Essentials of Analytical Geometry and Linear Algebra 1

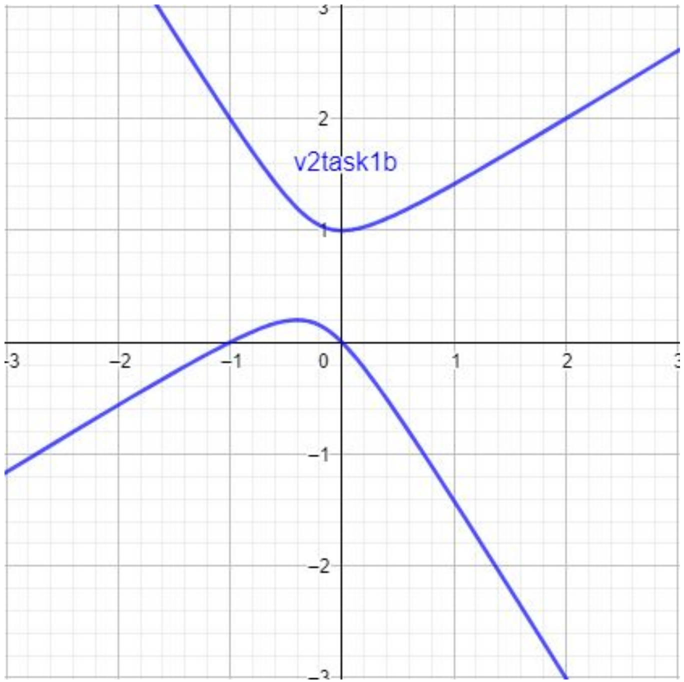
Polar coordinate

1B from midterm

Find the equations of directrices, the length of the latus rectum and coordinates of focus (or foci) of the following curves:

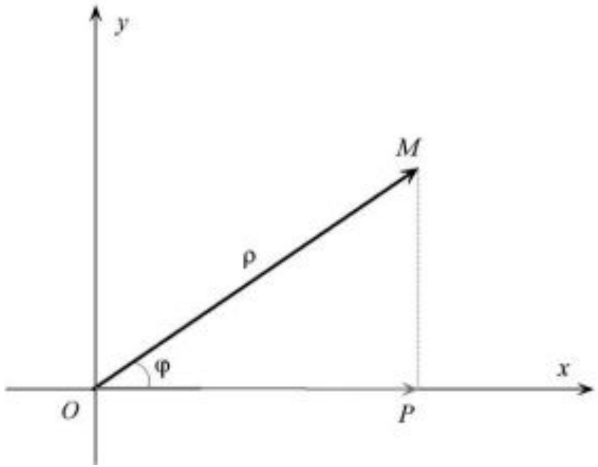
(a) $\frac{x^2}{72} - \frac{y^2}{8} = 2$

