APPENDIX A

Boundary Element Methods

Introduction

The Laplace equation is a *linear* differential equation, i.e. arbitrary linear combinations (superpositions) of elementary solutions of the Laplace equation will again form a possible solution. This chapter is devoted to various elementary solutions used in the computation of ship flows. It is not really necessary to understand the given formulae, but the concepts should be understood. Fortran subroutines for elements are in the public domain and may be obtained on the internet (see Preface).

Consider the case if still water is seen from a passing airplane with speed V, or from a razor blade ship not disturbing the flow. Here the water appears to flow uniformly in the negative direction x with the speed V. The water has no velocity component in the y or z direction, and everywhere uniformly the velocity is $-V$ in the x direction. The corresponding potential is:

$$\phi = -Vx$$

Another elementary potential is that of an undisturbed incident wave as given in Section 4.3.1, Chapter 4.

Various elements (elementary solutions) exist to approximate the disturbance effect of the ship. These more or less complicated mathematical expressions are useful to model displacement ('sources') or lift ('vortices', 'dipoles'). The common names indicate a graphical physical interpretation of the abstract mathematical formulae and will be discussed in the following.

The basic idea of all the related boundary element methods is to superimpose elements in an unbounded fluid. Since the flow does not cross a streamline just as it does not cross a real fluid boundary (such as the hull), any unbounded flow field in which a streamline coincides with the actual flow boundaries in the bounded problem can be interpreted as a solution for the bounded flow problem in the limited fluid domain. Outside this fluid domain, the flow cannot (and should not) be interpreted as a physical flow, even though the computation can give velocities, pressures, etc. everywhere.

The velocity at a field point $\vec{x}$ induced by some typical panel types and some related formula work is given in the following. Expressions are often derived in a local coordinate system. The derivatives of the potential are transformed from the local $x-y-z$ system to a global $\bar{x}-\bar{y}-\bar{z}$

system. In two dimensions, we limit ourselves to x and z as coordinates, as these are the typical coordinates for a strip in a strip method. $\vec{n} = (n_x, n_z)$ is the outward unit normal in global coordinates, coinciding with the local z vector. $\vec{t}$ and $\vec{s}$ are unit tangential vectors, coinciding with the local x and y vectors, respectively. The transformation from the local to the global system is as follows:

1. *Two-dimensional case*

$$\phi_{\bar{x}} = n_z \cdot \phi_x + n_x \cdot \phi_z$$

$$\phi_{\bar{z}} = -n_x \cdot \phi_x + n_z \cdot \phi_z$$

$$\phi_{\overline{xx}} = \left(n_z^2 - n_x^2\right) \cdot \phi_{xx} + (2n_x n_z) \cdot \phi_{xz}$$

$$\phi_{\overline{xz}} = \left(n_z^2 - n_x^2\right) \cdot \phi_{xz} - (2n_x n_z) \cdot \phi_{xx}$$

$$\phi_{\overline{xxz}} = n_z\left(1 - 4n_x^2\right) \cdot \phi_{xxz} - n_x\left(1 - 4n_z^2\right) \cdot \phi_{xzz}$$

$$\phi_{\overline{xzz}} = n_x\left(1 - 4n_z^2\right) \cdot \phi_{xxz} + n_z\left(1 - 4n_x^2\right) \cdot \phi_{xzz}$$

2. *Three-dimensional case*

$$\phi_{\bar{x}} = t_1 \cdot \phi_x + s_1 \cdot \phi_y + n_1 \cdot \phi_z$$

$$\phi_{\bar{y}} = t_2 \cdot \phi_x + s_2 \cdot \phi_y + n_2 \cdot \phi_z$$

$$\phi_{\bar{z}} = t_3 \cdot \phi_x + s_3 \cdot \phi_y + n_3 \cdot \phi_z$$

$$\phi_{\bar{x}\bar{x}} = t_1^2 \phi_{xx} + s_1^2 \phi_{yy} + n_1^2 \phi_{zz} + 2(s_1 t_1 \phi_{xy} + n_1 t_1 \phi_{xz} + n_1 s_1 \phi_{yz})$$

$$\begin{aligned}\phi_{\bar{x}\bar{y}} = {} & t_1 t_2 \phi_{xx} + t_1 s_2 \phi_{xy} + t_1 n_2 \phi_{xz} + s_1 t_2 \phi_{xy} + s_1 s_2 \phi_{yy} + s_1 n_2 \phi_{yz} + n_1 t_2 \phi_{xz} \\ & + n_1 s_2 \phi_{yz} + n_1 n_2 \phi_{zz}\end{aligned}$$

$$\begin{aligned}\phi_{\bar{x}\bar{z}} = {} & t_1 t_3 \phi_{xx} + t_1 s_3 \phi_{xy} + t_1 n_3 \phi_{xz} + s_1 t_3 \phi_{xy} + s_1 s_3 \phi_{yy} + s_1 n_3 \phi_{yz} + n_1 t_3 \phi_{xz} \\ & + n_1 s_3 \phi_{yz} + n_1 n_3 \phi_{zz}\end{aligned}$$

$$\phi_{\bar{y}\bar{y}} = t_2^2 \phi_{xx} + s_2^2 \phi_{yy} + n_2^2 \phi_{zz} + 2(s_2 t_2 \phi_{xy} + n_2 t_2 \phi_{xz} + n_2 s_2 \phi_{yz})$$

$$\begin{aligned}\phi_{\bar{y}\bar{z}} = {} & t_2 t_3 \phi_{xx} + t_2 s_3 \phi_{xy} + t_2 n_3 \phi_{xz} + s_2 t_3 \phi_{xy} + s_2 s_3 \phi_{yy} + s_2 n_3 \phi_{yz} + n_2 t_3 \phi_{xz} \\ & + n_2 s_3 \phi_{yz} + n_2 n_3 \phi_{zz}\end{aligned}$$

$$\begin{aligned}\phi_{\bar{x}\bar{x}\bar{z}} = {} & t_1^2(t_3 \phi_{xxx} + s_3 \phi_{xxy} + n_3 \phi_{xxz}) + s_1^2(t_3 \phi_{xyy} + s_3 \phi_{yyy} + n_3 \phi_{yyz}) \\ & + n_1^2(t_3 \phi_{xzz} + s_3 \phi_{yzz} + n_3 \phi_{zzz}) + 2(s_1 t_1(t_3 \phi_{xxy} + s_3 \phi_{xyy} + n_3 \phi_{xyz}) \\ & + n_1 t_1(t_3 \phi_{xxz} + s_3 \phi_{xyz} + n_3 \phi_{xzz}) + n_1 s_1(t_3 \phi_{xyz} + s_3 \phi_{yyz} + n_3 \phi_{yzz}))\end{aligned}$$

$$\begin{aligned}
\phi_{\bar{x}\bar{y}\bar{z}} = {} & t_1 t_2 (t_3 \phi_{xxx} + s_3 \phi_{xxy} + n_3 \phi_{xxz}) + (t_1 s_2 + s_1 t_2) \\
& \times (t_3 \phi_{xxy} + s_3 \phi_{xyy} + n_3 \phi_{xyz}) + s_1 s_2 (t_3 \phi_{xyy} + s_3 \phi_{yyy} + n_3 \phi_{yyz}) \\
& + (t_1 n_2 + n_1 t_2)(t_3 \phi_{xxz} + s_3 \phi_{xyz} + n_3 \phi_{xzz}) \\
& + n_1 n_2 (t_3 \phi_{xzz} + s_3 \phi_{yzz} + n_3 \phi_{zzz}) + (s_1 n_2 + n_1 s_2) \\
& \times (t_3 \phi_{xyz} + s_3 \phi_{yyz} + n_3 \phi_{yzz})
\end{aligned}$$

$$\begin{aligned}
\phi_{\bar{x}\bar{z}\bar{z}} = {} & t_1 t_3 (t_3 \phi_{xxx} + s_3 \phi_{xxy} + n_3 \phi_{xxz}) + (t_1 s_3 + s_1 t_3) \\
& \times (t_3 \phi_{xxy} + s_3 \phi_{xyy} + n_3 \phi_{xyz}) + s_1 s_3 (t_3 \phi_{xyy} + s_3 \phi_{yyy} + n_3 \phi_{yyz}) \\
& + (t_1 n_3 + n_1 t_3)(t_3 \phi_{xxz} + s_3 \phi_{xyz} + n_3 \phi_{xzz}) \\
& + n_1 n_3 (t_3 \phi_{xzz} + s_3 \phi_{yzz} + n_3 \phi_{zzz}) + (s_1 n_3 + n_1 s_3) \\
& \times (t_3 \phi_{xyz} + s_3 \phi_{yyz} + n_3 \phi_{yzz})
\end{aligned}$$

$$\begin{aligned}
\phi_{\bar{y}\bar{y}\bar{z}} = {} & t_2^2 (t_3 \phi_{xxx} + s_3 \phi_{xxy} + n_3 \phi_{xxz}) + s_2^2 (t_3 \phi_{xyy} + s_3 \phi_{yyy} + n_3 \phi_{yyz}) \\
& + n_2^2 (t_3 \phi_{xzz} + s_3 \phi_{yzz} + n_3 \phi_{zzz}) + 2(s_2 t_2 (t_3 \phi_{xxy} + s_3 \phi_{xyy} + n_3 \phi_{xyz}) \\
& + n_2 t_2 (t_3 \phi_{xxz} + s_3 \phi_{xyz} + n_3 \phi_{xzz}) + n_2 s_2 (t_3 \phi_{xyz} + s_3 \phi_{yyz} + n_3 \phi_{yzz}))
\end{aligned}$$

$$\begin{aligned}
\phi_{\bar{y}\bar{z}\bar{z}} = {} & t_2 t_3 (t_3 \phi_{xxx} + s_3 \phi_{xxy} + n_3 \phi_{xxz}) + (t_2 s_3 + s_2 t_3) \\
& \times (t_3 \phi_{xxy} + s_3 \phi_{xyy} + n_3 \phi_{xyz}) + s_2 s_3 (t_3 \phi_{xyy} + s_3 \phi_{yyy} + n_3 \phi_{yyz}) \\
& + (t_2 n_3 + n_2 t_3)(t_3 \phi_{xxz} + s_3 \phi_{xyz} + n_3 \phi_{xzz}) \\
& + n_2 n_3 (t_3 \phi_{xzz} + s_3 \phi_{yzz} + n_3 \phi_{zzz}) + (s_2 n_3 + n_2 s_3) \\
& \times (t_3 \phi_{xyz} + s_3 \phi_{yyz} + n_3 \phi_{yzz})
\end{aligned}$$

Source Elements

The most common elements used in ship flows are source elements which are used to model the displacement effect of a body. Elements used to model the lift effect such as vortices or dipoles are also employed if lift plays a significant role, e.g. in yawed ships for maneuvering.

