Theoretical Computer Science

Lab Session 8

March 25, 2021

Innopolis University

Agenda

- Turing Machine:
  - formal definition;
  - example;
  - exercises.

Turing Machine.

Turing Machine

Formal Definition

A Turing Machine (TM) with k-tapes is a tuple

$$T = \langle Q, I, \Gamma, \delta, q_0, Z_0, F \rangle$$

where

Q is a finite set of states;
I is the input language;
Γ is the memory alphabet;
δ is the transition function;
$q_0 \in Q$ is the initial state;
$Z_0 \in \Gamma$ is the initial memory symbol;
$F \subseteq Q$ is the set of final states.

Transition Function

The transition function is defined as

$$\delta : (Q - F) \times (I \cup \{\_\}) \times (\Gamma \cup \{\_\})^k \to Q \times (\Gamma \cup \{\_\})^k \times \{R, L, S\}^{k+1}$$

where elements of $\{R, L, S\}$ indicate "directions" of the head of the TM:

$$\begin{aligned} R &: \text{ move the head one position to the right;} \\ L &: \text{ move the head one position to the left;} \\ S &: \text{ stand still.} \end{aligned}$$

Remarks:

- the transition function can be partial;
- no transition outgoing from the final states;
- the symbol $\_ \notin \Gamma \cup I$ is a special blank symbol on the tapes.

Moves

Moves are based on

- one symbol read from the input tape,
- k symbols, one for each memory tape,
- state of the control device.

Actions

- Change state.
- Write a symbol replacing the one read on each memory tape.
- Move the $k + 1$ heads.

Moves: Graphically

$$q \xrightarrow{i, \langle A_1, A_2, \ldots, A_k \rangle / \langle A'_1, A'_2, \ldots, A'_k \rangle, \langle M_0, M_1, \ldots, M_k \rangle} q'$$

- $q \in Q - F$ and $q' \in Q$
- i is the input symbol,
- A_j is the symbol read from the j^{th} memory tape,
- A'_j is the symbol replacing A_j,
- M_0 is the direction of the head of the input tape,
- M_j is the direction of the head of the j^{th} memory tape.

where $1 \leq j \leq k$

Configuration

A configuration (a snapshot) c of a TM with k memory tapes is the following $(k+2)$-tuple:

$$c = \langle q, x{\uparrow}y, \alpha_1{\uparrow}\beta_1, \dots, \alpha_k{\uparrow}\beta_k \rangle$$

where

- $q \in Q$
- $x \in (I \cup \{\_\})^*$, $y = y' \cdot \_$ with $y' \in I^*$
- $\alpha_r \in (\Gamma \cup \{\_\})^*$ and $\beta'_r = \beta'_r \cdot \_$ with $\beta'_r \in \Gamma^*$ and $1 \leq r \leq k$
- $\uparrow \notin I \cup \Gamma$

Acceptance Condition

If $T = \langle Q, I, \Gamma, \delta, q_0, Z_0, F \rangle$ is a TM and $s \in I^*$, s is accepted by T if

$$c_0 \vdash^* c_F$$

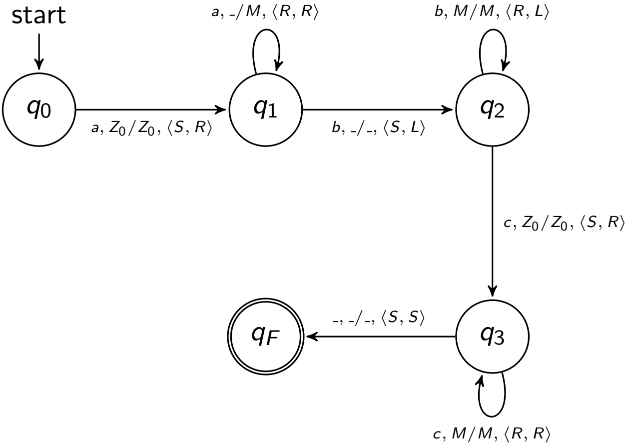
where

1. c_0 is an initial configuration defined as $c_0 = \langle q_0, \uparrow s, \uparrow Z_0, \ldots, \uparrow Z_0 \rangle$ where
   - $x = \epsilon$
   - $y = s_{\_}$
   - $\alpha_r = \epsilon$, $\beta_r = Z_0$, for any $1 \leq r \leq k$.
2. c_F is a final configuration defined as $c_F = \langle q, s' \uparrow y, \alpha_1 \uparrow \beta_1, \ldots, \alpha_k \uparrow \beta_k \rangle$ where
   - $q \in F$
   - $x = s'$

