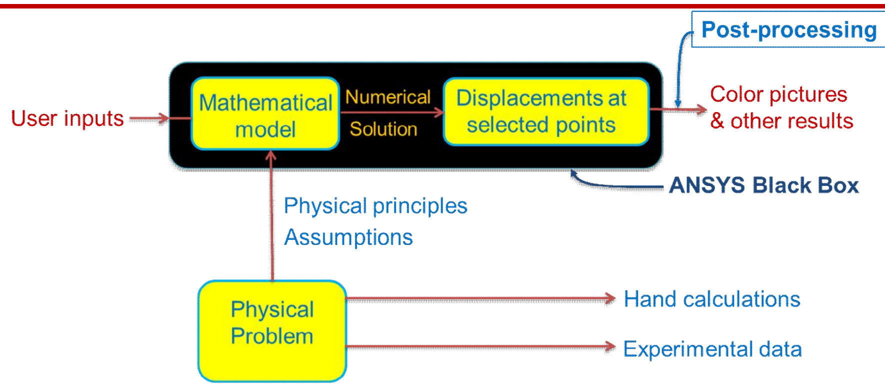
Pre-Analysis

Cornell Engineering

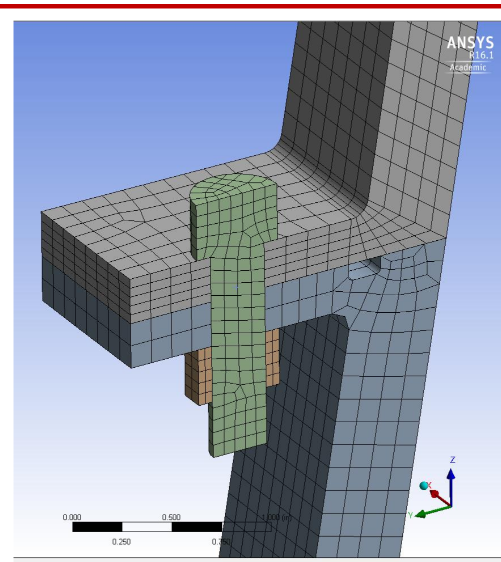


- Mathematical model: 3D Elasticity
- Numerical solution strategy: Finite-element method
- Hand-calculations of expected results/trends

Domain

Cornell Engineering

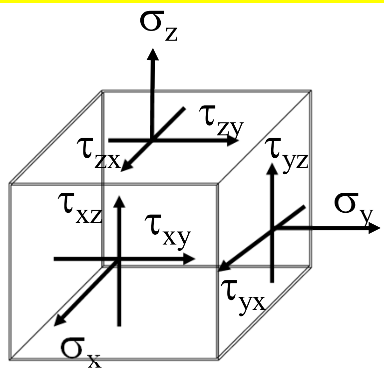
- Model half of a bolt-and-nut assuming symmetry
- Four parts
  - – Mid nozzle
  - – Lower nozzle
  - – Bolt
  - – Nut



Mathematical Model: Governing Equations

Physical principle: Equilibrium of infinitesimal element

$\vec{F} = m\,\vec{a}$ or $\Sigma\overrightarrow{F_i} = 0$



3D Differential Equations of Equilibrium

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \cancel{f_x} = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \cancel{f_y} = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \cancel{f_z} = 0$$

- 3 eqs.: Force balance in x, y, z
- 6 unknowns: $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz}$

Additional Equations: Constitutive Model

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\upsilon & \upsilon & \upsilon & 0 & 0 & 0 \\ \upsilon & 1-\upsilon & \upsilon & 0 & 0 & 0 \\ \upsilon & \upsilon & 1-\upsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\upsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\upsilon & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\upsilon \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix}$$

$$- \frac{E}{1-2\nu} \begin{bmatrix} \alpha\Delta T \\ \alpha\Delta T \\ \alpha\Delta T \\ 0 \\ 0 \\ 0 \end{bmatrix} - [Factor] \begin{bmatrix} \varepsilon_{x,bolt} \\ \varepsilon_{y,bolt} \\ \varepsilon_{z,bolt} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Mathematical Model: Additional Equations

Cornell Engineering

Strain-Displacement Relations

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \qquad \gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

