Mixed Strategy Nash Equilibria

Pulkit Gupta, Maths and Physics Club

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§1 Introduction

We learnt about pure strategies in week 2 and different concepts regarding it like domination and Nash equilibrium. But when each player has multiple strategies, the best strategy might not be a single one of them, sometimes it can be a combination of more than one strategies each with a specific probability. The elements of the new set of strategies are called mixed stratey while the orignal ones are called Pure strategies.
Thus we have

Definition 1.1. A mixed strategy is a probability distribution over pure strategies

A point to be noted is that every pure strategy is also a mixed strategy with the probability of that strategy being 1 and probability for the rest of the strategies being 0. For a strategic-form game with finitely many pure strategies for each player we define the mixed extension of the game, which is a game in strategic form in which the set of strategies of each player is his set of mixed strategies, and his payoff function is the multilinear extension of his payoff function in the original game.
The main result of this week is the Nash equilibrium, which is a major milestone in game theory. It states that a mixed extension of a game always has a Nash equilibrium; i.e a Nash equilibrium in mixed strategies always exists in strategic form games where all players have a finite number of pure strategies. In this week we will start by discussing the need of mixed strategies along with some examples, then we will move on to the Nash theorum and prove it, we also learn how to find the Nash equilibrium in mixed strategies for some special classes of games, although computing the Nash equilibrium for a general mixed extension of a game is an NP hard problem.

§2 Why mixed strategies?

Intuitively if a player has more than one strategy, one would think that one of them must be better in some cases while some other can be better in other cases. But even when we have perfect information about the game, we cant be sure of what strategies other players would use, so in general one might think that there would be one strategy which would be better than rest of the strategies in an average case and they might question the need of choosing a mixed strategy which incorporates randomness into it.
Here I will give some example where having a degree of randomness in your selection can improve your rewards rather than a wholly deterministic strategy.

1. In cricket pace bowlers ball slower ball sometimes to add an element of surprise for the batsmen, it the batsman knew that a slower ball in about to be bowled then he would have an easier time hitting it that a fast ball but the element of surprise makes it difficult for the batsman to play that ball/

2. If a traffic police car is situated at the same junction everyday, the effectiveness would be reduced

There are many more real world examples like these, but how do we incorporate these situations into a mathematical model.

Example 2.1

Consider the two-player zero-sum game depicted below:

	L	R
T	4	1
B	2	3

Player I's security level is 2; if he plays B he guarantees himself a payoff of at least 2. Player II's security level is 3; if he plays R he guarantees himself a payoff of at most 3.

This is written as

$$\underline{v} = \max_{s_I \in \{\text{T,B}\}} \min_{s_{II} \in \{\text{L,R}\}} u(s_I, s_{II}) = 2 \tag{1}$$

$$\bar{v} = \min_{s_{II} \in \{\text{L,R}\}} \max_{s_I \in \{\text{T,B}\}} u(s_I, s_{II}) = 3 \tag{2}$$

Since

$$\bar{v} = 3 > 2 = \underline{v}, \tag{3}$$

the game has no value.

Can one of the players, say Player I, guarantee a "better outcome" by playing "unpredictably"?

Suppose that Player I tosses a coin with parameter $\frac{1}{4}$, that is, a coin that comes up heads with probability $\frac{1}{4}$ and tails with probability $\frac{3}{4}$. Suppose furthermore that Player I plays T if the result of the coin toss is heads and B if the result is tails. Lets investigate what would happen in this case First of all, the payoffs would no longer be definite, but instead would be probabilistic payoffs.
If Player II plays L, the result is a lottery $[\frac{1}{4}(4), \frac{3}{4}(2)]$; that is, with probability $\frac{1}{4}$ Player II pays 4, and with probability $\frac{3}{4}$ pays 2.

If these payoffs are the utilities of a player whose preference relation satisfies the von Neumann–Morgenstern axioms, then Player I's utility from this lottery is

$$\frac{1}{4} \cdot 4 + \frac{3}{4} \cdot 2 = 2\tfrac{1}{2}.$$

If, however, Player II plays R, the result is the lottery $[\frac{1}{4}(1), \frac{3}{4}(3)]$. In this case, if the payoffs are utilities, Player I's utility from this lottery is

$$\frac{1}{4} \cdot 1 + \frac{3}{4} \cdot 3 = 2\tfrac{1}{2}.$$

This shows us that by playing "unpredictably" player 1 can achieve a better security value which would be $2\frac{1}{2}$

Similarly if player 2 chooses to play both L and R with equal probability, he can guarantee a max payoff of $2\frac{1}{2}$ which is better than what he would have achieved by playing a pure strategy.

§3 The mixed extension of a strategic-form game

In this section we will assume that the utilities of the players satisfy the von Neumann–Morgenstern axioms(refer to the appendix); hence their utility functions are linear (in probabilities). In other words, the payoff (=utility) to a player from a lottery is the expected payoff of that lottery. With this definition of what a payoff is, Player I can guarantee that no matter what happens his expected payoff will be at least $2\frac{1}{2}$, in contrast to a security level of 2 if he does not base his strategy on the coin toss.

Definition 3.1. Let $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a strategic-form game in which the set of strategies of each player is finite. A mixed strategy of player i is a probability distribution over his set of strategies S_i.

Denote by

$$\Sigma_i := \left\{ \sigma_i : S_i \to [0,1] \;\middle|\; \sum_{s_i \in S_i} \sigma_i(s_i) = 1 \right\} \tag{4}$$

the set of mixed strategies of player i.

A mixed strategy is therefore, a probability distribution over $S_i : \sigma_i = (\sigma_i(s_i))_{s_i\in S_i}$. The number $\sigma_i(s_i)$ is the probability of playing the strategy s_i
We now define the mixed extension of the game.

Definition 3.2. Let $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a strategic-form game in which, for every player $i \in N$, the set of pure strategies S_i is nonempty and finite.

Denote by

$$S := S_1 \times S_2 \times \cdots \times S_n$$

the set of pure strategy vectors.

The mixed extension of G is the game

$$\Gamma := (N, (\sigma_i)_{i\in N}, (U_i)_{i\in N}) \tag{5}$$

in which, for each $i \in N$, player i's set of strategies is $\sigma_i :=$ probability distributions over S_i, and his payoff function is the function $U_i : \sigma \to \mathbb{R}$, which associates each strategy vector $\sigma = (\sigma_1, \ldots, \sigma_n) \in \sigma = \sigma_1 \times \cdots \times \sigma_n$ with the payoff

$$U_i(\sigma) = \mathbb{E}_\sigma[u_i(\mathbf{s})] = \sum_{(s_1,\ldots,s_n)\in S} u_i(s_1, \ldots, s_n)\sigma_1(s_1)\sigma_2(s_2)\ldots\sigma_n(s_n) \tag{6}$$

> **Remark 3.3.** Mixed strategies were defined above only for the case in which the sets of pure strategies are finite. It follows that the mixed extension of a game is only defined when the set of pure strategies of each player is finite. However, the concept of mixed strategy, and hence the mixed extension of a game, can be defined when the set of pure strategies of a player is a countable set. In that case the set $\Sigma_i = \Delta(S_i)$ is an n infinite-dimensional set. It is possible to extend the definition of mixed strategy further to the case in which the set of strategies is any set in a measurable space, but that requires making use of concepts from measure theory that go beyond the background in mathematics assumed for this course

The mixed extension Γ of a strategic-form game G is itself a strategic-form game, in which the set of strategies of each player is of the cardinality of the continuum. It follows that all the concept s we defined in Week 2, such as dominant strategy, security level, and equilibrium, are also defined for Γ, and all the results we proved in Week 2 apply to mixed extensions of games.

Definition 3.4. Let G be a game in strategic form, and let Γ be its mixed extension. Every equilibrium of Γ is called an equilibrium in mixed strategies of G. If G is a two-player zero-sum game, and if Γ has value v, then v is called the value of G in mixed strategies.

Now we look at a formal way of calculating the equilibrium in mixed strategies for the game we saw in example 2.1

Example 3.5

Consider the two-player zero-sum game continued from example 2.1

	L	R
T	4	1
B	2	3

Figure 3.1: The game in strategic form

When Player I's strategy set contains two actions, T and B, we identify the mixed strategy $[x(T),(1-x)(B)]$ with the probability x of selecting the pure strategy T.