$L(T) = \{s \in I^* \mid x \text{ is accepted by } T\}$

Example: Language $A^nB^nC^n$

A TM T that recognises the language $A^nB^nC^n = \{a^nb^nc^n \mid n > 0\}$



Example: Language $A^nB^nC^n$

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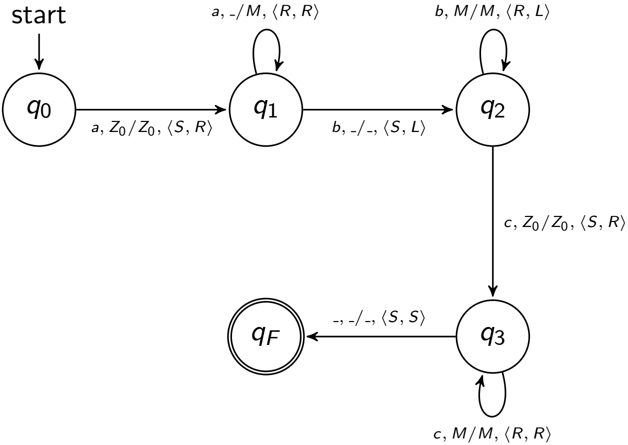
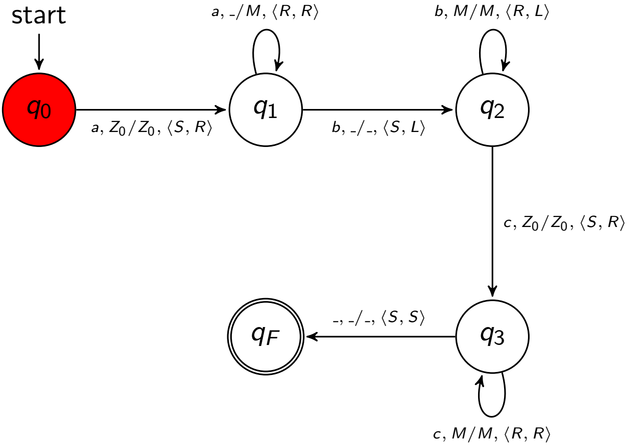


Is the string *aabbcc* recognised by T?

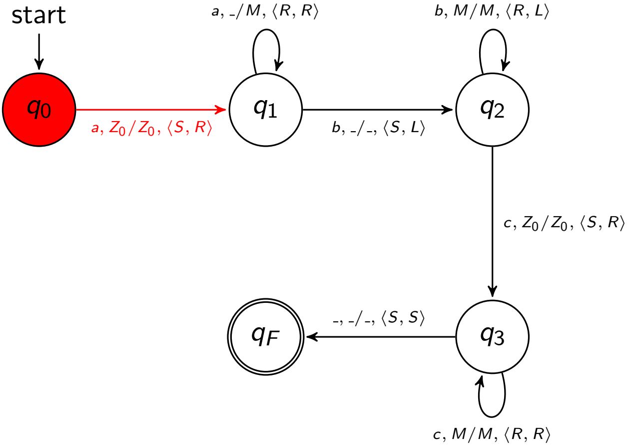
Example: Language $A^n B^n C^n$



Initial Configuration:

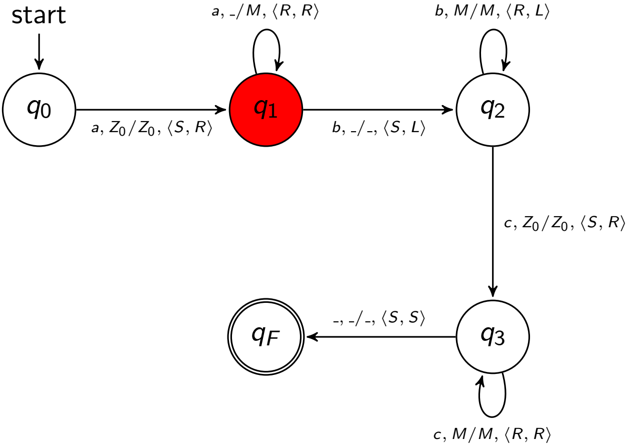
$$\langle q_0, \uparrow aabbcc, \uparrow Z_0\rangle$$

Example: Language $A^nB^nC^n$



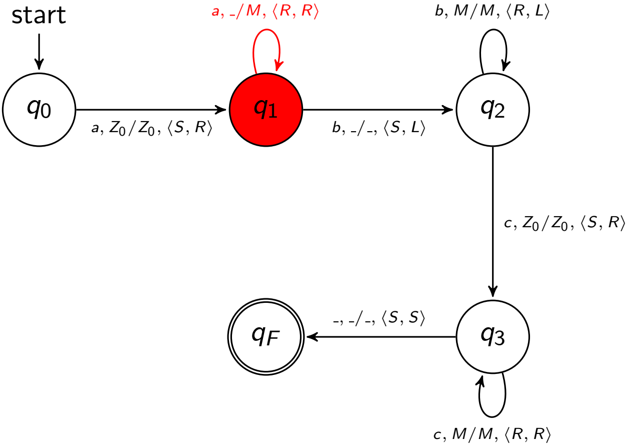
$$\langle q_0, \uparrow aabbcc, \uparrow Z_0\rangle \vdash$$

Example: Language $A^n B^n C^n$



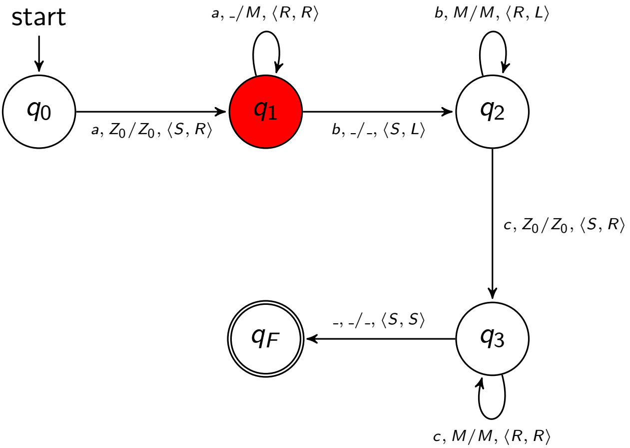
$$\langle q_0, \uparrow aabbcc, \uparrow Z_0\rangle \vdash \langle q_1, \uparrow aabbcc, Z_0\uparrow\rangle$$

