Essentials of Analytical Geometry and Linear Algebra. Lecture 11.

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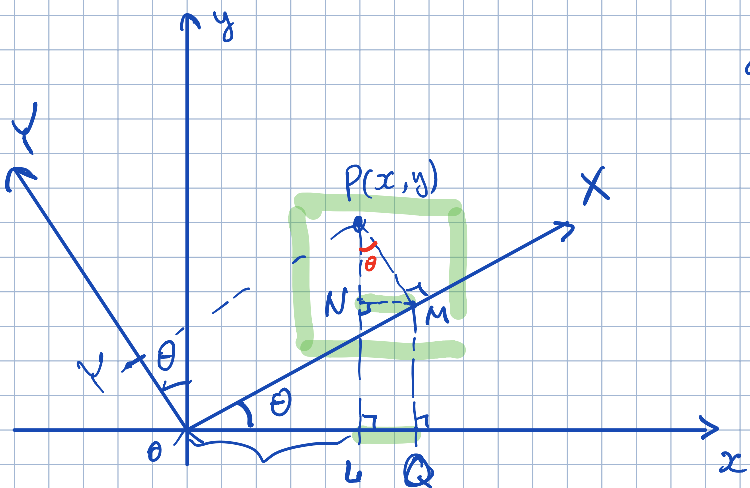
Lecture 11. Outline

- Part 1. Changes of coordinates. Invariants
- Part 2. Polar coordinates. Equation of lines and conics

Part 1. Changes of coordinates. Invariants



$\angle xOX = \angle yOY = \theta$

$PL \perp Ox$

$PM \perp OX$

$MN \perp PL$

$\angle NPM = \theta$

$P(X, Y)$

$$OL = OQ - LQ = OQ - NM =$$

$$= OM \cdot \cos\theta - NM =$$

$$= OM \cos\theta - PM \cdot \sin\theta$$

$$x = X\cos\theta - Y\sin\theta$$

$$y = X\sin\theta + Y\cos\theta$$

$$X = x\cos\theta + y\sin\theta$$

$$Y = -x\sin\theta + y\cos\theta$$

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

goal: transform to remove XY-term with angle θ

$$x = \underbrace{X \cdot \cos\theta - Y \cdot \sin\theta}$$

$$y = \underbrace{X \cdot \sin\theta + Y \cdot \cos\theta}$$

$$a(X\cos\theta - Y\sin\theta)^2 + 2h(X \cdot \cos\theta - Y \cdot \sin\theta) \cdot (X \cdot \sin\theta + Y \cdot \cos\theta) + b((X \cdot \sin\theta + Y \cdot \cos\theta))^2 +$$

$$+ 2g(X\cos\theta - Y\sin\theta) + 2f(X\sin\theta + Y\cos\theta) + c = 0$$

....

$$(a\cos^2\theta + 2h \cdot \sin\theta \cdot \cos\theta + b^2 \sin^2\theta)X^2 -$$

$$-2\underbrace{\left[(a-b)\sin\theta \cdot \cos\theta - h(\cos^2\theta - \sin^2\theta)\right]}_{=0} XY ..$$

. ..

$$(a-b)(\sin\theta\cdot\cos\theta) - \underbrace{h(\cos^2\theta - \sin^2\theta)} = 0$$

$$:(a-b)\cdot(\cos^2\theta - \sin^2\theta)$$

$$\boxed{\frac{\sin\theta\cdot\cos\theta}{\cos^2\theta-\sin^2\theta}} = \frac{h}{a-b}$$

$$tg\,2\theta = \frac{2\,tg\,\theta}{1-tg^2\theta} \quad t$$

$$\frac{1}{2}tg\,2\theta = \frac{h}{a-b}$$

$$tg\,2\theta = \boxed{\frac{2h}{a-b}}$$

$$\underbrace{ax^2 + 2hxy + by^2} + 2gx + 2fy + c = 0$$

θ s.t. $\tan 2\theta = \dfrac{2h}{a-b}$

$$AX^2 + BY^2 + 2GX + 2FY = 0$$

$a+b = A+B$, $\boxed{a\cdot b - h^2} = A\cdot B$?

$a\cdot b - h^2 = A\cdot B + \cancel{H^2}$

Cas1: $ab - h^2 = 0$

$A\cdot B = 0 \Rightarrow A = 0$ or $B = 0$

$A = 0 \Rightarrow$ parabola $(G \neq 0)$

$G = 0 \Rightarrow$

$BY^2 + 2FY = 0$

2 Lines

Case 2 $\quad ab - h^2 \neq 0$

$ab - h^2 < 0 \Rightarrow AB < 0$

$\Rightarrow$ hyperbola.

$A = -B$ rectangular.

$a \cdot b - h^2 > 0 \Rightarrow AB > 0$

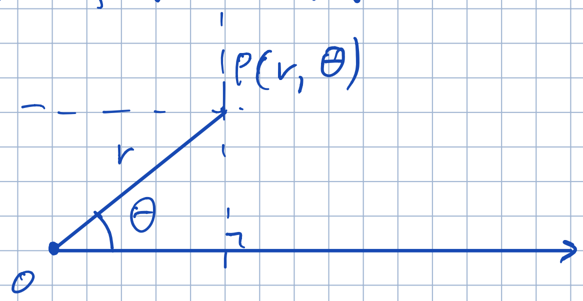
$\Rightarrow$ ellipse

1). Lines $\Rightarrow$ . . .