Similarly, when Player II's strategy set contains two actions, L and R, we identify the mixed strategy $[y(L),(1-y)(R)]$ with the probability y of selecting the pure strategy L.

For each pair of mixed strategies $x, y \in [0,1]$ (with the identifications $x \equiv [x(T),(1-x)(B)]$ and $y \equiv [y(L),(1-y)(R)]$), the payoff is:

$$U(x,y) = 4xy + 1x(1-y) + 2(1-x)y + 3(1-x)(1-y) \tag{7}$$

$$U(x,y) = 3 - 2x - y + 4xy \tag{8}$$

As we showed here, the game has the value $2\frac{1}{2}$, and its optimal strategies are $x = \frac{1}{4}$ and $y = \frac{1}{2}$.

It follows that the value in mixed strategies of the game in Figure 3.1 is $2\frac{1}{2}$, and the optimal strategies of the players are

$$x^* = \left[\tfrac{1}{4}(T),\ \tfrac{3}{4}(B)\right], \qquad y^* = \left[\tfrac{1}{2}(L),\ \tfrac{1}{2}(R)\right].$$

The payoff function defined in Equation (7) is a linear function over x for each fixed y and, similarly, a linear function over y for each fixed x. Such a function is called a bilinear function. The analysis we conducted in the example can be generalized to all two-player games where each player has two pure strategies. The extension to mixed strategies of such a game is a game on the unit square with bilinear payoff functions. In the converse direction, every zero-sum two-player game over the unit square with bilinear payoff functions is the extension to mixed strategies of a two-player zero-sum game in which each player has two pure strategies.

The next theorem states that this property can be generalized to any number of players and any number of actions, as long as we properly generalize the concept of bilinearity to multilinearity.

Theorem 3.6

Let $G = (N, (S_i)_{i\in N}, (u_i)_{i\in N})$ be a game in strategic form in which the set of strategies S_i of every player is finite, and let $\Gamma = (N, (\Sigma_i)_{i\in N}, (U_i)_{i\in N})$ be its mixed extension. Then for each player $i \in N$, the function U_i is a multilinear function in the n variables $(\sigma_i)_{i\in N}$; that is, for every player i, every $\sigma_i, \sigma_i' \in \sigma_i$, and every $\lambda \in [0, 1]$,

$$U_i(\lambda\sigma_i + (1-\lambda)\sigma_i', \sigma_{-i}) = \lambda U_i(\sigma_i, \sigma_{-i}) + (1-\lambda)U_i(\sigma_i', \sigma_{-i}), \quad \forall \sigma_{-i} \in \Sigma_{-i}.$$

Proof. Recall that

$$U_i(\sigma) = \sum_{(s_1,\ldots,s_n)\in S} u_i(s_1, \ldots, s_n) \cdot \sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_n(s_n). \tag{9}$$

The function U_i is a function of $\sum_{i=1}^{n} m_i$ variables:

$$\sigma_1(s_1^1), \sigma_1(s_1^2), \ldots, \sigma_1(s_1^{m_1}),\ \sigma_2(s_2^1), \ldots, \sigma_2(s_2^{m_2}),\ \ldots,\ \sigma_n(s_n^{m_n}). \tag{10}$$

For each $i \in N$, for all j, $1 \le j \le m_i$, and for each $s = (s_1, \ldots, s_n) \in S$, the function

$$\sigma_i(s_i^j) \mapsto u_i(s_1, \ldots, s_n) \cdot \sigma_1(s_1)\sigma_2(s_2)\cdots\sigma_n(s_n) \tag{11}$$

is a constant function if $s_i \neq s_i^j$, and a linear function of $\sigma_i(s_i^j)$ with slope

$$u_i(s_1, \ldots, s_n) \cdot \sigma_1(s_1)\cdots\sigma_{i-1}(s_{i-1}) \cdot \sigma_{i+1}(s_{i+1})\cdots\sigma_n(s_n), \tag{12}$$

if $s_i = s_i^j$.

Thus, the function U_i, as the sum of linear functions in $\sigma_i(s_i^j)$, is also linear in each $\sigma_i(s_i^j)$.

It follows that for every $i \in N$, the function $U_i(\cdot, \sigma_{-i})$ is linear in each coordinate $\sigma_i(s_i^j)$, for all $\sigma_{-i} \in \Sigma_{-i}$:

$$U_i(\lambda\sigma_i + (1-\lambda)\sigma_i', \sigma_{-i}) = \lambda U_i(\sigma_i, \sigma_{-i}) + (1-\lambda)U_i(\sigma_i', \sigma_{-i}), \tag{13}$$

for every $\lambda \in [0, 1]$, and every $\sigma_i, \sigma_i' \in \Sigma_i$. □

Since a multilinear function over Σ is a continuous function (see Exercise 5.4), we have the following corollary.

Corollary 3.7

The payoff function U_i of player i is a continuous function in the extension to mixed strategies of every finite strategic-form game

$$G = (N, (S_i)_{i\in N}, (u_i)_{i\in N}).$$

We can also derive a second corollary from Theorem 3.6, which can be used to determine whether a particular mixed-strategy vector is an equilibrium.

Corollary 3.8

Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a strategic-form game, and let Γ be its mixed extension.

A mixed-strategy vector σ^* is an equilibrium in mixed strategies of Γ if and only if for every player $i \in N$ and every pure strategy $s_i \in S_i$,

$$U_i(\sigma^*) \geq U_i(s_i, \sigma^*_{-i}). \tag{14}$$

Proof. If σ^* is an equilibrium in mixed strategies of Γ, then $U_i(\sigma^*) \geq U_i(\sigma_i, \sigma^*_{-i})$ for all $\sigma_i \in \Sigma_i$. Since every pure strategy is in particular a mixed strategy, it follows that $U_i(\sigma^*) \geq U_i(s_i, \sigma^*_{-i})$ for every $_i \in S_i$, and hence Equation (14) holds.

To show the converse, suppose σ^* satisfies Equation (14) for all $i \in N$, $s_i \in S_i$. Then for each mixed strategy σ_i of player i,

$$U_i(\sigma_i, \sigma^*_{-i}) = \sum_{s_i \in S_i} \sigma_i(s_i) \cdot U_i(s_i, \sigma^*_{-i}) \tag{15}$$

$$\leq \sum_{s_i \in S_i} \sigma_i(s_i) \cdot U_i(\sigma^*) \tag{16}$$

$$= U_i(\sigma^*) \cdot \sum_{s_i \in S_i} \sigma_i(s_i) = U_i(\sigma^*), \tag{17}$$

where Equation (15) follows from the multilinearity of U_i, and Equation (16) follows from (14).

Hence σ^* is an equilibrium in mixed strategies of Γ. □

Example 3.9

A mixed extension of a two-player game that is not zero-sum.

Consider the two-player non-zero-sum game given by the payoff matrix shown in Figure 3.2.

	L	R
T	$1, -1$	$0, 2$
B	$0, 1$	$2, 0$

Figure 3.2: A two-player, non-zero-sum game without an equilibrium in pure strategies

As we now show, this game has no equilibrium in pure strategies (you can follow the arrows in Figure 3.2 to see why this is so):

- (T, L) is not an equilibrium, since Player II can gain by deviating to R.
- (T, R) is not an equilibrium, since Player I can gain by deviating to B.
- (B, L) is not an equilibrium, since Player I can gain by deviating to T.
- (B, R) is not an equilibrium, since Player II can gain by deviating to L.

Does this game have an equilibrium in mixed strategies? To answer this question, we first write out the mixed extension of the game:

- The set of players is the same as in the original game: $N = \{\mathrm{I}, \mathrm{II}\}$.
- Player I's set of strategies is $\sigma_I = \{[x(\mathrm{T}), (1-x)(\mathrm{B})] : x \in [0,1]\}$, which can be identified with the interval $[0,1]$.
- Player II's set of strategies is $\sigma_{II} = \{[y(\mathrm{L}), (1-y)(\mathrm{R})] : y \in [0,1]\}$, which can also be identified with the interval $[0,1]$.
- Player I's payoff function is

$$U_I(x,y) = xy + 2(1-x)(1-y) = 3xy - 2x - 2y + 2. \tag{18}$$

- Player II's payoff function is

$$U_{II}(x,y) = -xy + 2x(1-y) + y(1-x) = -4xy + 2x + y. \tag{19}$$

Hence we found a unique equilibrium for this game:

$$x^* = \frac{1}{4}, \quad y^* = \frac{2}{3}.$$

The unique equilibrium in mixed strategies of the game given in Figure 3.2 is therefore:

$$\left[\tfrac{1}{4}(\mathrm{T}),\ \tfrac{3}{4}(\mathrm{B})\right], \quad \left[\tfrac{2}{3}(\mathrm{L}),\ \tfrac{1}{3}(\mathrm{R})\right]. \tag{20}$$

We have seen two examples of two-player games, one a zero-sum game and the other a non-zero-sum game, both of them had exactly two strategies for either of the players. Neither of them has an equilibrium in pure strategies, but they both have equilibria in mixed strategies. So which classes of games have equilibria in mixed strategies?