Point Source

1. *Two-dimensional case*

   The coordinates of the source are (x_q, z_q). The distance between source point and field point (x, y) is $r = \sqrt{(x - x_q)^2 + (z - z_q)^2}$. The potential induced at the field point is then:

$$\phi = \frac{\sigma}{2\pi} \ln r = \frac{\sigma}{4\pi} \ln((x - x_q)^2 + (z - z_q)^2)$$

This yields the velocities:

$$\vec{v} = \begin{Bmatrix} \phi_x \\ \phi_z \end{Bmatrix} = \frac{\sigma}{2\pi r^2}\begin{Bmatrix} x - x_q \\ z - z_q \end{Bmatrix}$$

The absolute value of the velocity is then:

$$v = \frac{\sigma}{2\pi r^2}\sqrt{(x - x_q)^2 + (z - z_q)^2} = \frac{\sigma}{2\pi r}$$

The absolute value of the velocity is thus the same for all points on a radius r around the point source. The direction of the velocity is pointing radially away from the source point and the velocity decreases with distance as $1/r$. Thus the flow across each concentric ring around the source point is constant. The element can be physically interpreted as a source of water which constantly pours water flowing radially in all directions. σ is the strength of this source. For negative σ, the element acts like a sink with water flowing from all directions into the center. Figure A.1 illustrates the effect of the element.

Higher derivatives are:

$$\phi_{xx} = -\phi_{zz} = \frac{\sigma}{2\pi}\frac{1}{r^2} - \phi_x \cdot \frac{2(x - x_q)}{r^2}$$

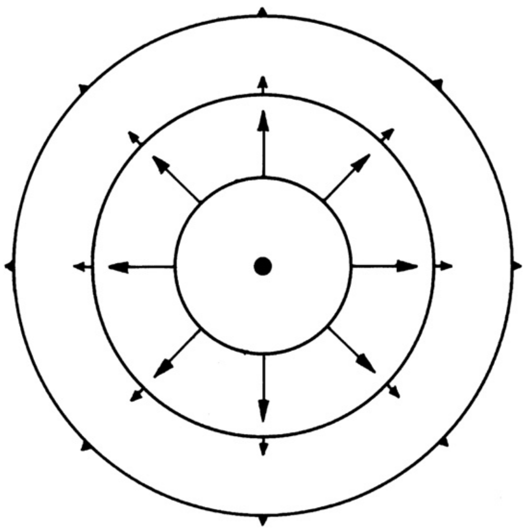

Figure A.1:
Effect of a point source

$$\phi_{xz} = -\phi_x \cdot \frac{2(z - z_q)}{r^2}$$

$$\phi_{xxz} = -\phi_{zzz} = -2 \cdot \left(\frac{(x - x_q)}{r^2} \phi_{xz} + \frac{(z - z_q)}{r^2} \phi_{xx} \right)$$

$$\phi_{xzz} = -2 \cdot \left(\frac{(x - x_q)}{r^2} \phi_{zz} + \frac{(z - z_q)}{r^2} \phi_{xz} \right)$$

2. *Three-dimensional case*

The corresponding expressions in three dimensions are:

$$r = \sqrt{(x - x_q)^2 + (y - y_q)^2 + (z - z_q)^2}$$

$$\phi = -\sigma \frac{1}{4\pi r}$$

$$\phi_x = \sigma \frac{1}{2\pi r^3} (x - x_q)$$

$$\phi_y = \sigma \frac{1}{2\pi r^3} (y - y_q)$$

$$\phi_z = \sigma \frac{1}{2\pi r^3} (z - z_q)$$

$$\phi_{xx} = (-3\phi_x (x - x_q) - \phi)/r^2$$

$$\phi_{xy} = (-3\phi_x (y - y_q))/r^2$$

$$\phi_{xz} = (-3\phi_x (z - z_q))/r^2$$

$$\phi_{yy} = (-3\phi_y (y - y_q) - \phi)/r^2$$

$$\phi_{yz} = (-3\phi_y (z - z_q))/r^2$$

$$\phi_{xxz} = -(2(\phi/r^2) + 5\phi_{xx})\mathrm{d}z/r^2$$

$$\phi_{xyz} = -5\phi_{xy}\mathrm{d}z/r^2$$

$$\phi_{xzz} = -5\phi_{xz}\mathrm{d}z/r^2 - 3\phi_x/r^2$$

$$\phi_{yyz} = -(2(\phi/r^2) + 5\phi_{yy})\mathrm{d}z/r^2$$

$$\phi_{yzz} = -5\phi_{yz}\mathrm{d}z/r^2 - 3\phi_y/r^2$$

Regular First-Order Panel

1. *Two-dimensional case*
For a panel of constant source strength we formulate the potential in a local coordinate system. The origin of the local system lies at the center of the panel. The panel lies on the local x-axis; the local z-axis is perpendicular to the panel pointing outward. The panel extends from $x = -d$ to $x = d$. The potential is then

$$\phi = \int_{-d}^{d} \frac{\sigma}{2\pi} \cdot \ln \sqrt{(x - x_q)^2 + z^2 \mathrm{d}x_q}$$

With the substitution $t = x - x_q$ this becomes:

$$\begin{aligned}\phi &= \frac{1}{2} \int_{x-d}^{x+d} \frac{\sigma}{2\pi} \cdot \ln(t^2 + \ z^2)\mathrm{d}t \\ &= \frac{\sigma}{4\pi} \left[t \ln\left(t^2 + z^2\right) \ + 2z \arctan \frac{t}{z} - 2t \right]_{x-d}^{x+d}\end{aligned}$$

Additive constants can be neglected, giving:

$$\phi = \frac{\sigma}{4\pi} \left(x \ln \frac{r_1}{r_2} + d \ln(r_1 r_2) + z2 \arctan \frac{2\mathrm{d}z}{x^2 + z^2 - d^2} + 4d \right)$$

with $r_1 = \ (x + d)^2 + z^2$ and $r_2 = (x - d)^2 + z^2$. The derivatives of the potential (still in local coordinates) are:

$$\phi_x = \frac{\sigma}{2\pi} \cdot \frac{1}{2} \ln \frac{r_1}{r_2}$$

$$\phi_z = \frac{\sigma}{2\pi} \cdot \arctan \frac{2\mathrm{d}z}{x^2 + z^2 - d^2}$$

$$\phi_{xx} = \frac{\sigma}{2\pi} \cdot \left(\frac{x + d}{r_1} - \frac{x - d}{r_2} \right)$$

$$\phi_{xz} = \frac{\sigma}{2\pi} \cdot z \cdot \left(\frac{1}{r_1} - \frac{1}{r_2} \right)$$

$$\phi_{xxz} = \frac{\sigma}{2\pi} \cdot (-2z) \cdot \left(\frac{x + d}{r_1^2} - \frac{x - d}{r_2^2} \right)$$

$$\phi_{xzz} = \frac{\sigma}{2\pi} \cdot \left(\frac{(x+d)^2 - z^2}{r_1^2} - \frac{(x-d)^2 - z^2}{r_2^2} \right)$$

ϕ_x cannot be evaluated (is singular) at the corners of the panel. For the center point of the panel itself ϕ_z is:

$$\phi_z(0,0) = \lim_{z \to 0} \phi_z(0,z) = \frac{\sigma}{2}$$

If the ATAN2 function in Fortran is used for the general expression of ϕ_z, this is automatically fulfilled.

2. *Three-dimensional case*
   In three dimensions the corresponding expressions for an arbitrary panel are rather complicated. Let us therefore consider first a simplified case, namely a plane rectangular panel of constant source strength (Fig. A.2). We denote the distances of the field point to the four corner points by:

$$r_1 = \sqrt{x^2 + y^2 + z^2}$$

$$r_2 = \sqrt{(x-\ell)^2 + y^2 + z^2}$$

$$r_3 = \sqrt{x^2 + (y-h)^2 + z^2}$$

$$r_4 = \sqrt{(x-\ell)^2 + (y-h)^2 + z^2}$$

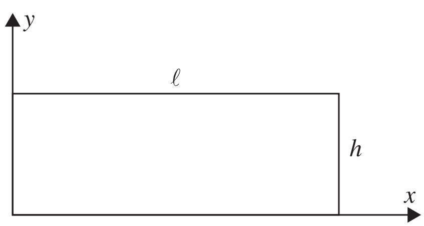

Figure A.2:
Simple rectangular flat panel of constant strength; origin at center of panel

The potential is:

$$\phi = -\frac{\sigma}{4\pi}\int_0^h \int_0^\ell \frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}}\,\mathrm{d}\xi\,\mathrm{d}\eta$$

The velocity in the x direction is:

$$\begin{aligned}
\frac{\partial\phi}{\partial x} &= \frac{\sigma}{4\pi}\int_0^h \int_0^\ell \frac{x-\xi}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}^3}\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&= \frac{\sigma}{4\pi}\int_0^h -\frac{1}{\sqrt{(x-\ell)^2 + (y-\eta)^2 + z^2}} + \frac{1}{\sqrt{x^2 + (y-\eta)^2 + z^2}}\,\mathrm{d}\eta \\
&= \frac{\sigma}{4\pi}\ln\frac{(r_3 - (y-h))(r_1 - y)}{(r_2 - y)(r_4 - (y-h))}
\end{aligned}$$

The velocity in the y direction is, in similar fashion:

$$\frac{\partial\phi}{\partial y} = \frac{\sigma}{4\pi}\ln\frac{(r_2 - (x-\ell))(r_1 - x)}{(r_3 - x)(r_4 - (x-\ell))}$$

