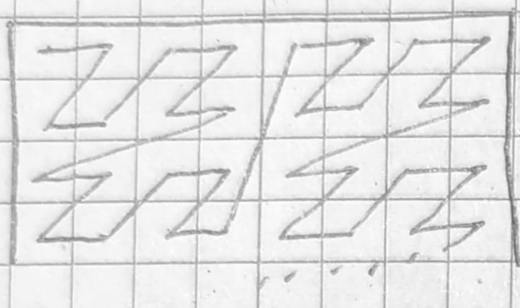
2 - Images and transformation

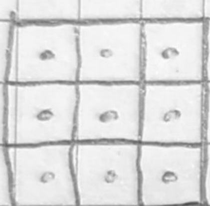
Raster images

Have width and height, stored interleaved:

R|G|B R|G|B R|G|B ..., typically in 1D array ~~and~~ or in scanline order or swizzled order:



Example of z-swizzled order

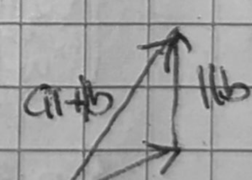


"A pixel is not a little square", but rather the single point in the middle. When we have to few pixels, we can't folly represent the image, it can then get blurry. This creates a low-frequency image. A high frequency has sharp color changes.

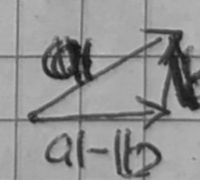
We also have alpha, α, which represents opacity, storing RGBA with 8-bits per channel $\rightarrow$ 32 bits, in interleaved or scanline order.

Vectors and matrices

vector $\mathbf{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$, with operations $\mathbf{a}+\mathbf{b}$

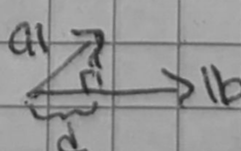


$\mathbf{a}-\mathbf{b}$



Matrix $\mathbb{A} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix}$

$\mathbf{a}\cdot\mathbf{b}$

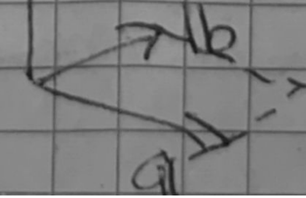


if $|\mathbf{b}|=0$
$\Rightarrow d = \mathbf{a}\cdot\mathbf{b}$
else
$\Rightarrow d = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$

$\mathbb{AB} \neq \mathbb{BA}$
$\mathbb{C} = \mathbb{A}\mathbf{b}$

$\mathbf{a}\cdot\mathbf{b} = a_x b_x + a_y b_y + a_z b_z$
$= |\mathbf{a}||\mathbf{b}|\cos\theta$

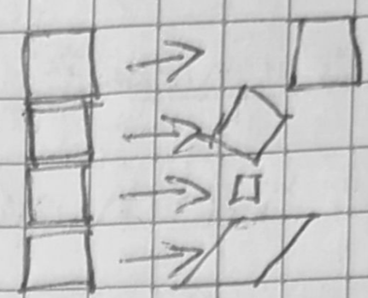
$\mathbf{a}\times\mathbf{b}$



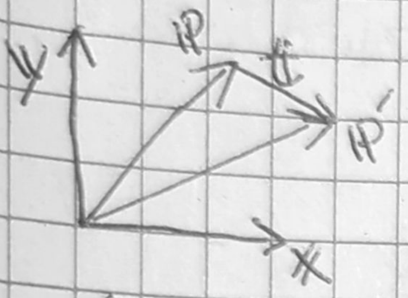
Transformations

Affine transformations:
- Translation
- Rotation
- scale
- ~~skew~~

Combination of rotations and scale



Translation:

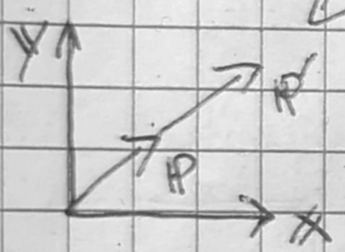


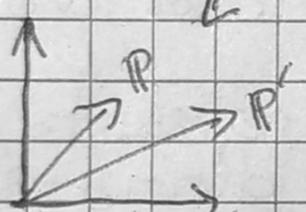
$P' = P + t$

scale:

uniform

non-uniform



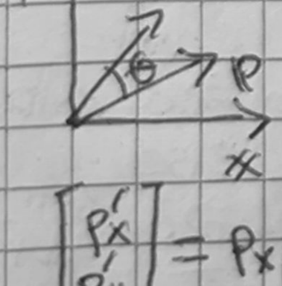


$P' = S P$

$$\begin{bmatrix} P'_x \\ P'_y \end{bmatrix} = \begin{bmatrix} S_x \cdot P_x \\ S_y \cdot P_y \end{bmatrix}$$

$$S P = \begin{bmatrix} S_x & 0 \\ 0 & S_y \end{bmatrix} \begin{bmatrix} P_x \\ P_y \end{bmatrix}$$