Example: Language $A^nB^nC^n$



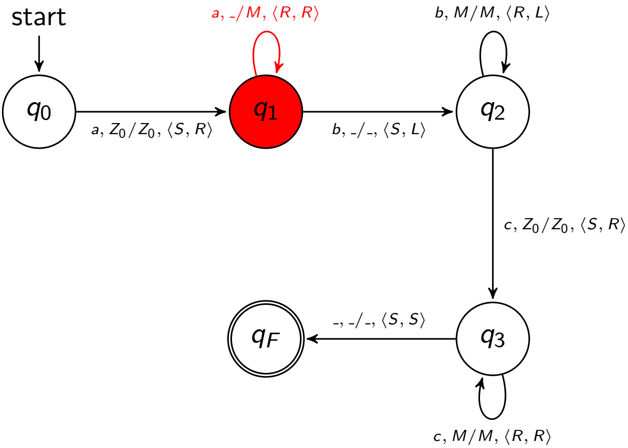
$$\ldots \vdash \langle q_1, \uparrow aabbcc, Z_0\uparrow \rangle \vdash$$

Example: Language $A^n B^n C^n$



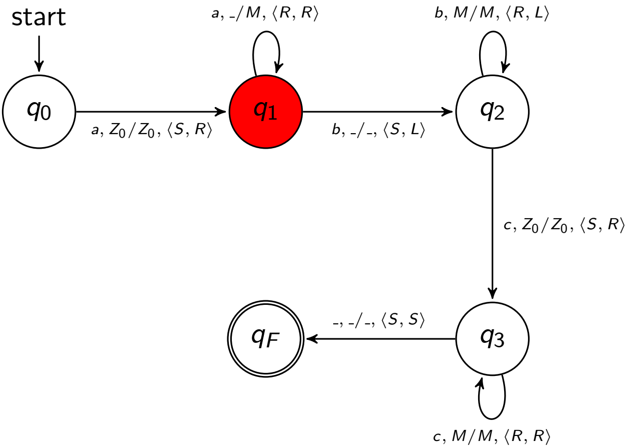
$$\ldots \vdash \langle q_1, \uparrow aabbcc, Z_0\uparrow\rangle \vdash \langle q_1, a\uparrow abbcc, Z_0 M\uparrow\rangle$$

Example: Language $A^n B^n C^n$



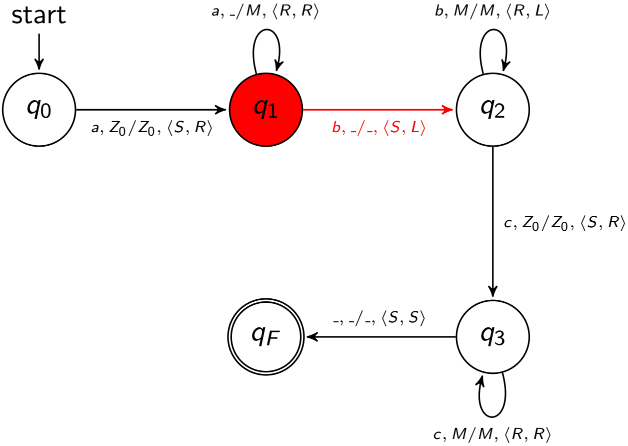
$$\ldots \vdash \langle q_1, a{\uparrow}abbcc, Z_0 M{\uparrow}\rangle$$

Example: Language $A^nB^nC^n$



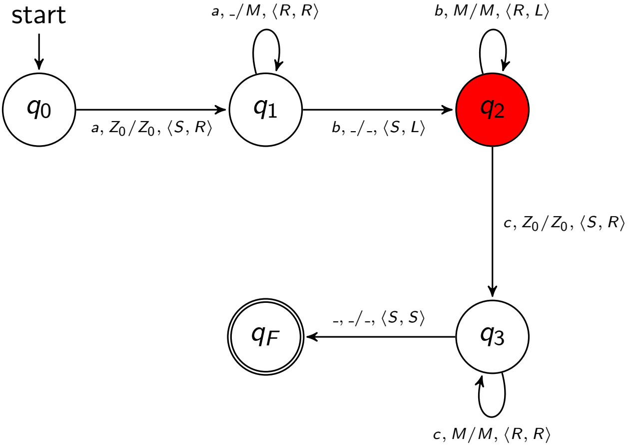
$$\ldots \vdash \langle q_1, a{\uparrow}abbcc, Z_0M{\uparrow}\rangle \vdash \langle q_1, aa{\uparrow}bbcc, Z_0MM{\uparrow}\rangle$$