The velocity in the z direction is:

$$\begin{aligned}
\frac{\partial\phi}{\partial y} &= \frac{\sigma}{4\pi}\int_0^h \int_0^\ell \frac{z}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}^3}\,\mathrm{d}\xi\,\mathrm{d}\eta \\
&= \frac{\sigma}{4\pi}\int_0^h -\frac{z(x-\ell)}{((y-\eta)^2 + z^2)\sqrt{(x-\ell)^2 + (y-\eta)^2 + z^2}} \\
&\quad + \frac{zx}{((y-\eta)^2 + z^2)\sqrt{x^2 + (y-\eta)^2 + z^2}}\,\mathrm{d}\eta
\end{aligned}$$

Substituting:

$$t = \frac{\eta - y}{\sqrt{x^2 + (\eta - y)^2 + z^2}}$$

yields:

$$\frac{\partial\phi}{\partial z} = \frac{\sigma}{4\pi}\left[\int_{-y/r_2}^{(h-y)/r_4} \frac{-z(x-\ell)}{z^2+(x-\ell)^2t^2}\,\mathrm{d}t + \int_{-y/r_1}^{(h-y)/r_3} \frac{zx}{z^2+x^2t^2}\,\mathrm{d}t\right]$$

$$= \frac{\sigma}{4\pi}\left[-\arctan\frac{x-\ell}{z}\frac{h-y}{r_4} + \arctan\frac{x-\ell}{z}\frac{-y}{r_2} - \arctan\frac{x}{z}\frac{-y}{r_1} + \arctan\frac{x}{z}\frac{h-y}{r_3}\right]$$

The derivation used:

$$\int \frac{1}{\sqrt{x^2+a^2}}\,\mathrm{d}x = \ln(x + \sqrt{x^2+a^2}) + C$$

$$\int \frac{x}{\sqrt{x^2+a^2}^3}\,\mathrm{d}x = -\frac{1}{\sqrt{x^2+a^2}} + C$$

$$\int \frac{1}{\sqrt{x^2+a^2}^3}\,\mathrm{d}x = \frac{x}{a^2\sqrt{x^2+a^2}} + C$$

$$\int \frac{1}{\sqrt{a+bx^2}}\,\mathrm{d}x = \frac{1}{\sqrt{ab}}\arctan\frac{bx}{\sqrt{ab}} \quad \text{for} \quad b>0$$

The numerical evaluation of the induced velocities has to consider some special cases. As an example: the finite accuracy of computers can lead to problems for the above given expression of the x component of the velocity, when for small values of x and z the argument of the logarithm is rounded off to zero. Therefore, for ($\sqrt{x^2+z^2} \ll y$) the term $r_1 - y$ must be substituted by the approximation $(x^2 + z^2)/2x$. The other velocity components require similar special treatment.

Hess and Smith (1964) pioneered the development of boundary element methods in aeronautics, thus also laying the foundation for most subsequent work for BEM applications to ship flows. Their original panel used constant source strength over a plane polygon, usually a quadrilateral. This panel is still the most popular choice in practice.

The velocity is again given in a local coordinate system (Fig. A.3). For quadrilaterals of unit source strength, the induced velocities are:

$$\frac{\partial\phi}{\partial x} = \frac{y_2-y_1}{d_{12}}\ln\left(\frac{r_1+r_2-d_{12}}{r_1+r_2+d_{12}}\right) + \frac{y_3-y_2}{d_{23}}\ln\left(\frac{r_2+r_3-d_{23}}{r_2+r_3+d_{23}}\right)$$

$$+ \frac{y_4-y_3}{d_{34}}\ln\left(\frac{r_3+r_4-d_{34}}{r_3+r_4+d_{34}}\right) + \frac{y_1-y_4}{d_{41}}\ln\left(\frac{r_4+r_1-d_{41}}{r_4+r_1+d_{41}}\right)$$

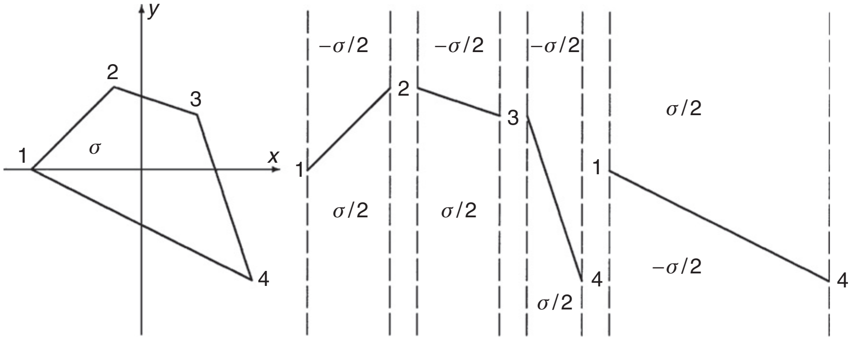

Figure A.3:
A quadrilateral flat panel of constant strength is represented by Hess and Smith as superposition of four semi-infinite strips

$$\frac{\partial\phi}{\partial y} = \frac{x_2 - x_1}{d_{12}} \ln\left(\frac{r_1 + r_2 - d_{12}}{r_1 + r_2 + d_{12}}\right) + \frac{x_3 - x_2}{d_{23}} \ln\left(\frac{r_2 + r_3 - d_{23}}{r_2 + r_3 + d_{23}}\right)$$
$$+ \frac{x_4 - x_2}{d_{34}} \ln\left(\frac{r_3 + r_4 - d_{34}}{r_3 + r_4 + d_{34}}\right) + \frac{x_1 - x_4}{d_{41}} \ln\left(\frac{r_4 + r_1 - d_{41}}{r_4 + r_1 + d_{41}}\right)$$

$$\frac{\partial\phi}{\partial z} = \arctan\left(\frac{m_{12}e_1 - h_1}{zr_1}\right) - \arctan\left(\frac{m_{12}e_2 - h_2}{zr_2}\right)$$
$$+ \arctan\left(\frac{m_{23}e_2 - h_2}{zr_2}\right) - \arctan\left(\frac{m_{23}e_2 - h_3}{zr_3}\right)$$
$$+ \arctan\left(\frac{m_{34}e_3 - h_3}{zr_3}\right) - \arctan\left(\frac{m_{34}e_4 - h_4}{zr_4}\right)$$
$$+ \arctan\left(\frac{m_{41}e_4 - h_4}{zr_4}\right) - \arctan\left(\frac{m_{41}e_1 - h_1}{zr_1}\right)$$

x_i, y_i are the local coordinates of the corner points i, r_i the distance of the field point (x, y, z) from the corner point i, d_{ij} the distance of the corner point i from the corner point $j, m_{ij} = (y_j - y_i)/(x_j - x_i), e_i = z^2 + (x - x_i)^2$ and $h_i = (y - y_i)(x - x_i)$. For larger distances between field point and panel, the velocities are approximated by a multipole expansion consisting of a point source and a point quadrupole. For large distances the point source alone approximates the effect of the panel.

For real ship geometries, four corners on the hull often do not lie in one plane. The panel corners are then constructed to lie within one plane approximating the four points on the actual hull: the normal on the panel is determined from the cross-product of the two ‘diagonal’ vectors. The center of the panel is determined by simple averaging of the coordinates of the four corners. This point and the normal define the plane of the panel. The four points on the hull are projected normally on this plane. The panels thus created do not form a closed body. As long as the gaps are small, the resulting errors are negligible compared to other sources of errors, e.g. the assumption of constant strength, constant pressure, constant normal over each panel, or enforcing the boundary condition only in one point of the panel. Hess and Smith (1964) comment on this issue:

> ‘Nevertheless, the fact that these openings exist is sometimes disturbing to people hearing about the method for the first time. It should be kept in mind that the elements are simply devices for obtaining the surface source distribution and that the polyhedral body… has no direct physical significance, in the sense that the flow eventually calculated is not the flow about the polyhedral-type body. Even if the edges of the adjacent elements are coincident, the normal velocity is zero at only one point of each element. Over the remainder of the element there is flow through it. Also, the computed velocity is infinite on the edges of the elements, whether these are coincident or not.’

Jensen Panel

Jensen (1988) developed a panel of the same order of accuracy, but much simpler to program, which avoids the evaluation of complicated transcendental functions and in it implementation relies largely on just a repeated evaluation of point source routines. As the original publication is little known and difficult to obtain internationally, the theory is repeated here. The approach requires, however, closed bodies. Then the velocities (and higher derivatives) can be computed by simple numerical integration if the integrands are transformed analytically to remove singularities. In the formulae for this element, $\vec{n}$ is the unit normal pointing outward from the body into the fluid, $\oint$ the integral over S excluding the immediate neighborhood of $\vec{x}_q$, and ∇ the Nabla operator with respect to $\vec{x}$.

1. *Two-dimensional case*

   A Rankine source distribution on a closed body induces the following potential at a field point $\vec{x}$:

$$\phi(\vec{x}) = \int_S \sigma(\vec{x}_q) G(\vec{x}, \vec{x}_q) \, \mathrm{dS}$$

   S is the surface contour of the body, σ the source strength, and $G(\vec{x}, \vec{x}_q) = (1/2\pi) \ln|\vec{x} - \vec{x}_q|$ is the Green function (potential) of a unit point source. Then the induced normal velocity component is:

$$v_n(\vec{x}) = \vec{n}(\vec{x}) \nabla \phi(\vec{x}) = \oint \sigma(\vec{x}_q) \vec{n}(\vec{x}) \nabla G(\vec{x}, \vec{x}_q) \, \mathrm{d}S + \frac{1}{2} \sigma(\vec{x}_q)$$

   Usually the normal velocity is given as boundary condition. Then the important part of the solution is the tangential velocity on the body:

$$v_t(\vec{x}) = \vec{t}(\vec{x}) \nabla \phi(\vec{x}) = \oint \sigma(\vec{x}_q) \vec{t}(\vec{x}) \nabla G(\vec{x}, \vec{x}_q) \, \mathrm{d}S$$

Without further proof, the tangential velocity (circulation) induced by a distribution of point sources of the same strength at point $\bar{x}_q$ vanishes:

$$\oint_{S(\vec{x})} \nabla G(\vec{x},\vec{x}_q)\vec{t}(\vec{x})\,\mathrm{d}\mathcal{S} = 0$$

Exchanging the designations $\vec{x}$ and $\vec{x}_q$ and using $\nabla G(\vec{x},\vec{x}_q) = -\nabla G(\vec{x}_q,\vec{x})$, we obtain:

$$\oint_s \nabla G(\vec{x},\vec{x}_q)\vec{t}(\vec{x}_q)\mathrm{d}\mathcal{S} = 0$$

We can multiply the integrand by $\sigma(\vec{x})$ — which is a constant as the integration variable is $\vec{x}_q$ — and subtract this zero expression from our initial integral expression for the tangential velocity:

$$\begin{aligned} v_t(\vec{x}) &= \oint_S \sigma(\vec{x}_q)\vec{t}(\vec{x})\nabla G(\vec{x},\vec{x}_q)\mathrm{d}S - \underbrace{\oint_S \sigma(\vec{x})\nabla G(\vec{x},\vec{x}_q)\vec{t}(\vec{x}_q)\mathrm{d}\mathcal{S}}_{=0} \\ &= \oint_S \left[\sigma(\vec{x}_q)\vec{t}(\vec{x}) - \sigma(\vec{x})\vec{t}(\vec{x}_q)\right]\nabla G(\vec{x},\vec{x}_q)\mathrm{d}\mathcal{S} \end{aligned}$$

For panels of constant source strength, the integrand in this formula tends to zero as $\vec{x} \rightarrow \vec{x}_q$, i.e. at the previously singular point of the integral. Therefore this expression for v_t can be evaluated numerically. Only the length ΔS of the contour panels and the first derivatives of the source potential for each $\vec{x}, \vec{x}_q$ combination are required.