Rotation:



$$\begin{bmatrix} P'_x \\ P'_y \end{bmatrix} = P_x \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + P_y \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} P_x \\ P_y \end{bmatrix}$$

Any square matrix: $\mathbb{M} = \mathbb{U}\,\mathbb{S}\,\mathbb{V}^T$

- $\mathbb{U}$ → orthogonal (rotation)
- $\mathbb{S}$ → diagonal (scale)
- $\mathbb{V}^T$ → orthogonal (rotation)

Singular value decomposition (SVD)

Any series of rotation & scale:

$\mathbb{p}' = \underbrace{\mathbb{R}\mathbb{S}\mathbb{R}\mathbb{S}\mathbb{R}}\,\mathbb{p} = \mathbb{M}\mathbb{p}$, what for translation?

Homogeneous coordinatse:

$$\mathbb{p}' = \mathbb{p} + \mathbb{t} = \mathbb{T}\mathbb{p} \Longleftrightarrow \begin{bmatrix} p'_x \\ p'_y \end{bmatrix} = \begin{bmatrix} p_x + t_x \\ p_y + t_y \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix} = \begin{bmatrix} p'_x \\ p'_y \\ 1 \end{bmatrix}$$

"Hack"

can just throw away

so $\mathbb{p}' = \mathbb{T}\mathbb{S}\mathbb{R}\mathbb{T}\mathbb{S}\mathbb{R}\mathbb{T}\mathbb{p} = \mathbb{M}\mathbb{p}$

$$\begin{bmatrix} p'_x \\ p'_y \\ 1 \end{bmatrix} = \begin{bmatrix} a & c & e \\ b & d & f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ 1 \end{bmatrix}$$

<u>Affine 3D transformations:</u> with homogeneous coordinates

scale: 2D: $\begin{bmatrix} S_x & 0 & 0 \\ 0 & S_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 3D: $\begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Translation: $\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$

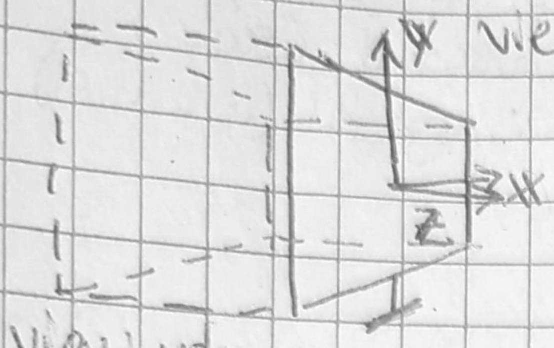
Rotation:

$\mathbb{R}_x$: $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\mathbb{R}_y$: $\begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\mathbb{R}_z$: $\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\sin\theta$ sign switches based on rotation direction