Example: Language $A^n B^n C^n$



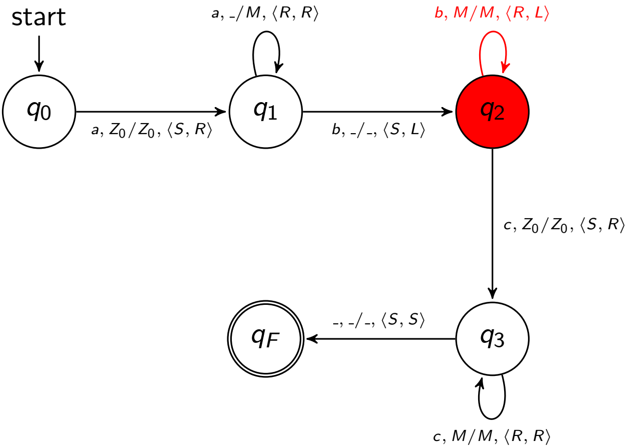
$$\ldots \vdash \langle q_1, aa{\uparrow}bbcc, Z_0MM{\uparrow}\rangle$$

Example: Language $A^n B^n C^n$



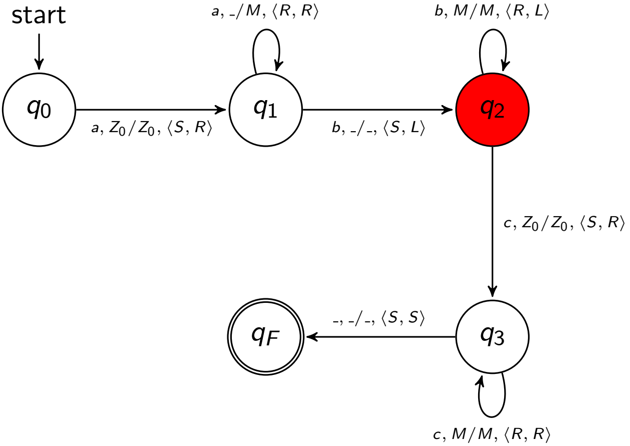
$$\ldots \vdash \langle q_1, aa{\uparrow}bbcc, Z_0MM{\uparrow}\rangle \vdash \langle q_2, aa{\uparrow}bbcc, Z_0M{\uparrow}M\rangle$$

Example: Language $A^n B^n C^n$



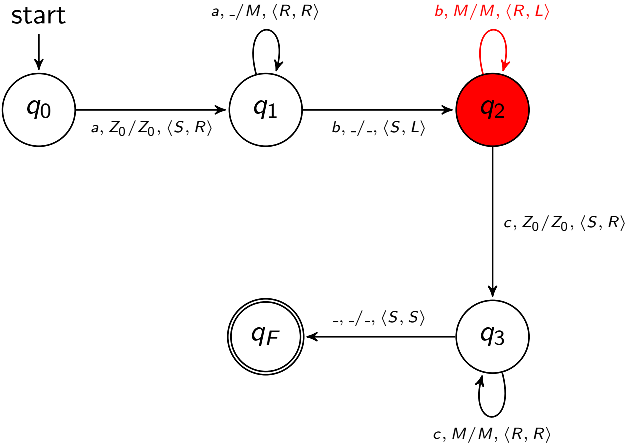
$$\ldots \vdash \langle q_2, aa{\uparrow}bbcc, Z_0 M{\uparrow}M\rangle$$

Example: Language $A^n B^n C^n$



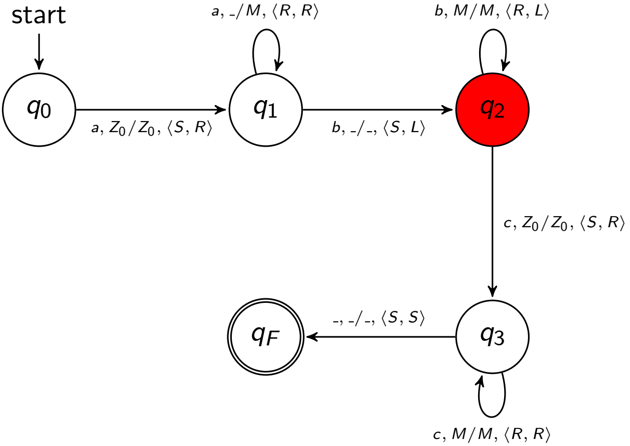
$$\ldots \vdash \langle q_2, aa{\uparrow}bbcc, Z_0M{\uparrow}M\rangle \vdash \langle q_2, aab{\uparrow}bcc, Z_0{\uparrow}MM\rangle$$

Example: Language $A^nB^nC^n$



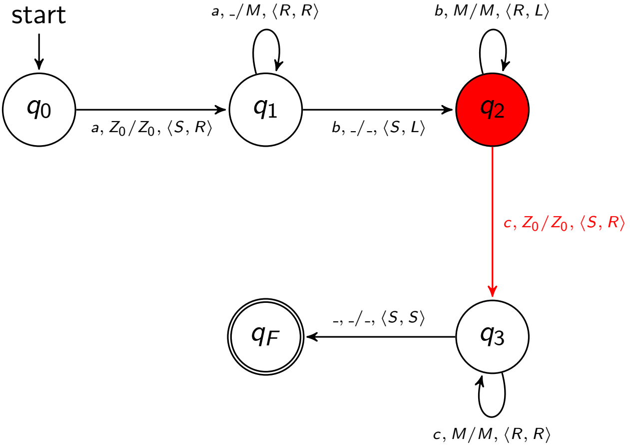
$$\ldots \vdash \langle q_2, aab{\uparrow}bcc, Z_0{\uparrow}MM\rangle$$