John Nash answered this question in 1951 with perhaps the single most important result in game theory

Theorem 3.10 (Nash Theorem (1951))

Every game in strategic form G, with a finite number of players and in which every player has a finite number of pure strategies, has an equilibrium in mixed strategies.

A special case of this theorem was proven by von Neumann 22 years ago

Theorem 3.11 (von Neumann's Minmax Theorem (1928))

Every two-player zero-sum game in which every player has a finite number of pure strategies has a value in mixed strategies.

In other words, in every two-player zero-sum game the minmax value in mixed strategies is equal to the maxmin value in mixed strategies. This is, in fact, a generalization of the Minmax Theorem here to two-player games that may not be zero-sum, and to games with any finite number of players.

Recall that the value in mixed strategies of a two-player zero-sum game, if it exists, is given by

$$v := \max_{\sigma_I \in \Sigma_I} \min_{\sigma_{II} \in \Sigma_{II}} U(\sigma_I, \sigma_{II}) = \min_{\sigma_{II} \in \Sigma_{II}} \max_{\sigma_I \in \Sigma_I} U(\sigma_I, \sigma_{II}). \tag{21}$$

In the next example we show a case where Nash theorem and Minmax theorem don't hold for an infinite number of pure strategies for 2 players.

Example 3.12

Choosing the largest number

Consider the following two-player zero-sum game. Two players simultaneously and independently choose a positive integer. The player who chooses the smaller number pays a dollar to the player who chooses the larger number. If the two players choose the same integer, no exchange of money occurs.

We model this as a game in strategic form and show that it has no value in mixed strategies.

Both players have the same set of pure strategies:

$$S_I = S_{II} = \mathbb{N} = \{1, 2, 3, \dots\}. \tag{22}$$

This set is not finite; it is a countably infinite set.

The payoff function is

$$u(s_I, s_{II}) = \begin{cases} 1 & \text{if } s_I > s_{II}, \\ 0 & \text{if } s_I = s_{II}, \\ -1 & \text{if } s_I < s_{II}. \end{cases} \tag{23}$$

A mixed strategy in this game is a probability distribution over the set of positive integers:

$$\Sigma_I = \Sigma_{II} = \left\{ (x_1, x_2, \dots) \;\middle|\; \sum_{k=1}^{\infty} x_k = 1,\ x_k \geq 0 \text{ for all } k \in \mathbb{N} \right\}. \tag{24}$$

We will show that

$$\sup_{\sigma_I \in \Sigma_I} \inf_{\sigma_{II} \in \Sigma_{II}} U(\sigma_I, \sigma_{II}) = -1, \tag{25}$$

and

$$\inf_{\sigma_{II} \in \Sigma_{II}} \sup_{\sigma_I \in \Sigma_I} U(\sigma_I, \sigma_{II}) = 1. \tag{26}$$

It then follows from equations (25) and (26) that the game has no value in mixed strategies.

Let σ_I be the strategy of Player I, and let $\varepsilon \in (0, 1)$. Since σ_I is a distribution over $\mathbb{N}$, there exists a sufficiently large $k \in \mathbb{N}$ such that

$$\sigma_I(\{1, 2, \dots, k\}) > 1 - \varepsilon. \tag{27}$$

In words, the probability that Player I chooses a number less than or equal to k is greater than $1 - \varepsilon$. But then, if Player II chooses the pure strategy $k + 1$, we have

$$U(\sigma_I, k + 1) < (1 - \varepsilon)(-1) + \varepsilon(1) = -1 + 2\varepsilon, \tag{28}$$

because with probability greater than $1 - \varepsilon$, Player I loses (payoff -1), and with probability less than ε, he wins (payoff 1).

Since this is true for any $\varepsilon \in (0, 1)$, equation (25) holds. Equation (26) is proved in a similar manner.

We defined extensive-form games with the use of finite games; in particular, in every extensive-form game every player has a finite number of pure strategies. We therefore have the following corollary of Theorem 3.10:

Theorem 3.13

Every extensive-form game has an equilibrium in mixed strategies

§4 Computing Equilibria in Mixed Strategies

Before we proceed to the proof of Nash's Theorem, we consider the subject of computing equilibria in mixed strategies.

When the number of players is large — and similarly, when the number of strategies is large — finding an equilibrium, let alone all equilibria, is a very difficult problem, both theoretically and computationally.

In this section, we present only a few examples of computing equilibria in special cases of games.

§4.1 The Direct Approach

The **direct approach** to finding equilibria is to write down the mixed extension of the strategic-form game, and then compute the equilibria in that mixed extension (assuming we can do so).

In the case of a two-player game where each player has two pure strategies, the mixed extension is a game over the unit square with bilinear payoff functions). Although this approach works well in two-player games with two strategies each, it becomes quite complicated when there are more strategies or more players.

We will now go over two player zero sum games where finding equilibria is equivalent to finding the value of the game and where equilibrium strategies are optimal strategies

Using Equation (21), we can find the value of the game by computing

$$\max_{\sigma_I \in \Sigma_I} \min_{s_{II} \in S_{II}} U(\sigma_I, s_{II}) \quad \text{or} \quad \min_{\sigma_{II} \in \Sigma_{II}} \max_{s_I \in S_I} U(s_I, \sigma_{II}),$$

which also enables us to find the optimal strategies of the players.

In particular:

- Every strategy σ_I at which the maximum in

$$\max_{\sigma_I \in \Sigma_I} \min_{s_{II} \in S_{II}} U(\sigma_I, s_{II})$$

  is attained is an optimal strategy of Player I.

- Every strategy σ_{II} at which the minimum in

$$\min_{\sigma_{II} \in \Sigma_{II}} \max_{s_I \in S_I} U(s_I, \sigma_{II})$$

  is attained is an optimal strategy of Player II.

The first game we analyze is a game over the unit square.

Example 4.1

A two-player zero-sum game, in which each player has two pure strategies.

Consider the two-player zero-sum game in Figure 5.4.

	L	R
T	5	0
B	3	4

A two-player zero-sum game

We begin by computing

$$\max_{\sigma_I \in \Sigma_I} \min_{s_{II} \in S_{II}} U(\sigma_I, s_{II}).$$

Suppose Player I plays the mixed strategy $[x(T), (1-x)(B)]$. Then his payoff depends on the strategy of Player II:

- If Player II plays L:

$$U(x, L) = 5x + 3(1-x) = 2x + 3.$$

- If Player II plays R:

$$U(x, R) = 0x + 4(1-x) = -4x + 4.$$

The lower envelope of these two payoffs is $\min_{s_{II} \in S_{II}} U(x, s_{II})$. The value of the game in mixed strategies is the maximum of this lower envelope:

$$\max_{x \in [0,1]} \min\{2x + 3,\ -4x + 4\}.$$

This maximum is attained at the point of intersection of the two functions:

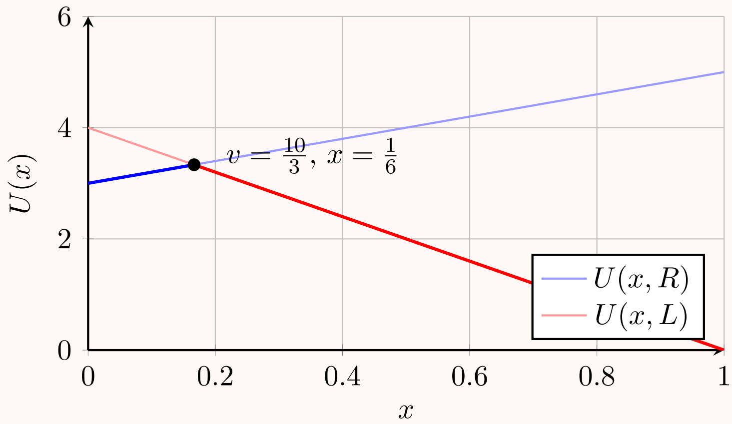
$$2x + 3 = -4x + 4. \tag{29}$$

Solving gives $x = \frac{1}{6}$. Thus, Player I's optimal strategy is:

$$x^* = \left[\tfrac{1}{6}(T),\ \tfrac{5}{6}(B)\right].$$

The value of the game is the height of the intersection point:

$$v = 2 \cdot \tfrac{1}{6} + 3 = \tfrac{1}{3} + 3 = \tfrac{10}{3}.$$



The payoff functions $U(x, L)$, $U(x, R)$, and their lower envelope

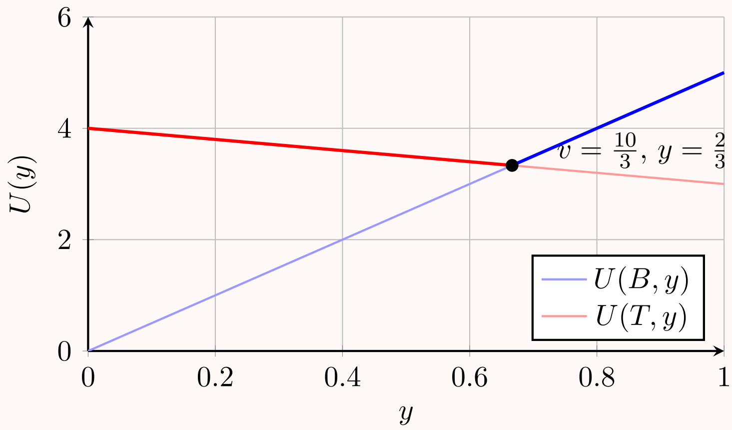
We now compute the optimal strategy for Player II. Let y denote the probability with which Player II plays L, so their strategy is $[y(L), (1 - y)(R)]$. The expected payoff to Player I is:

- If Player I plays T:
$$U(T, y) = 5y.$$

- If Player I plays B:
$$U(B, y) = 3y + 4(1 - y) = -y + 4.$$

We now take the *maximum* over Player I's pure strategies, and Player II aims to minimize this maximum:

$$\min_{y \in [0,1]} \max\{5y,\ -y + 4\}.$$



The payoff functions $U(T, y)$, $U(B, y)$, and their upper envelope

The minimum of the upper envelope occurs at the point of intersection:

$$5y = -y + 4.$$

Solving gives $y = \frac{2}{3}$. Thus, the optimal strategy for Player II is:

$$y^* = \left[\tfrac{2}{3}(L),\ \tfrac{1}{3}(R)\right].$$

The value of the game is:

$$U(B, y^*) = -\tfrac{2}{3} + 4 = \tfrac{10}{3}. \tag{30}$$

This procedure can be used to compute optimal strategies for any game in which each player has two pure strategies.

Note that the value v computed for player 1 (maxmin value) equals the value computed for player 2 (minmax value), both being $\frac{10}{3}$. This equality follows from Theorem 3.11, which states that the game has a value in mixed strategies. □

The graphical representation in the above example is very convenient, in fact it can also be extended to two player zero-sum games where one player has 2 strategies while the other player has a finite number of strategies.

Suppose Player I has two strategies while Player II has a finite number of strategies. Instead of creating an envelope using two lines, we can make it using multiple lines and find the maxmin value for Player I. Let the two(or more) lines which intersect at the maxmin value of player I correspond to strategies $\sigma_1, \sigma_2 ... \sigma_n$
We can pick any two of these strategies for player two and calculate the probabilities with which these strategies need to played to achieve the minmax value equal to that of Player I.

> **Remark 4.2.** *Try to think why you can achieve the optimal minmax value by using atmost two of these n strategies, and also think what would happen if you choose any of the other strategies with a non-zero probability*

This method can also be extended to all two player zero sum games where both players have a finite number of pure strategies. Instead of using 2d graphs, we can consider an $n-$dimensional graph where n is the minimum of the number of pure strategies of the players.

And hence we arrive at a more general case of the remark 4.2

Theorem 4.3

For every two-player zero-sum game where Player I has m_1 pure strategies and Player II has m_2 pure strategies, if $m_1 \leq m_2$ then Player II has an optimal mixed strategy that chooses, with positive probability, at most m_1 pure strategies.

We will look at ways of solving for the optimal mixed strategies in such two player games later, for now we will look into computing the equilibrium points of a two-player non zero-sum game.

§4.2 Computing equilibrium points

When we are dealing with non zero-sum games, the Nash equilibrium solution concept is not equivalent to the maxmin value. The computational procedure above will therefore not lead to Nash equilibrium points in that case, and we need other procedures.

The most straightforward and natural way to develop such a procedure is to build on the definition of the Nash equilibrium in terms of the "best reply". We present an example here of a two player non zero-sum game which we saw in week 2, in which there are more than one equilibrium points.

Example 4.4 (Battle of the Sexes)

Consider the Battle of the Sexes game shown below.

	F	C
F	$2,1$	$0,0$
C	$0,0$	$1,2$

Battle of the Sexes

Recall that for each mixed strategy $x = [\text{x}(F), (1-x)(C)]$ of Player I, we denoted the set of best replies of Player II as:

$$\text{br}_{II}(x) = \arg\max_{y\in[0,1]} u_{II}(x,y) = \{y \in [0,1] : u_{II}(x,y) \geq u_{II}(x,z)\ \forall z \in [0,1]\} \quad (31)$$

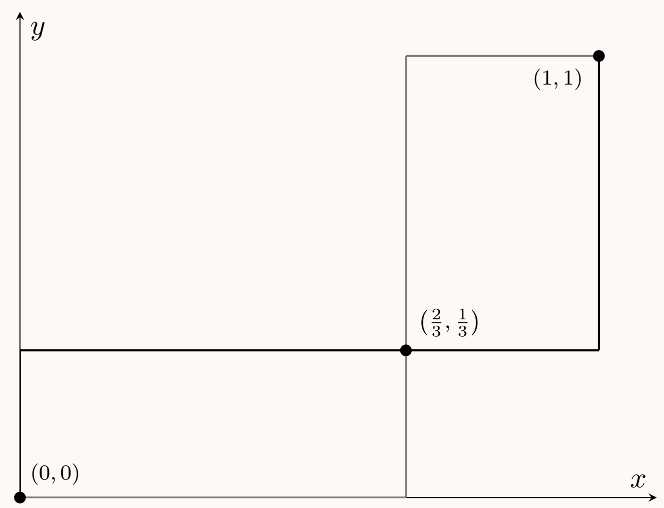
Similarly, for each mixed strategy $y = [\text{y}(F), (1-y)(C)]$ of Player II, we denoted Player I's best replies as:

$$\text{br}_{I}(y) = \arg\max_{x\in[0,1]} u_{I}(x,y) = \{x \in [0,1] : u_{I}(x,y) \geq u_{I}(z,y)\ \forall z \in [0,1]\} \quad (32)$$

In this game, the best response correspondences are:

$$\text{br}_{II}(x) = \begin{cases} 0 & \text{if } x < \frac{2}{3}, \\ [0,1] & \text{if } x = \frac{2}{3}, \\ 1 & \text{if } x > \frac{2}{3}, \end{cases} \qquad \text{br}_{I}(y) = \begin{cases} 0 & \text{if } y < \frac{1}{3}, \\ [0,1] & \text{if } y = \frac{1}{3}, \\ 1 & \text{if } y > \frac{1}{3}. \end{cases}$$

The graph below depicts the graphs of these two set-valued functions, br_I and br_{II}. The graph of br_{II} is the lighter line, and the graph of br_I is the darker line. The two graphs are shown on the same set of axes, where the $x-axis$ is the horizontal line and the $y-axis$ is the vertical line. For each $x \in [0,1]$, $br_{II}(x)$ is a point or a line located above x. For each $y \in [0,1]$, $br_I(y)$ is a point or a line located to the right of y.



