Essentials of Analytical Geometry and Linear Algebra. Lecture 12.

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Main question for today's lecture

Quadratic surfaces in 3D or **Quadrics**

Lecture 12. Outline

- Part 1. Types of Quadrics and their equations
- Part 2. Cylindrical / Spherical coordinates.
- Part 3. Applications

Part 1. Types of Quadrics and their equations

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In matrix/vector form the **same** equation can be presented as:

$$\begin{bmatrix} x & y & z \end{bmatrix} Q \begin{bmatrix} x \\ y \\ z \end{bmatrix} + P \begin{bmatrix} x \\ y \\ z \end{bmatrix} + R = 0$$

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Matrix of the **quadratic form** is Q

Demo in Geogebra

In the demo, have you recognised any types of 3D surfaces?

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Which (make a list)?

Let us check the types:

Should be 9 types

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6. Hyperbolic paraboloid

Let us check the types:

Should be 9 types

1. Ellipsoid
2. Hyperboloid of 1 sheet
3. Cone
4. Hyperboloid of 2 sheets
5. Elliptic paraboloid
6. Hyperbolic paraboloid
7. Elliptic cylinder
8. Hyperbolic cylinder
9. Parabolic cylinder

All the difference is in the equation (in the canonical form).

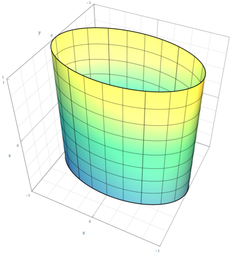
Cylinders

Properties

In the equation, some variable is missing...

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(for any z this will be an ellipse)



Example

Example

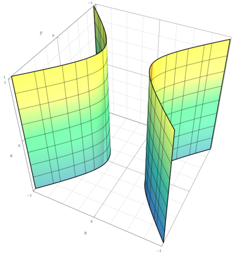
Cylinders...

Properties

In the equation, some variable is missing...

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

(for any z this will be a hyperbola)



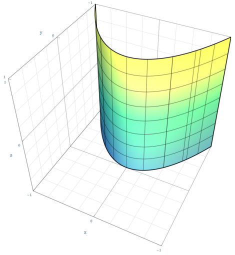
Cylinders...

Properties

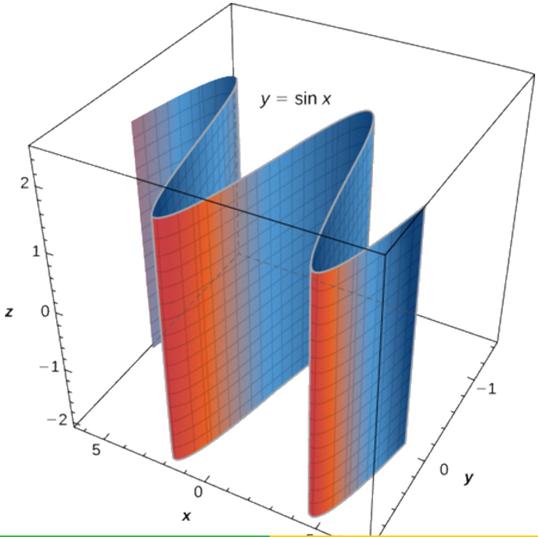
In the equation, some variable is missing...

$$\frac{x^2}{a^2} + 2hy = 0$$

(for any z this will be a parabola)



Other cylindrical surfaces

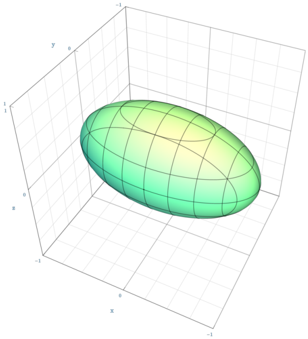


Ellipsoid

Properties

We have all ”pluses”:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$



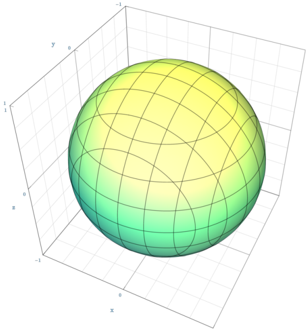
Example

Sphere is a special case of an ellipsoid

Properties

We have all "pluses" and $a = b = c$:

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{a^2} = 1$$



Example

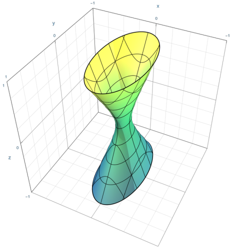
Hyperboloids and Cone

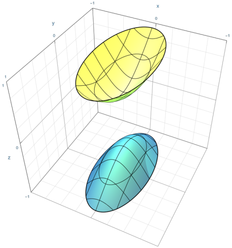
Hyperboloids

Properties

We have either one or two minuses:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$





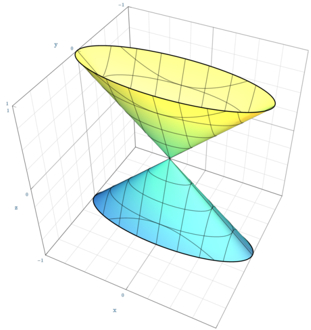
Example

Cone as a limiting case of hyperbolois

Properties

We have zero on the right-hand side of the equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$



Example

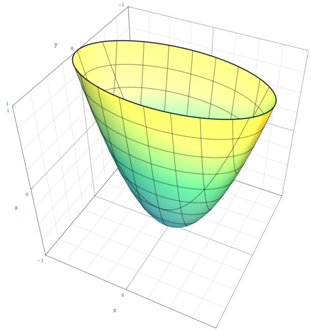
Paraboloids

(Elliptic) Paraboloid

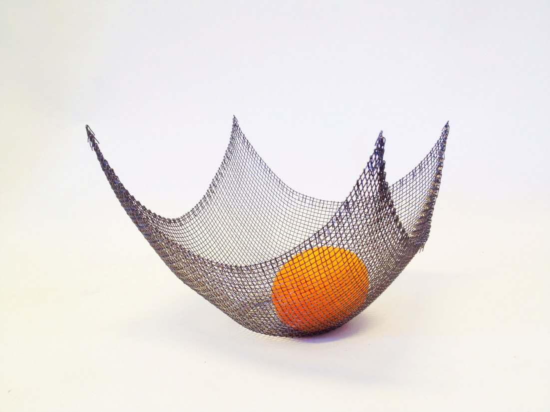
Properties

We have a linear term (without a quadratic one), two other are positive:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - z = 0$$



Example

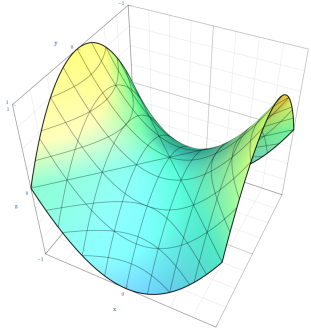


Hyperbolic Paraboloid

Properties

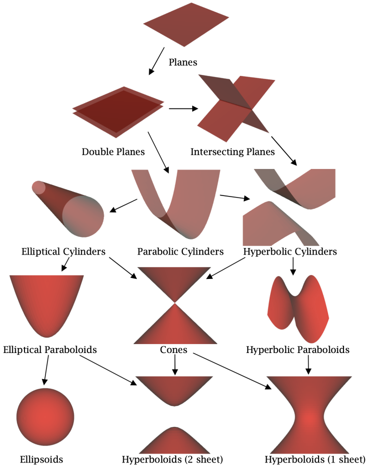
We have a linear term (without a quadratic one), but one of the quadratic terms has a minus:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - z = 0$$



Example

Break, 5-10 min.