(b) $x^2 - xy - y^2 + x + y = 0$

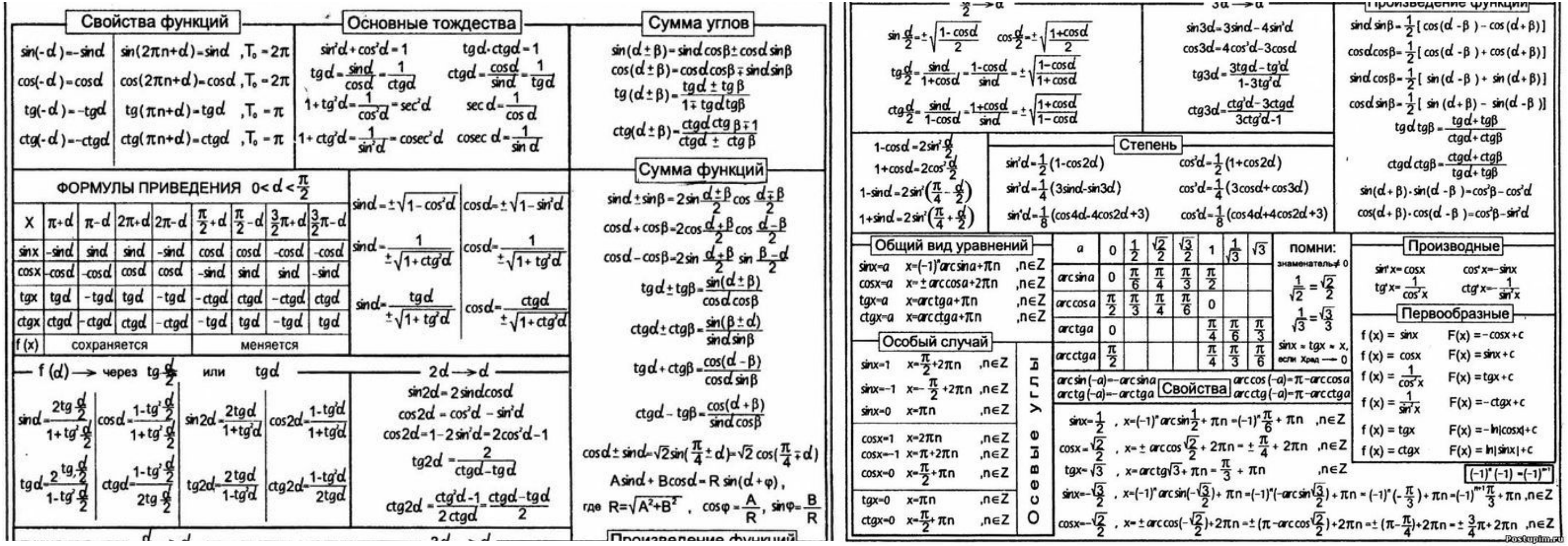


Polar coordinates (1)

$$\begin{cases} x = \rho\cos\varphi, \\ y = \rho\sin\varphi, \end{cases} \quad \begin{cases} \rho = \sqrt{x^2 + y^2}; \\ \cos\varphi = \dfrac{x}{\sqrt{x^2+y^2}};\ \sin\varphi = \dfrac{y}{\sqrt{x^2+y^2}}. \end{cases}$$



Trigonometric formulas



Свойства функций

$\sin(-\alpha)=-\sin\alpha$; $\sin(2\pi n+\alpha)=\sin\alpha$, $T_0=2\pi$

$\cos(-\alpha)=\cos\alpha$; $\cos(2\pi n+\alpha)=\cos\alpha$, $T_0=2\pi$

$\mathrm{tg}(-\alpha)=-\mathrm{tg}\,\alpha$; $\mathrm{tg}(\pi n+\alpha)=\mathrm{tg}\,\alpha$, $T_0=\pi$

$\mathrm{ctg}(-\alpha)=-\mathrm{ctg}\,\alpha$; $\mathrm{ctg}(\pi n+\alpha)=\mathrm{ctg}\,\alpha$, $T_0=\pi$

Основные тождества

$\sin^2\alpha+\cos^2\alpha=1$; $\mathrm{tg}\,\alpha\cdot\mathrm{ctg}\,\alpha=1$

$\mathrm{tg}\,\alpha=\frac{\sin\alpha}{\cos\alpha}=\frac{1}{\mathrm{ctg}\,\alpha}$; $\mathrm{ctg}\,\alpha=\frac{\cos\alpha}{\sin\alpha}=\frac{1}{\mathrm{tg}\,\alpha}$

$1+\mathrm{tg}^2\alpha=\frac{1}{\cos^2\alpha}=\sec^2\alpha$; $\sec\alpha=\frac{1}{\cos\alpha}$

$1+\mathrm{ctg}^2\alpha=\frac{1}{\sin^2\alpha}=\mathrm{cosec}^2\alpha$; $\mathrm{cosec}\,\alpha=\frac{1}{\sin\alpha}$

Сумма углов

$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$

$\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$

$\mathrm{tg}(\alpha\pm\beta)=\frac{\mathrm{tg}\,\alpha\pm\mathrm{tg}\,\beta}{1\mp\mathrm{tg}\,\alpha\,\mathrm{tg}\,\beta}$

$\mathrm{ctg}(\alpha\pm\beta)=\frac{\mathrm{ctg}\,\alpha\,\mathrm{ctg}\,\beta\mp1}{\mathrm{ctg}\,\alpha\pm\mathrm{ctg}\,\beta}$