Graphs of best response correspondences br_I (black) and br_{II} (gray)

A point (x^*, y^*) is a mixed-strategy Nash equilibrium if and only if

$$x^* \in \mathrm{br}_I(y^*) \quad \text{and} \quad y^* \in \mathrm{br}_{II}(x^*),$$

i.e., an intersection point of the two graphs. From the graph, we identify the following equilibria:

- $(x^*, y^*) = (0, 0)$, corresponding to the pure strategy equilibrium (C, C).
- $(x^*, y^*) = (1, 1)$, corresponding to the pure strategy equilibrium (F, F).
- $(x^*, y^*) = \left(\frac{2}{3}, \frac{1}{3}\right)$, corresponding to the mixed strategy equilibrium

$$x^* = \left(\frac{2}{3}(F), \frac{1}{3}(C)\right), \quad y^* = \left(\frac{1}{3}(F), \frac{2}{3}(C)\right).$$

We can make two interesting observations from this:-

- The payoff at the mixed-strategy equilibrium is $\left(\frac{2}{3}, \frac{2}{3}\right)$. For each player, this is *worse* than the worst-case payoff from either pure strategy equilibrium.
- This value $\frac{2}{3}$ is the *security level* (maxmin value) of each player. However, the strategies that guarantee this value — namely,

$$\text{Player I: } \left(\frac{1}{3}(F), \frac{2}{3}(C)\right), \quad \text{Player II: } \left(\frac{2}{3}(F), \frac{1}{3}(C)\right),$$

— are not equilibrium strategies.

This geometric procedure for computing equilibrium points, as intersection points of the graphs of the best replies of the players, is not applicable if there are more than two players or if each player has more than two pure strategies. But there are cases in which this procedure can be mimicked by finding solutions of algebraic equations corresponding to the intersections of best-response graphs.

§4.3 Indifference principle

One effective tool for finding equilibria is the *indifference principle.* The indifference principle says that if a mixed equilibrium calls for a player to use two distinct pure strategies with positive probability, then the expected payoff to that player for using one of those pure strategies equals the expected payoff to him for using the other pure strategy, assuming that the other players are playing according to the equilibrium.
This can be formulated as follows

Theorem 4.5 (Indifference principle)

Let σ^* be an equilibrium in mixed strategies of a strategic-form game, and let s_i and s_i' be two pure strategies of player i. If $\sigma_i^*(s_i) > 0$ and $\sigma_i^*(s_i') > 0$, then

$$U_i(s_i, \sigma_{-i}^*) = U_i(s_i', \sigma_{-i}^*). \tag{33}$$

The proof is trivial and is left to the reader

A straightforward corollary of the indifference principle is as follows

Corollary 4.6

Let σ^* be an equilibrium in a strategic-form game and let s_i and s_i' be two pure strategies of player i.

(a) If $U_i(s_i, \sigma^*_{-i}) < U_i(\sigma^*)$, then $\sigma^*_i(s_i) = 0$.

(b) If $U_i(s_i, \sigma^*_{-i}) < U_i(s_i', \sigma^*_{-i})$, then $\sigma^*_i(s_i) = 0$.

(c) If $\sigma^*_i(s_i) > 0$ and $\sigma^*_i(s_i') > 0$, then $U_i(s_i, \sigma^*_{-i}) = U_i(s_i', \sigma^*_{-i})$

(d) If s_i is strictly dominated by s_i', then $\sigma^*_i(s_i) = 0$.

The proof is trivial and is left to the reader

Now we will define what a completely mixed strategy is

Definition 4.7. A mixed strategy σ_i of player i is called a **completely mixed strategy** if

$$\sigma_i(s_i) > 0 \quad \text{for every } s_i \in S_i.$$

An equilibrium $\sigma^* = (\sigma^*_i)_{i \in N}$ is called a **completely mixed equilibrium** if for every player $i \in N$, the strategy σ^*_i is a completely mixed strategy.

And as a consequence of the indifference principle

Corollary 4.8

Let σ^* be an equilibrium in a strategic-form game and let $i \in N$ be a player. If σ^*_i is a completely mixed strategy, then

$$U_i(s_i, \sigma^*_{-i}) = U_i(s_i', \sigma^*_{-i})$$

for every two pure strategies $s_i, s_i' \in S_i$.

Try to compute the equilibrium of the game in example 3.9 using the indifference principle and compare the methods

§4.4 Effect of domination in MSNE

The concept of strict dominance is a useful tool for computing equilibrium points. Much like the pure strategies, we can eliminate strictly dominated pure strategies from our mixed strategies, i.e choose them with a probability of 0

Theorem 4.9

Let $G = (N, (S_i)_{i \in N}, (u_i)_{i \in N})$ be a game in strategic form in which the sets $(S_i)_{i \in N}$ are all finite. If a pure strategy $s_i \in S_i$ of player i is strictly dominated by a mixed strategy $\sigma_i \in \Sigma_i$, then in every equilibrium of the game, the pure strategy s_i is chosen by player i with probability zero.

The proof of this theorem can be done by using the corollary 4.6(a), for a detailed proof, you can check the references at the end of this document.

Now we will see an example where the concept of domination helps us in finding equilibrium points in a game

Example 4.10

Consider the two-player game shown below, where $N = \{\mathrm{I}, \mathrm{II}\}$.

	L	C	R
T	$6,2$	$0,6$	$4,4$
M	$2,12$	$4,3$	$2,5$
B	$0,6$	$10,0$	$2,2$

The strategic-form game for example 4.10

In this game, no pure strategy is dominated by another pure strategy (verify this). However, strategy M of Player I is strictly dominated by the mixed strategy $\left(\frac{1}{2}(T), \frac{1}{2}(B)\right)$ (verify this).

It follows from Theorem 4.9 that the deletion of strategy M has no effect on the set of equilibria in the game. After removing M, we are left with:

	L	C	R
T	$6,2$	$0,6$	$4,4$
B	$0,6$	$10,0$	$2,2$

The game after eliminating strategy M

In this reduced game, strategy R of Player II is strictly dominated by the mixed strategy $\left(\frac{5}{12}(L), \frac{7}{12}(C)\right)$. We then delete R, resulting in:

	L	C
T	$6,2$	$0,6$
B	$0,6$	$10,0$

The game after eliminating strategies M and R

The game in the above figure has no pure-strategy equilibria (verify this). The only mixed-strategy equilibrium, which can be computed using the indifference principle, is:

$$\left(\frac{3}{5}(T), \frac{2}{5}(B)\right), \quad \left(\frac{5}{8}(L), \frac{3}{8}(C)\right),$$

which yields payoffs:

$$\left(\frac{15}{4}, \frac{18}{5}\right).$$

Since all eliminated strategies were strictly dominated, this mixed-strategy equilibrium is also the only equilibrium of the original game.

§4.5 Two-player zero-sum games and linear programming

Now we will look at a formal method of finding the equilibrium points of games as mentioned in theorem 4.3 Computing the value of two-player zero-sum games, where each player has a finite number of strategies, and finding optimal strategies in such games, can be presented as a linear programming problem. It follows that these computations can be accomplished using known linear programming algorithms. In this section, we will look at linear programs that correspond to finding the value and optimal strategies of a two-player zero-sum game.

Let $(N, (S_i)_{i \in N}, u)$ be a two-player zero-sum game, where $N = \{I, II\}$. As usual, let U denote the multilinear extension of u.

Theorem 4.11

Denote by Z_P the value of the following linear program in the variables $(x_{s_I})_{s_I \in S_I}$:

$$\begin{aligned} \text{Compute:} \quad & Z_P := \max z \\ \text{Subject to:} \quad & \sum_{s_I \in S_I} x(s_I)\, u(s_I, s_{II}) \geq z, \quad \forall s_{II} \in S_{II}; \\ & \sum_{s_I \in S_I} x(s_I) = 1; \\ & x(s_I) \geq 0, \qquad\qquad\qquad \forall s_I \in S_I \end{aligned}$$

Then Z_P is the value in mixed strategies of the game.

Proof. Let v be the value of the game $(N, (S_i)_{i \in N}, u)$ in mixed strategies. We will prove that $Z_P = v$ by showing both $Z_P \geq v$ and $Z_P \leq v$

Step 1: $Z_P \geq v$

Suppose v is the value of the game. Then Player I has an optimal mixed strategy σ_I^* such that

$$U(\sigma_I^*, \sigma_{II}) \geq v, \quad \forall \sigma_{II} \in \Sigma_{II}. \tag{34}$$

In particular, the inequality holds for each pure strategy $s_{II} \in S_{II}$, so the vector $(x, z) = (\sigma_I^*, v)$ satisfies the constraints of the linear program. Since Z_P is the largest z for which the constraints hold, it must be that $Z_P \geq v$.

