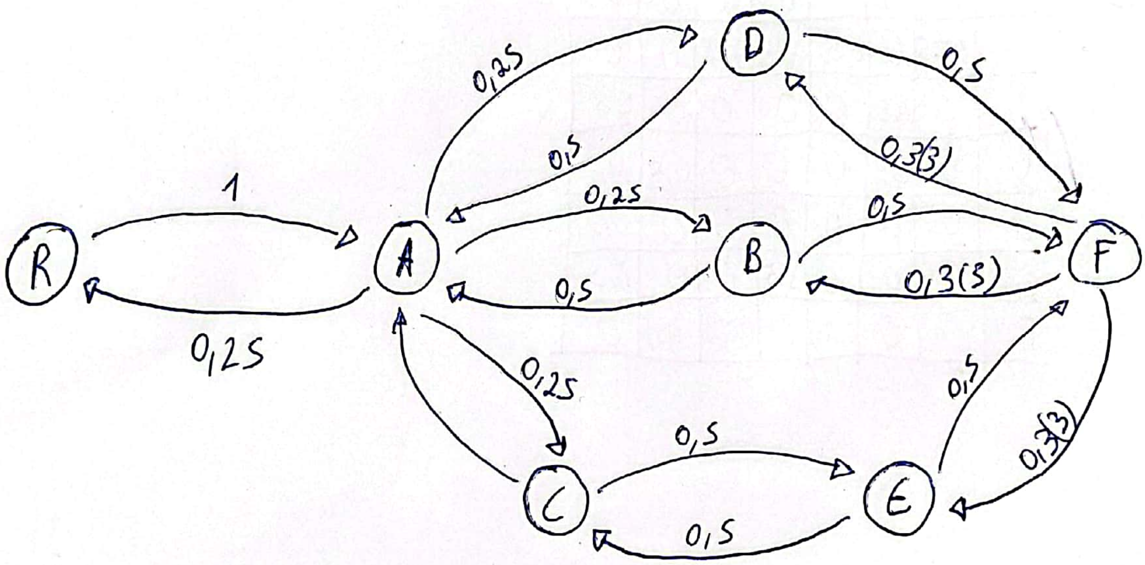
Homework 1

Exercise 1 (a)

R → Recycling plant
A → stop A
B → stop B
C → stop C
D → stop D
E → stop E
F → stop F

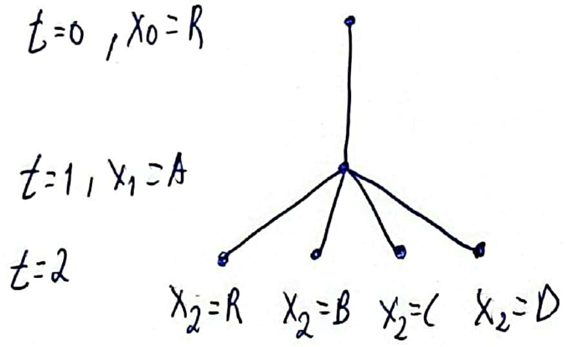
Markov Chain Model with space states and transition probabilities



Space state: $\mathcal{X} = \{R, A, B, C, D, E, F\}$

	R	A	B	C	D	E	F
R	0	1	0	0	0	0	0
A	0,25	0	0,25	0,25	0,25	0	0
B	0	0,5	0	0	0	0	0,5
C	0	0,5	0	0	0	0,5	0
D	0	0,5	0	0	0	0	0,5
E	0	0	0	0,5	0	0	0,5
F	0	0	0,3(3)	0	0,3(3)	0,3(3)	0

