|\psi\rangle$, it would take forever for Alice to describe the state to Bob. It's not looking good for Alice. Fortunately for Alice, quantum teleportation is a way of utilizing the entangled EPR pair in order to send $|\psi\rangle$ to Bob, with only a small overhead of classical communication.

In outline, the steps of the solution are as follows: Alice interacts the qubit $|\psi\rangle$ with her half of the EPR pair, and then measures the two qubits in her possession, obtaining one of four possible classical results, 00, 01, 10, and 11. She sends this information to Bob. Depending on Alice's classical message, Bob performs one of four operations on his half of the EPR pair. Amazingly, by doing this he can recover the original state $|\psi\rangle$!

The quantum circuit shown in Figure 1.13 gives a more precise description of quantum teleportation. The state to be teleported is $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where α and β are unknown amplitudes. The state input into the circuit $|\psi_0\rangle$ is

$$|\psi_0\rangle = |\psi\rangle|\beta_{00}\rangle \tag{1.28}$$

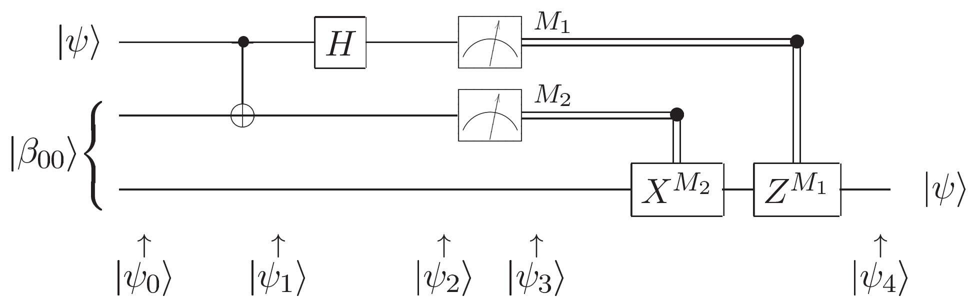

Figure 1.13. Quantum circuit for teleporting a qubit. The two top lines represent Alice's system, while the bottom line is Bob's system. The meters represent measurement, and the double lines coming out of them carry classical bits (recall that single lines denote qubits).

$$= \frac{1}{\sqrt{2}} \Big[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|00\rangle + |11\rangle)\Big] , \tag{1.29}$$

where we use the convention that the first two qubits (on the left) belong to Alice, and the third qubit to Bob. As we explained previously, Alice's second qubit and Bob's qubit start out in an EPR state. Alice sends her qubits through a CNOT gate, obtaining

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} \Big[\alpha|0\rangle(|00\rangle + |11\rangle) + \beta|1\rangle(|10\rangle + |01\rangle)\Big] . \tag{1.30}$$

She then sends the first qubit through a Hadamard gate, obtaining

$$|\psi_2\rangle = \frac{1}{2} \Big[\alpha(|0\rangle + |1\rangle)(|00\rangle + |11\rangle) + \beta(|0\rangle - |1\rangle)(|10\rangle + |01\rangle)\Big] . \tag{1.31}$$

This state may be re-written in the following way, simply by regrouping terms:

$$\begin{aligned} |\psi_2\rangle = \frac{1}{2} \Big[&|00\rangle \big(\alpha|0\rangle + \beta|1\rangle\big) + |01\rangle \big(\alpha|1\rangle + \beta|0\rangle\big) \\ &+ |10\rangle \big(\alpha|0\rangle - \beta|1\rangle\big) + |11\rangle \big(\alpha|1\rangle - \beta|0\rangle\big)\Big] . \end{aligned} \tag{1.32}$$

This expression naturally breaks down into four terms. The first term has Alice's qubits in the state $|00\rangle$, and Bob's qubit in the state $\alpha|0\rangle + \beta|1\rangle$ – which is the original state $|\psi\rangle$. If Alice performs a measurement and obtains the result 00 then Bob's system will be in the state $|\psi\rangle$. Similarly, from the previous expression we can read off Bob's post-measurement state, given the result of Alice's measurement:

$$00 \longmapsto |\psi_3(00)\rangle \equiv \Big[\alpha|0\rangle + \beta|1\rangle\Big] \tag{1.33}$$

$$01 \longmapsto |\psi_3(01)\rangle \equiv \Big[\alpha|1\rangle + \beta|0\rangle\Big] \tag{1.34}$$

$$10 \longmapsto |\psi_3(10)\rangle \equiv \Big[\alpha|0\rangle - \beta|1\rangle\Big] \tag{1.35}$$

$$11 \longmapsto |\psi_3(11)\rangle \equiv \Big[\alpha|1\rangle - \beta|0\rangle\Big] . \tag{1.36}$$

Depending on Alice's measurement outcome, Bob's qubit will end up in one of these four possible states. Of course, to know which state it is in, Bob must be told the result of Alice's measurement – we will show later that it is this fact which prevents teleportation from being used to transmit information faster than light. Once Bob has learned the measurement outcome, Bob can 'fix up' his state, recovering $|\psi\rangle$, by applying the appropriate quantum gate. For example, in the case where the measurement yields 00, Bob doesn't need to do anything. If the measurement is 01 then Bob can fix up his state by applying the X gate. If the measurement is 10 then Bob can fix up his state by applying the Z gate. If the measurement is 11 then Bob can fix up his state by applying first an X and then a Z gate. Summing up, Bob needs to apply the transformation $Z^{M_1}X^{M_2}$ (note how time goes from left to right in circuit diagrams, but in matrix products terms on the *right* happen *first*) to his qubit, and he will recover the state $|\psi\rangle$.

There are many interesting features of teleportation, some of which we shall return to later in the book. For now we content ourselves with commenting on a couple of aspects. First, doesn't teleportation allow one to transmit quantum states faster than light? This would be rather peculiar, because the theory of relativity implies that faster than light information transfer could be used to send information backwards in time. Fortunately, quantum teleportation does not enable faster than light communication, because to complete the teleportation Alice must transmit her measurement result to Bob over a classical communications channel. We will show in Section 2.4.3 that without this classical communication, teleportation does not convey *any* information at all. The classical channel is limited by the speed of light, so it follows that quantum teleportation cannot be accomplished faster than the speed of light, resolving the apparent paradox.

A second puzzle about teleportation is that it appears to create a copy of the quantum state being teleported, in apparent violation of the no-cloning theorem discussed in Section 1.3.5. This violation is only illusory since after the teleportation process only the target qubit is left in the state $|\psi\rangle$, and the original data qubit ends up in one of the computational basis states $|0\rangle$ or $|1\rangle$, depending upon the measurement result on the first qubit.

What can we learn from quantum teleportation? Quite a lot! It's much more than just a neat trick one can do with quantum states. Quantum teleportation emphasizes the interchangeability of *different* resources in quantum mechanics, showing that one shared EPR pair together with two classical bits of communication is a resource at least the equal of one qubit of communication. Quantum computation and quantum information has revealed a plethora of methods for interchanging resources, many built upon quantum teleportation. In particular, in Chapter 10 we explain how teleportation can be used to build quantum gates which are resistant to the effects of noise, and in Chapter 12 we show that teleportation is intimately connected with the properties of quantum error-correcting codes. Despite these connections with other subjects, it is fair to say that we are only beginning to understand *why* it is that quantum teleportation is possible in quantum mechanics; in later chapters we endeavor to explain some of the insights that make such an understanding possible.