ФОРМУЛЫ ПРИВЕДЕНИЯ $0<\alpha<\frac{\pi}{2}$

x	$\pi+\alpha$	$\pi-\alpha$	$2\pi+\alpha$	$2\pi-\alpha$	$\frac{\pi}{2}+\alpha$	$\frac{\pi}{2}-\alpha$	$\frac{3}{2}\pi+\alpha$	$\frac{3}{2}\pi-\alpha$
sinx	$-\sin\alpha$	$\sin\alpha$	$\sin\alpha$	$-\sin\alpha$	$\cos\alpha$	$\cos\alpha$	$-\cos\alpha$	$-\cos\alpha$
cosx	$-\cos\alpha$	$-\cos\alpha$	$\cos\alpha$	$\cos\alpha$	$-\sin\alpha$	$\sin\alpha$	$\sin\alpha$	$-\sin\alpha$
tgx	$\mathrm{tg}\,\alpha$	$-\mathrm{tg}\,\alpha$	$\mathrm{tg}\,\alpha$	$-\mathrm{tg}\,\alpha$	$-\mathrm{ctg}\,\alpha$	$\mathrm{ctg}\,\alpha$	$-\mathrm{ctg}\,\alpha$	$\mathrm{ctg}\,\alpha$
ctgx	$\mathrm{ctg}\,\alpha$	$-\mathrm{ctg}\,\alpha$	$\mathrm{ctg}\,\alpha$	$-\mathrm{ctg}\,\alpha$	$-\mathrm{tg}\,\alpha$	$\mathrm{tg}\,\alpha$	$-\mathrm{tg}\,\alpha$	$\mathrm{tg}\,\alpha$
f (x)	сохраняется				меняется			

$\sin\alpha=\pm\sqrt{1-\cos^2\alpha}$; $\cos\alpha=\pm\sqrt{1-\sin^2\alpha}$

$\sin\alpha=\frac{1}{\pm\sqrt{1+\mathrm{ctg}^2\alpha}}$; $\cos\alpha=\frac{1}{\pm\sqrt{1+\mathrm{tg}^2\alpha}}$

$\sin\alpha=\frac{\mathrm{tg}\,\alpha}{\pm\sqrt{1+\mathrm{tg}^2\alpha}}$; $\cos\alpha=\frac{\mathrm{ctg}\,\alpha}{\pm\sqrt{1+\mathrm{ctg}^2\alpha}}$

Сумма функций

$\sin\alpha\pm\sin\beta=2\sin\frac{\alpha\pm\beta}{2}\cos\frac{\alpha\mp\beta}{2}$

$\cos\alpha+\cos\beta=2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$\cos\alpha-\cos\beta=2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}$

$\mathrm{tg}\,\alpha\pm\mathrm{tg}\,\beta=\frac{\sin(\alpha\pm\beta)}{\cos\alpha\cos\beta}$

$\mathrm{ctg}\,\alpha\pm\mathrm{ctg}\,\beta=\frac{\sin(\beta\pm\alpha)}{\sin\alpha\sin\beta}$

$\mathrm{tg}\,\alpha+\mathrm{ctg}\,\beta=\frac{\cos(\alpha-\beta)}{\cos\alpha\sin\beta}$

$\mathrm{ctg}\,\alpha-\mathrm{tg}\,\beta=\frac{\cos(\alpha+\beta)}{\sin\alpha\cos\beta}$

$\cos\alpha\pm\sin\alpha=\sqrt{2}\sin(\frac{\pi}{4}\pm\alpha)=\sqrt{2}\cos(\frac{\pi}{4}\mp\alpha)$

$A\sin\alpha+B\cos\alpha=R\sin(\alpha+\varphi)$, где $R=\sqrt{A^2+B^2}$, $\cos\varphi=\frac{A}{R}$, $\sin\varphi=\frac{B}{R}$

f (α) → через $\mathrm{tg}\frac{\alpha}{2}$ или $\mathrm{tg}\,\alpha$

$\sin\alpha=\frac{2\mathrm{tg}\frac{\alpha}{2}}{1+\mathrm{tg}^2\frac{\alpha}{2}}$; $\cos\alpha=\frac{1-\mathrm{tg}^2\frac{\alpha}{2}}{1+\mathrm{tg}^2\frac{\alpha}{2}}$; $\sin2\alpha=\frac{2\mathrm{tg}\,\alpha}{1+\mathrm{tg}^2\alpha}$; $\cos2\alpha=\frac{1-\mathrm{tg}^2\alpha}{1+\mathrm{tg}^2\alpha}$