2. *Three-dimensional case*
   The potential at a field point $\vec{x}$ due to a source distribution on a closed body surface S is:

$$\phi(\vec{x}) = \int_s \sigma(\vec{x}_q)G(\vec{x},\vec{x}_q)\mathrm{d}\mathcal{S}$$

σ is the source strength and $G(\vec{x},\vec{x}_q) = -(4\pi|\vec{x}-\vec{x}_q|)^{-1}$ is the Green function (potential) of a unit point source. Then the induced normal velocity component on the body is:

$$v_n(\vec{x}) = \vec{n}(\vec{x})\nabla\phi(\vec{x}) = \oint_S \sigma(\vec{x}_q)\vec{n}(\vec{x})\nabla G(\vec{x},\vec{x}_q)\ \mathrm{d}\mathcal{S} + \frac{1}{2}\sigma(\vec{x}_q)$$

Usually the normal velocity is prescribed by the boundary condition. Then the important part of the solution is the velocity in the tangential directions $\vec{t}$ and $\vec{s}$. $\vec{t}$ can be chosen arbitrarily, $\vec{s}$ forms a right-handed coordinate system with $\vec{n}$ and $\vec{t}$. We will treat here only the velocity in the t direction, as the velocity in the s direction has the same form. The original, straightforward form is:

$$v_t(\vec{x}) = \vec{t}(\vec{x})\nabla\phi(\vec{x}) = \oint_S \sigma(\vec{x}_q)\vec{t}(\vec{x})\nabla G(\vec{x},\vec{x}_q)\mathrm{d}\mathcal{S}$$

A source distribution of constant strength on the surface $\mathcal{S}$ of a sphere does not induce a tangential velocity on $\mathcal{S}$:

$$\oint_S \vec{t}(\vec{x})\nabla G(\vec{x},\vec{k})\mathrm{d}\mathcal{S} = 0$$

for $\vec{x}$ and $\vec{k}$ on $\mathcal{S}$. The sphere is placed touching the body tangentially at the point $\vec{x}$. The center of the sphere must lie within the body. (The radius of the sphere has little influence on the results within wide limits. A rather large radius is recommended.) Then every point $\vec{x}_q$ on the body surface can be projected to a point $\vec{k}$ on the sphere surface by passing a straight line through $\vec{k}, \vec{x}_q$, and the sphere's center. This projection is denoted by $\vec{k} = P(\vec{x}_q)$. d$\mathcal{S}$ on the body is projected on d$\mathcal{S}$ on the sphere. R denotes the relative size of these areas: d$\mathcal{S}$ = R d$\mathcal{S}$. Let R be the radius of the sphere and $\vec{c}$ be its center. Then the projection of $\vec{x}_q$ is:

$$P(\vec{x}_q) = \frac{\vec{x}_q - \vec{c}}{|\vec{x}_q - \vec{c}|}\, R + \vec{c}$$

The area ratio R is given by:

$$R = \frac{\vec{n}\cdot(\vec{x}_q - \vec{c})}{|\vec{x}_q - \vec{c}|}\left(\frac{R}{|\vec{x}_q - \vec{c}|}\right)^2$$

With these definitions, the contribution of the sphere ('fancy zero') can be transformed into an integral over the body surface:

$$\oint_S \vec{t}(\vec{x})\nabla G(\vec{x}, P(\vec{x}_q))R\, \mathrm{d}\mathcal{S} = 0$$

We can multiply the integrand by $\sigma(\vec{x})$ − which is a constant as the integration variable is $\vec{x}_q$ − and subtract this zero expression from our original expression for the tangential velocity:

$$\begin{aligned} v_t(\vec{x}) &= \oint_S \sigma(\vec{x}_q)\vec{t}(\vec{x})\nabla G(\vec{x},\vec{x}_q)\mathrm{d}\mathcal{S} - \underbrace{\oint_S \sigma(\vec{x})\, \vec{t}(\vec{x})\nabla G(\vec{x}, P(\vec{x}_q))R\, \mathrm{d}\mathcal{S}}_{=0} \\ &= \oint_S \left[\sigma(\vec{x}_q)\vec{t}(\vec{x})\nabla G(\vec{x},\vec{x}_q) - \sigma(\vec{x})\vec{t}(\vec{x})\nabla G(\vec{x}, P(\vec{x}_q))R\right]\mathrm{d}\mathcal{S} \end{aligned}$$

For panels of constant source strength, the integrand in this expression tends to zero as $\vec{x} \rightarrow \vec{x}_q$, i.e. at the previously singular point of the integral. Therefore this expression for v_t can be evaluated numerically.

Higher-Order Panel

The panels considered so far are 'first-order' panels, i.e. halving the grid spacing will halve the error in approximating a flow (for sufficiently fine grids). Higher-order panels (these are invariably second-order panels) will quadratically decrease the error for grid refinement. Second-order panels need to be at least quadratic in shape and linear in source distribution. They give much better results for simple geometries which can be described easily by analytical terms, e.g. spheres or Wigley parabolic hulls. For real ship geometries, first-order panels are usually sufficient and may even be more accurate for the same effort, as higher-order panels require more care in grid generation and are prone to 'overshoot' in regions of high curvature as in the aftbody. For some applications, however, second derivatives of the potential are needed on the hull and these are evaluated simply by second-order panels, but not by first-order panels.

1. *Two-dimensional case*
   We want to compute derivatives of the potential at a point (x, y) induced by a given curved portion of the boundary. It is convenient to describe the problem in a local coordinate system (Fig. A.4). The x- or ξ-axis is tangential to the curve and the perpendicular projections on the x-axis of the ends of the curve lie equal distances d to the right and the left of the origin. The y- or η-axis is normal to the curve. The arc length along the curve is denoted by s, and a general point on the curve is (ξ, η). The distance between (x, y) and (ξ, η) is:

$$r = \sqrt{(x-\xi)^2 + (y-\eta)^2}$$

The velocity induced at (x, y) by a source density distribution $\sigma(s)$ along the boundary curve is:

$$\nabla\phi = \frac{1}{2\pi}\int_{-d}^{d}\begin{Bmatrix} x-\xi \\ y-\eta \end{Bmatrix}\frac{\sigma(s)}{r^2}\frac{ds}{d\xi}\,d\xi$$

The boundary curve is defined by $\eta = \eta(\xi)$. In the neighborhood of the origin, the curve has a power series:

$$\eta = c\xi^2 + d\xi^3 + \cdots$$

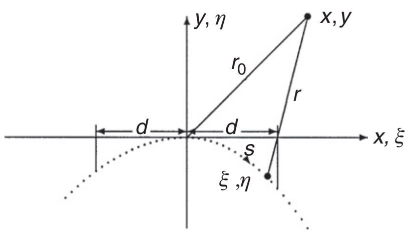

Figure A.4:
Coordinate system for higher-order panel (two-dimensional)

There is no term proportional to ξ, because the coordinate system lies tangentially to the panel. Similarly the source density has a power series:

$$\sigma(s) = \sigma^{(0)} + \sigma^{(1)}s + \sigma^{(2)}s^2 + \cdots$$

Then the integrand in the above expression for ∇_ϕ can be expressed as a function of ξ and then expanded in powers of ξ. The resulting integrals can be integrated to give an expansion for ∇_ϕ in powers of *d*. However, the resulting expansion will not converge if the distance of the point (*x, y*) from the origin is less than *d*. Therefore, a modified expansion is used for the distance *r*:

$$r^2 = \left[\left(x-\xi\right)^2 + y^2\right] - 2y\eta\ + \eta^2\ = r_f^2 - 2y\eta + \eta^2$$

$r_f = \sqrt{(x-\xi)^2 + y^2}$ is the distance (x, y) from a point on the flat element. Only the latter terms in this expression for r^2 are expanded:

$$r^2 = r_f^2\ - 2yc\xi^2\ + O(\xi^3)$$

Powers $O(\xi^3)$ and higher will be neglected from now on:

$$\frac{1}{r^2} = \frac{1}{r_f^2 - 2yc\xi^2}\cdot\frac{r_f^2 + 2yc\xi^2}{r_f^2 + 2yc\xi^2} = \frac{1}{r_f^2} + \frac{2yc\xi^2}{r_f^4}$$
$$\frac{1}{r^4} = \frac{1}{r_f^4} + \frac{4yc\xi^2}{r_f^6}$$

The remaining parts of the expansion are straightforward:

$$s = \int_0^\xi \sqrt{1 + \left(\frac{\mathrm{d}\eta}{\mathrm{d}\xi}\right)^2}\mathrm{d}\xi = \int_0^\xi \sqrt{1 + (2c\xi)^2}\mathrm{d}\xi$$
$$\approx \int_0^\xi 1 + 2c^2\xi^2\mathrm{d}\xi = \xi + \frac{2}{3}c^2\xi^3$$

