Theoretical Computer Science

Lab Session 2

February 4, 2021

Innopolis University

Agenda

- Operations on Formal Languages
- Exercises

Operations on Languages

- Union
- Intersection
- Set difference
- Complement: if L is a language over Σ,

$$\overline{L} = \Sigma^* \backslash L$$

- Concatenation: if L_1 and L_2 are both languages over Σ,

$$L_1 L_2 = \{xy \mid x \in L_1, y \in L_2\}$$

- Power of n

$$L^n = \{x_1 x_2 ... x_n \mid x_i \in L \text{ for all } 1 \leq i \leq n\}$$

- Kleene Star

$$L^* = \{x_1 x_2 ... x_n \mid n \in \mathbb{N}, x_1, x_2, ..., x_n \in L\} = \bigcup_{n \in \mathbb{N}} L^n$$

Exercise Session - Operations on Languages

Exercises (1)

1. Let $L = \{a^i, i \geq 0\}$ be a language over $\Sigma = \{a, b\}$. Find $\overline{L}$ and L^*
2. Let L_1, L_2 be languages over $\Sigma = \{a, b\}$. Find $L_1 L_2$
   a) $L_1 = \{\epsilon, a, aa\}$, $L_2 = \{aa, aaa\}$
   b) $L_1 = \{a, a^2, a^4\}$, $L_2 = \{b^0, b^2, b^3\}$
3. Let $L = \{0, 01, 001\}$. Find L^2.
4. Describe in plain English the following languages over $\Sigma = \{a, b\}$:
   a) $L = \{a, b\}^*$
   b) $L = \{a\}^* \cup \{b\}^*$
   c) $L = \{a\}^* \cap \{b\}^*$
   d) $L = \{aa\}^* \backslash \{aaaa\}^*$
5. Write out in full the strings $0^5, 0^3 1^3, (010)^2, (01)^3 0, 1^0$

Solutions (1)

1. Let $L = \{a^i, i \geq 0\}$, $\Sigma = \{a, b\}$.
   $\overline{L}$ = all nonempty strings containing at least one b
   $L^* = \{a^i, i \geq 0\}$
2. Let L_1, L_2 be languages over $\Sigma = \{a, b\}$. Find $L_1 L_2$
   a) $\{\epsilon, a, aa\}\{aa, aaa\} = \{aa, aaa, aaaa, aaaaa\}$
   b) $\{a, a^2, a^4\}\{b^0, b^2, b^3\} = \{a, aa, aaaa\}\{\epsilon, bb, bbb\} =$
   $\{a, aa, aaaa, abb, aabb, aaaabb, abbb, aabbb, aaaabbb\}$
3. $\{0, 01, 001\}^2 =$
   $\{00, 001, 0001, 010, 0101, 01001, 0010, 00101, 001001\}$
4. a) $L = \{a, b\}^*$ - all strings of a's and b's, including empty string
   b) $L = \{a\}^* \cup \{b\}^*$ - empty string and strings of only a's or only b's
   c) $L = \{a\}^* \cap \{b\}^*$ - empty string
   d) $L = \{aa\}^* \backslash \{aaaa\}^*$ - strings of even number of a's which is not a multiple of 4
5. $00000, 000111, 010010, 0101010, \epsilon$

Exercises (2)

Perform operations on the languages over $\Sigma = \{0, 1\}$:
$L_1 = \{0, 1, 00, 11, 000, 111, ...\}$,
$L_2 = \{0, 1\}^*$,
$L_3 = \{w \mid w \in \Sigma^*, |w| = 1\}$,
$L_4 = \{w \mid w \in \Sigma^*, |w| = 2\}$,
$L_5 = \{w \mid w \in \Sigma^*, |w| \geq 1\}$

1. $L_1 \cup L_2, \quad L_3 \cup L_4$
2. $L_1 \cap L_2, \quad L_1 \cap L_3, \quad L_1 \cap L_4, \quad L_1 \cap L_5, \quad L_3 \cap L_4$
3. $L_1 \backslash L_2, \quad L_1 \backslash L_3, \quad L_3 \backslash L_4, \quad L_4 \backslash L_5, \quad L_5 \backslash L_4$
4. $\overline{L_1}, \quad \overline{L_2}, \quad \overline{L_3}, \quad \overline{L_5 \backslash L_4}$
5. $L_1 L_2, \quad L_3 L_4, \quad L_4 L_3$
6. $L_2^*, \quad L_3^*, \quad L_4^*$

Solution (2)

Perform operations on the languages over $\Sigma = \{0, 1\}$:
$L_1 = \{0, 1, 00, 11, 000, 111, ...\}$,
$L_2 = \{0, 1\}^* =$
$\{\epsilon, 0, 1, 00, 11, 01, 10, 000, 111, 010, 011, 100, 101, ...\}$,
$L_3 = \{w \mid w \in \Sigma^*, |w| = 1\} = \{0, 1\}$,
$L_4 = \{w \mid w \in \Sigma^*, |w| = 2\} = \{00, 11, 01, 10\}$,
$L_5 = \{w \mid w \in \Sigma^*, |w| \geq 1\} =$
$\{0,1,00,11,01,10,000,111,010,011,100,101,...\} = L_2 \setminus \{\epsilon\}$,

1. $L_1 \cup L_2 = L_2$
   $L_3 \cup L_4 = \{w \mid w \in \Sigma^*, |w| = 1 \text{ or } |w| = 2\}$
   $= \{0,1,00,11,01,10\}$