Example: Language $A^n B^n C^n$



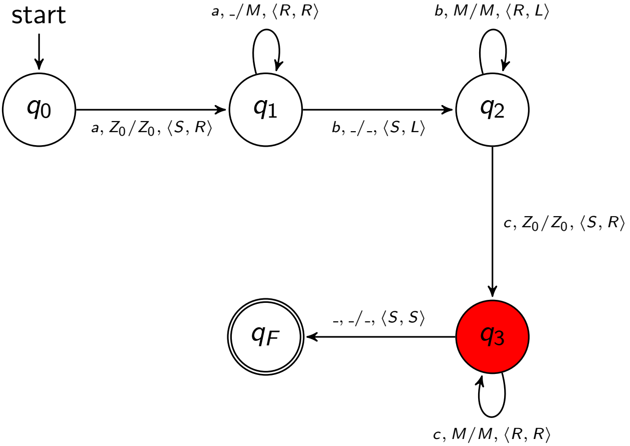
$$\ldots \vdash \langle q_2, aab{\uparrow}bcc, Z_0{\uparrow}MM\rangle \vdash \langle q_2, aabb{\uparrow}cc, {\uparrow}Z_0MM\rangle$$

Example: Language $A^n B^n C^n$



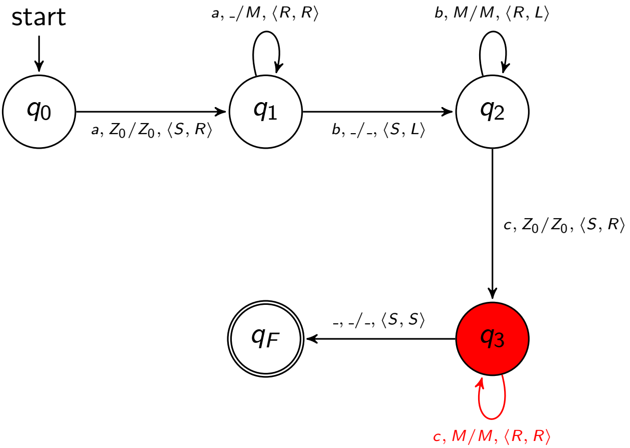
$$\ldots \vdash \langle q_2, aabb{\uparrow}cc, {\uparrow}Z_0MM\rangle$$

Example: Language $A^n B^n C^n$



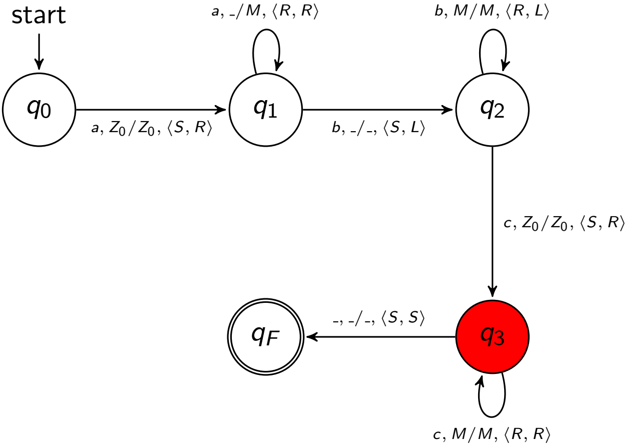
$$\ldots \vdash \langle q_2, aabb{\uparrow}cc, {\uparrow}Z_0MM\rangle \vdash \langle q_3, aabb{\uparrow}cc, Z_0{\uparrow}MM\rangle$$

Example: Language $A^nB^nC^n$



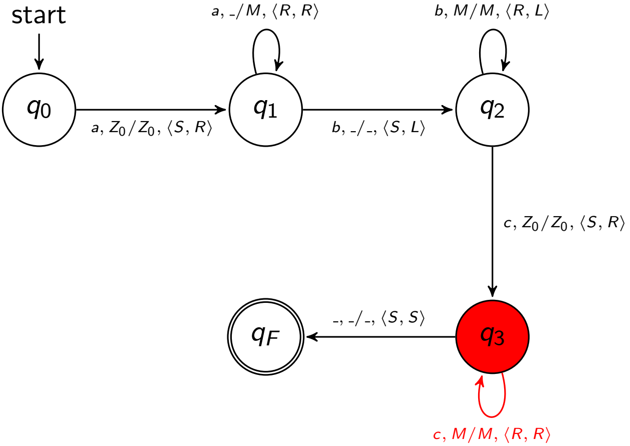
$$\ldots \vdash \langle q_3, aabb{\uparrow}cc, Z_0{\uparrow}MM\rangle$$

Example: Language $A^nB^nC^n$



$$\ldots \vdash \langle q_3, aabb{\uparrow}cc, Z_0{\uparrow}MM\rangle \vdash \langle q_3, aabbc{\uparrow}c, Z_0M{\uparrow}M\rangle$$

