Manoeuvring to required approach parameters–CPA distance and time

Article *in* Annual of Navigation · December 1999

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MANOEUVRING TO REQUIRED APPROACH PARAMETERS - CPA DISTANCE AND TIME

ABSTRACT

The predicted object CPA (Closest Point of Approach) distance D_{CPA} and, to a lesser extent, the time interval to its occurrence T_{CPA} are well established criteria for collision threat. They are approach parameters widely used as well in collision avoidance systems featuring computer - aided tracking (ARPAs) as in manual radar plots. The scope of this paper is aimed at the problem which although it can be and it is connected with collision avoidance manoeuvres, but it is rather reversed and can be applied for intentional approaches or in naval tactical manoeuvres - what own speed and/or course manoeuvre should be undertaken to achieve the required CPA distance and/or time?

ASSUMPTIONS AND INPUT PARAMETERS

For the purposes of this analysis, own vessel and extraneous objects of interest are regarded as if the mass of each object was concentrated at a point. It will be assumed that all moving external objects are travelling at constant speed and course. In the movable plane tangential to the Earth's surface Cartesian coordinates system Ox, Oy (Fig. 1) with Oy pointing North O is at the present position of own vessel. It will also be assumed that manual plots or the radar processing and tracking has yielded the present relative position of the extraneous object X, Y and components of its true V_{tx}, V_{ty} or relative V_{rx}, V_{ry} speed. The relationship of the own and the object speeds can be described by equations

$$V_{tx} = V_{rx} + V_x \tag{1}$$

$$V_{ty} = V_{ry} + V_y \tag{2}$$

where: V_x, V_y - own speed components,

$$V_x = V \sin \psi \tag{3}$$

$$V_y = V \cos \psi \tag{4}$$

$$V = \sqrt{V_x^2 + V_y^2} \tag{5}$$

where: ψ - own course (the angle measured clockwise from Oy to V).

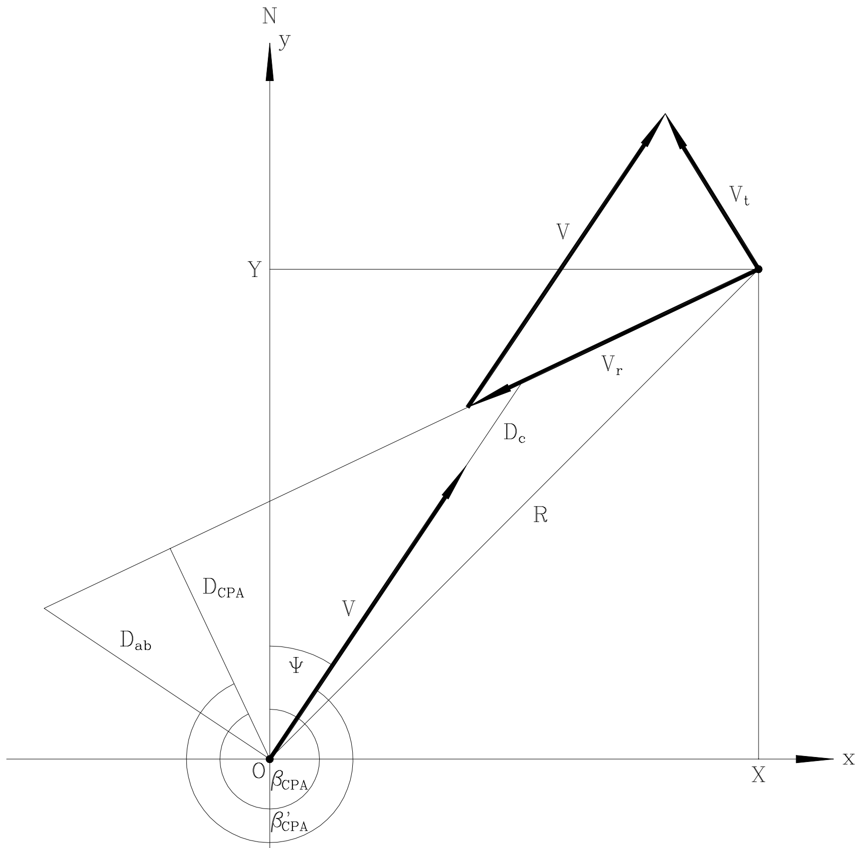

Fig. 1. Input parameters

The own and the object's motion parameters should be either ground or sea referenced and a drift angle is assumed to be zero.