(b) Objective: $P[X_2 = y \mid X_0 = R]$



$\mu_0 \to t=0$, beginning of recycling plant

$\mu_0 = [1\ 0\ 0\ 0\ 0\ 0\ 0]$

$$P[\text{states}] = \begin{array}{c} \\ R \\ A \\ B \\ C \\ D \\ E \\ F \end{array} \begin{array}{c} \begin{array}{ccccccc} R & A & B & C & D & E & F \end{array} \\ \left[\begin{array}{ccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0{,}25 & 0 & 0{,}25 & 0{,}25 & 0{,}25 & 0 & 0 \\ 0 & 0{,}5 & 0 & 0 & 0 & 0 & 0{,}5 \\ 0 & 0{,}5 & 0 & 0 & 0 & 0{,}5 & 0 \\ 0 & 0{,}5 & 0 & 0 & 0 & 0 & 0{,}5 \\ 0 & 0 & 0 & 0{,}5 & 0 & 0 & 0{,}5 \\ 0 & 0 & 0{,}3(3) & 0 & 0{,}3(3) & 0{,}3(3) & 0 \end{array}\right] \end{array}$$

$P[X_1 = y \mid X_0 = R] = \mu_0 \cdot P[\text{states}]$

$$\mu_0 \cdot P[\text{states}] = \underset{R\ A\ B\ C\ D\ E\ F}{[0\ 1\ 0\ 0\ 0\ 0\ 0]} = \mu_1, \quad \text{when } t=1$$

$P[X_1 = A \mid X_0 = R] = 1$

The other cases are equal to 0.

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$P[X_2 = y \mid X_1 = A] = \mu_1 \cdot P[\text{states}] = \mu_2$, When $t = 2$

$\mu_1 \cdot P[\text{states}] = \begin{array}{c} [\,0{,}25 \quad 0 \quad 0{,}25 \quad 0{,}25 \quad 0{,}25 \quad 0\,0\,] \\ R \quad\; A \quad\; B \quad\;\; C \quad\;\; D \quad E\,F \end{array} = \mu_2$

$P[X_2 = R \mid X_1 = A] = 0{,}25$

$P[X_2 = B \mid X_1 = A] = 0{,}25$

$P[X_2 = C \mid X_1 = A] = 0{,}25$

$P[X_2 = D \mid X_1 = A] = 0{,}25$

The other cases are equal to 0.

Exercise 1

c) Let $T_y = \inf\{n \geq 1 : X_n = y\}$. Then $E(T_y | X_0 = y) = m_y$ is called the expected first return time for the state y.

In our case, we wanto to know $E(T_R | X_0 = R) = m_R$. where $R \in \mathcal{X}$.

Let's $E(X_n = y | X_0 = x) = m_{xy}$, be the expected time of starting in x and going to y, with $x, y \in \mathcal{X}$

. In our case, we want

$$m_R = 30 + m_{AR}$$

$$m_{AR} = \frac{1}{4}(30 + m_{ER}) + \frac{1}{4}(40 + m_{BR}) + \frac{1}{4}(70 + m_{DR})$$

$$+ \frac{1}{4}(55 + m_{CR}) = \frac{195}{4} + \frac{1}{4}(m_{BR} + m_{DR} + m_{CR})$$

$$m_{BR} = \frac{1}{2}(40 + m_{AR}) + \frac{1}{2}(80 + m_{FR}) = 60 + \frac{1}{2}(m_{AR} + m_{FR})$$

$$m_{DR} = \frac{1}{2}(70 + m_{AR}) + \frac{1}{2}(70 + m_{FR}) = 70 + \frac{1}{2}(m_{AR} + m_{FR})$$

$$m_{CR} = \frac{1}{2}(55 + m_{AR}) + \frac{1}{2}(55 + m_{ER}) = 55 + \frac{1}{2}(m_{AR} + m_{ER})$$

$$m_{ER} = \frac{1}{2}(55 + m_{CR}) + \frac{1}{2}(20 + m_{FR}) = \frac{75}{2} + \frac{1}{2}(m_{CR} + m_{FR})$$

$$m_{FR} = \frac{1}{3}(80 + m_{BR}) + \frac{1}{3}(70 + m_{DR}) + \frac{1}{3}(20 + m_{ER})$$

$$= \frac{170}{3} + \frac{1}{3}(m_{BR} + m_{DR} + m_{ER})$$

$$\to m_{ER} = \frac{75}{2} + \frac{1}{2}\left(55 + \frac{1}{2}(m_{AR} + m_{ER}) + m_{FR}\right)$$