Example: Language $A^n B^n C^n$



$$\ldots \vdash \langle q_3, aabbc{\uparrow}c, Z_0M{\uparrow}M\rangle$$

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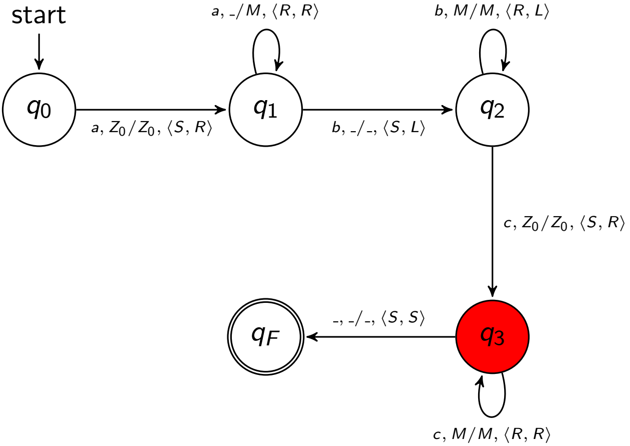
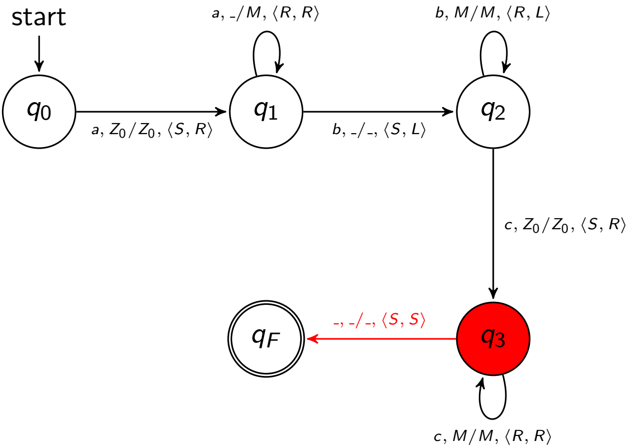


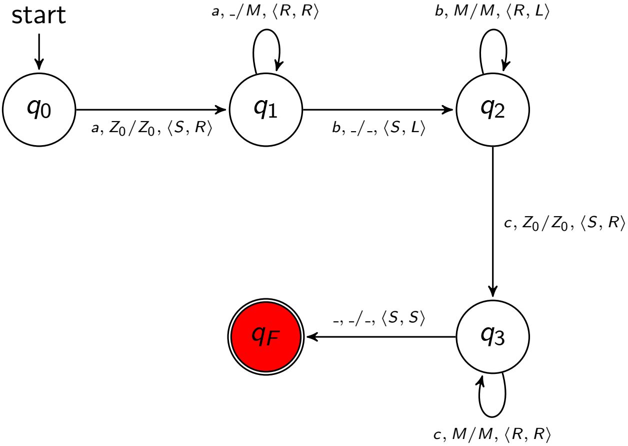
$$\ldots \vdash \langle q_3, aabbc{\uparrow}c, Z_0M{\uparrow}M\rangle \vdash \langle q_3, aabbcc{\uparrow}, Z_0MM{\uparrow}\rangle$$

Example: Language $A^n B^n C^n$



$$\ldots \vdash \langle q_3, aabbcc\uparrow, Z_0MM\uparrow\rangle$$

Example: Language $A^n B^n C^n$



$$\ldots \vdash \langle q_3, aabbcc\uparrow, Z_0MM\uparrow\rangle \vdash \langle q_F, aabbcc\uparrow, Z_0MM\uparrow\rangle$$

Example: Language $A^n B^n C^n$

Is the string *aabbcc* recognised by T?
Yes, we found:

$$\langle q_0, \uparrow aabbcc, \uparrow Z_0 \rangle \vdash^* \langle q_F, aabbcc\uparrow, Z_0 MM\uparrow \rangle$$

Turing Machine in JFLAP

Formal Definition

In JFLAP a Turing Machine is single-taped

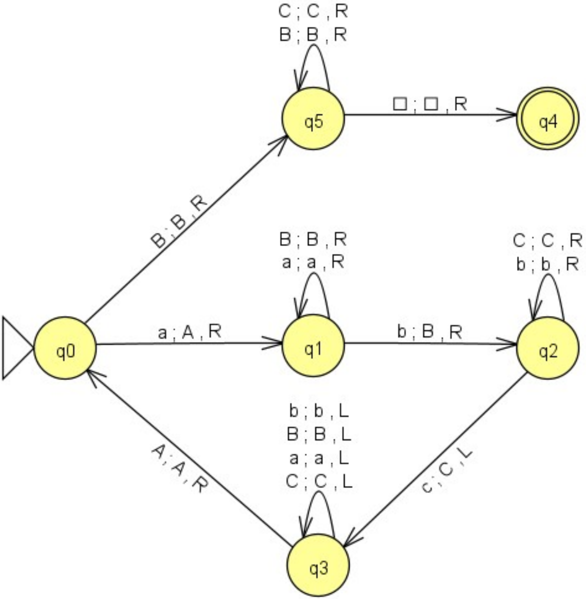
$$T = \langle Q, I, \Gamma, \delta, q_0, \square, F \rangle$$

where

Q is a finite set of states;
I is the input alphabet;
Γ is the tape alphabet (I is always a subset of of Γ);
δ is the transition function;
$q_0 \in Q$ is the initial state;
$\square$ is the blank symbol;
$F \subseteq Q$ is the set of final states.