$\mathrm{tg}\,\alpha=\frac{2\mathrm{tg}\frac{\alpha}{2}}{1-\mathrm{tg}^2\frac{\alpha}{2}}$; $\mathrm{ctg}\,\alpha=\frac{1-\mathrm{tg}^2\frac{\alpha}{2}}{2\mathrm{tg}\frac{\alpha}{2}}$; $\mathrm{tg}\,2\alpha=\frac{2\mathrm{tg}\,\alpha}{1-\mathrm{tg}^2\alpha}$; $\mathrm{ctg}\,2\alpha=\frac{1-\mathrm{tg}^2\alpha}{2\mathrm{tg}\,\alpha}$

2α → α

$\sin2\alpha=2\sin\alpha\cos\alpha$

$\cos2\alpha=\cos^2\alpha-\sin^2\alpha$

$\cos2\alpha=1-2\sin^2\alpha=2\cos^2\alpha-1$

$\mathrm{tg}\,2\alpha=\frac{2}{\mathrm{ctg}\,\alpha-\mathrm{tg}\,\alpha}$

$\mathrm{ctg}\,2\alpha=\frac{\mathrm{ctg}^2\alpha-1}{2\,\mathrm{ctg}\,\alpha}=\frac{\mathrm{ctg}\,\alpha-\mathrm{tg}\,\alpha}{2}$

$\frac{\alpha}{2}\to\alpha$

$\sin\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{2}}$; $\cos\frac{\alpha}{2}=\pm\sqrt{\frac{1+\cos\alpha}{2}}$

$\mathrm{tg}\frac{\alpha}{2}=\frac{\sin\alpha}{1+\cos\alpha}=\frac{1-\cos\alpha}{\sin\alpha}=\pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$

$\mathrm{ctg}\frac{\alpha}{2}=\frac{\sin\alpha}{1-\cos\alpha}=\frac{1+\cos\alpha}{\sin\alpha}=\pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}$

3α → α

$\sin3\alpha=3\sin\alpha-4\sin^3\alpha$

$\cos3\alpha=4\cos^3\alpha-3\cos\alpha$

$\mathrm{tg}\,3\alpha=\frac{3\mathrm{tg}\,\alpha-\mathrm{tg}^3\alpha}{1-3\mathrm{tg}^2\alpha}$

$\mathrm{ctg}\,3\alpha=\frac{\mathrm{ctg}^3\alpha-3\mathrm{ctg}\,\alpha}{3\mathrm{ctg}^2\alpha-1}$

Произведение функций

$\sin\alpha\sin\beta=\frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$

$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$

$\sin\alpha\cos\beta=\frac{1}{2}[\sin(\alpha-\beta)+\sin(\alpha+\beta)]$

$\cos\alpha\sin\beta=\frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$

$\mathrm{tg}\,\alpha\,\mathrm{tg}\,\beta=\frac{\mathrm{tg}\,\alpha+\mathrm{tg}\,\beta}{\mathrm{ctg}\,\alpha+\mathrm{ctg}\,\beta}$

$\mathrm{ctg}\,\alpha\,\mathrm{ctg}\,\beta=\frac{\mathrm{ctg}\,\alpha+\mathrm{ctg}\,\beta}{\mathrm{tg}\,\alpha+\mathrm{tg}\,\beta}$

$\sin(\alpha+\beta)\cdot\sin(\alpha-\beta)=\cos^2\beta-\cos^2\alpha$

$\cos(\alpha+\beta)\cdot\cos(\alpha-\beta)=\cos^2\beta-\sin^2\alpha$

$1-\cos\alpha=2\sin^2\frac{\alpha}{2}$

$1+\cos\alpha=2\cos^2\frac{\alpha}{2}$

$1-\sin\alpha=2\sin^2(\frac{\pi}{4}-\frac{\alpha}{2})$

$1+\sin\alpha=2\sin^2(\frac{\pi}{4}+\frac{\alpha}{2})$

Степень

$\sin^2\alpha=\frac{1}{2}(1-\cos2\alpha)$; $\cos^2\alpha=\frac{1}{2}(1+\cos2\alpha)$