Step 2: $Z_P \leq v$**.**

First, we show that $Z_P < \infty$. Let (x, z) be any feasible solution to the linear program. Then for every $s_{II} \in S_{II}$, we have:

$$\sum_{s_I \in S_I} x(s_I) u(s_I, s_{II}) \geq z, \quad \text{and} \quad \sum_{s_I \in S_I} x(s_I) = 1.$$

Therefore,

$$z \leq \sum_{s_I \in S_I} x(s_I) u(s_I, s_{II}) \leq \max_{s_I \in S_I} \max_{s_{II} \in S_{II}} |u(s_I, s_{II})| \times \sum_{s_I \in S_I} x(s_I) \tag{35}$$

$$= \max_{s_I \in S_I} \max_{s_{II} \in S_{II}} |u(s_I, s_{II})| < \infty \tag{36}$$

(36) is implied by the fact that the number of pure strategies of both players are finite

Now, since Z_P is the value of the linear program, there exists a vector x such that (x, Z_P) satisfies the constraints. In particular, $x \in \Sigma_I$ is a mixed strategy for Player I, and

$$u(x, s_{II}) \geq Z_P, \quad \forall s_{II} \in S_{II}.$$

By multilinearity of U, it follows that

$$U(x, \sigma_{II}) \geq Z_P, \quad \forall \sigma_{II} \in \Sigma_{II}. \tag{37}$$

Thus, Player I has a strategy that guarantees a payoff of at least Z_P, implying that $v \geq Z_P$. Combining both steps, we conclude that $Z_P = v$. $\square$

The fact that the value of a game in mixed strategies can be found using linear programming is an expression of the strong connection that exists between the Minmax Theorem and the Duality Theorem.

§4.6 Two-player games that are not zero-sum

Computing the value of a two-player zero-sum game can be accomplished by solving a linear program. Similarly, computing equilibria in a two-player game that is not zero-sum can be accomplished by solving a quadratic program. However, while there are efficient algorithms for solving linear programs, there are no known efficient algorithms for solving generic quadratic programs.

A straightforward method for finding equilibria in two-player games that are not zero-sum is based on the following idea.

Let $(\sigma_I^*, \sigma_{II}^*)$ be a Nash equilibrium in mixed strategies. Define the *support* of the mixed strategies as:

$$\operatorname{supp}(\sigma_I^*) := \{s_I \in S_I : \sigma_I^*(s_I) > 0\}, \tag{38}$$
$$\operatorname{supp}(\sigma_{II}^*) := \{s_{II} \in S_{II} : \sigma_{II}^*(s_{II}) > 0\}. \tag{39}$$

The sets $\operatorname{supp}(\sigma_I^*)$ and $\operatorname{supp}(\sigma_{II}^*)$ contain all pure strategies that are played with positive probability under σ_I^* and σ_{II}^*, respectively.

By the *Indifference Principle* (see Theorem 4.5), at equilibrium, any two pure strategies played with positive probability yield the same payoff. Let $s_I^0 \in \operatorname{supp}(\sigma_I^*)$, and $s_{II}^0 \in \operatorname{supp}(\sigma_{II}^*)$. Then the equilibrium $(\sigma_I^*, \sigma_{II}^*)$ must satisfy:

$$U_I(s_I^0, \sigma_{II}^*) = U_I(s_I, \sigma_{II}^*), \quad \forall s_I \in \operatorname{supp}(\sigma_I^*), \tag{40}$$
$$U_{II}(\sigma_I^*, s_{II}^0) = U_{II}(\sigma_I^*, s_{II}), \quad \forall s_{II} \in \operatorname{supp}(\sigma_{II}^*). \tag{41}$$

At equilibrium, neither player can benefit from a unilateral deviation. Thus:

$$U_I(s_I^0, \sigma_{II}^*) \geq U_I(s_I, \sigma_{II}^*), \quad \forall s_I \in S_I \setminus \operatorname{supp}(\sigma_I^*), \tag{42}$$
$$U_{II}(\sigma_I^*, s_{II}^0) \geq U_{II}(\sigma_I^*, s_{II}), \quad \forall s_{II} \in S_{II} \setminus \operatorname{supp}(\sigma_{II}^*). \tag{43}$$

Since U_I and U_{II} are multilinear functions, the above defines a system of linear equations in σ_I^* and σ_{II}^*, together with the fact that they are probability distributions.

Every pair of mixed strategies $(\sigma_I^*, \sigma_{II}^*)$ that solves equations (40)-(43) is a Nash equilibrium

This leads to a direct algorithm for finding equilibria in a two-player game that is not zero-sum:

> For every nonempty subset $Y_I \subseteq S_I$ and every nonempty subset $Y_{II} \subseteq S_{II}$, determine whether there exists an equilibrium $(\sigma_I^*, \sigma_{II}^*)$ such that
>
> $$Y_I = \operatorname{supp}(\sigma_I^*), \qquad Y_{II} = \operatorname{supp}(\sigma_{II}^*).$$

The set of equilibria whose support is Y_I and Y_{II} is the set of solutions of the system of equations comprised of Equations (44)–(53), in which s_I^0 and s_{II}^0 are any two pure strategies in Y_I and Y_{II}, respectively.

$$\sum_{s_{II}\in S_{II}} \sigma_{II}(s_{II})\, u_I(s_I^0, s_{II}) = \sum_{s_{II}\in S_{II}} \sigma_{II}(s_{II})\, u_I(s_I, s_{II}), \quad \forall s_I \in Y_I, \tag{44}$$

$$\sum_{s_I\in S_I} \sigma_I(s_I)\, u_I(s_I, s_{II}^0) = \sum_{s_I\in S_I} \sigma_I(s_I)\, u_I(s_I, s_{II}), \quad \forall s_{II} \in Y_{II}, \tag{45}$$

$$\sum_{s_{II}\in S_{II}} \sigma_{II}(s_{II})\, u_I(s_I^0, s_{II}) \geq \sum_{s_{II}\in S_{II}} \sigma_{II}(s_{II})\, u_I(s_I, s_{II}), \quad \forall s_I \in S_I \setminus Y_I, \tag{46}$$

$$\sum_{s_I\in S_I} \sigma_I(s_I)\, u_I(s_I, s_{II}^0) \geq \sum_{s_I\in S_I} \sigma_I(s_I)\, u_I(s_I, s_{II}), \quad \forall s_{II} \in S_{II} \setminus Y_{II}, \tag{47}$$

$$\sum_{s_I\in S_I} \sigma_I(s_I) = 1, \tag{48}$$

$$\sum_{s_{II}\in S_{II}} \sigma_{II}(s_{II}) = 1, \tag{49}$$

$$\sigma_I(s_I) > 0, \quad \forall s_I \in Y_I, \tag{50}$$

$$\sigma_{II}(s_{II}) > 0, \quad \forall s_{II} \in Y_{II}, \tag{51}$$

$$\sigma_I(s_I) = 0, \quad \forall s_I \in S_I \setminus Y_I, \tag{52}$$

$$\sigma_{II}(s_{II}) = 0, \quad \forall s_{II} \in S_{II} \setminus Y_{II}. \tag{53}$$

Determining whether this system has a solution can be accomplished using linear programming.

However, since the number of nonempty subsets of S_I is $2^{m_I} - 1$, and similarly $2^{m_{II}} - 1$ for S_{II}, the complexity of this algorithm is exponential in m_I and m_{II}, making it computationally inefficient.

§5 Proof of Nash's Theorem

This section is devoted to proving Nash's Theorem (Theorem 3.10), which states that every finite game has an equilibrium in mixed strategies. The proof of the theorem makes use of the following result.

Theorem 5.1 (Brouwer's Fixed Point Theorem)

Let X be a convex and compact set in a d-dimensional Euclidean space, and let $f : X \to X$ be a continuous function. Then there exists a point $x \in X$ such that $f(x) = x$. Such a point x is called a *fixed point* of f.

Brouwer's Fixed Point Theorem states that every continuous function from a convex and compact set to itself has a fixed point, that is, a point that the function maps to itself.

In one dimension, Brouwer's Theorem takes an especially simple form. A convex and compact space in one dimension is a closed interval $[a, b]$. When $f : [a, b] \to [a, b]$ is a continuous function, one of the following three alternatives must hold:

1. $f(a) = a$, hence a is a fixed point of f.
2. $f(b) = b$, hence b is a fixed point of f.