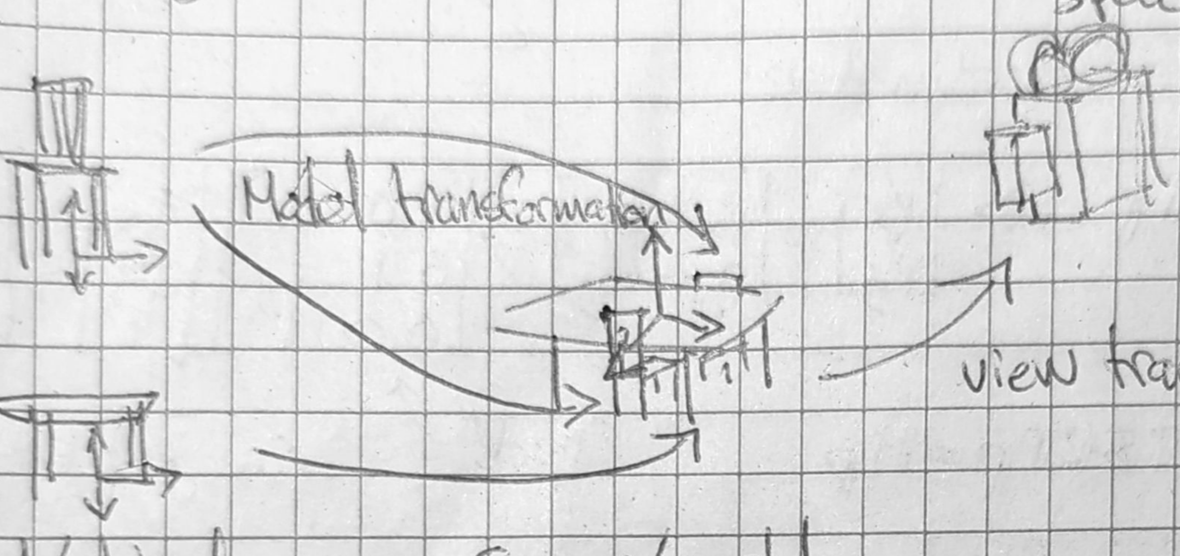
order matters, right to left and $\mathbb{R}_x\mathbb{R}_y\mathbb{R}_z \neq \mathbb{R}_y\mathbb{R}_x\mathbb{R}_z$

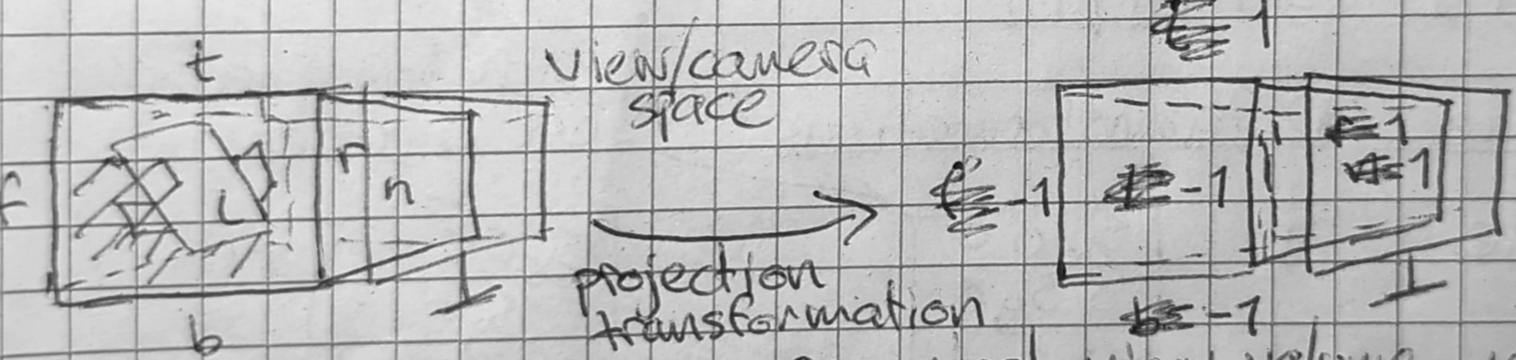
Viewing



The goal is to project a 3D-object onto the 2D-screen.

Viewing transformations:





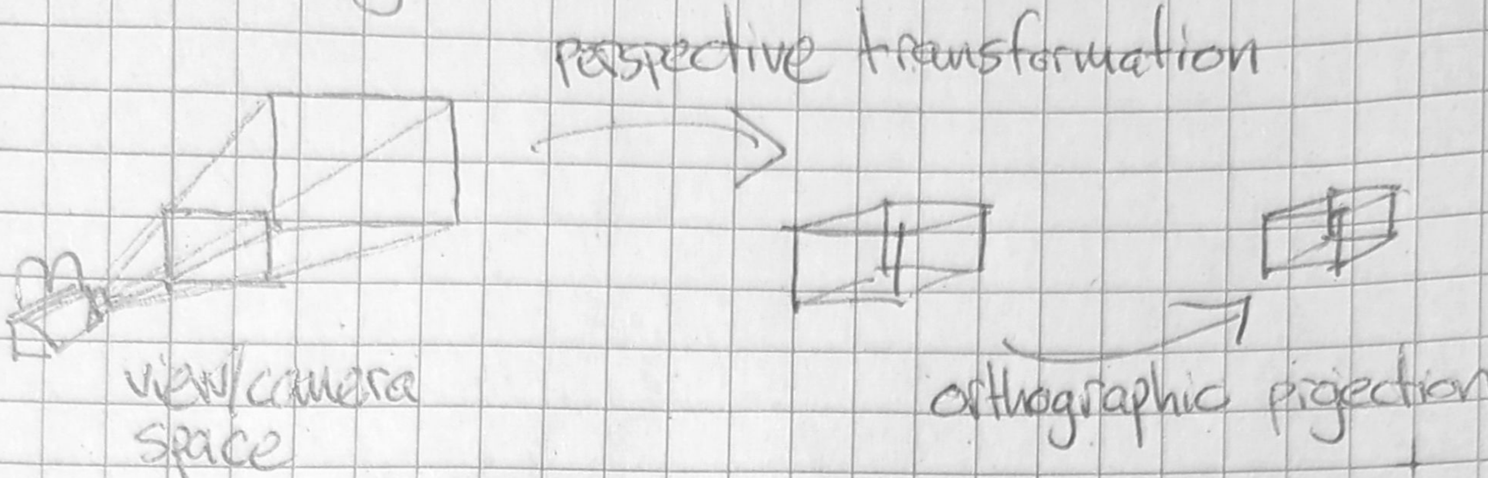
canonical view volume, not same as view/camera space, and not touching screen.