$\sin^3\alpha=\frac{1}{4}(3\sin\alpha-\sin3\alpha)$; $\cos^3\alpha=\frac{1}{4}(3\cos\alpha+\cos3\alpha)$

$\sin^4\alpha=\frac{1}{8}(\cos4\alpha-4\cos2\alpha+3)$; $\cos^4\alpha=\frac{1}{8}(\cos4\alpha+4\cos2\alpha+3)$

Общий вид уравнений

$\sin x=a$ $x=(-1)^n\arcsin a+\pi n$, $n\in Z$

$\cos x=a$ $x=\pm\arccos a+2\pi n$, $n\in Z$

$\mathrm{tg}\,x=a$ $x=\mathrm{arctg}\,a+\pi n$, $n\in Z$

$\mathrm{ctg}\,x=a$ $x=\mathrm{arcctg}\,a+\pi n$, $n\in Z$

a	0	$\frac{1}{2}$	$\frac{\sqrt2}{2}$	$\frac{\sqrt3}{2}$	1	$\frac{1}{\sqrt3}$	$\sqrt3$
arcsin a	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$		
arccos a	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$	0		
arctg a	0				$\frac{\pi}{4}$	$\frac{\pi}{6}$	$\frac{\pi}{3}$
arcctg a	$\frac{\pi}{2}$				$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{6}$

помни:
знаменатель $\neq 0$

$\frac{1}{\sqrt2}=\frac{\sqrt2}{2}$; $\frac{1}{\sqrt3}=\frac{\sqrt3}{3}$

$\sin x\approx\mathrm{tg}\,x\approx x$, если $x_{рад}\to0$

Свойства

$\arcsin(-a)=-\arcsin a$; $\arccos(-a)=\pi-\arccos a$

$\mathrm{arctg}(-a)=-\mathrm{arctg}\,a$; $\mathrm{arcctg}(-a)=\pi-\mathrm{arcctg}\,a$

Особый случай (Осевые углы)

$\sin x=1$ $x=\frac{\pi}{2}+2\pi n$, $n\in Z$

$\sin x=-1$ $x=-\frac{\pi}{2}+2\pi n$, $n\in Z$

$\sin x=0$ $x=\pi n$, $n\in Z$

$\cos x=1$ $x=2\pi n$, $n\in Z$

$\cos x=-1$ $x=\pi+2\pi n$, $n\in Z$

$\cos x=0$ $x=\frac{\pi}{2}+\pi n$, $n\in Z$

$\mathrm{tg}\,x=0$ $x=\pi n$, $n\in Z$

$\mathrm{ctg}\,x=0$ $x=\frac{\pi}{2}+\pi n$, $n\in Z$

$\sin x=\frac{1}{2}$, $x=(-1)^n\arcsin\frac{1}{2}+\pi n=(-1)^n\frac{\pi}{6}+\pi n$, $n\in Z$

$\cos x=\frac{\sqrt2}{2}$, $x=\pm\arccos\frac{\sqrt2}{2}+2\pi n=\pm\frac{\pi}{4}+2\pi n$, $n\in Z$

$\mathrm{tg}\,x=\sqrt3$, $x=\mathrm{arctg}\sqrt3+\pi n=\frac{\pi}{3}+\pi n$, $n\in Z$

$\sin x=-\frac{\sqrt3}{2}$, $x=(-1)^n\arcsin(-\frac{\sqrt3}{2})+\pi n=(-1)^n(-\arcsin\frac{\sqrt3}{2})+\pi n=(-1)^n(-\frac{\pi}{3})+\pi n=(-1)^{n+1}\frac{\pi}{3}+\pi n$, $n\in Z$

$\cos x=-\frac{\sqrt2}{2}$, $x=\pm\arccos(-\frac{\sqrt2}{2})+2\pi n=\pm(\pi-\arccos\frac{\sqrt2}{2})+2\pi n=\pm(\pi-\frac{\pi}{4})+2\pi n=\pm\frac{3}{4}\pi+2\pi n$, $n\in Z$