3. $f(a) > a$ and $f(b) < b$. Consider the function $g(x) = f(x) - x$, which is continuous. Then $g(a) > 0$ and $g(b) < 0$. By the Intermediate Value Theorem, there exists $x \in [a, b]$ such that $g(x) = 0$, i.e., $f(x) = x$. Every such x is a fixed point of f.

If the dimension is two or greater, the proof of Brouwer's Fixed Point Theorem is not simple. It can be proved in several different ways, using a variety of mathematical tools.

We will now proceed to prove Nash's Theorem using Brouwer's Fixed Point Theorem. The proofs of the following two claims are left to the reader.

Theorem 5.2

If player i's set of pure strategies S_i is finite, then his set of mixed strategies Σ_i is convex and compact.

Theorem 5.3

If $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ are compact sets, then the set $A \times B$ is a compact subset of $\mathbb{R}^{n+m}$. If A and B are convex sets, then $A \times B$ is a convex subset of $\mathbb{R}^{n+m}$.

Theorems 5.2 and 5.3 imply that the set

$$\Sigma := \Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_n$$

is a convex and compact subset of the Euclidean space $\mathbb{R}^{m_1+m_2+\cdots+m_n}$.
The proof of Nash's Theorem then proceeds as follows. We will define a function

$$f : \Sigma \to \Sigma,$$

and prove that it satisfies the following two properties:

- f is a continuous function.
- Every fixed point of f is an equilibrium of the game.

Since Σ is convex and compact, and f is continuous, it follows from Brouwer's Fixed Point Theorem that f has at least one fixed point. The second property then implies that the game has at least one equilibrium point.

The idea behind the definition of f is as follows. For each strategy vector $\sigma \in \Sigma$, we define

$$f(\sigma) := (f_i(\sigma))_{i \in N}$$

to be a vector of strategies, where $f_i(\sigma) \in \Sigma_i$ is a strategy of player i. The function $f_i(\sigma)$ is defined so that if σ_i is not a best reply to σ_{-i}, then $f_i(\sigma)$ is a strategy that shifts weight toward a "better reply" to σ_{-i}. It then follows that $f_i(\sigma) = \sigma_i$ if and only if σ_i is a best reply to σ_{-i}.

To define f, we first define an auxiliary function

$$g_i^j : \Sigma \to [0, \infty)$$

for each player i and each index j, where $1 \leq j \leq m_i$. That is, for each strategy profile σ, we define a nonnegative number $g_i^j(\sigma)$.

The payoff that player i receives under the strategy profile σ is $U_i(\sigma)$. The payoff player i would receive by switching to the pure strategy s_i^j while the other players play σ_{-i} is $U_i(s_i^j, \sigma_{-i})$. We define:

$$g_i^j(\sigma) := \max\left\{0, U_i(s_i^j, \sigma_{-i}) - U_i(\sigma)\right\}. \tag{54}$$

In words, $g_i^j(\sigma)$ equals 0 if player i cannot profit from deviating from σ_i to s_i^j. When $g_i^j(\sigma) > 0$, player i can improve their payoff by increasing the probability of playing the pure strategy s_i^j.

Because a player has a profitable deviation if and only if they have a profitable deviation to a pure strategy, we obtain the following result:

Claim 5.4 — The strategy vector σ is an equilibrium if and only if $g_i^j(\sigma) = 0$, for each player $i \in N$ and for all $j = 1, 2, \ldots, m_i$.

To proceed with the proof, we need the following claim:

Claim 5.5 — For every player $i \in N$, and every $j = 1, 2, \ldots, m_i$, the function g_i^j is continuous.

Proof. Let $i \in N$ be a player, and let $j \in \{1, 2, \ldots, m_i\}$. From Corollary 3.7, the function U_i is continuous. The function $\sigma_{-i} \mapsto U_i(s_i^j, \sigma_{-i})$, as a function of σ_{-i}, is therefore also continuous. In particular, the difference $U_i(s_i^j, \sigma_{-i}) - U_i(\sigma)$ is a continuous function. Since 0 is a continuous function, and since the maximum of continuous functions is a continuous function, we have that the function g_i^j is continuous. □

We can now define the function f. The function f must satisfy the property that every one of its fixed points is an equilibrium of the game. It then follows that if σ is not an equilibrium, it must be the case that $\sigma \neq f(\sigma)$.

How can we guarantee that? The main idea is to consider, for every player i, the indices j such that $g_i^j(\sigma) > 0$; these indices correspond to pure strategies at which $g_i^j(\sigma) > 0$, i.e., the strategies that will increase player i's payoff if he increases the probability that they will be played (and decreases the probability of playing pure strategies that do not satisfy this inequality). This idea leads to the following definition.

Because $f(\sigma)$ is an element in Σ, i.e., it is a vector of mixed strategies, $f_i^j(\sigma)$ is the probability that player i will play the pure strategy s_i^j. Define:

$$f_i^j(\sigma) := \frac{\sigma_i(s_i^j) + g_i^j(\sigma)}{1 + \sum_{k=1}^{m_i} g_i^k(\sigma)}. \tag{55}$$

In words, if s_i^j is a better reply than σ_i to σ_{-i}, we increase its probability by $g_i^j(\sigma)$, and then normalize the resulting numbers so that we obtain a probability distribution.

We now turn our attention to the proof that f satisfies all its required properties.

Claim 5.6 — The range of f is Σ, i.e., $f(\Sigma) \subseteq \Sigma$.

Proof. We need to show that $f(\sigma)$ is a vector of mixed strategies for every $\sigma \in \Sigma$, i.e.,

1. $f_i^j(\sigma) \geq 0$ for all i and for all $j \in \{1, 2, \ldots, m_i\}$,

2. $\sum_{j=1}^{m_i} f_i^j(\sigma) = 1$ for all players $i \in N$.

The first condition holds because $g_i^j(\sigma)$ is nonnegative by definition, and hence the denominator in Equation (55) is at least 1, and the numerator is nonnegative.

As for the second condition, because $\sum_{j=1}^{m_i} \sigma_i(s_i^j) = 1$, it follows that

$$\sum_{j=1}^{m_i} f_i^j(\sigma) = \sum_{j=1}^{m_i} \frac{\sigma_i(s_i^j) + g_i^j(\sigma)}{1 + \sum_{k=1}^{m_i} g_i^k(\sigma)} \tag{56}$$

$$= \frac{\sum_{j=1}^{m_i} \sigma_i(s_i^j) + \sum_{j=1}^{m_i} g_i^j(\sigma)}{1 + \sum_{j=1}^{m_i} g_i^j(\sigma)} \tag{57}$$

$$= \frac{1 + \sum_{j=1}^{m_i} g_i^j(\sigma)}{1 + \sum_{j=1}^{m_i} g_i^j(\sigma)} = 1 \tag{58}$$

□

Since the numerator and denominator of f_j^i are continuous functions by Claim 5.5 and it has a denominator greater than 1, f **is a *continuous* function**

Claim 5.7 — Let σ be a fixed point of f. Then

$$g_i^j(\sigma) = \sigma_i(s_i^j) \sum_{k=1}^{m_i} g_i^k(\sigma), \quad \forall i \in N, \quad \forall j \in \{1, 2, \ldots, m_i\}. \tag{59}$$

Proof. The strategy vector σ is a fixed point of f, and therefore $f(\sigma) = \sigma$. This is an equality between vectors, so every coordinate must match:

$$f_i^j(\sigma) = \sigma_i(s_i^j), \quad \forall i \in N, \quad \forall j \in \{1, 2, \ldots, m_i\}. \tag{60}$$

From the definition of f, we have:

$$\frac{\sigma_i(s_i^j) + g_i^j(\sigma)}{1 + \sum_{k=1}^{m_i} g_i^k(\sigma)} = \sigma_i(s_i^j), \quad \forall i, j. \tag{61}$$

Since the denominator is positive, multiply both sides by $1 + \sum_{k=1}^{m_i} g_i^k(\sigma)$ to get:

$$\sigma_i(s_i^j) + g_i^j(\sigma) = \sigma_i(s_i^j) + \sigma_i(s_i^j) \sum_{k=1}^{m_i} g_i^k(\sigma). \tag{62}$$

Subtracting $\sigma_i(s_i^j)$ from both sides yields the desired equation:

$$g_i^j(\sigma) = \sigma_i(s_i^j) \sum_{k=1}^{m_i} g_i^k(\sigma).$$

□

Claim 5.8 — Let σ be a fixed point of f. Then σ is a Nash equilibrium.