Combine this expression for ***s*** with the power series for $\sigma(s)$:

$$\sigma(s)\ = \sigma^{(0)} + \sigma^{(1)}\xi + \sigma^{(2)}\xi^2$$

Combine the expression of s with the above expression for $1/r^2$:

$$\frac{1}{r^2}\frac{\mathrm{d}s}{\mathrm{d}\xi} = \left(\frac{1}{r_f^2} + \frac{2cy\xi^2}{r_f^4}\right)\left(1\ + 2c^2\xi^2\right) = \left(\frac{1}{r_f^2} + \frac{2cy\xi^2}{r_f^4} + \frac{2c^2\xi^2}{r_f^2}\right)$$

Now the integrands in the expression for $\nabla\phi$ can be evaluated:

$$\begin{aligned}\phi_x &= \frac{1}{2\pi}\int_{-d}^{d}\sigma\,\frac{(x-\xi)}{r^2}\frac{\mathrm{d}s}{\mathrm{d}\xi}\,\mathrm{d}\xi \\ &= \frac{1}{2\pi}\int_{-d}^{d}(\sigma^{(0)} + \sigma^{(1)}\xi + \sigma^{(2)}\xi^2\ )(x-\xi)\left(\frac{1}{r_f^2} + \frac{2cy\xi^2}{r_f^4} + \frac{2c^2\xi^2}{r_f^2}\right)\mathrm{d}\xi \\ &= \frac{1}{4\pi}\Big[\phi_x^{(0)}\sigma^{(0)} + \phi_x^{(1)}\sigma^{(1)} + c\phi_x^{(c)}\sigma^{(0)} + \phi_x^{(2)}(\sigma^{(2)} + 2c^2\sigma^{(0)})\Big]\end{aligned}$$

$$\phi_x^{(0)} = \int_{-d}^{d}\frac{2(x-\xi)}{r_f^2}\,\mathrm{d}\xi = \int_{x-d}^{x+d}\frac{2t}{t^2+y^2}\,\mathrm{d}t = \left[\ln(t^2+y^2)\right]_{x-d}^{x+d} = \ln(r_1^2/r_2^2)$$

with $r_1 = \sqrt{(x+d)^2 + y^2}$ and $r_2 = \sqrt{(x-d)^2 + y^2}$.

$$\phi_x^{(1)} = \int_{-d}^{d}\frac{2\xi(x-\xi)}{r_f^2}\,\mathrm{d}\xi\ = 2\int_{x-d}^{x+d}\frac{t(x-t)}{t^2+y^2}\,\mathrm{d}t = x\phi_x^{(0)} + y\phi_y^{(0)} - 4d$$

$$\phi_x^{(c)} = \int_{-d}^{d}\frac{4(x-\xi)y\xi^2}{r_f^4}\,\mathrm{d}\xi = 4y\int_{x-d}^{x+d}\frac{t(t-x)^2}{(t^2+y^2)^2}\,\mathrm{d}t - 2\phi_y^{(1)} + \frac{(2d)^3xy}{r_1^2r_2^2}$$

$$\phi_x^{(2)} = \int_{-d}^{d}\frac{2(x-\xi)\xi^2}{r_f^2}\,\mathrm{d}\xi = 2\int_{x-d}^{x+d}\frac{t(t-x)^2}{t^2+y^2}\,\mathrm{d}t = x\phi_x^{(1)} + y\phi_y^{(1)}$$

Here the integrals were transformed with the substitution $t = (x-\xi)$.

$$
\begin{aligned}
\phi_y &= \frac{1}{2\pi}\int_{-d}^{d}\sigma\frac{(y-\eta)}{r^2}\frac{\mathrm{d}s}{\mathrm{d}\xi}\,\mathrm{d}\xi \\
&= \frac{1}{2\pi}\int_{-d}^{d}(\sigma^{(0)}+\sigma^{(1)}\xi+\sigma^{(2)}\xi^2)(y-c\xi^2)\ \times\ \left(\frac{1}{r_f^2}+\frac{2cy\xi^2}{r_f^4}+\frac{2c^2\xi^2}{r_f^2}\right)\mathrm{d}\xi \\
&= \frac{1}{4\pi}\left[\phi_y^{(0)}\sigma^{(0)}+\phi_y^{(1)}\sigma^{(1)}+c\phi_y^{(c)}\sigma^{(0)}+\phi_y^{(2)}(\sigma^{(2)}+2c^2\sigma^{(0)})\right]
\end{aligned}
$$

$$
\begin{aligned}
\phi_y^{(0)} &= \int_{-d}^{d}\frac{2y}{r_f^2}\,\mathrm{d}\xi = 2\int_{x-d}^{x+d}\frac{y}{t^2+y^2}\,\mathrm{d}t \\
&= 2\left[\arctan\frac{t}{y}\right]_{x-d}^{x+d} = 2\ \arctan\frac{2dy}{x^2+y^2-d^2}
\end{aligned}
$$

$$
\phi_y^{(1)} = \int_{-d}^{d}\frac{2\xi y}{r_f^2}\,\mathrm{d}\xi = 2y\int_{x-d}^{x+d}\frac{(x-t)}{t^2+y^2}\,\mathrm{d}t = x\phi_y^{(0)}-y\phi_x^{(0)}
$$

$$
\begin{aligned}
\phi_y^{(c)} &= \int_{-d}^{d}4y^2\frac{\xi^2}{r_f^4}-\frac{2\xi^2}{r_f^2}\,\mathrm{d}\xi \\
&= 4y^2\int_{x-d}^{x+d}\frac{(t-x)^2}{(t^2+y^2)^2}\,\mathrm{d}t-2\int_{x-d}^{x+d}\frac{(t-x)^2}{t^2+y^2}\,\mathrm{d}t \\
&= 2\phi_x^{(1)}-4d^3\frac{x^2-y^2-d^2}{r_1^2r_2^2}
\end{aligned}
$$

$$
\phi_y^{(2)} = \int_{-d}^{d}\frac{2y\xi^2}{r_f^2}\,\mathrm{d}\xi = 2y\int_{x-d}^{x+d}\frac{(t-x)^2}{t^2+y^2}\,\mathrm{d}t = x\phi_y^{(1)}-y\phi_x^{(1)}
$$

The original formulae for the first derivatives of these higher-order panels were analytically equivalent, but less suited for programming involving more arithmetic operations than the formulae given here. Higher derivatives of the potential are given in Bertram (1999).

2. *Three-dimensional case*
   The higher-order panels are parabolic in shape with a bi-linear source distribution on each panel. The original procedure of Hess was modified by Hughes and Bertram (1995) to also include higher derivatives of the potential. The complete description of the formulae used to determine the velocity induced by the higher-order panels would be rather lengthy.

So only the general procedure is described here. The surface of the ship is divided into panels as in a first-order panel method. However, the surface of each panel is approximated by a parabolic surface, as opposed to a flat surface. The geometry of a panel in the local panel coordinate system is described as:

$$\zeta = C + A\xi + B\eta + P\xi^2 + 2Q\xi\eta + R\eta^2$$

The ξ-axis and η-axis lie in the plane tangential to the panel at the panel control point, and the ζ-axis is normal to this plane. This equation can be written in the form:

$$\zeta - \zeta_0 = P(\xi - \xi_0)^2 + 2Q(\xi - \xi_0)(\eta - \eta_0) + R(\eta - \eta_0)^2$$

The point (ξ_0, η_0, ζ_0) is used as a collocation point and origin of the local panel coordinate system. In the local panel coordinate system, terms depending on *A, B,* and C do not appear in the formulae for the velocity induced by a source distribution on the panel. *R* and *P* represent the local curvatures of the ship in the two coordinate directions, *Q* the local 'twist' in the ship form.

The required input consists of the coordinates of panel corner points lying on the body surface and information concerning how the corner points are connected to form the panels. In our implementation, each panel is allowed to have either three or four sides. The first and third sides of the panel should be (nearly) parallel. Otherwise, the accuracy of the panels deteriorates. For a given panel, the information available to determine the coefficients *A* … *R* consists of the three or four panel corner points of the panel and the corner points of the panels which border the panel in question. For a quadrilateral panel with neighboring panels on all sides, eight 'extra' vertex points are provided by the corner points of the adjacent panels (Fig. A.5). For triangular panels and panels lying on the edges of the body, fewer extra vertex points will be available. The panel corner points will be listed as x_i, y_i, z_i, $i = 1 \ldots 4$, and the extra vertex points as $\tilde{x}_j$, $\tilde{y}_j$, $\tilde{z}_j$, $j = 1 \ldots N_v$, where N_v is the number of extra vertex points ($3 < N_v < 8$). For triangular panels four corner points are also specified, but either the first and second or the third and fourth corner points are identical (i.e. the first or third side of the panel has zero length). The curved panel is required to pass exactly through all of its corner points and to pass as closely as possible to the extra vertex points of the neighboring panels. The order in which the corner points are specified is important, in that this determines whether the normal vector points into the fluid domain or into the body. In our method, the corner points should be ordered clockwise when viewed from the fluid domain, so that the normal vector points into the fluid domain.

The source strength on each panel is represented by a bi-linear distribution, as opposed to a constant distribution as in a first-order method:

$$\sigma(\xi, \eta) = \sigma_0 + \sigma_x(\xi - \xi_0) + \sigma_y(\eta - \eta_0)$$

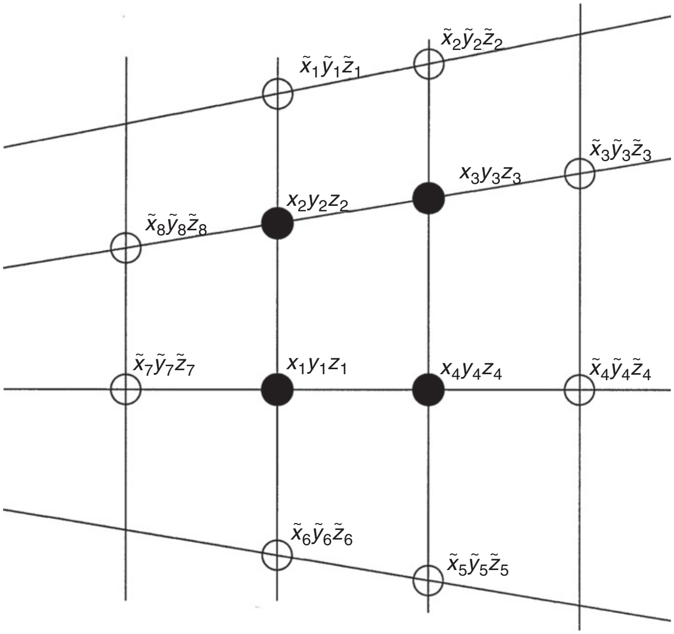

Figure A.5:
Additional points used for computing the local surface curvature of a panel (seen from the fluid domain)

σ_x and σ_y are the slopes of the source strength distribution in the ξ and η directions respectively. In the system of linear equations set up in this method, only the value of the source strength density at the collocation point on each panel, the σ_0 term, is solved for directly. The derivatives of the source density (σ_x and σ_y terms) are expressed in terms of the source strength density at the panel collocation point and at the collocation points of panels bordering the panel in question. First the collocation points of the adjacent panels are transformed into the local coordinate system of the panel in question. Then the above equation for the source strength is fitted in a least squares sense to the values of source density at the collocation points of the adjacent panels to determine σ_0, σ_x, and σ_y. For a four-sided panel which does not lie on a boundary of the body, four adjacent panel collocation points will be available for performing the least squares fit (Fig. A.6). In other cases only three or possibly two adjacent panels will be available. The procedure expresses the unknown source strength derivatives in terms of the source density at the collocation point of the adjacent panels. If the higher-order terms are set to zero, the element reduces to the regular first-order panel. A corresponding option is programmed in our version of the panel.