R : move the head one position to the right;
L : move the head one position to the left.

Example in JFLAP: Language $A^n B^n C^n$



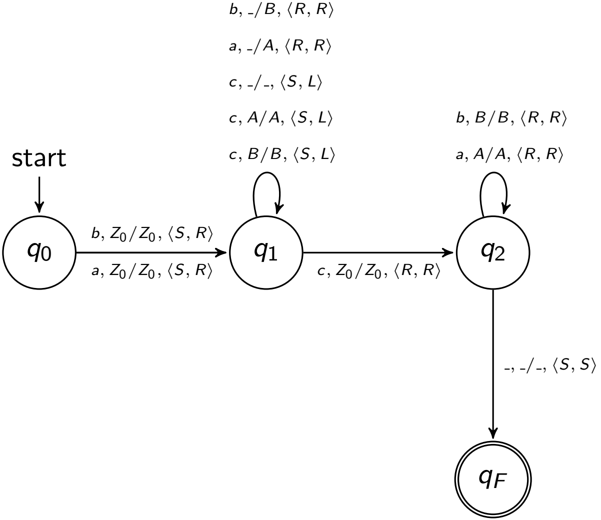
Exercises

Build TMs that recognise the following languages:

- $L_1 = \{wcw \mid w \in \{a, b\}^+\}$
- $L_2 = \{wcw^R \mid w \in \{a, b\}^+\}$, where w^R is the reversed string w.
- $L_3 = \{w \mid w \in \{a, b\}^*\}$, where w is a palindrome (Construct a 1-tape TM for this task).
- $L_4 = \{a^n b^n | n \geq 0\} \cup \{a^n b^{2n} | n \geq 0\}$

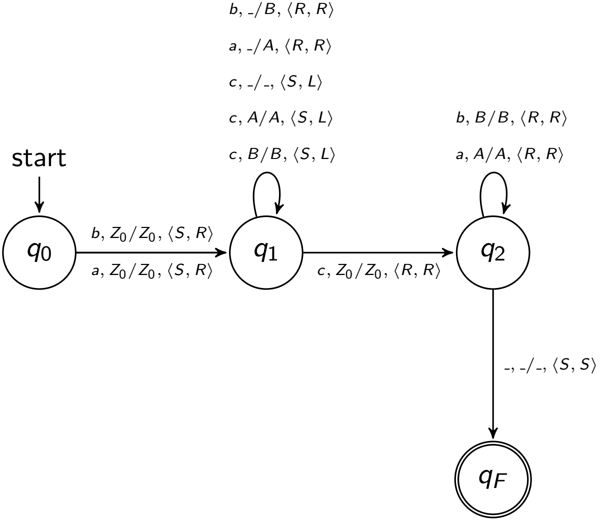
Solution (1)

TM that recognises the language $L_1 = \{wcw \mid w \in \{a, b\}^+\}$



Solution (1)

TM that recognises the language $L_1 = \{wcw \mid w \in \{a,b\}^+\}$



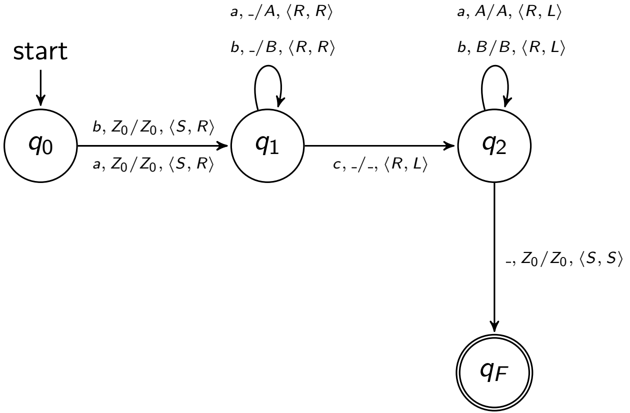
Is the string *abbcabb* recognised by the TM above?

Is the string *abbcabb* recognised by the TM?

$$
\begin{array}{l}
\langle q_0, \uparrow abbcabb, \uparrow Z_0 \rangle \vdash \\
\langle q_1, \uparrow abbcabb, Z_0 \uparrow \rangle \vdash \\
\langle q_1, a \uparrow bbcabb, Z_0 A \uparrow \rangle \vdash \\
\langle q_1, ab \uparrow bcabb, Z_0 AB \uparrow \rangle \vdash \\
\langle q_1, abb \uparrow cabb, Z_0 ABB \uparrow \rangle \vdash \\
\langle q_1, abb \uparrow cabb, Z_0 AB \uparrow B \rangle \vdash \\
\langle q_1, abb \uparrow cabb, Z_0 A \uparrow BB \rangle \vdash \\
\langle q_1, abb \uparrow cabb, Z_0 \uparrow ABB \rangle \vdash \\
\langle q_1, abb \uparrow cabb, \uparrow Z_0 ABB \rangle \vdash \\
\langle q_2, abbc \uparrow abb, Z_0 \uparrow ABB \rangle \vdash \\
\langle q_2, abbca \uparrow bb, Z_0 A \uparrow BB \rangle \vdash \\
\langle q_2, abbcab \uparrow b, Z_0 AB \uparrow B \rangle \vdash \\
\langle q_2, abbcabb \uparrow, Z_0 ABB \uparrow \rangle \vdash \\
\langle q_F, abbcabb \uparrow, Z_0 ABB \uparrow \rangle
\end{array}
$$

