Theoretical Computer Science

Tutorial - week 5

February 18, 2021

Innopolis University

Agenda

- Recap
- Pumping lemma
- Examples

Recap

- What is proposition?

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- What is predicate?

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- Quantifiers in predicate logic:
  - $\exists$

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- What is Pumping lemma for regular languages?

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- What is Pumping lemma for regular languages?
  - Can we use this theorem to prove that a language is regular?

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- What is Pumping lemma for regular languages?
  - Can we use this theorem to prove that a language is regular?
  - Can we use this theorem to prove that a language is not regular?

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- What is proposition?
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- Quantifiers in predicate logic:
  - $\exists$ - existential quantifier, "there exists ..., for at least one..."
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- What is Pumping lemma for regular languages?
  - Can we use this theorem to prove that a language is regular?
  - Can we use this theorem to prove that a language is not regular? How?

Pumping lemma

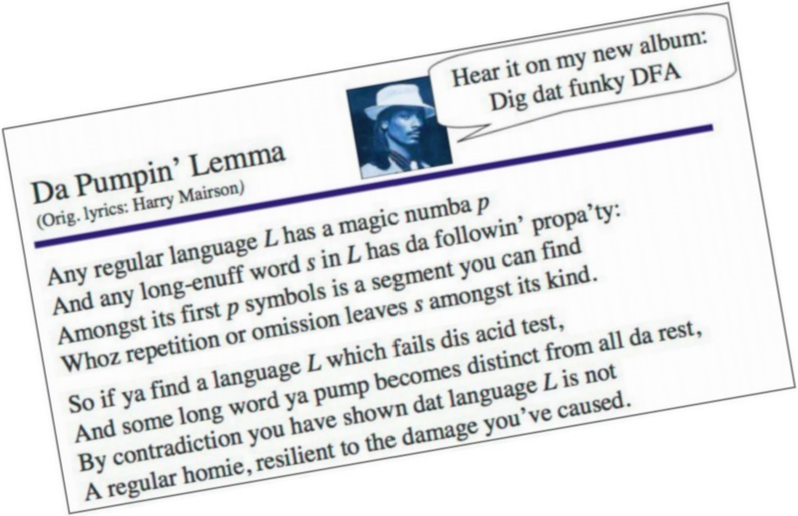
Let $L \subseteq \Sigma^*$ be a regular language. Then there exists $m \geq 1$ such that, for any $w \in L$ where $\mid w \mid \geq m$, there exist $x, y, z \in \Sigma^*$ with $\mid y \mid \geq 1$ and $\mid xy \mid \leq m$ such that $w = xyz$ and, for any $i \geq 0$, we have $xy^i z \in L$.

Pumping lemma: formally

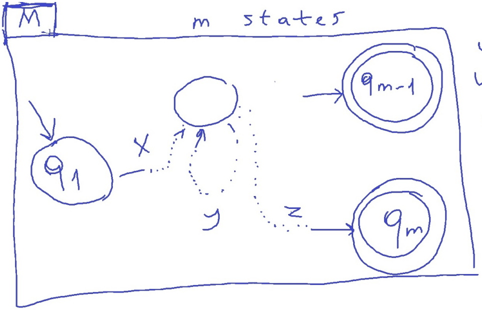
$$
\begin{array}{l}
\forall L \subseteq \Sigma^* \bullet \mathit{regular}(L) \implies \\
\quad (\exists m \in \mathbb{N} \bullet m \geq 1 \wedge \\
\quad\quad (\forall w \in L \bullet \mid w \mid \geq m \implies \\
\quad\quad\quad (\exists x, y, z \in \Sigma^* \bullet w = xyz \wedge \mid y \mid \geq 1 \wedge \mid xy \mid \leq m \wedge \\
\quad\quad\quad\quad (\forall i \geq 0 \bullet xy^i z \in L))))
\end{array}
$$

Pumping lemma: intuition

Pumping lemma: intuition :-)



Pumping lemma: intuition[1]



$w \in L(M)$

$w = w_1 \ldots\ldots w_m w_{m+1} \ldots$

$|w| \geq m$

$\Downarrow$

some state(s) is(are) repeated

$xz \in L$

$xyz \in L$ where $|y| \geq 1$

$\ldots\ldots$

$xy^iz \in L$ for all $i \geq 0$

[1]A tool JFLAP is used

How Pumping lemma is useful?