Vortex Elements

Vortex elements are useful to model lifting flows, e.g. in the lifting-line method for propellers and foils (see Section 2.3, Chapter 2).

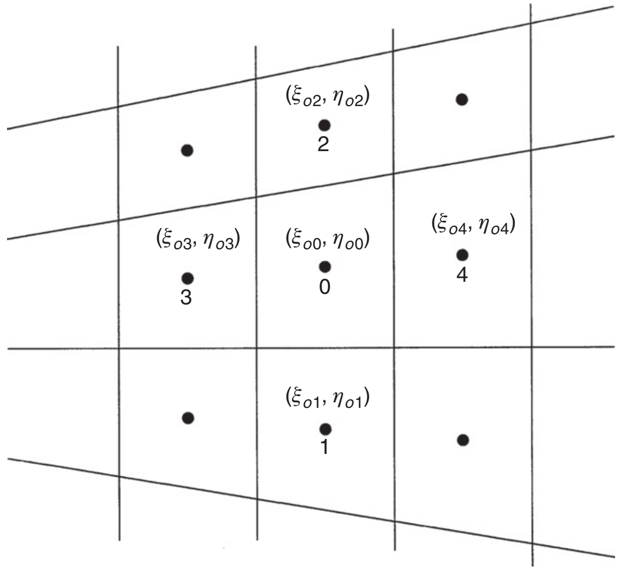

Figure A.6:
Adjacent panels used in the least-squares fit for the source density derivatives

1. *Two-dimensional case*
Consider a vortex of strength Γ at x_w, z_w and a field point x, z. Denote $\Delta x = x - x_w$ and $\Delta z = z - z_w$. The distance between the two points is $r = \sqrt{\Delta x^2 + \Delta z^2}$. The potential and velocities induced by this vortex are:

$$\phi = -\frac{\Gamma}{2\pi}\arctan\frac{z - z_w}{x - x_w}$$

$$\phi_x = \frac{\Gamma}{2\pi}\frac{\Delta z}{r^2}$$

$$\phi_z = -\frac{\Gamma}{2\pi}\frac{\Delta x}{r^2}$$

The absolute value of the velocity is then $(\Gamma/2\pi)1/r$, i.e. the same for each point on a concentric ring around the center x_w, z_w. The velocity decays with distance to the center. So far, the vortex has the same features as the source. The difference is the direction of the velocity. The vortex induces velocities that are always tangential to the concentric ring (Fig. A.7), while the source produced radial velocities. The formulation given here produces counter-clockwise velocities for positive Γ.

The strength of the vortex is the 'circulation'. In general, the circulation is defined as the integral of the tangential velocities about any closed curve. For the definition given above, this integral about any concentric ring will indeed yield Γ as a result.

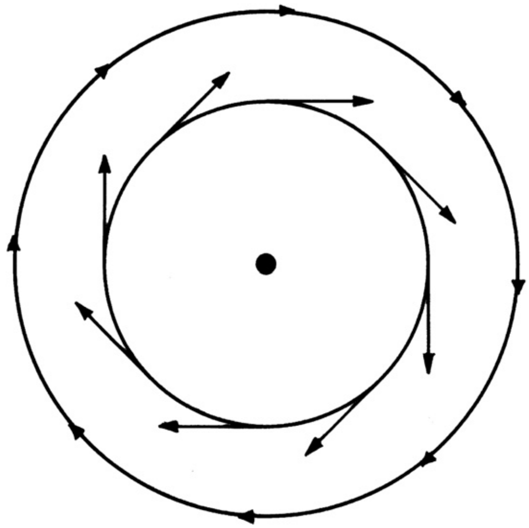

Figure A.7:
Velocities induced by vortex

This point vortex of strength Γ leads to similar expressions for velocities and higher derivatives as a point source of strength σ. One can thus express one by the other as follows:

$$\frac{2\pi}{\Gamma}\Phi_x = \frac{2\pi}{\sigma}\phi_z$$

$$\frac{2\pi}{\Gamma}\Phi_z = -\frac{2\pi}{\sigma}\phi_x$$

$$\frac{2\pi}{\Gamma}\Phi_{xx} = \frac{2\pi}{\sigma}\phi_{xz}$$

$$\frac{2\pi}{\Gamma}\Phi_{xz} = -\frac{2\pi}{\sigma}\phi_{xx}$$

$$\frac{2\pi}{\Gamma}\Phi_{xxz} = \frac{2\pi}{\sigma}\phi_{xzz}$$

$$\frac{2\pi}{\Gamma}\Phi_{xzz} = -\frac{2\pi}{\sigma}\phi_{xxz}$$

Φ is the potential of the vortex and ϕ the potential of the source. The same relations hold for converting between vortex panels and source panels of constant strength. It is thus usually not necessary to program vortex elements separately. One can rather call the source subroutines with a suitable rearrangement of the output parameters in the call of the subroutine.

A vortex panel of constant strength – i.e. all panels have the same strength – distributed on the body coinciding geometrically with the source panels (of individual strength) enforces automatically a Kutta condition, e.g. for a hydrofoil.

2. *Three-dimensional case*
The most commonly used three-dimensional vortex element is the horseshoe vortex. A three-dimensional vortex of strength Γ, lying on a closed curve C, induces a velocity field:

$$\vec{v} = \nabla\phi = \frac{\Gamma}{4\pi}\int_C \frac{d\vec{s}\times\vec{D}}{D^3}$$

We use the special case that a horseshoe vortex lies in the plane $y = y_w =$ const., from $x = -\infty$ to $x = x_w$. Arbitrary cases may be derived from this case using a coordinate transformation. The vertical part of the horseshoe vortex runs from $z = z_l$ to $z = z_2$ (Fig. A.8). Consider a field point (x, y, z). Then: $\Delta x = x - x_w,\ \Delta y = y - y_w,\ \Delta z_1 = z - z_1,$ $\Delta z_2 = z - z_2,\ t_1 = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z_1^2}$ and $t_2 = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z_2^2}$.
The horseshoe vortex then induces the following velocity:

$$\vec{v} = \frac{\Gamma}{4\pi}\left[\left(\frac{\Delta z_1}{t_1} - \frac{\Delta z_2}{t_2}\right)\frac{1}{\Delta x^2 + \Delta y^2}\begin{Bmatrix} -\Delta y \\ \Delta x \\ 0 \end{Bmatrix} + \left(1 - \frac{\Delta x}{t_1}\right)\right.$$
$$\left.\times\frac{1}{\Delta y^2 + \Delta z_1^2}\begin{Bmatrix} 0 \\ -\Delta z_1 \\ \Delta y \end{Bmatrix} i - \left(1 - \frac{\Delta x}{t_2}\right)\frac{1}{\Delta y^2 + \Delta z_2^2}\begin{Bmatrix} 0 \\ s\mathrm{d}z_2 \\ \Delta y \end{Bmatrix}\right]$$

The derivation used $\int (t^2 + a^2)^{-3/2}\mathrm{d}t = t/(a^2\sqrt{t^2 + a^2})$. For $\Delta x^2 + \Delta y^2 \ll |\Delta z_1||\Delta z_2|$ or $\Delta y^2 + \Delta z_1^2 \ll \Delta x^2$ special formulae are used. Bertram (1992) gives details and expressions for higher derivatives.

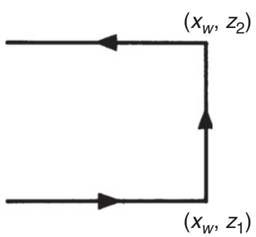

Figure A.8:
Horseshoe vortex

Dipole Elements

Point Dipole

The dipole (or doublet) is the limit of a source and sink of equal strength brought together along some direction (usually x) keeping the product of distance and source strength constant. The result is formally the same as differentiating the source potential in the required direction. The strength of a dipole is usually denoted by m. Again, r denotes the distance between field point $\vec{x}$ and the dipole at $\vec{x}_d$. We consider a dipole in the x direction here. We define $\Delta\vec{x} = \vec{x} - \vec{x}_d$.

1. *Two-dimensional case*
   The potential and derivatives for a dipole in the x direction are:

$$\phi = \frac{m}{2\pi r^2}\,\Delta x$$

$$\phi_x = \frac{m}{2\pi r^2} - 2\phi\,\frac{\Delta x}{r^2}$$

$$\phi_z = -2\phi\,\frac{\Delta z}{r^2}$$

$$\phi_{xx} = \left(-6 + 8\,\frac{\Delta x^2}{r^2}\right)\cdot\frac{\phi}{r^2}$$

$$\phi_{xz} = -2\,\frac{\Delta z\cdot\phi_x + \Delta x\cdot\phi_z}{r^2}$$

The streamlines created by this dipole are circles (Fig. A.9).