$\boxed{(-1)^n(-1)=(-1)^{n+1}}$

Производные

$\sin'x=\cos x$; $\cos'x=-\sin x$

$\mathrm{tg}'x=\frac{1}{\cos^2x}$; $\mathrm{ctg}'x=-\frac{1}{\sin^2x}$

Первообразные

$f(x)=\sin x$ $F(x)=-\cos x+c$

$f(x)=\cos x$ $F(x)=\sin x+c$

$f(x)=\frac{1}{\cos^2x}$ $F(x)=\mathrm{tg}\,x+c$

$f(x)=\frac{1}{\sin^2x}$ $F(x)=-\mathrm{ctg}\,x+c$

$f(x)=\mathrm{tg}\,x$ $F(x)=-\ln|\cos x|+c$

$f(x)=\mathrm{ctg}\,x$ $F(x)=\ln|\sin x|+c$

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Polar coordinates (2): straight line

The general equation of a straight line in Cartesian coordinates is *Ax* + *By* + *C* = 0, where *A*, *B* and *C* are constants. Let (r, θ) be polar

$$A(r\cos\theta) + B(r\sin\theta) + C = 0$$

$$\text{(i.e.)} \quad A\cos\theta + B\sin\theta + C/r = 0$$

The equation of the line joining the points (r_1, θ_1) and (r_2, θ_2) is

$$\frac{1}{r}\sin(\theta_2 - \theta_1) = \frac{1}{r_1}\sin(\theta_2 - \theta) + \frac{1}{r_2}\sin(\theta - \theta_1).$$

Note 9.5.1: Polar equation of the straight line perpendicular to $A\cos\theta + B\sin\theta = \frac{l}{r}$ is of the form $A\cos\left(\theta + \frac{\pi}{2}\right) + B\sin\left(\theta + \frac{\pi}{2}\right) = k\frac{l}{r}$.

$$\text{(i.e.)} \quad -A\sin\theta + B\cos\theta = \frac{kl}{r}$$

Note 9.5.2: The polar equation of the straight line parallel to $A\cos\theta + B\sin\theta = \frac{l}{r}$ is $A\cos\theta + B\sin\theta = \frac{kl}{r}$, where *k* is a constant.

Task 4

Find the equation of the line joining the points $\left(2, \frac{\pi}{3}\right)$ and $\left(3, \frac{\pi}{6}\right)$ and deduce that this line also passes through the point $\left(\frac{6}{3\sqrt{3}-2}, \frac{\pi}{2}\right)$.

Task 4 (solution)

The equation of the line joining the points (r_1, θ_1) and (r_2, θ_2) is

$$\frac{1}{r}\sin(\theta_2-\theta_1)=\frac{1}{r_1}\sin(\theta_2-\theta)+\frac{1}{r_2}\sin(\theta-\theta_1).$$

Therefore, the equation of the line joining the points $\left(2,\frac{\pi}{3}\right)$ and $\left(3,\frac{\pi}{6}\right)$ is

$$\frac{1}{r}\sin\left(\frac{\pi}{6}-\frac{\pi}{3}\right)=\frac{1}{2}\sin\left(\frac{\pi}{6}-\theta\right)+\frac{1}{3}\sin\left(\theta-\frac{\pi}{3}\right).$$

$$-\frac{1}{r}\sin\left(\frac{\pi}{6}\right)=\frac{-3\sin\left(\theta-\frac{\pi}{6}\right)+2\sin\left(\theta-\frac{\pi}{3}\right)}{6}$$

when $\theta=\frac{\pi}{2}$

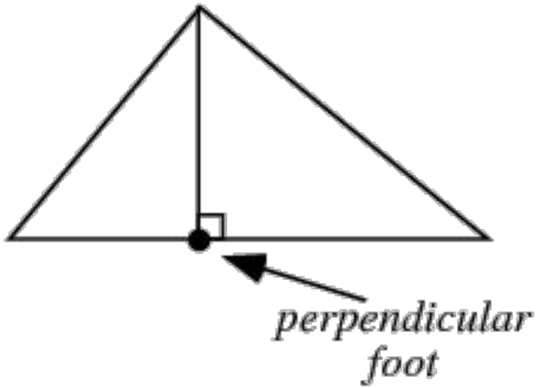
$$\therefore \frac{1}{r}=\frac{-3\cos\frac{\pi}{6}+2\cos\frac{\pi}{3}}{-6\sin\frac{\pi}{6}}=\frac{-3\frac{\sqrt{3}}{2}+2\frac{1}{2}}{-6\times\frac{1}{2}}=\frac{3\sqrt{3}-2}{6}$$

$$r=\frac{6}{3\sqrt{3}-2}$$

Hence, the point $\left(\frac{6}{3\sqrt{3}-2},\frac{\pi}{2}\right)$ lies on the straight line.

