Maximum Likelihood Estimation, Logistic Regression

Course: Data Mining

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Subject: Maximum Likelihood Estimation Problem and Solution

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Problem 1: MLE for Gaussian Parameters. Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\theta_1, \theta_2)$, where

$$\Theta = \{(\theta_1, \theta_2) : -\infty < \theta_1 < \infty,\, 0 < \theta_2 < \infty\}.$$

Here, let $\theta_1 = \mu$ and $\theta_2 = \sigma^2$. Then the likelihood function is:

$$L(\theta_1, \theta_2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} \exp\left(-\frac{(x_i - \theta_1)^2}{2\theta_2}\right).$$

Equivalently,

$$L(\theta_1, \theta_2) = (\frac{1}{\sqrt{2\pi\theta_2}})^n \exp\left(-\frac{\sum_{i=1}^{n}(x_i - \theta_1)^2}{2\theta_2}\right), \quad (\theta_1, \theta_2) \in \Theta.$$

The natural logarithm of the likelihood function is:

$$\ln L(\theta_1, \theta_2) = -\frac{n}{2}\ln(2\pi\theta_2) - \frac{\sum_{i=1}^{n}(x_i - \theta_1)^2}{2\theta_2}.$$

The partial derivatives with respect to θ_1 and θ_2 are:

$$\frac{\partial(\ln L)}{\partial\theta_1} = \frac{1}{\theta_2}\sum_{i=1}^{n}(x_i - \theta_1),$$

and

$$\frac{\partial(\ln L)}{\partial\theta_2} = -\frac{n}{2\theta_2} + \frac{1}{2\theta_2^2}\sum_{i=1}^{n}(x_i - \theta_1)^2.$$

Setting $\frac{\partial(\ln L)}{\partial\theta_1} = 0$ yields the solution $\theta_1 = \bar{x}$, where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$.
Setting $\frac{\partial(\ln L)}{\partial\theta_2} = 0$ and replacing θ_1 with $\bar{x}$, we find:

$$\theta_2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2.$$

By considering the usual condition on the second-order partial derivatives, we see that these solutions provide a maximum. Thus, the maximum likelihood estimators of $\mu = \theta_1$ and $\sigma^2 = \theta_2$ are:

$$\hat{\theta}_1 = \bar{X}, \quad \hat{\theta}_2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 = V.$$

Problem 2: MLE for Exponential Parameters. Let $X_1, X_2, \ldots, X_n$ be a random sample from the exponential distribution with pdf:

$$f(x;\theta) = \frac{1}{\theta}e^{-x/\theta}, \quad 0 < x < \infty,\, \theta \in \Theta = \{\theta : 0 < \theta < \infty\}.$$

The likelihood function is given by:

$$L(\theta) = L(\theta; x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n}\frac{1}{\theta}e^{-x_i/\theta} = \frac{1}{\theta^n}\exp\left(-\frac{\sum_{i=1}^{n} x_i}{\theta}\right), \quad 0 < \theta < \infty.$$

The natural logarithm of $L(\theta)$ is:

$$\ln L(\theta) = -n\ln(\theta) - \frac{1}{\theta}\sum_{i=1}^{n} x_i, \quad 0 < \theta < \infty.$$

The derivative of $\ln L(\theta)$ with respect to θ is:

$$\frac{d[\ln L(\theta)]}{d\theta} = -\frac{n}{\theta} + \frac{\sum_{i=1}^{n} x_i}{\theta^2} = 0.$$

The solution of this equation for θ is:

$$\theta = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}.$$

To confirm the maximum, note that:

$$\frac{d[\ln L(\theta)]}{d\theta} = \begin{cases} \frac{1}{\theta}\left(-n + \frac{n\bar{x}}{\theta}\right) > 0, & \theta < \bar{x}, \\ 0, & \theta = \bar{x}, \\ \frac{1}{\theta}\left(-n + \frac{n\bar{x}}{\theta}\right) < 0, & \theta > \bar{x}. \end{cases}$$

Hence, $\ln L(\theta)$ does have a maximum at $\bar{x}$, and it follows that the maximum likelihood estimator for θ is:

$$\hat{\theta} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$