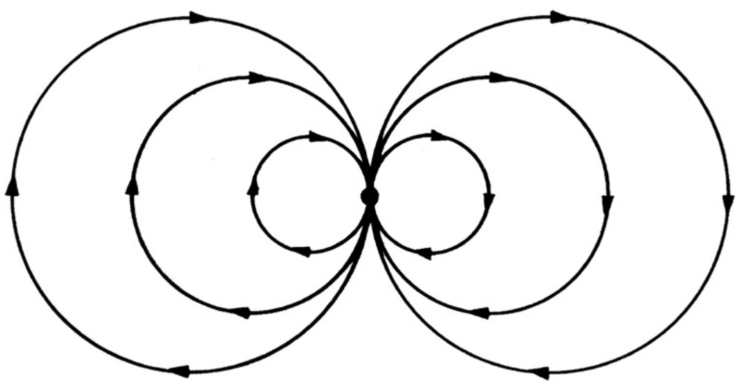

Figure A.9:
Velocities induced by point dipole

2. *Three-dimensional case*
The three-dimensional point dipole in the x direction is correspondingly given by:

$$\phi = -\frac{m}{4\pi r^3}\,\Delta x$$

$$\phi_x = -\frac{m}{4\pi r^3} - 3\phi\,\frac{\Delta x}{r^2}$$

$$\phi_y = -3\phi\,\frac{\Delta y}{r^2}$$

$$\phi_z = -3\phi\,\frac{\Delta z}{r^2}$$

$$\phi_{xx} = \frac{-5\phi_x\Delta x - 4\phi}{r^2}$$

$$\phi_{xy} = \frac{-5\phi_x\Delta y + 2\Delta y\cdot(-m/4\pi r^3)}{r^2}$$

$$\phi_{xz} = \frac{-5\phi_x\Delta z + 2\Delta z\cdot(-m/4\pi r^3)}{r^2}$$

$$\phi_{yy} = \frac{-5\phi_x\Delta y - 3\phi}{r^2}$$

$$\phi_{yz} = \frac{-5\phi_z\Delta y}{r^2}$$

$$\phi_{xx} = \frac{-5\phi_x\Delta x - 4\phi}{r^2}$$

The expressions for the dipole can be derived formally by differentiation of the corresponding source expression in x. Therefore usually source subroutines (also for distributed panels) are used with a corresponding redefinition of variables in the parameter list of the call. This avoids double programming. Dipoles like vortices can be used (rather equivalently) to generate lift in flows.

Thiart Element

The ship including the rudder can be considered as a vertical foil of considerable thickness and extremely short span. For a steady yaw angle, i.e. a typical maneuvering application, one would certainly enforce a Kutta condition at the trailing edge, either employing vortex or dipole elements. For harmonic motions in waves, i.e. a typical seakeeping problem, one should similarly employ a Kutta condition, but this is often omitted. If a Kutta condition is employed in frequency-domain computations, the wake will oscillate harmonically in strength. This can be modeled by discrete dipole elements of constant strength, but for high frequencies this approach requires many elements. The use of special elements which consider the oscillating strength analytically is more efficient and accurate, but also more complicated. Such a 'Thiart element' has been developed by Professor Gerhard Thiart of Stellen-bosch University and is described in detail in Bertram (1998), and Bertram and Thiart (1998). The oscillating ship creates a vorticity. The problem is similar to that of an oscillating airfoil. The circulation is assumed constant within the ship. Behind the ship, vorticity is shed downstream with ship speed V. Then:

$$\left(\frac{\partial}{\partial t} - V\frac{\partial}{\partial x}\right)\gamma(x,z,t) = 0$$

γ is the vortex density, i.e. the strength distribution for a continuous vortex sheet. The following distribution fulfills the above condition:

$$\gamma(x,z,t) = \mathrm{Re}(\widehat{\gamma}_a(z)\cdot \mathrm{e}^{i(\omega_e/V)(x-x_a)}\cdot \mathrm{e}^{i\omega_e t}) \quad \text{for } x \le x_a$$

Here $\widehat{\gamma}_a$ is the vorticity density at the trailing edge x_a (stern of the ship). We continue the vortex sheet inside the ship at the symmetry plane $y = 0$, assuming a constant vorticity density:

$$\gamma(x,z,t) = \mathrm{Re}(\widehat{\gamma}_a(z)\cdot \mathrm{e}^{i\omega_e t}) \quad \text{for } x_a \le x \le x_f$$

x_f is the leading edge (forward stem of the ship). This vorticity density is spatially constant within the ship.

A vortex distribution is equivalent to a dipole distribution if the vortex density γ and the dipole density m are coupled by:

$$\gamma = \frac{\mathrm{d}m}{\mathrm{d}x}$$

The potential of an equivalent semi-infinite strip of dipoles is then obtained by integration. This potential is given (except for a so far arbitrary 'strength' constant) by:

$$\Phi(x,y,z) = \mathrm{Re}\left(\int_{-\infty}^{x_f}\int_{z_m}^{z_o} \widehat{m}(\xi)\frac{y}{r^3}\,\mathrm{d}\xi\mathrm{d}\zeta\mathrm{e}^{i\omega_e t}\right) = \mathrm{Re}(\varphi(x,y,z)\cdot \mathrm{e}^{i\omega_e t})$$

with $r = \sqrt{(x-\xi)^2 + y^2 + (z-\zeta)^2}$ and:

$$\widehat{m}(\xi) = \begin{cases} x_f - \xi & \text{for } x_a \le \xi \le x_f \\ \dfrac{V}{i\omega_e}(1 - \mathrm{e}^{i(\omega_e/V)}(\xi - x_a)) + (x_f - x_a) & \text{for } -\infty \le \xi \le x_a \end{cases}$$

It is convenient to write φ as:

$$\varphi(x,y,z) = y\int_{z_n}^{z_o}\int_{x_a}^{x_f}\frac{x-\xi}{r^3}\,\mathrm{d}\xi\mathrm{d}\zeta + (x_f - x)\int_{z_m}^{z_o}\int_{x_a}^{x_f}\frac{y}{r^3}\,\mathrm{d}\xi\,\mathrm{d}\zeta$$

$$+\left(\frac{V}{i\omega_e} + (x_f - x_a)\right)\int_{z_u}^{z_o}\int_{-\infty}^{x_a}\frac{y}{r^3}\,\mathrm{d}\xi\,\mathrm{d}\zeta$$

$$-\left(\frac{V}{i\omega_e}\,\mathrm{e}^{-i\omega_e x_a/V}\right)y\int_{z_u}^{z_o}\int_{-\infty}^{x_a}\mathrm{e}^{i\omega_e\xi/V}\frac{1}{r^3}\,\mathrm{d}\xi\,\mathrm{d}\zeta$$

The velocity components and higher derivatives are then derived by differentiation of Φ, which can be reduced to differentiation of φ. The exact formulae are given in Bertram (1998), and Bertram and Thiart (1998). The expressions involve integrals with integrands of the form 'arbitrary smooth function' · 'harmonically oscillating function'. These are accurately and efficiently evaluated using a modified Simpson's method developed by Söding:

$$\int_{x_1}^{x_1+2h} f(x)\mathrm{e}^{ikx}\mathrm{d}x = \frac{\mathrm{e}^{ikx_1}}{k}\left[\mathrm{e}^{2ikh}\left(\frac{0.5f_1 - 2f_2 + 1.5f_3}{kh} - i\left(f_3 - \frac{\Delta^2 f}{k^2h^2}\right)\right)\right.$$

$$\left. + \frac{1.5f_1 - 2f_2 + 0.5f_3}{kh} + i\left(f_1 - \frac{\Delta^2 f}{h^2k^2}\right)\right]$$

where $f_1 = f(x_1)$, $f_2 = f(x_1 + h)$, $f_3 = f(x_1 + 2h)$, $\Delta^2 f = f_1 - 2f_2 + f_3$.

Special Techniques

Desingularization

The potential and its derivatives become singular directly on a panel, i.e. infinite terms appear in the usual formulae which prevent straightforward evaluation. For the normal velocity, this singularity can be removed analytically for the collocation point on the panel itself, but the resulting special treatment makes parallelization of codes difficult. When the element is placed somewhat outside the domain of the problem (Fig. A.10), it is 'desingularized', i.e. the singularity is removed. This has several advantages:

- In principle the same expression can be evaluated everywhere. This facilitates parallel algorithms in numerical evaluation and makes the code generally shorter and easier.
- Numerical experiments show that desingularization improves the accuracy as long as the depth of submergence is not too large.

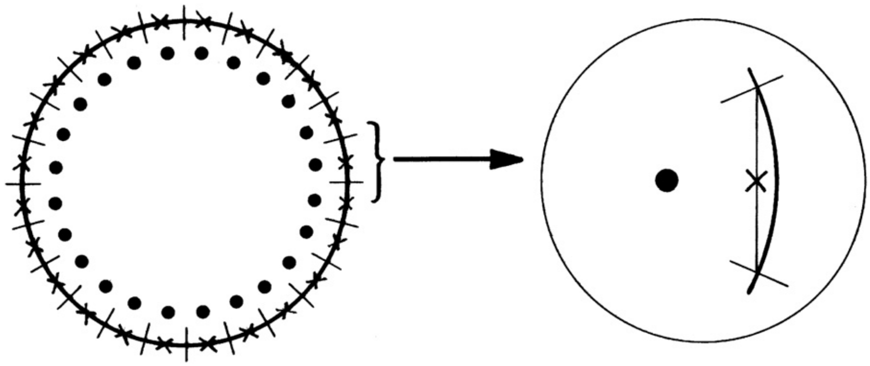

Figure A.10:
Desingularization

The last point surprised some mathematicians. Desingularization results in a Fredholm integral equation of the first kind. (Otherwise a Fredholm equation of second kind results.) This can lead in principle to problems with uniqueness and existence of solutions, which in practice manifest themselves first by an ill-conditioned matrix for the unknowns (source strengths or directly potential). For engineers, the problems are directly apparent without going into mathematical classification:

- If the individual elements (sources) are too far from the collocation points, they will all have almost the same influence. Then they will not be able to represent arbitrary local flow patterns.
- If the individual elements are somewhat removed, the individual sharp local steepness in flow pattern (singularity) will smooth out rapidly, forming a relatively smooth flow distribution which can relatively smoothly approximate arbitrary flows.
- If the individual elements are very close, an uneven cobblestone flow distribution results due to the discontinuity between the individual elements.