2) Circle $\Rightarrow$ $A = B$

Part 2. Polar coordinates. Equation of lines and conics

Definition.



(r, θ) - coordinates of point P

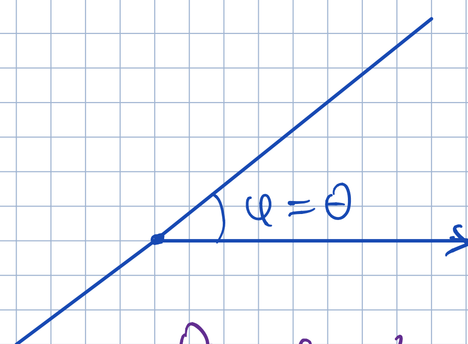
$x = r \cos\theta$

$y = r \sin\theta$

$r^2 = x^2 + y^2$

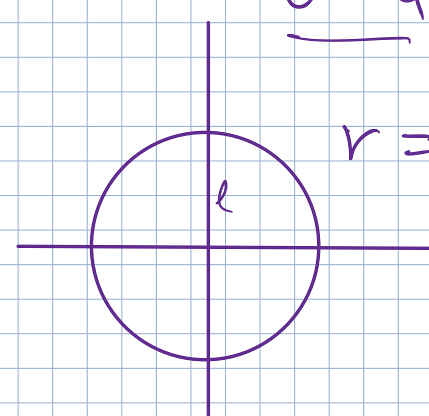
$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} \qquad \sin\theta = \frac{y}{\sqrt{x^2+y^2}}$$

Eq-n of a Line.



$\theta = \text{const}$

$\underline{\theta = \varphi}$ line



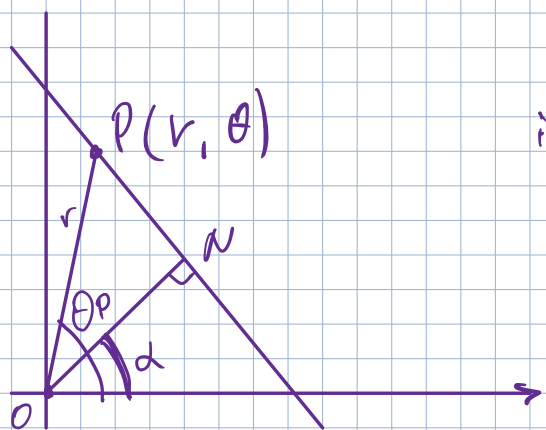
$r = \ell$

$$Ax + By + C = 0$$

$$Ar\cos\theta + Br\sin\theta + C = 0$$

$$A\cos\theta + B\sin\theta = \frac{\ell}{r}$$

line in P.C.



in △ OPN : $\frac{ON}{OP} = \cos(\theta - \alpha)$

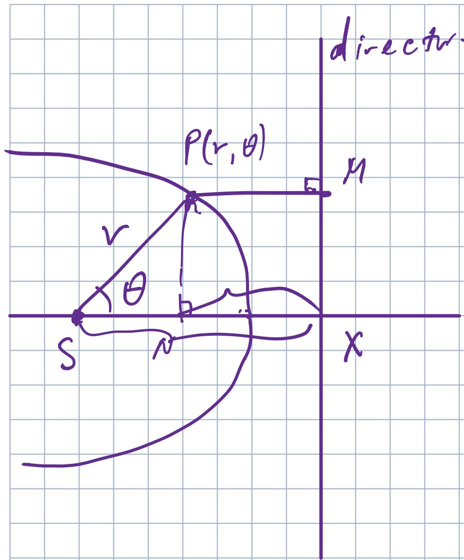
$$\frac{p}{r} = \cos(\theta - \alpha)$$

if new Line is $\perp$ to $A\cos\theta + B\sin\theta = \frac{\ell}{r}$

$\Leftrightarrow$ its eq-n : $A\cos\left(\theta + \frac{\pi}{2}\right) + B\sin\left(\theta + \frac{\pi}{2}\right) = \frac{k\ell}{r}$

directr..

S – focus



$$\frac{SP}{PN} = \varepsilon$$

$\angle XSP = \theta$ $\qquad SX = \dfrac{\ell}{\varepsilon}$

$SP = r$

$PN = r \cdot \sin\theta$

$SN = r \cdot \cos\theta$

$$r = SP = \varepsilon\, PM = \varepsilon \cdot NX =$$

$$= \varepsilon\,(SX - SN) =$$

$$= \varepsilon\left(\frac{\ell}{\varepsilon} - r\cos\theta\right)$$

$$r = \varepsilon \cdot \left(\frac{\ell}{\varepsilon} - r\cos\theta\right)$$

$$1 = \frac{\ell}{r} - \varepsilon\cos\theta$$

$$\boxed{\frac{\ell}{r} = 1 + \varepsilon \cdot \cos\theta}$$

eq-n of a conic in Polar coord-tes.

$4x^2 + 9y^2 = 36 \longrightarrow$ P.C. eq-n.

Given a conic

$$Q(x, y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

- $B^2 - 4AC < 0$, the equation represents an ellipse;
  $A = C$ and $B = 0$, the equation represents a circle,
- $B^2 - 4AC = 0$, the equation represents a parabola;
- $B^2 - 4AC > 0$, the equation represents a hyperbola;
  $A + C = 0$, the equation represents a rectangular hyperbola

Useful links

- https://www.geogebra.org
- https://youtu.be/fNk_zzaMoSs
- http://immersivemath.com/ila