The relative position of an extraneous object, at time t, is given by

$$X(t) = X + V_{rx}\,t \tag{6}$$

$$Y(t) = Y + V_{ry}\,t \tag{7}$$

and then [Lenart, 1986]

$$D_{CPA} = \left| \frac{XV_{ry} - YV_{rx}}{V_r} \right| \tag{8}$$

$$T_{CPA} = -\frac{XV_{rx} + YV_{ry}}{V_r^2} \tag{9}$$

where:

$$V_r = \sqrt{V_{rx}^2 + V_{ry}^2} \tag{10}$$

DERIVATION OF EQUATION $V = F(\psi, D_{CPA})$

From equations (8) and (10) squaring both sides and rearranging terms we obtain a quadratic equation in V_{ry}

$$(X^2 - D^2_{CPA})V^2_{ry} - 2XYV_{rx}V_{ry} + (Y^2 - D^2_{CPA})V^2_{rx} = 0 \tag{11}$$

whose solution is

$$V_{ry} = A_{DCPA}\, V_{rx} \tag{12}$$

where:

$$A_{DCPA} = \frac{XY \pm D_{CPA}\sqrt{R^2 - D_{CPA}^2}}{X^2 - D_{CPA}^2} \tag{13}$$

$$R = \sqrt{X^2 + Y^2} \tag{14}$$

From equations (1) through (4)

$$V_{rx} = V_{tx} - V \sin \psi \tag{15}$$

$$V_{ry} = V_{ty} - V \cos \psi \tag{16}$$

Substitution in equation (12) and rearranging yields

$$V = \frac{B_{DCPA}}{A_{DCPA} \sin\psi - \cos\psi} \tag{17}$$

where:

$$B_{DCPA} = A_{DCPA} V_{tx} - V_{ty} \tag{18}$$

and real solutions exist if

$$R \geq D_{CPA} \tag{19}$$

Equation (17) gives the speed V which own vessel must adopt to achieve the required CPA distance D_{CPA} (in respect to the selected object) for different assumed own courses ψ, but we should search for solution

$$V \geq 0 \tag{20}$$

and V, ψ for which

$$T_{CPA} \geq 0 \tag{21}$$

Condition (21) means that the closest approach is at present or will be in the future and not in the past. Equation (21) (from equations (9), (15) and (16)) can be rearranged to the form

$$V(X \sin\psi + Y \cos\psi) \geq XV_{tx} + YV_{ty} \tag{22}$$

A graphical interpretation of A_{DCPA} and B_{DCPA} can be obtained in Cartesian coordinates of own speed V_x , V_y substituting in equation (17) equations (3) and (4)

$$V_y = A_{DCPA} V_x - B_{DCPA} \tag{23}$$

In these coordinates all points corresponding to the required value of D_{CPA} will lie on two straight lines having slopes A_{DCPA} (A_{DCPA} and B_{DCPA} can have two values) and cutting the V_y axis at $-B_{DCPA}$.

A conventional PPI displays the position of each object by plotting them in polar (r, ψ) or Cartesian (x, y) coordinates. If we apply a scaling factor τ to the speed coordinates (V, ψ) or (V_x, V_y) such that

$$r = V\,\tau \tag{24}$$

$$x = V_x\,\tau \tag{25}$$

$$y = V_y\,\tau \tag{26}$$

then the position and speed coordinates can be plotted on a common display.

Equations (17) and (23) then transform respectively to

$$r = \frac{B_{DCPA}\,\tau}{A_{DCPA}\sin\psi - \cos\psi} \tag{27}$$

and

$$y = A_{DCPA}\,x - B_{DCPA}\,\tau \tag{28}$$

In the combined coordinates frame for plotting position and speed can also be plotted positions and speed vectors of objects and the own speed vector (real or simulated). Figure 2 illustrates a family of lines (23) or (27) and (28) for various required D_{CPA} and an exemplary object.