Mathematical Model: Governing Equations Summary

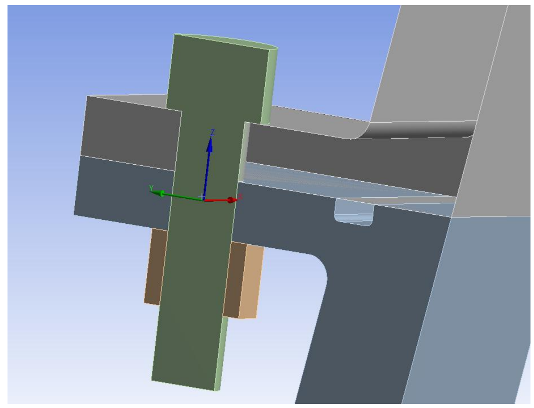
Cornell Engineering

- Equations
  - 3: Force balance on infinitesimal element in x, y, z directions
  - 6: Constitutive Model
  - 6: Strain-displacement relations
  - Total = 15
- Unknowns
  - 6 stress components: $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz}$
  - 6 strain components: $\epsilon_x, \epsilon_y, \epsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{xz}$
  - 3 displacement components: u, v, w
  - Total = 15

Mathematical Model: Boundary Conditions

Cornell Engineering

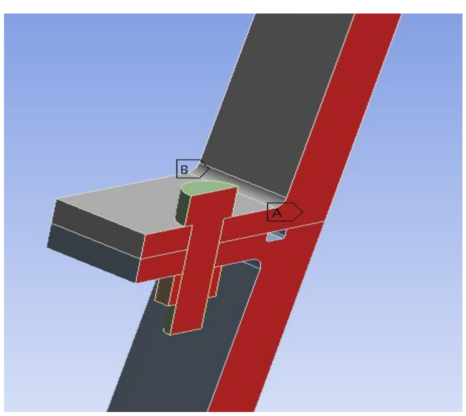
- At every point on the boundary, the traction or displacement has to be defined
  - Normal as well as 2 tangential directions

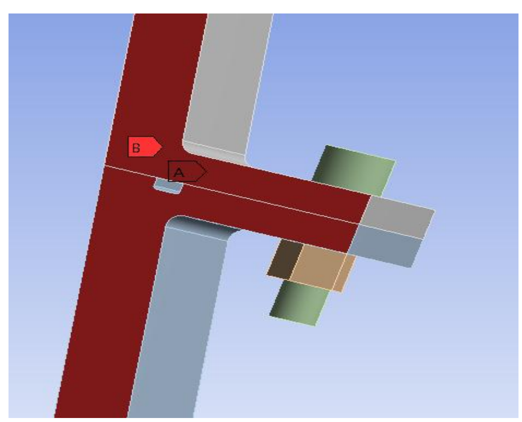


Displacement or Essential Boundary Conditions (1/2)

Cornell Engineering

- Symmetry condition from periodicity





Displacement or Essential Boundary Conditions (2/2)

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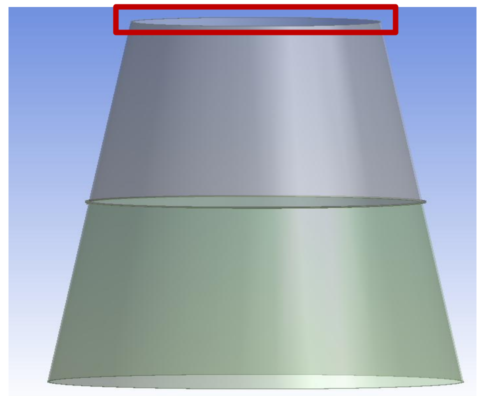
- "Frictionless support" at top surface of mid nozzle
  - Normal displacement = 0
  - Tangential traction = 0
- Approximates connection to upper nozzle (which is not included in the model)



Traction or Natural Boundary Conditions (1/2)

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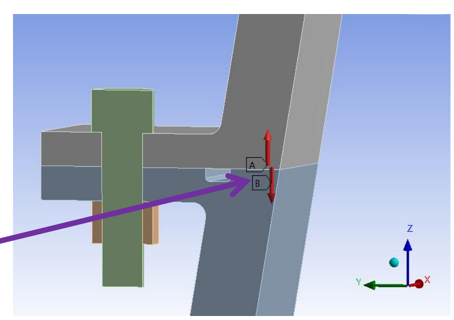
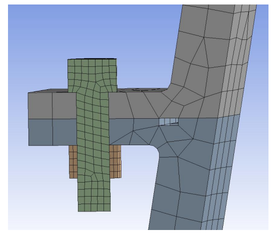
- Pressure due to propellant
  - Calculated using 1D gas dynamics
  - Varies in axial direction ("z")
- Force from regeneration channels
  - Pulls apart the mid and lower nozzles

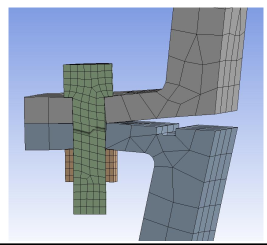


Traction or Natural Boundary Conditions (2/2)

