140C Summary of Definitions and Results - Part II

1 Lebesgue Theory

1.1 Set functions

Definition 1.1 *A family of sets $\mathcal{R}$ is called a ring if*

$$A, B \in \mathcal{R} \implies A \cup B \in \mathcal{R} \text{ and } A \setminus B \in \mathcal{R}.$$

Thus, a ring is closed under finite unions and set differences (as well as finite intersections).

Definition 1.2 *A σ-ring is a ring that is also closed under countable unions, i.e.,*

$$A_n \in \mathcal{R}, n = 1, 2, ... \implies \cup_n A_n \in \mathcal{R}.$$

It can be deduced that a σ-ring is also closed under countable intersections.

Definition 1.3 *A set function on a ring (or σ-ring) assigns to every element of $\mathcal{R}$ a number (in the extended reals).*

Definition 1.4 *A set function f is additive if for disjoint sets $A, B \in \mathcal{R}$,*

$$f(A \cup B) = f(A) + f(B).$$

Definition 1.5 *A set function f is countably additive if for disjoint sets $A_i \in \mathcal{R}$, $i = 1, 2, ...$*

$$f(\cup A_i) = \sum_{i=1}^{\infty} f(A_i).$$

Theorem 1.6 *Suppose f is a countably additive function on a σ-ring $\mathcal{R}$. Let $A_n \in \mathcal{R}$, $n = 1, 2, ...$ with $A_1 \subset A_2 \subset ...$ and suppose that $A := \bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$.*

Then

$$\lim_{n\to\infty} f(A_n) = f(A).$$

1.2 Construction of the Lebesgue measure

Definition 1.7 *An interval in* $\mathbb{R}^p$ *is a set of the form* $I = [a_1, b_1] \times \ldots \times [a_p, b_p]$.

Definition 1.8 *An elementary set is a finite union of intervals.*

Definition 1.9 *For an interval* I*, define the set function* m*, via*

$$m(I) := \prod_{i=1}^{p} (b_i - a_i).$$

Definition 1.10 *For a finite disjoint union of intervals* I_i*, set*

$$m(\bigcup_{i=1}^{n} I_i) := \sum_{i=1}^{n} m(I_i).$$

Definition 1.11 *Denote by* $\mathcal{E}$ *the collection of all elementary subsets of* $\mathbb{R}^p$

Remark 1.12 $\mathcal{E}$ *is a ring, but not a* σ*-ring. Elements of* $\mathcal{E}$ *can be decomposed as finite, disjoint unions of intervals.*

Remark 1.13 *The function* m *defined above is additive on* $\mathcal{E}$.

Definition 1.14 (Regularity) *A non-negative, additive set function* f *defined on* $\mathcal{E}$ *is regular if*

$$\forall A \in \mathcal{E}, \epsilon > 0, \exists F, G, \in \mathcal{E}, \text{ where } F \text{ is closed and } G \text{ is open,}$$

and

$$F \subset A \subset G,$$

with

$$f(G) - \epsilon \leq f(A) \leq f(F) + \epsilon.$$

Example 1.15 *The set function* m *defined above is regular on* $\mathcal{E}$.

Definition 1.16 (Outer measure) *Let* μ *be a non-negative, additive, finite, regular set function defined on* $\mathcal{E}$*. The outer measure of* $E \subset \mathbb{R}^p$ *is given by*

$$\mu^*(E) = \inf \sum_{n=1}^{\infty} \mu(A_n)$$

where the infimum is taken over all countable covers of E by elementary open sets.

Fact 1.17 *The following are simple to deduce from the definition:*

$$\mu^*(E) \geq 0,$$

$$E_1 \subset E_2 \implies \mu^*(E_1) \leq \mu^*(E_2).$$

Theorem 1.18 *Let μ be finite, non-negative, additive, regular, then μ^* agrees with μ on elementary sets, and it is countably sub-additive. That is,*

(a) $A \in \mathcal{E} \implies \mu^*(A) = \mu(A)$,

(b) $E = \bigcup_{i=1}^{\infty} E_i \implies \mu^*(E) \leq \sum_{i=1}^{\infty} \mu^*(E_i)$.

Definition 1.19 (Convergence) *We say that a sequence of sets A_n converges to A if*

$$\lim_{n\to\infty} d(A, A_n) = 0,$$

where for two sets A and B we define

$$d(A, B) := \mu^*\big((A \setminus B) \cup (B \setminus A)\big).$$

Notice that the notion of covergence above depends on the choice of μ.

Definition 1.20 (Finitely μ-measurable sets) *If there is a sequence of elementary sets converging to A, we say A is finitely μ-measurable; we write $A \in \mathcal{M}_F(\mu)$.*

Definition 1.21 (μ-measurable sets) *If A is a countable union of finitely μ-measurable sets, we say that A is μ-measurable; we write $A \in \mathcal{M}(\mu)$.*

Theorem 1.22 *$\mathcal{M}(\mu)$ is a σ-ring and μ^* is countably additive on $\mathcal{M}(\mu)$.*

Thus we may now replace $\mu^*(A)$ by $\mu(A)$ – and we can call μ a measure. When $\mu = m$, we call it the Lebesgue measure.

Definition 1.23 *A Borel set is a set that can be obtained via countable unions, countable intersections, and/or set differences, complements, of open sets.*

Remark 1.24 *The collection of Borel sets in $\mathbb{R}^p$ is a σ-ring. In fact, it is the smallest σ-ring containing all open sets.*

Remark 1.25 *Every $A \in \mathcal{M}(\mu)$ is the union of a Borel set and a set of measure zero.*

2 Measure spaces

Definition 2.1 *Let X be some set. If there exists a σ-ring $\mathcal{M}$ of subsets of X (called measurable sets) and a non-negative countable additive set function μ (called a measure) defined on μ, then X is called a* measure space. *We often write $(X, \mathcal{M}, \mu)$ to identify the σ-ring and measure associated with X.*

Definition 2.2 *We say that a function f defined on a measurable space $\mathcal{M}$ is measurable if*

$$\{x|f(x) > a\}$$

is measurable (i.e., belongs to $\mathcal{M}$) for all a.

Theorem 2.3 *The following are equivalent:*

- $\{x|f(x) > a\}$ *is measurable for all real a.*
- $\{x|f(x) \geq a\}$ *is measurable for all real a.*
- $\{x|f(x) < a\}$ *is measurable for all real a.*
- $\{x|f(x) \leq a\}$ *is measurable for all real a.*

Theorem 2.4 *The* $\inf$, $\sup$, $\liminf$, *and* $\limsup$ *of a sequence of measurable functions are measurable. The limit of a converging sequence of measurable functions is measurable.*

Theorem 2.5 *Suppose that f and g are measurable functions, then*

- $|f|$ *is also measurable*
- $\max(f, g)$, $\min(f, g)$, $f^+ = \max(f, 0)$, $f^- = -\min(f, 0)$ *are measurable.*

Theorem 2.6 *If f, g are measurable real-valued functions on X, and $F : \mathbb{R}^2 \to \mathbb{R}$ is continuous, then h defined via*

$$h(x) = F(f(x), g(x)), x \in X$$

is measurable. For example, this implies the sum and product of measurable functions is measurable.

Remark 2.7 *Notice that the way we define measurable functions does not really require a measure, only a σ-ring $\mathcal{M}$.*