Dominance and Stability

Shivansh Nair, Maths and Physics Club

June 2025

§1 Introduction

Last week we looked at the prisoners dilemma, an example of a strategic form game. This week we will look at how you define a general strategic form game, and how one can compare two strategies.
First let us define some key concepts of game theory: strategies and utility functions.

§2 Strategies and Utility Functions

Definition 2.1. A strategic-form game consists of:

- A finite set of players $N = \{1, 2, \ldots, n\}$.
- For each player $i \in N$, a set of strategies S_i, not necessarily finite. S is defined as $S_1 \times \cdots \times S_n$
- A **utility function** $u_i : S \to \mathbb{R}$, which gives the payoff for player i based on the strategy profile.

Each player chooses a strategy to maximize their utility, anticipating the strategies of the others.

For example in last week, prisoners dilemma was a strategic form game where **C** and **D** where 2 strategies available to both players. We want a utility function that gives higher points for lesser years served, for example it can be 10 - years served.

	D	C
D	$4, 4$	$10, 0$
C	$0, 10$	$9, 9$

This payoff matrix illustrates the utility function, where it gives the reward for each player for each element of S. For n players, we will have a n dimensional payoff matrix.

Now that we have defined what a strategic game is, let us try to formally compare strategies. An important concept to do this is **domination**.

§3 Strict Domination

Let s_{-i} represent the set of each player's strategy except the ith player. Then:

Definition 3.1. A strategy s_i is **strictly dominated** by another strategy t_i for player i if:

$$u_i(t_i, s_{-i}) > u_i(s_i, s_{-i}) \quad \text{for all } s_{-i} \in S_{-i}.$$

What this means is, no matter what strategies the other players except i choose, player i choosing strategy t_i instead of s_i will always give them a strictly higher reward.

Last week, we assumed that all players are rational and that this fact is common knowledge among the players. We now also assume that a rational player will never play a strictly dominated strategy. This is of course because, playing the strategy that dominates it will be better for them in every scenario.

Let us look at an example game where Player I has 2 strategies and player II has 3 strategies:

	A	B	C
D	$1,0$	$1,2$	$0,1$
E	$0,3$	$0,1$	$2,0$

Here the 1st and 2nd element represent the 1st and 2nd player's reward for each combination of strategies.

We see that strategy C gives a worse reward than B to player II, irrespective of wether player I chooses D or E. hence C is strictly dominated by B.

Since we have assumed all players are rational, and a rational player will not play a strictly dominated strategy, player II will not play C. But can player I be sure of this fact, while he is choosing his own strategy? This is where the fact that rationality is common knowledge comes in. Since player I knows that player II is rational, he knows that player II will not play C. Similarly Player II knows that player I knows that Player II will not play C, and similarly for any finite chain. Hence our earlier assumptions make it common knowledge that any player will not play a strictly dominated strategy, and allow us to eliminate that strategy.

Eliminating C from the table gives:

	A	B
D	$1,0$	$1,2$
E	$0,3$	$0,1$

We can see that E is also now strictly dominated by D, whereas earlier it was not strictly dominated, due to C. Hence after eliminating C, we can eliminate E as well. Note that we could not have eliminated E, if rationality was not common knowledge among the players.

	A	B
D	$1,0$	$1,2$

Now finally A is strictly dominated by B, hence we can eliminate A. So for each player we are left with only a single strategy B and D.

Definition 3.2. This process is known as *iterated elimination of strictly dominated strategies*. If we end up with only one possible strategy for all players at the end, that is called a solution to the game.

> **Remark 3.3.** A special case where this process guarantees a solution is when each player has a strategy that strictly dominates all of their other strategies. In this case the elimination leaves each player with that strictly dominant strategy.

Now that we know how to compare strategies, let us solve the prisoners dilemma.

	D	C
D	$4, 4$	$10, 0$
C	$0, 10$	$9, 9$

We see that for both players, C is strictly dominated by D. hence according to Remark 3.3, the solution of the game is for both players to choose D. However, this is very counter intuitive. C, C is an outcome that both players would prefer to D, D even though solving the game returns D, D. Hence we see, a rational set of strategies might not necessarily be the conventional best outcome.