Task 5

Show that the feet of the perpendiculars from the origin on the sides of the triangle formed by the points with vectorial angles α, β, γ and which lie on the circle $r = 2a \cos \theta$ lie on the straight line $2a \cos \alpha \cos \beta \cos \gamma = r \cos (\pi - \alpha - \beta - \gamma)$.



Task 5 (solution)

The equation of the circle is $r = 2a \cos\theta$.
Let the vectorial angles of P, Q, R be α, β, γ respectively.

The equations of the chord PQ, QR and RP are

$$2a\cos\alpha\cos\beta = r\cos(\theta - \overline{\alpha+\beta})$$
$$2a\cos\beta\cos\gamma = r\cos(\theta - \overline{\beta+\gamma})$$
$$2a\cos\gamma\cos\alpha = r\cos(\theta - \overline{\gamma+\alpha})$$

Let L, M and N be the feet of the perpendiculars from O on the lines PQ, QR and RP

Then from the above equations, we infer that the coordinates of L, M and N are

$$(2a\cos\alpha\cos\beta, \alpha+\beta)$$
$$(2a\cos\beta\cos\gamma, \beta+\gamma)$$
$$(2a\cos\gamma\cos\alpha, \gamma+\alpha)$$

These three points satisfy the equation

$$2a\cos\alpha\cos\beta\cos\gamma = r\cos(\theta - \alpha - \beta - \gamma)$$

Hence L, M and N lies on the above line.

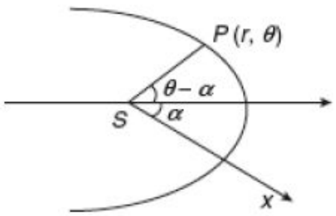
Polar coordinates: conic sections

Note 9.7.1.1: If the axis of the conic is inclined at an angle α to the

initial line then the polar equation of conic is $\frac{l}{r} = 1 + e\cos(\theta - \alpha)$.

However, the equation of tangent is given as

$$\frac{l}{r} = A\cos\theta + B\sin\theta$$



To trace the conic, $\frac{l}{r} = 1 + e\cos\theta$.

polar equation of the directrix is $\frac{l}{r} = e\cos\theta$.

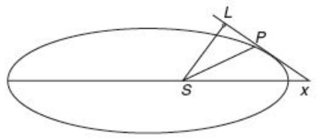
Task 6

A focal chord SP of an ellipse is inclined at an angle α to the major axis. Prove that the perpendicular from the focus on the tangent at P makes with the axis an angle $\tan^{-1}\left(\frac{\sin\alpha}{e+\cos\alpha}\right)$.

Task 6 (solution)

Let the equation of the conic be

$$\frac{l}{r} = 1 + e\cos\theta \tag{9.57}$$



The equation of tangent at *P* is

$$\frac{l}{r} = 1 + e\cos\theta + \cos(\theta - \alpha) \tag{9.58}$$

The equation of the perpendicular line to the tangent at *P* is

$$\frac{k}{r} = e\cos\left(\theta + \frac{\pi}{2}\right) + \cos\left(\theta + \frac{\pi}{2} - \alpha\right)$$

$$\text{(i.e.)} \quad \frac{k}{r} = -e\sin\theta - \sin(\theta - \alpha)$$

If the perpendicular passes through the focus then $k = 0$

$$-e\sin\theta - \sin(\theta - \alpha) = 0$$

$$\text{(i.e.)} \quad e\sin\theta + \sin\theta\cos\alpha - \cos\theta\sin\alpha = 0$$

$$\tan\theta = \frac{\sin\alpha}{e + \cos\alpha}$$

$$\text{or} \quad \theta = \tan^{-1}\left(\frac{\sin\alpha}{e + \cos\alpha}\right)$$

Deserve “A” grade!

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Room 105 (Underground robotics lab)