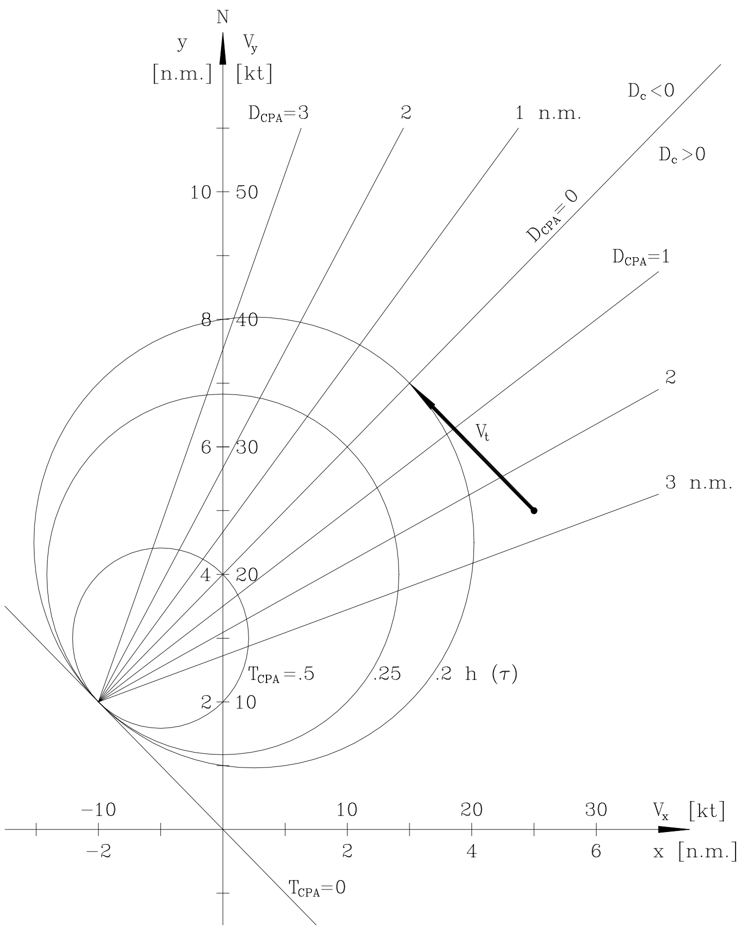

Fig. 2. Lines D_{CPA}=const. and circles T_{CPA}=const.
τ=0.2 h, X=Y=5 n.m., V_{tx}= -10 kt, V_{ty}=10 kt

Equation (17) can yield up to two real own speeds V. Let D_c be the distance at which the object crosses the course of own vessel. It can be proved (Lenart, 1986), that if for an assumed own course ψ exists $V=f(\psi, D_{CPA}=0)$ with $T_{CPA}>0$ (own speed which will lead to a collision) then for $V<V=f(\psi, D_{CPA}=0)$ the object will pass ahead ($D_c>0$), and for $V>V=f(\psi, D_{CPA}=0)$ the object will pass astern ($D_c<0$).

DERIVATION OF EQUATION ψ = G(V, D_{CPA})

If we search for own course ψ which will lead to the required CPA distance D_{CPA} at an assumed own speed V then we can get an inverse function $\psi=g(V, D_{CPA})$ to the function $V=f(\psi, D_{CPA})$ by substitution in equation (17) trigonometric identities

$$\sin\psi = \frac{2\tan\frac{\psi}{2}}{1+\tan^2\frac{\psi}{2}} \tag{29}$$

$$\cos\psi = \frac{1-\tan^2\frac{\psi}{2}}{1+\tan^2\frac{\psi}{2}} \tag{30}$$

which will result in equation

$$(V - B_{DCPA})\tan^2\frac{\psi}{2} + 2\,A_{DCPA}\,V\tan\frac{\psi}{2} - (V + B_{DCPA}) \tag{31}$$

and its solution

$$\tan\frac{\psi}{2} = \frac{A_{DCPA}V \pm \sqrt{(A^2_{DCPA}+1)V^2 - B^2_{DCPA}}}{B_{DCPA} - V} \tag{32}$$

Real solutions exist if

$$V^2 \ge \frac{B^2_{DCPA}}{A^2_{DCPA}+1} \;\text{ and }\; R \ge D_{CPA} \tag{33}$$

and equation (32) can give up to four own courses ψ which will lead to the required CPA distance D_{CPA} at an assumed own speed V if they additionally fulfil condition (22).