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  No, because it gives only a necessary condition for a language to be regular (and not a sufficient condition).

- **We can use it to prove that a language is not regular. How?**

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We can use it to prove that a language is not regular. How?

Proof by contrapositive

$R \implies P$

$\neg P \implies \neg R$

Pumping lemma: formally

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\qquad (\forall w \in L \bullet \mid w \mid \geq m \implies \\
\qquad\quad (\exists x, y, z \in \Sigma^* \bullet w = xyz \wedge (\mid y \mid \geq 1 \wedge \mid xy \mid \leq m \wedge \\
\qquad\qquad (\forall i \geq 0 \bullet xy^i z \in L))))
\end{array}$$

Example 1

Let's consider language L_1

$$L_1 = \{a^n b^m \mid n \leq m\}$$

Is L_1 a regular language?

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$$\begin{array}{l}
\mathit{regular}(L_1) \implies \\
\quad (\exists m \in \mathbb{N} \bullet m \geq 1 \wedge \\
\quad\quad (\forall w \in L_1 \bullet \mid w \mid \geq m \implies \\
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Is L_1 a regular language?

Which is equivalent to ...

$$\begin{array}{l}\neg(\exists m \in \mathbb{N} \bullet m \geq 1 \wedge \\ \quad (\forall w \in L_1 \bullet \mid w \mid \geq m \implies \\ \quad\quad (\exists x, y, z \in \Sigma^* \bullet w = xyz \wedge \mid y \mid \geq 1 \wedge \mid xy \mid \leq m \wedge \\ \quad\quad\quad (\forall i \geq 0 \bullet xy^i z \in L_1)))) \implies \neg regular(L_1)\end{array}$$

Negation

The negation of a universal quantifier:

$$\neg(\forall x \bullet P(x)) \text{ is logically equivalent to } \exists x \bullet \neg P(x)$$

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De Morgan's law:

$$\neg(P \wedge Q) \text{ is logically equivalent to } \neg P \vee \neg Q$$
$$\neg(P \vee Q) \text{ is logically equivalent to } \neg P \wedge \neg Q$$

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&\quad \neg(\forall w \in L_1 \bullet \mid w \mid \geq m \implies \\
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&\quad\quad\quad (\forall i \geq 0 \bullet xy^i z \notin L_1)))) \implies \neg regular(L_1)
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Negation of an Implication

The negation of an implication is a conjunction:

$$\neg(P \implies Q) \text{ is logically equivalent to } P \wedge \neg Q$$

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&\quad (\exists w \in L_1 \bullet \mid w \mid \geq m \wedge \\
&\quad\quad (\forall x, y, z \in \Sigma^* \bullet (w \neq xyz) \vee (\mid y \mid < 1) \vee (\mid xy \mid > m) \vee \\
&\quad\quad\quad (\exists i \geq 0 \bullet xy^i z \notin L_1)))) \implies \neg regular(L_1)
\end{aligned}$$

Disjunction elimination

But before eliminating $\neg$, let us eliminate $\vee$'s

$$P \vee Q \text{ is logically equivalent to } \neg P \implies Q$$

Or, more generally:

$$Q_1 \vee \cdots \vee Q_{n-1} \vee Q_n$$

is logically equivalent to

$$\neg Q_1 \implies (\cdots \implies (\neg Q_{n-1} \implies Q_n) \ldots)$$

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$L_1 = \{a^n b^k \mid n \leq k\}$

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Let $m \in \mathbb{N}$. We set $w = a^m b^m$; notice that $w \in L_1$ and $|w| = 2m$ which is $|w| > m$.

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By applying the Pumping lemma for regular languages, we can conclude that that the language L_1 is not regular.

Wrap up

- What have you learnt today?

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- What have you learnt today?
- What for this could be useful?