Essentials of Analytical Geometry and Linear Algebra. Lecture 13.

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Main questions for today's lecture

Parametrization of surfaces. Normals to surfaces

Lecture 13. Outline

- Part 1. Recap. Parametric equations
- Part 2. Orientation of surfaces
- Part 3. Summary

Recap. Parametric Equation of a line in 3D

Equation of a line: $\begin{cases} x(t) = x_0 + q_x t \\ y(t) = y_0 + q_y t \\ z(t) = z_0 + q_z t \end{cases}$

Recap. Parametric Equation of a line in 3D

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In vector form:

$$\mathbf{r} = [x(t), y(t), z(t)]^\top = [x_0, y_0, z_0]^\top + [q_x t, q_y t, q_y t]^\top =$$

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$$= \mathbf{r_0} + t\mathbf{q}$$

Parametric equation of plane in 3D

Equation of a plane: $\begin{cases} x(t,s) = x_0 + q_x t + k_x s \\ y(t,s) = y_0 + q_y t + k_y s \\ z(t,s) = z_0 + q_z t + k_z s \end{cases}$

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In vector form:

$$\mathbf{r} = \mathbf{r_0} + t\mathbf{q} + s\mathbf{k}$$

Recap. Parametric Equation of a curve in 3D

$$\text{Equation of a curve: } \begin{cases} x(t) = f_1(t) \\ y(t) = f_2(t) \\ z(t) = f_3(t) \end{cases}$$

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In vector form:

$$\mathbf{r} = [x(t), y(t), z(t)]^\top$$

Example. Ellipse

In vector form:

$$\mathbf{r}(t) = [2\cos(t), 3\sin(t), 0]^\top$$

$$t \in [0, 2\pi]$$

Example. Spiral

In vector form:

$$\mathbf{r}(t) = [3\cos(t), 2\sin(t), t]^\top$$

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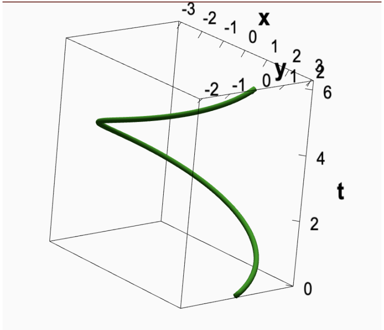

Recap. Parametric Equation of a surface in 3D

Equation of a surface: $\begin{cases} \mathsf{x(t,s)} = \mathsf{f}_1(t,s) \\ \mathsf{y(t,s)} = \mathsf{f}_2(t,s) \\ \mathsf{z(t,s)} = \mathsf{f}_3(t,s) \end{cases}$

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In vector form:

$$\mathbf{r} = [x(t,s), y(t,s), z(t,s)]^\top$$

Example. Ellipsoid

In vector form:

$$\mathbf{r}(t, s) = [2\cos(t)\sin(s), 3\sin(t)\sin(s), 5\cos(s)]^\top$$

$$t \in [0, 2\pi]$$

$$s \in [-\pi/2, \pi/2]$$

Which Greek letters we would probably use for parameters (s, t) in spherical coordinates?

Example. Helicoid

In vector form:

$$\mathbf{r}(t, s) = [t\cos(s), t\sin(s), s]^\top$$

$$t \in [0, 1]$$

$$s \in [0, 2\pi]$$

Example. Helicoid

In vector form:

$$\mathbf{r}(t, s) = [t\cos(s), t\sin(s), s]^\top$$

$$t \in [0, 1]$$

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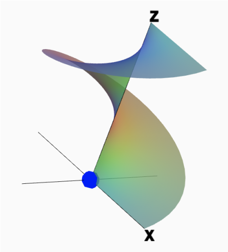

Break.

Checking the visualization...

https://mathinsight.org/parametrized_surface_introduction

Orientation of surfaces

Normal to a parametric surface

The tangent plane is given by equation $ax + by + cz + d = 0$,
or in parametric form:

$$\mathbf{r}(s,t) = \mathbf{r_0} + s\mathbf{q} + t\mathbf{k}$$

$\mathbf{q}$ and $\mathbf{k}$ are vectors on the plane. What is the normal vector?

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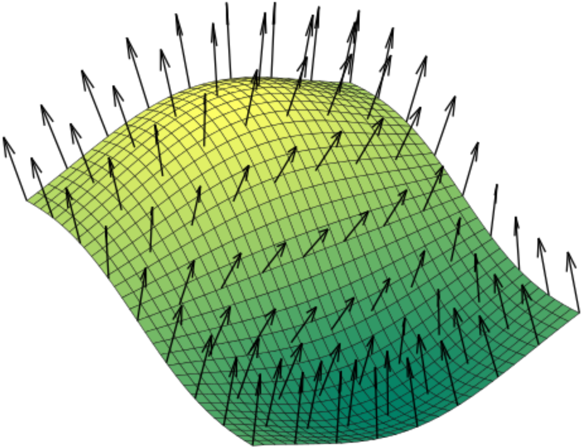
At the same time, one can see that

$$\mathbf{q} = \frac{\partial \mathbf{r}}{\partial s}; \quad \mathbf{k} = \frac{\partial \mathbf{r}}{\partial t}; \quad \mathbf{n} = \frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}$$

Orientation of surfaces

Normal vector defines orientation of a surface



Normal to a surface in general form

For an implicit surface, satisfying general equation $F(x, y, z) = 0$, a normal vector at a point (x_0, y_0, z_0) on the surface is given by the **gradient** (a vector):

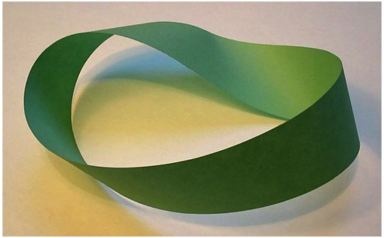
$$\mathbf{n} = \nabla F(x, y, z)$$

evaluated at that point (x_0, y_0, z_0).

Assignment

Determine the unit normal vector to $x^2 + y^2 + z^2 = 6$ at point $(2, 1, 1)$

Mobius (or Moebius strip/loop), is a surface with only one side.



In cylindrical coordinates (r, θ, z), a Möbius strip can be represented by the equation:

$$\log(r)\sin\left(\frac{1}{2}\theta\right) = z\cos\left(\frac{1}{2}\theta\right)$$

Summary of the course

- Write 3-5 main things (takeaways) of the EAGLA-1 course.

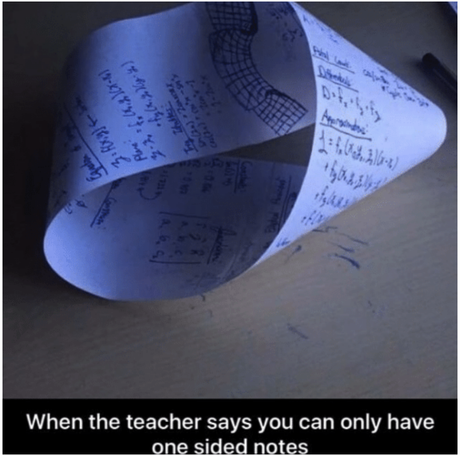
- For you, what is the result of the course?

What's next?

- Before 11th of December:
- - Retake of Test 2
- On 18th of December:
- - Final Written for offline students
- - Final Oral for online students

You can take 1 (one) A4 (hand-written) sheet with formulas (aka, cheat-sheet).

Final meme



Useful links

- https://www.geogebra.org
- https://youtu.be/fNk_zzaMoSs
- http://immersivemath.com/ila
- https://en.wikipedia.org/wiki/Quadric