DERIVATION OF EQUATION $V = F(\psi, T_{CPA})$

Substitution in equation (9) equations (10), (15) and (16) gives a quadratic equation in V

$$T_{CPA}V^2-[(X+2V_{tx}T_{CPA})\sin\psi+(Y+2V_{ty}T_{CPA})\cos\psi]V+(V_t^2T_{CPA}+XV_{tx}+YV_{ty})=0 \quad (34)$$

whose solution is

$$V = A_{TCPA}\sin\psi + B_{TCPA}\cos\psi \pm \sqrt{(A_{TCPA}\sin\psi + B_{TCPA}\cos\psi)^2 - C_{TCPA}} \quad (35)$$

where:

$$A_{TCPA} = V_{tx} + \frac{X}{2T_{CPA}} \quad (36)$$

$$B_{TCPA} = V_{ty} + \frac{Y}{2T_{CPA}} \quad (37)$$

$$C_{TCPA} = V_t^2 + \frac{XV_{tx} + YV_{ty}}{T_{CPA}} \quad (38)$$

$$V_t^2 = V_{tx}^2 + V_{ty}^2 \quad (39)$$

Real solutions exist if

$$(A_{TCPA}\sin\psi + B_{TCPA}\cos\psi)^2 \geq C_{TCPA} \quad (40)$$

Equation (35) can yield up to two speeds $V \geq 0$ which own vessel must adopt to achieve the required time to CPA T_{CPA} (in respect to the selected object) for different assumed own courses ψ.

A graphical interpretation of solutions given by equation (35) can be obtained in Cartesian coordinates of own speed V_x, V_y substituting in equation (34) equations (3) and (4)

$$(V_x - A_{TCPA})^2 + (V_y - B_{TCPA})^2 = \left(\frac{R}{2\,T_{CPA}}\right)^2 \quad (41)$$

The above equation reveals that the locus of points for which T_{CPA} is a constant is a circle centred at (A_{TCPA}, B_{TCPA}) and having radius $\left|R/(2\,T_{CPA})\right|$. Figure 2 illustrates a family of circles for various values of T_{CPA}.

DERIVATION OF EQUATION $\psi = G(V, T_{CPA})$

If we search for own course ψ which will lead to the required time to CPA T_{CPA} at an assumed own speed V then we can get an inverse function $\psi=g(V, T_{CPA})$ to the function $V=f(\psi, T_{CPA})$ by substitution in equation (34) identities (29) and (30) which after solving yields

$$\tan\frac{\psi}{2} = \frac{A_{\psi TCPA} \pm \sqrt{A^2_{\psi TCPA} + B^2_{\psi TCPA} - C^2_{\psi TCPA}}}{B_{\psi TCPA} + C_{\psi TCPA}} \tag{42}$$

where:

$$A_{\psi TCPA} = X + 2\, V_{tx}\, T_{CPA} = 2\, A_{TCPA} T_{CPA} \tag{43}$$

$$B_{\psi TCPA} = Y + 2\, V_{ty}\, T_{CPA} = 2\, B_{TCPA}\, T_{CPA} \tag{44}$$

$$C_{\psi TCPA} = \frac{(V^2 + V_t^2)T_{CPA} + XV_{tx} + YV_{ty}}{V} = \left(V + \frac{C_{TCPA}}{V}\right)T_{CPA} \tag{45}$$

Real solutions exist if

$$A^2_{\psi TCPA} + B^2_{\psi TCPA} \geq C^2_{\psi TCPA} \tag{46}$$

and equation (42) can give up to two own courses ψ which will lead to the required time to CPA T_{CPA} at an assumed own speed V.

DERIVATION OF EQUATIONS V, $\psi = F(D_{CPA}, T_{CPA})$

From equation (9), taking into consideration equations (10) and (12), we can get

$$T_{CPA}V^2_{rx} + T_{CPA}V^2_{ry} + XV_{rx} + YV_{ry} = 0 \tag{47}$$

$$V_{ry} = A_{DCPA}\, V_{rx} \tag{48}$$

This system of equations has two solutions

$$V_{rx} = 0,\ V_{ry} = 0 \tag{50}$$

and

$$V_{rx} = -\frac{X + A_{DCPA} Y}{(A_{DCPA}^2 + 1) T_{CPA}} \tag{51}$$

$$V_{ry} = -\frac{A_{DCPA}(X + A_{DCPA} Y)}{(A_{DCPA}^2 + 1) T_{CPA}} \tag{52}$$

The first solution is a consequence of the fact that D_{CPA} (equation (8)) and T_{CPA} (equation (9)) are mathematically indeterminate if V_{rx}=0, V_{ry}=0 (in Fig. 2 all lines D_{CPA} and circles T_{CPA} crosses the point V_{rx}=0, V_{ry}=0 i. e. V_x=V_{tx}, V_y=V_{ty}) but we can assume that in that case D_{CPA}=R and T_{CPA}=0 [Lenart, 1986].