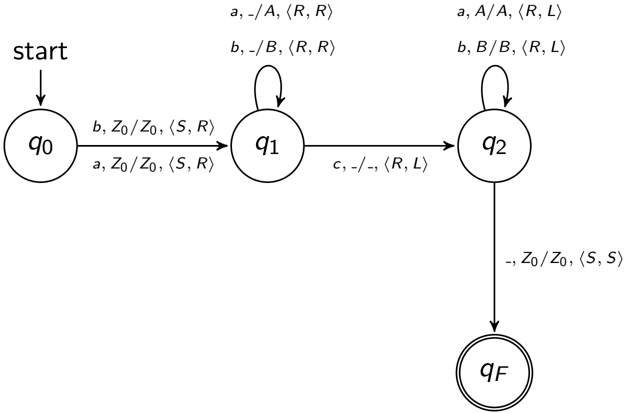
Solution (2)

TM that recognises the language $L_2 = \{wcw^R \mid w \in \{a, b\}^+\}$, where w^R is the reversed string w.



Solution (2)

TM that recognises the language $L_2 = \{wcw^R \mid w \in \{a, b\}^+\}$, where w^R is the reversed string w.

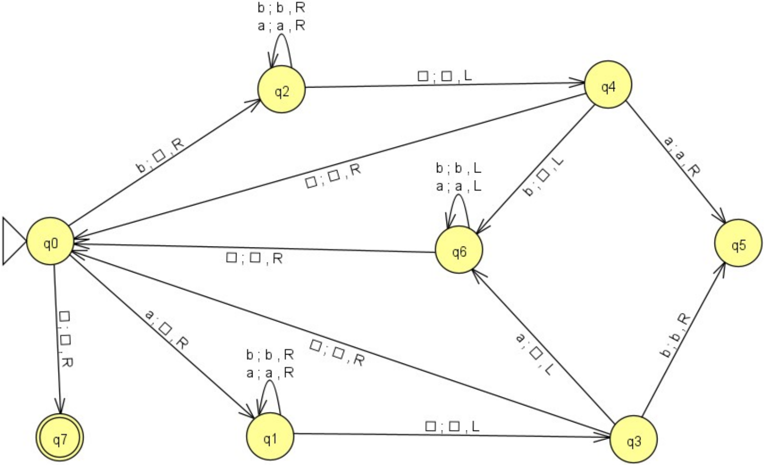


Is the string *abbcbba* recognised by the TM above?

Is the string *abbcbba* recognised by the TM?

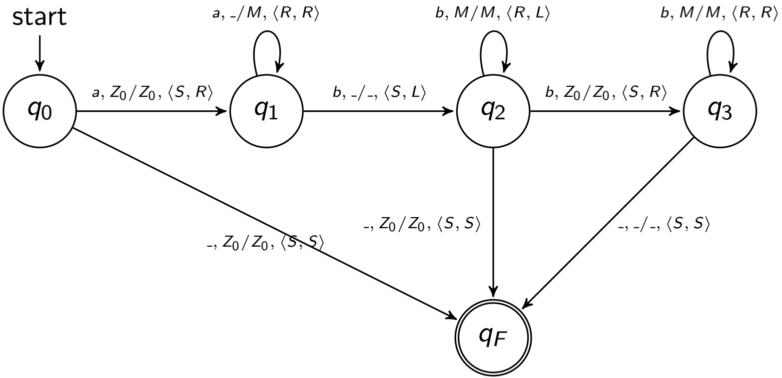
$$
\begin{aligned}
&\langle q_0, \uparrow abbcbba, \uparrow Z_0 \rangle \vdash \\
&\langle q_1, \uparrow abbcbba, Z_0 \uparrow \rangle \vdash \\
&\langle q_1, a \uparrow bbcbba, Z_0 A \uparrow \rangle \vdash \\
&\langle q_1, ab \uparrow bcbba, Z_0 AB \uparrow \rangle \vdash \\
&\langle q_1, abb \uparrow cbba, Z_0 ABB \uparrow \rangle \vdash \\
&\langle q_2, abbc \uparrow bba, Z_0 AB \uparrow B \rangle \vdash \\
&\langle q_2, abbcb \uparrow ba, Z_0 A \uparrow BB \rangle \vdash \\
&\langle q_2, abbcbb \uparrow a, Z_0 \uparrow ABB \rangle \vdash \\
&\langle q_2, abbcbba \uparrow, \uparrow Z_0 ABB \rangle \vdash \\
&\langle q_F, abbcbba \uparrow, \uparrow Z_0 ABB \rangle
\end{aligned}
$$

Solution (3): ab Palindrome



Solution (4)

TM TT that recognises the language
$L_4 = \{a^n b^n | n \geq 0\} \cup \{a^n b^{2n} | n \geq 0\}$



Homework Exercises

Build TMs that recognise the following languages:

- $L_5 = \{(ab)^n, n \geq 0\}$
- $L_6 = \{a^n b^{2n} c^{3n}, n \geq 0\}$