Intro to Quantum Computing

MnP

Week #2 Assignment

Deadline: 24 June, 11:59 PM

1. **Time Evolution under a Spin $\frac{1}{2}$ Hamiltonian**
   A spin-$\frac{1}{2}$ particle evolves under

$$H = \omega\big(\cos\alpha\,\sigma_x + \sin\alpha\,\sigma_z\big).$$

   (a) Find the unitary time-evolution operator $U(t) = e^{-iHt}$.

   (b) If the initial state is $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, compute $|\psi(t)\rangle = U(t)\ |+\rangle$.

   (c) Compute $\langle\sigma_z\rangle_t = \langle\psi(t)|\,\sigma_z\,|\psi(t)\rangle$.

2. **Amplitude Amplification**
   Create a sub-circuit acting on a 4-dimensional Hilbert space of 2 qubits which functions as an inverter of the phase of only the $|11\rangle$ or $|3\rangle$ state. Follow this up with a sub-circuit that implements the following : $2\,|s\rangle\,\langle s| - \mathbb{I}$, where $|s\rangle = \frac{1}{\sqrt{4}}\sum_{i=0}^{3}|i\rangle$. For the input, give an equal superposition of all states ($|s\rangle$).

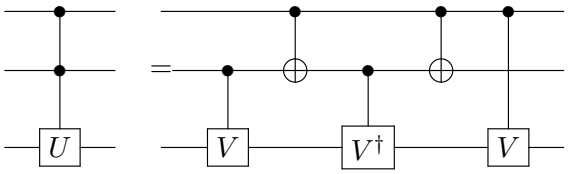
   (a) What does this entire circuit achieve?

   (b) Code this up in Qiskit, and print out the unitaries of both the sub-circuits. (Use `Operator` method from `qiskit.quantum_info`).

3. **Toffoli Gate Decomposition**
   The *Toffoli gate*, also known as the Controlled-Controlled-NOT gate or $C^2(X)$, is a three-qubit gate that flips the third qubit (target) if and only if the first two qubits (controls) are both 1.

   A general construction of a controlled-controlled-unitary gate $C^2(U)$ is

shown below:



(a) Identify a suitable unitary V such that the above circuit implements a Toffoli gate.

(b) Implement the constructed circuit using Qiskit.

4. **3-bit Counter**
Given a three-digit binary number represented by qubits a, b, c, where the full binary number is interpreted as abc (with a as the most significant bit), design a quantum circuit that increments this number by 1 modulo 8.

- Your circuit must perform the increment *in-place*, using only the original three qubits.
- The input state is given as:

$$|a\rangle \otimes |b\rangle \otimes |c\rangle\,, \quad \text{with } a, b, c \in \{0, 1\}.$$

- The increment is modulo 8, i.e., $111 \mapsto 000$, $000 \mapsto 001$, etc.

Code this circuit in Qiskit and verify that it works correctly for all 8 input combinations.