The second solution is real if (equation (19))

$$R \geq D_{CPA} \tag{53}$$

and with regard to equations (1) through (5)

$$V_x = V_{tx} + \frac{X + A_{DCPA} Y}{(A_{DCPA}^2 + 1) T_{CPA}} \tag{54}$$

$$V_y = V_{ty} + \frac{A_{DCPA}(X + A_{CPDA} Y)}{(A_{DCPA}^2 + 1) T_{CPA}} \tag{55}$$

$$V = \sqrt{V_x^2 + V_y^2} \tag{56}$$

$$\tan\psi = \frac{V_x}{V_y} \tag{57}$$

Equations (54) through (57) and (13), (14) can give up to two own speeds V and own courses ψ which will lead to the required CPA distance D_{CPA} at the required time T_{CPA}.

POSITION OF CPA

At the closest point of approach the relative position of the object (in respect to our vessel) is (X_{CPA}, Y_{CPA}) or in polar coordinates (D_{CPA}, β_{CPA}) or (D_{CPA}, β'_{CPA}) where β_{CPA} and β'_{CPA} are true and relative bearings to the object at CPA respectively. These parameters are given by equations

$$X_{CPA} = X + V_{rx} T_{CPA} = X + (V_{tx} - V \sin\psi) T_{CPA} \tag{58}$$

$$Y_{CPA} = Y + V_{ry} T_{CPA} = Y + (V_{ty} - V \cos\psi) T_{CPA} \tag{59}$$

$$D_{CPA} = \sqrt{X_{CPA}^2 + Y_{CPA}^2} \tag{60}$$

$$\tan\beta_{CPA} = \frac{X_{CPA}}{Y_{CPA}} \tag{61}$$

$$\beta'_{CPA} = \beta_{CPA} - \psi \tag{62}$$

and D_{CPA} and T_{CPA} are either required or calculated from equation (60) or (8) and (9) transformed to true speeds by substitution equations (15) and 16)

$$D_{CPA} = \left| \frac{XV_{ty} - YV_{tx} - (X\cos\psi - Y\sin\psi)V}{\sqrt{V^2 + V_t^2 - 2V(V_{tx}\sin\psi + V_{ty}\cos\psi)}} \right| \tag{63}$$

$$T_{CPA} = \frac{(X\sin\psi + Y\cos\psi)V - (XV_{tx} + YV_{ty})}{V^2 + V_t^2 - 2V(V_{tx}\sin\psi + V_{ty}\cos\psi)} \tag{64}$$

Having calculated β'_{CPA} we can also calculate the distance on course D_c and the distance abeam D_{ab} [Lenart, 1986]

$$D_c = \frac{D_{CPA}}{\cos\beta'_{CPA}} \tag{65}$$

$$D_{ab} = \frac{D_{CPA}}{\sin\beta'_{CPA}} \tag{66}$$

TIME TO MANOEUVRE

It has to be emphasized that the calculated above manoeuvres are kinematic and should be undertaken immediately. If we require to have the time lapse Δt for calculations, for the decision to initiate a manoeuvre and for the execution of the calculated manoeuvre then X, Y in the above equations should be replaced by $X_{\Delta t}$, $Y_{\Delta t}$ respectively, given by equations

$$X_{\Delta t} = X + V_{rx}\Delta t \tag{67}$$

$$Y_{\Delta t} = Y + V_{ry}\Delta t \tag{68}$$

REFERENCES

1. Lenart A. S. Some selected problems in analysis and synthesis of shipboard collision avoidance systems. Zeszyty Naukowe Politechniki Gdańskiej Nr 405 - Budownictwo Okrętowe Nr XLIV, Gdańsk 1986 (in Polish).