http://graphics.berkeley.edu/papers/Andrews-TCD-2013-06/Andrews-TCD-2013-06.pdf

Part 2. Cylindrical / Spherical coordinates

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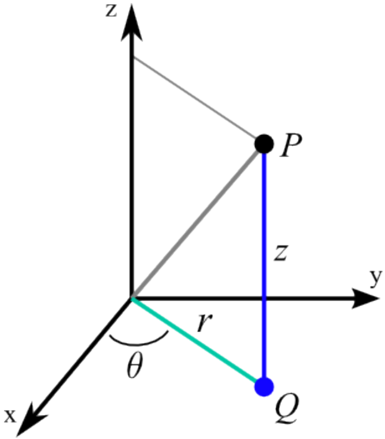
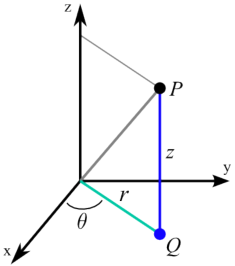
Cylindrical coordinates



$$P(r, \theta, z)$$

Cylindrical coordinates. Change to Cartesian



- x =
- y =
- z =

Example

Given the cylindrical coordinates, identify each of the three surfaces

$r = 5$

$r^2 + z^2 = 100$

$z = r$

Example

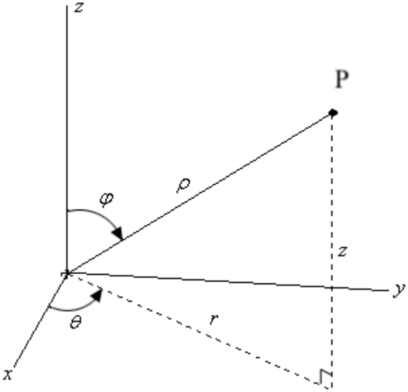
Transform point P from cylindrical coordinates to Cartesian

$P(2, 2\pi/3, 9)$

Transform point P from Cartesian coordinates to cylindrical

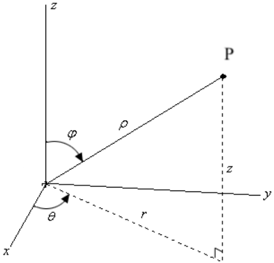
$P(3, -3, 8)$

Spherical coordinates



$$P(\rho, \theta, \phi)$$

Spherical coordinates. Change to Cartesian



- x =
- y =
- z =

Example

Transform point P from Spherical coordinates to Cartesian

$P(2, \pi/4, \pi/3)$

Transform point P from Cartesian coordinates to Spherical

$P(0, 2\sqrt{3}, -2)$

MEME time: Cylindrical / Spherical coordinates

Cylindrical / Spherical coordinates

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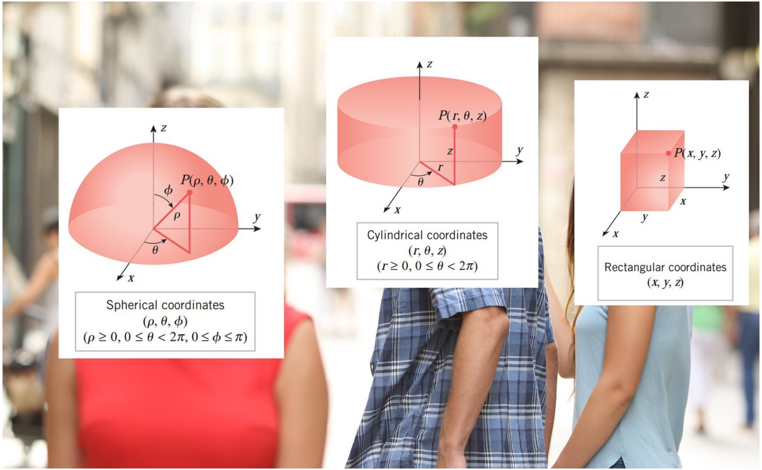

Part 3. Applications

Simplify of integrals

Measurements of distances (e.g. on sphere)

...

Extra. Tangent planes to quadrics

Given a function $f(x, y)$ that defines a surface, you need to define a tangent plane at a point with coordinates (x_0, y_0)

The tangent plane is given by the first-order Taylor polynomial approximation:

$$z = f(x_0, y_0) + f'_x(x - x_0) + f'_y(y - y_0)$$

You can vary the point of tangency.
Find an equation of tangent plane to the surface $z = -x^2 - y^2$ at point $(x_0, y_0) = (-1, -1)$

Useful links

- https://www.geogebra.org
- https://youtu.be/fNk_zzaMoSs
- http://immersivemath.com/ila
- https://en.wikipedia.org/wiki/Quadric