Solution (2)

Perform operations on the languages over $\Sigma = \{0, 1\}$:
$L_1 = \{0, 1, 00, 11, 000, 111, ...\}$,
$L_2 = \{0, 1\}^* =$
$\{\epsilon, 0, 1, 00, 11, 01, 10, 000, 111, 010, 011, 100, 101, ...\}$,
$L_3 = \{w \mid w \in \Sigma^*, |w| = 1\} = \{0, 1\}$,
$L_4 = \{w \mid w \in \Sigma^*, |w| = 2\} = \{00, 11, 01, 10\}$,
$L_5 = \{w \mid w \in \Sigma^*, |w| \geq 1\} =$
$\{0,1,00,11,01,10,000,111,010,011,100,101,...\} = L_2 \backslash \{\epsilon\}$,

2. $L_1 \cap L_2 = L_1$
   $L_1 \cap L_3 = L_3$
   $L_1 \cap L_4 = \{00, 11\}$
   $L_1 \cap L_5 = L_1$
   $L_3 \cap L_4 = \phi$

Solution (2)

Perform operations on the languages over $\Sigma = \{0, 1\}$:
$L_1 = \{0, 1, 00, 11, 000, 111, ...\}$,
$L_2 = \{0, 1\}^* =$
$\{\epsilon, 0, 1, 00, 11, 01, 10, 000, 111, 010, 011, 100, 101, ...\}$,
$L_3 = \{w \mid w \in \Sigma^*, |w| = 1\} = \{0, 1\}$,
$L_4 = \{w \mid w \in \Sigma^*, |w| = 2\} = \{00, 11, 01, 10\}$,
$L_5 = \{w \mid w \in \Sigma^*, |w| \geq 1\} =$
$\{0,1,00,11,01,10,000,111,010,011,100,101,...\} = L_2 \backslash \{\epsilon\}$,

3. $L_1 \backslash L_2 = \phi$
   $L_1 \backslash L_3 = \{00, 11, 000, 111, ...\}$
   $L_3 \backslash L_4 = L_3 = \{0, 1\}$
   $L_4 \backslash L_5 = \phi$
   $L_5 \backslash L_4 = \{w \mid w \in \Sigma^*, |w| = 1 \text{ or } |w| \geq 3\}$

Solution (2)

Perform operations on the languages over $\Sigma = \{0, 1\}$:
$L_1 = \{0, 1, 00, 11, 000, 111, ...\}$,
$L_2 = \{0, 1\}^* =$
$\{\epsilon, 0, 1, 00, 11, 01, 10, 000, 111, 010, 011, 100, 101, ...\}$,
$L_3 = \{w \mid w \in \Sigma^*, |w| = 1\} = \{0, 1\}$,
$L_4 = \{w \mid w \in \Sigma^*, |w| = 2\} = \{00, 11, 01, 10\}$,
$L_5 = \{w \mid w \in \Sigma^*, |w| \geq 1\} =$
$\{0,1,00,11,01,10,000,111,010,011,100,101,...\} = L_2 \backslash \{\epsilon\}$,

4. $\overline{L_1} = \{\epsilon, 01, 10, 010, 011, 100, 101, ...\}$
   $\overline{L_2} = \phi$
   $\overline{L_3} = \{\epsilon, 00, 11, 01, 10, 000, 111, 010, 011, 100, 101, ..\}$
   $\overline{L_5 \backslash L_4} = L_4 \cup \{\epsilon\}$

Solution (2)

Perform operations on the languages over $\Sigma = \{0, 1\}$:
$L_1 = \{0, 1, 00, 11, 000, 111, ...\}$,
$L_2 = \{0, 1\}^* =$
$\{\epsilon, 0, 1, 00, 11, 01, 10, 000, 111, 010, 011, 100, 101, ...\}$,
$L_3 = \{w \mid w \in \Sigma^*, |w| = 1\} = \{0, 1\}$,
$L_4 = \{w \mid w \in \Sigma^*, |w| = 2\} = \{00, 11, 01, 10\}$,
$L_5 = \{w \mid w \in \Sigma^*, |w| \geq 1\} =$
$\{0,1,00,11,01,10,000,111,010,011,100,101,...\} = L_2 \setminus L_0$,

5. $L_1 L_2 = L_2 \setminus \{\epsilon\}$
   $L_3 L_4 = \{000, 011, 001, 010, 100, 111, 101, 110\}$
   $L_4 L_3 = L_3 L_4 = \{000, 001, 110, 111, 010, 011, 100, 101\}$

Solution (2)

Perform operations on the languages over $\Sigma = \{0, 1\}$:
$L_1 = \{0, 1, 00, 11, 000, 111, ...\}$,
$L_2 = \{0, 1\}^* =$
$\{\epsilon, 0, 1, 00, 11, 01, 10, 000, 111, 010, 011, 100, 101, ...\}$,
$L_3 = \{w \mid w \in \Sigma^*, |w| = 1\} = \{0, 1\}$,
$L_4 = \{w \mid w \in \Sigma^*, |w| = 2\} = \{00, 11, 01, 10\}$,
$L_5 = \{w \mid w \in \Sigma^*, |w| \geq 1\} =$
$\{0,1,00,11,01,10,000,111,010,011,100,101,...\} = L_2 \backslash \{\epsilon\}$,

6. $L_2^* = L_2$
   $L_3^* = L_2$
   $L_4^* = \{w \mid w \in \Sigma^*, |w| = 2k, k \in \mathbb{N}\}$