Proof. Suppose, for contradiction, that σ is not an equilibrium. Then by Claim 5.4, there exists a player $i \in N$ and some index $\ell \in \{1, 2, \ldots, m_i\}$ such that $g_i^\ell(\sigma) > 0$. This implies:

$$\sum_{k=1}^{m_i} g_i^k(\sigma) > 0.$$

From Equation (59), it follows that:

$$\sigma_i(s_i^j) > 0 \iff g_i^j(\sigma) > 0, \quad \forall j. \tag{63}$$

In particular, since $g_i^\ell(\sigma) > 0$, we have $\sigma_i(s_i^\ell) > 0$. Since U_i is multilinear, $U_i(\sigma) = \sum_{j=1}^{m_i} \sigma_i(s_i^j) U_i(s_i^j, \sigma_{-i})$. This yields:

$$0 = \sum_{j=1}^{m_i} \sigma_i(s_i^j) \left(U_i(s_i^j, \sigma_{-i}) - U_i(\sigma) \right) \tag{64}$$

$$= \sum_{\substack{j: \\ \sigma_i(s_i^j)>0}} \sigma_i(s_i^j) \left(U_i(s_i^j, \sigma_{-i}) - U_i(\sigma) \right) \tag{65}$$

$$= \sum_{\substack{j: \\ \sigma_i(s_i^j)>0}} \sigma_i(s_i^j) \cdot g_i^j(\sigma). \tag{66}$$

Equation (66) is a sum of positive terms, since $\sigma_i(s_i^\ell) > 0$, and all $g_i^j(\sigma) > 0$ whenever $\sigma_i(s_i^j) > 0$ (by Equation (63)). Thus the sum is strictly positive, contradicting the assumption that it is 0.

This contradiction implies that σ must be a Nash equilibrium. □

§6 Generalizing Nash's Theorem

There are situations in which, due to various constraints, a player cannot make use of some mixed strategies. For example, there may be situations in which player i cannot choose two pure strategies s_i and s_i' with different probability, and is then forced to limit himself to mixed strategies σ_i in which $\sigma_i(s_i) = \sigma_i(s_i')$. A player may also find himself in a situation in which he must choose a particular pure strategy s_i with probability greater than or equal to some given number $p_i(s_i)$, and is thus forced to limit himself to mixed strategies σ_i satisfying $\sigma_i(s_i) \geq p_i(s_i)$. In both of these examples, the constraints can be translated into linear inequalities.

A bounded set that is defined by the intersection of a finite number of half-spaces is called a *polytope*. The number of extreme points of every polytope S is finite, and every polytope is the convex hull of its extreme points: if $x_1, x_2, \ldots, x_K$ are the extreme points of S, then S is the smallest convex set containing $x_1, x_2, \ldots, x_K$. In other words, for each $s \in S$, there exist nonnegative numbers $\{\alpha_\ell\}_{\ell=1}^K$ whose sum is 1, such that

$$s = \sum_{\ell=1}^{K} \alpha_\ell x_\ell;$$

conversely, for each vector of nonnegative numbers $\{\alpha_\ell\}_{\ell=1}^K$ whose sum is 1, the vector $\sum_{\ell=1}^K \alpha_\ell x_\ell \in S$.

The space of mixed strategies Σ_i is a *simplex*, which is a polytope whose extreme points are the unit vectors $e^1, e^2, \ldots, e^{m_i}$, where

$$e^k = (0, \ldots, 0, 1, 0, \ldots, 0)$$

is an m_i-dimensional vector whose k-th coordinate is 1 and all other coordinates are 0.

We now show that Nash's Theorem still holds when the strategy space of a player is a polytope, and not necessarily a simplex. We note that Nash's Theorem holds under even more generalized conditions, but we do not present those generalizations in this book.

Theorem 6.1

Let $G = (N, \{X_i\}_{i\in N}, \{U_i\}_{i\in N})$ be a strategic-form game in which, for each player i,

- The set X_i is a polytope in $\mathbb{R}^{d_i}$,
- The function U_i is a multilinear function over the variables $\{s_j\}_{j\in N}$.

Then G has an equilibrium.

Nash's Theorem is a special case of Theorem 6.1, where $X_i = \Sigma_i$ for every player $i \in N$.

Proof. The set of strategies X_i of player i in the game G is a polytope. Denote the extreme points of this set by $\{x_i^1, x_i^2, \ldots, x_i^{K_i}\}$. Define an auxiliary strategic-form game G' in which:

- The set of players is N.
- The set of pure strategies of player $i \in N$ is $L_i := \{1, 2, \ldots, K_i\}$. Denote $L := \times_{i\in N} L_i$.
- For each vector of pure strategies $\ell = (\ell_1, \ell_2, \ldots, \ell_n) \in L$, the payoff to player i is defined as
$$v_i(\ell) := U_i(x_1^{\ell_1}, x_2^{\ell_2}, \ldots, x_n^{\ell_n}). \tag{67}$$

Thus, in the auxiliary game, every player i chooses an extreme point in their strategy set X_i, and their payoff is given by U_i.

For each $i \in N$, denote by V_i the multilinear extension of v_i. Since U_i is a multilinear function, player i's payoff function in the mixed extension of G' is

$$V_i(\alpha) = \sum_{\ell_1=1}^{K_1}\sum_{\ell_2=1}^{K_2}\cdots\sum_{\ell_n=1}^{K_n} \alpha^1_{\ell_1}\alpha^2_{\ell_2}\cdots\alpha^n_{\ell_n}\cdot v_i(\ell_1, \ell_2, \ldots, \ell_n) \tag{68}$$
$$= \sum_{\ell_1=1}^{K_1}\sum_{\ell_2=1}^{K_2}\cdots\sum_{\ell_n=1}^{K_n} \alpha^1_{\ell_1}\alpha^2_{\ell_2}\cdots\alpha^n_{\ell_n}\cdot U_i(x_1^{\ell_1}, x_2^{\ell_2}, \ldots, x_n^{\ell_n}) \tag{69}$$
$$= U_i\left(\sum_{\ell_1=1}^{K_1}\alpha^1_{\ell_1}x_1^{\ell_1}, \ldots, \sum_{\ell_n=1}^{K_n}\alpha^n_{\ell_n}x_n^{\ell_n}\right). \tag{70}$$

The auxiliary game G' satisfies the conditions of Nash's Theorem, so it has a Nash equilibrium in mixed strategies α^*. This means that for every player i,

$$V_i(\alpha^*) \geq V_i(\alpha_i, \alpha^*_{-i}) \quad \forall \alpha_i \in \Delta(L_i). \tag{71}$$

Let $\alpha_i^* = (\alpha_i^{*,\ell_i})_{\ell_i=1}^{K_i}$ be the strategy of player i in the equilibrium α^*. Since X_i is convex, define

$$s_i^* := \sum_{\ell_i=1}^{K_i} \alpha_i^{*,\ell_i} x_i^{\ell_i}. \tag{72}$$

Then $s^* := (s_i^*)_{i\in N}$ is a point in X_i. We now show that s^* is an equilibrium of the game G.

Let $i \in N$, and let $s_i \in X_i$ be any strategy. Since $\{x_i^1, x_i^2, \ldots, x_i^{K_i}\}$ are the extreme points of X_i, there exists a distribution $\alpha_i = (\alpha_i^{\ell_i})_{\ell_i=1}^{K_i}$ over L_i such that

$$s_i = \sum_{\ell_i=1}^{K_i} \alpha_i^{\ell_i} x_i^{\ell_i}.$$

Using equations (72), (68), and (71), we get:

$$U_i(s^*) = V_i(\alpha^*) \geq V_i(\alpha_i, \alpha_{-i}) = U_i(s_i, s_{-i}). \tag{73}$$

That is, no player $i \in N$ can gain by deviating to $s_i \quad \forall s_i \in S_i$, so s^* is an equilibrium. $\square$

§7 Further Reading

1. To get a better understanding of how we can calculate equilibria for two players, read the section of ”games over unit squares” chapter 4 of Game Theory, by Maschler,Solan and Zamir.
2. Most of the content of this week is taken from the chapter 5 of them same book, a lot of proofs left to be done by the reader can be found there.