§4 Weak Domination

It is not always possible to find strictly dominated strategies and eliminate them. Hence we slightly loosen our constraints and define weakly dominated strategies.

Definition 4.1. A strategy s_i is **weakly dominated** by another strategy t_i for player i if:

$$u_i(t_i, s_{-i}) \geq u_i(s_i, s_{-i}) \quad \text{for all } s_{-i} \in S_{-i}$$

and

$$u_i(t_i, t_{-i}) > u_i(s_i, t_{-i}) \quad \text{for some } t_{-i} \in S_{-i}.$$

What this means is, no matter what strategies the other players except i choose, player i choosing strategy t_i instead of s_i will always give them an atleast equal reward, and there will be atleast one combination of strategies that the other players can choose for which t_i will have a higher reward.

Since strict domination also implies weak domination, from now on we will use the term domination to refer to weak domination, unless the word strictly is explicitly used.
Earlier we assumed that rational players do not use strictly dominated strategies, but stating the same for dominated strategies in general is a very strong assumption. One way to justify this is with the trembling hand principle.

For *iterated elimination of strictly dominated strategies*, the order of elimination does not matter. However for *iterated elimination of weakly dominated strategies* it can infact make a difference.

§5 Stability

While domination is a very important concept, it has limitations and can't always predict the rational result. Now we will look at another important concept, *stability*.

Take this game for example, where no player has any dominated strategy. Depending on the strategy used by the other player, each player has a best strategy they could use.

	D	E	F
A	$0,6$	$6,0$	$4,3$
B	$6,0$	$0,6$	$4,3$
C	$3,3$	$3,3$	$5,5$

If player I chooses A player II will choose D
If player I chooses B player II will choose E
If player I chooses C player II will choose F
If player II chooses D player I will choose B
If player II chooses E player I will choose A
If player II chooses F player I will choose C

Now look at the pair C, F. the best response to C is F and the best response to F is C. This is a pair satisfying the stability property, as defined by John Nash. Let us define this formally:

Definition 5.1. A *strategy vector* $s^* = (s_1^*, \ldots, s_n^*)$ is a *Nash equilibrium if for each player* $i \in N$ *and each strategy* $s_i \in S_i$*, the following is satisfied:*

$$u_i(s^*) \geq u_i(s_i, s_{-i}^*).$$

The payoff vector $u(s^*)$ is the equilibrium payoff corresponding to the Nash equilibrium s^*.

The strategy $\hat{s}_i \in S_i$ *is a profitable deviation* of player i at a strategy vector $s \in S$ if

$$u_i(\hat{s}_i, s_{-i}) > u_i(s).$$

A Nash equilibrium is a strategy vector at which no player has a profitable deviation.

In the above example C, F is in Nash equilibrium, while A, D is not as B, D is a profitable deviation for player II.

Now lets look at some different situations to analyze their equilibrium:

§5.1 Prisoners dilemma

First let us once again talk about the prisoners dilemma. Choosing cooperation by either player is unstable, as changing to deflection is a profitable deviation. Hence D, D is the only Nash equilibrium. Similarly, can we say that the solution in Remark 3.3 is the only point of Nash equilibrium in such cases?

§5.2 Coordination

	a	b
A	$1,1$	$0,0$
B	$0,0$	$3,3$

This is an example of a broad class of games, known as *coordination games*. It is the best interest of both players to coordinate their strategies, and attain the best reward together. Here A, a and B, b are both equilibrium points, although B, b is better. A, b or B, a are points of miscoordination with payoff 0, 0.

§5.3 Battle of the sexes

There is a couple which want to go out together, and the man prefers watching football(F) while the woman wants to go to a concert(C), but they would never want to go alone.

	F	C
F	$2,1$	$0,0$
C	$0,0$	$1,2$

This has 2 different equilibrium points F, F and C, C. The first one favors the man while the second one favors the woman.

§5.4 Security dilemma

After the second world war, both US and Russia were considering producing nuclear weapons. The best scenario for both of them is if neither country produces weapons, as they are expensive and can cause war(payoff 4). A less desirable outcome is for only themselves to have weapons while the other country doesn't(payoff 3). An even less desirable outcome is for both countries to produce weapons(payoff 2), and the worst outcome is for only the other country to have weapons(payoff 1).