$$\Leftrightarrow m_{ER} = \frac{75}{2} + \frac{55}{2} + \frac{m_{AR}}{4} + \frac{m_{ER}}{4} + \frac{m_{FR}}{4}$$

$$\Leftrightarrow \frac{3}{4} m_{ER} = \frac{130}{2} + \frac{m_{AR}}{4} + \frac{m_{FR}}{2} \Leftrightarrow m_{ER} = \frac{260}{3} + \frac{m_{AR}}{3} + \frac{2m_{FR}}{3}$$

$$= \frac{260}{3} + \frac{1}{3}(m_{AR} + 2m_{FR}$$

$$\to m_{CR} = 55 + \frac{1}{2}\left(m_{AR} + \frac{260}{3} + \frac{1}{3}(m_{AR} + 2m_{FR})\right)$$

$$= 55 + \frac{260}{6} + \frac{m_{AR}}{2} + \frac{m_{AR}}{6} + \frac{2m_{AR}}{6}$$

$$= \frac{590}{6} + \frac{2}{3} m_{AR} + \frac{m_{FR}}{3} = \frac{295}{3} + \frac{2}{3} m_{AR} + \frac{m_{FR}}{3}$$

$$\to m_{FR} = \frac{1}{3}\left(80 + 60 + \frac{1}{2}(m_{AR} + m_{FR})\right) + \frac{1}{3}\left(70 + 70 + \frac{1}{2}(m_{AR} + m_{FR})\right)$$

$$+ \frac{1}{3}\left(20 + \frac{260}{3} + \frac{1}{3}(m_{AR} + 2m_{FR})\right)$$

$$= \frac{1160}{9} + \frac{m_{AR}}{6} + \frac{m_{AR}}{6} + \frac{m_{AR}}{9} + \frac{m_{FR}}{6} + \frac{m_{FR}}{6} + \frac{2m_{FR}}{9}$$

$$= \frac{1160}{9} + \frac{4m_{AR}}{9} + \frac{5m_{FR}}{9}$$

$$m_{FR} = \frac{1160}{9} + \frac{4m_{AR}}{9} + \frac{5m_{FR}}{9} \Leftrightarrow 4m_{FR} = 1160 + 4m_{AR} \Leftrightarrow m_{FR} = 290 + m_{AR}$$

Exercício 11C (3)

$$m_{AR} = \frac{30}{4} + \frac{1}{4}\left(40+60+\frac{1}{2}\left(m_{AR}+m_{AR}+290\right)\right)$$
$$+\frac{1}{4}\left(70+70+\frac{1}{2}\left(m_{AR}+m_{AR}+290\right)\right)$$
$$+\frac{1}{4}\left(55+\frac{295}{3}+\frac{2}{3}m_{AR}+\frac{m_{AR}}{3}+\frac{290}{3}\right)$$
$$=\frac{30}{4}+\frac{100}{4}+\frac{140}{4}+\frac{55}{4}+\frac{295}{12}+\frac{290}{8}+\frac{290}{8}+\frac{290}{12}$$
$$+\frac{m_{AR}}{4}+\frac{m_{AR}}{4}+\frac{3m_{AR}}{12}$$
$$=\frac{1845}{12}+\frac{585}{12}+\frac{3m_{AR}}{4}=\frac{2430}{12}+\frac{3m_{AR}}{4}=\frac{810}{4}+\frac{3m_{AR}}{4}$$

$$\rightarrow m_{AR} = \frac{810}{4}+\frac{3m_{AR}}{4} \Leftrightarrow m_{AR} = 810$$

$$\longrightarrow m_R = 30 + m_{AR} = 840 \text{ minutes}$$

- Let's transform m_R in hours $m_R = \frac{840}{60} = 14$ ~~hours~~ hours

- Taking into account that the truck departed the recycling plant at 10am on Monday, ~~the~~ the truck is expected to return to the plant at 12pm on Tuesday.