- Traction at contact surfaces
  - It is not known a priori where the parts are going to come into contact at the interfaces
  - Traction is dependent on the displacement
    - $\vec{t} = \vec{t}(u, v, w)$
    - Highly nonlinear



Undeformed



Deformed

Numerical Solution Strategy

Mathematical Model (Coupled Boundary Value Problems) → Set of algebraic equations in nodal displacements

Piecewise polynomial approximation for u, v, w

$\{G(d)\} = \{f\}$

- System of algebraic equations are nonlinear
  - $\{G(d)\} = \{f\}$
- Compare to linear case:
  - $[K]\{d\} = \{f\}$

Source of nonlinearity:
- Contact

Newton-Rhapson Method for Solving Nonlinear Algebraic Equations

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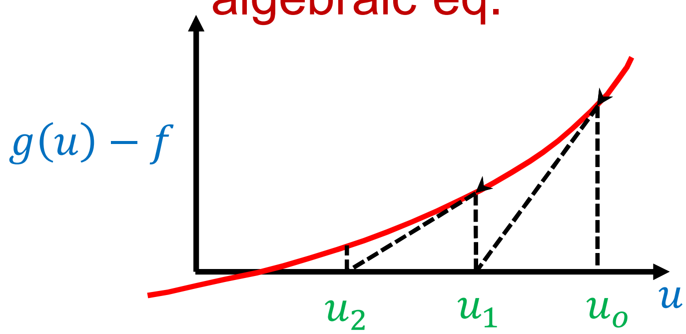
- We need to find $\{d\}$ such that
  - $\{G(d)\} - \{f\} = 0$
- Scalar analog:
  - $g(u) - f = 0$
  - Eg. $u^3 - 20 = 0$
  - Update eq.
    - $g'(u_o)\, u_1 = g'(u_o)\, u_o - g(u_o) + f$

Newton-Rhapson for single nonlinear algebraic eq.



Newton-Rhapson Method for Solving Nonlinear Algebraic Equations

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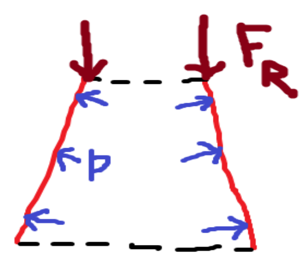
- Need to solve $\{G(d)\} - \{f\} = 0$
- Initial guess $\{d^0\}$
- Update using Newton-Rhapson to get $\{d^1\}$
  - Update eq.: $[K(d^0)]\,\{d^1\} = \{f\} + \{\bar{f}(d^0)\}$
- Calculate residual:
  - $\{G(d^1)\} - \{f\} = \{f_{residual}\}$
- If residual is larger than tolerance, update guess and repeat
  - New update eq.: $[K(d^1)]\{d^2\} = \{f\} + \{\bar{f}(d^1)\}$
  - Solve to calculate $\{d^2\}$

Hand Calculations of Expected Results

1. Reaction in axial (*z)* direction where mid nozzle is attached to top nozzle (not modeled)
2. Hoop stress $\frac{pr}{t}$
3. Thermal strain and deformation
4. Bolt preload check $\Delta l = \frac{F\ l}{E\ A}$

Hand Calculations: Reaction in Axial (*z)* Direction

- Average gas pressure:
  - $\frac{12.17+4\ 772}{2} \approx 30\, psi$
- Top radius = 41.75 inches
  Bottom radius = 69.50 inches
- Projected area in *z* direction = $\pi(6$
  $41.75^2) = 9699\ in^2$
- Net pressure force in *z* direction = $30 * 9699$
- Net reaction force in -*z* direction on 1/400th model = $\frac{30*9699}{4\ 00} \approx 720\ lbf$



Hand Calculations: Hoop Stress

- $\sigma_\theta = \frac{pr}{t}$
- At exit:
- $p = 12.17\ psi$
- $r = 69.5\ in$
- $t = 0.5\ in$
- $\Rightarrow \sigma_\theta \sim 1692\ psi$

Hand Calculations: Thermal Strain

- Thermal strain = $\alpha\ \Delta T = \alpha(700F - 70F)$
- Recall that this term appears in the constitutive model

Hand Calculations: Bolt Preload

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- $\Delta l = \frac{F\ l}{E\ A}$
- $F = 2320\ lbf$
- $A = 3.8 \times 10^{-2}\ in^2$
- $E = 2.9 \times 10^{7} psi$
- $l \sim 0.5\ in$
- $\Rightarrow \Delta l = 0.001\ in$