	P	NP
P	$2,2$	$3,1$
NP	$1,3$	$4,4$

This also has 2 equilibrium points of P, P and NP, NP. While NP, NP is a more desirable outcome, it is also the more risky one in case the other country switches, the one that did not switch will be worse off. However in the less desirable P, P if the other country switches they will be worse off. This is why people claim P, P is the more reasonable equilibrium point, and that is infact the point obtained historically.

§6 Security and maxmin

As we have seen already, dominance and equilibrium can't always give the fully expected outcome of a rational player. Now we look at the concept of security, which helps us think of how rational players might prepare for worst case scenarios.

Consider the table below:

	D	E
A	$2,1$	$2,-20$
B	$3,0$	$-10,1$
C	$-100,2$	$3,3$

The unique Nash equilibrium in this game is C, E with a payoff of $(3,3)$. But this is extremely risky for player I. If player II chooses D, wether by irrationality or accident, player I would get a devestating payoff of -100.

Hence player I might hesitate to pick C and instead choose A where both cases have a safe payoff of 2. Player II further might anticipate player I wanting to play A, and hence waver towards D to not get the payoff of -20. This in turn will further motivate player I to play A instead of C.

This highlights an important aspect of rational behavior: the desire to *guarantee* the best possible result without depending on the rationality of the other players, or in other words *preparing for the worst-case scenario.*

The Maxmin Value

So, what can a player guarantee for themselves in general? If Player i chooses a strategy s_i, the worst possible payoff they can receive (considering all possible strategies of the other players) is:

$$\min_{s_{-i}} u_i(s_i, s_{-i})$$

To maximize their own guaranteed outcome, Player i can choose the strategy s_i that maximizes this minimum:

$$v_i = \max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$$

The value v_i is called the *maxmin value* (or security level) of Player i. A strategy s_i^* that achieves this value is called a *maxmin strategy*, and it satisfies:

$$u_i(s_i^*, s_{-i}) \geq v_i \quad \text{for all } t_{-i} \in S_{-i}$$

For the game above, the maxmin value of player I is 2, guaranteed by strategy A. Similarly, the maxmin value of player II is 0, guaranteed by strategy D.If both players choose their maxmin strategy, the payoff of A, D will be (2, 1), with player I getting his maxmin value and player II getting 1 higher.

A player can have multiple maxmin strategies for a game, and the strategy they choose can affect final payoff function. For example:

	C	D
A	$3, 1$	$0, 4$
B	$2, 3$	$1, 1$

Player I has maxmin of 1 with strategy B, while player II has maxmin of 1 with both C and D. for B, C the payoff is (2, 3) while for B, C the payoff is (1, 1).
For comparison, let us look apply our previous methods to this game as well. We see that there is no dominated strategy or Nash equilibrium, hence the concept of security can help us predict rational outcomes even when our earlier methods do not work.
Now let's further see the connection between dominated strategies and maxmin.

Theorem 6.1

A strategy of player i that dominates all his other strategies is a maxmin strategy for that player. Such a strategy, furthermore, is a best reply of player i to any strategy vector of the other players

This can be proved easily by considering the inequality between a dominant strategy with any other strategy.

This theorem can be used on the special case of Remark 3.3 earlier to imply that in a game where each player has a dominant strategy, the vector of dominant strategies is both an equilibrium point and a vector of maxmin strategies. In case there is strict domination, we have the following theorem:

Theorem 6.2

In a game in which every player i has a strategy s_i^* that strictly dominates all of his other strategies, the strategy vector $(s_1^*, \ldots, s_n^*)$ is the unique equilibrium point of the game as well as the unique vector of maxmin strategies.

We can also prove that any Nash equilibrium will have payoff for each player at least equal to their maxmin value. Since if a player had a higher maxmin value, that means they have a strategy where the payoff will at least equal the maxmin value, and hence be greater than the Nash equilibrium, which is not possible.

§7 Further reading

If you want to understand the topics mentioned in more detail, read chapter 4.1 to 4.10 of Game Theory, by Maschler,Solan and Zamir.