Orthographic projection:
vertices are transformed parallel from view space to canonical view volume, using matrix:

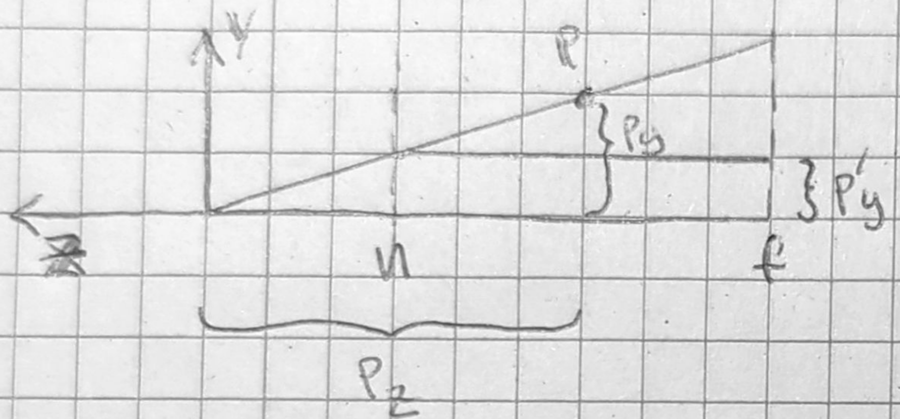
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & \frac{-2l}{r-l}-1 \\ 0 & \frac{2}{t-b} & 0 & \frac{-2b}{t-b}-1 \\ 0 & 0 & \frac{2}{f-n} & \frac{-2n}{f-n}-1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\xrightarrow{\text{simplification}} \quad -\frac{r+l}{r-l} \qquad -\frac{t+b}{t-b} \qquad -\frac{f+n}{f-n} \qquad 1$$

Perspective projection:



perspective transformation:



For all points along the line, p_y/p_z is the same, so

$$p'_y = \frac{p_y}{p_z} n, \quad p'_x = \frac{p_x}{p_z} n$$

$$\Rightarrow \begin{bmatrix} p'_x \\ p'_y \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \end{bmatrix} \frac{n}{p_z}$$

What about p_z? We need to extend the definition of homogeneous coordinates:

$$\begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \equiv \begin{bmatrix} \alpha p_x \\ \alpha p_y \\ \alpha p_z \\ \alpha \end{bmatrix} \Rightarrow \begin{bmatrix} p'_x \\ p'_y \\ p'_z \\ 1 \end{bmatrix} = \begin{bmatrix} p_x\, n/p_z \\ p_y\, n/p_z \\ ? \\ 1 \end{bmatrix} \equiv \begin{bmatrix} p_x n \\ p_y n \\ ? \\ p_z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} p'_x \\ p'_y \\ p'_z \\ 1 \end{bmatrix} = \begin{bmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -fn \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

because when $p_z = n$

$$\Rightarrow p'_z = ((n+f)p_z - fn)/p_z$$
$$= (n+f) - fn/p_z$$
$$= (n+f) - f$$
$$= n$$

and $p_z = f$

$$\Rightarrow p'_z = (n+f) - n = f$$

So they are the same at n and f, but differ in between $\Rightarrow$