Thus the desingularization distance has to be chosen appropriately within a bandwidth to yield acceptable results. The distance should be related to the grid size. As the grid becomes finer, the desingularized solution approaches the conventional singular-element solution. Fortunately, several researchers have shown that the results are relatively insensitive to the desingularization distance, as long as this ranges between 0.5 and 2 typical grid spacings.

The historical development of desingularization of boundary element methods is reviewed in Cao et al. (1991), and Raven (1998).

Desingularization is used in many ‘fully non-linear’ wave resistance codes in practice for the free-surface elements. Sometimes it is also used for the hull elements, but here narrow pointed bows introduce difficulties often requiring special effort in grid generation. Some close-fit routines for two-dimensional seakeeping codes (strip-method modules) also employ desingularization.

Patch Method

Traditional boundary element methods enforce the kinematic condition (no-penetration condition) on the hull exactly at one collocation point per panel, usually the panel center. The resistance predicted by these methods is for usual discretizations insufficient for practical requirements, at least if conventional pressure integration on the hull is used. Söding (1993) therefore proposed a variation of the traditional approach which differs in some details from the conventional approach. Since his approach also uses flat segments on the hull, but not as distributed singularities, he called the approach 'patch' method to distinguish it from the usual 'panel' methods.

For double-body flows the resistance in an ideal fluid should be zero. This allows the comparison of the accuracy of various methods and discretizations as the non-zero numerical resistance is then purely due to discretization errors. For double-body flows, the patch method reduces the error in the resistance by one order of magnitude compared to ordinary first-order panel methods, without increasing the computational time or the effort in grid generation. However, higher derivatives of the potential or the pressure directly on the hull cannot be computed as easily as for a regular panel method.

The patch method basically introduces three changes to ordinary panel methods:

- 'Patch condition'
  Panel methods enforce the no-penetration condition on the hull exactly at one collocation point per panel. The 'patch condition' states that the integral of this condition over one patch of the surface is zero. This averaging of the condition corresponds to the techniques used in finite element methods.
- Pressure integration
  Potentials and velocities are calculated at the patch corners. Numerical differentiation of the potential yields an average velocity. A quadratic approximation for the velocity using the average velocity and the corner velocities is used in pressure integration. The unit normal is still considered constant.
- Desingularization
  Single point sources are submerged to give a smoother distribution of the potential on the hull. As desingularization distance between patch center and point source, the minimum of 10% of the patch length, 50% of the normal distance from patch center to a line of symmetry is recommended.

Söding (1993) did not investigate the individual influence of each factor, but the higher-order pressure integration and the patch condition contribute approximately the same.

The patch condition states that the flow through a surface element (patch) (and not just at its center) is zero. Desingularized Rankine point sources instead of panels are used as elementary solutions. The potential of the total flow is:

$$\phi = -Vx + \sum_i \sigma_i \varphi_i$$

σ is the source strength, φ is the potential of a Rankine point source, r is the distance between source and field point. Let M_i be the outflow through a patch (outflow = flow from interior of the body into the fluid) induced by a point source of unit strength.

1. *Two-dimensional case*

   The potential of a two-dimensional point source is:

$$\varphi = \frac{1}{4\pi} \ln r^2$$

   The integral zero-flow condition for a patch is:

$$-V \cdot n_x \cdot l + \sum_i \sigma_i M_i = 0$$

   n_x is the x component of the unit normal (from the body into the fluid), l the patch area (length). The flow through a patch is invariant of the coordinate system. Consider a local coordinate system x, z (Fig. A.11). The patch extends in this coordinate system from $-s$ to s. The flow through the patch is:

$$M = -\int_{-s}^{s} \phi_z \mathrm{d}x$$

   A Rankine point source of unit strength induces at x, z the vertical velocity:

$$\phi_z = \frac{1}{2\pi} \frac{z - z_q}{(x - x_q)^2 + (z - z_q)^2}$$

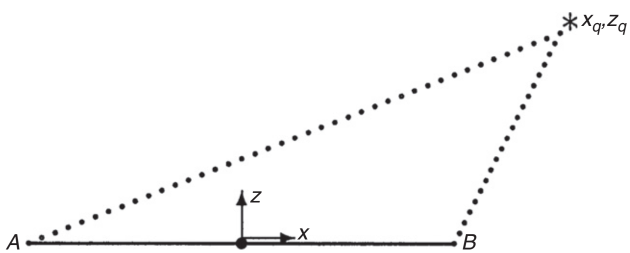

Figure A.11:
Patch in 2d

Since $z = 0$ on the patch, this yields:

$$M = \int_{-s}^{s} \frac{1}{2\pi} \frac{z_q}{(x - x_q)^2 + z_q^2} \, \mathrm{d}x = \frac{1}{2\pi} \arctan \frac{l z_q}{x_q^2 + z_q^2 - s^2}$$

The local z_q transforms from the global coordinates:

$$z_q = -n_x \cdot (\overline{x}_q - \overline{x}_c) - n_z \cdot (\overline{z}_q - \overline{z}_c)$$

$\overline{x}_c$, $\overline{z}_c$ are the global coordinates of the patch center, $\overline{x}_q, \overline{z}_q$ of the source.

From the value of the potential ϕ at the corners *A* and *B*, the average velocity within the patch is found as:

$$\overline{\vec{v}} = \frac{\phi_B - \phi_A}{|\vec{x}_B - \vec{x}_A|} \cdot \frac{\vec{x}_B - \vec{x}_A}{|\vec{x}_B - \vec{x}_A|}$$

i.e. the absolute value of the velocity is:

$$\frac{\Delta\phi}{\Delta s} = \frac{\phi_B - \phi_A}{|\vec{x}_B - \vec{x}_A|}$$

The direction is tangential to the body; the unit tangential is $(\vec{x}_B - \vec{x}_A)/|\vec{x}_B - \vec{x}_A|$. The pressure force on the patch is:

$$\Delta\vec{f} = \vec{n} \int p \mathrm{d}l = \vec{n} \frac{\rho}{2} \left(V^2 \cdot l - \int \vec{v}^2 \mathrm{d}l \right)$$

$\vec{v}$ is not constant! To evaluate this expression, the velocity within the patch is approximated by:

$$\vec{v} = a + bt + ct^2$$

t is the tangential coordinate directed from *A* to *B*. $\vec{v}_A$ and $\vec{v}_B$ are the velocities at the patch corners. The coefficients *a, b,* and *c* are determined from the conditions:

- The velocity at $t = 0$ is $\vec{v}_A$: $a = \vec{v}_A$.
- The velocity at $t = 1$ is $\vec{v}_B$: $a + b + c = \vec{v}_B$.
- The average velocity (integral over one patch) is $\overline{\vec{v}}$: $a + 1/2b + 1/3c = \overline{\vec{v}}$.

This yields:

$$a = \vec{v}_A$$

$$b = 6\overline{\vec{v}} - 4\vec{v}_A - 2\vec{v}_B$$

$$c = -6\overline{\vec{v}} + 3\vec{v}_A + 3\vec{v}_B$$

Using the above quadratic approximation for $\vec{v}$, the integral of $\overline{\vec{v}}^2$ over the patch area is found after some lengthy algebraic manipulations as:

$$\int \vec{v}^2 \mathrm{d}l = l \int_0^1 \vec{v}^{\,2} \mathrm{d}t = l \cdot \left(a^2 + ab + \frac{1}{3}(2ac + b^2) + \frac{1}{2} bc + \frac{1}{5} c^2 \right)$$

$$= l \cdot \left(\overline{\vec{v}}^{\,2} + \frac{2}{15}((\vec{v}_A - \overline{\vec{v}}) + (\vec{v}_B - \overline{\vec{v}}))^2 - \frac{1}{3}(\vec{v}_A - \overline{\vec{v}})(\vec{v}_B - \overline{\vec{v}}) \right)$$

Thus the force on one patch is:

$$\Delta\vec{f} = -\vec{n} \cdot l \cdot \left((\overline{\vec{v}}^2 - V^2) + \frac{2}{15}((\vec{v}_A - \overline{\vec{v}}) + (\vec{v}_B - \overline{\vec{v}}))^2 - \frac{1}{3}((\vec{v}_A - \overline{\vec{v}})(\vec{v}_B - \overline{\vec{v}})) \right)$$

2. *Three-dimensional case*

   The potential of a three-dimensional source is:

$$\varphi = -\frac{1}{4\pi |\vec{x} - \vec{x}_q|}$$

Figure A.12 shows a triangular patch ABC and a source S. Quadrilateral patches may be created by combining two triangles. The zero-flow condition for this patch is:

$$-V \frac{(\vec{a} \times \vec{b})_1}{2} + \sum_i \sigma_i M_i = 0$$

The first term is the volume flow through ABC due to the uniform flow; the index 1 indicates the x component (of the vector product of two sides of the triangle). The flow M through a patch ABC induced by a point source of unit strength is $-\alpha/(4\pi)$. α is the solid angle in which ABC is seen from S. The rules of spherical geometry give α as the sum of the angles between each pair of planes SAB, SBC, and SCA minus π:

$$\alpha = \beta_{SAB,SBC} + \beta_{SBC,SCA} + \beta_{SCA,SAB} - \pi$$

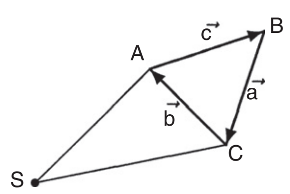

Figure A.12:
Source point S and patch ABC

where, e.g.,

$$\beta_{SAB,SBC} = \arctan \frac{-[(\vec{A}\times\vec{B})\times(\vec{B}\times\vec{C})]\cdot\vec{B}}{(\vec{A}\times\vec{B})\cdot(\vec{B}\times\vec{C})|\vec{B}|}$$

Here $\vec{A}$, $\vec{B}$, $\vec{C}$ are the vectors pointing from the source point S to the panel corners A, B, C. The solid angle may be approximated by A^*/d^2 if the distance d between patch center and source point exceeds a given limit. A^* is the patch area projected on a plane normal to the direction from the source to the patch center:

$$\vec{d} = \frac{1}{3}(\vec{A}+\vec{B}+\vec{C})$$
$$A^* = \frac{1}{2}(\vec{a}\times\vec{b})\,\frac{\vec{d}}{d}$$

With known source strengths σ_i, one can determine the potential ϕ and its derivatives $\nabla\phi$ at all patch corners. From the ϕ values at the corners A, B, C, the average velocity within the triangle is found as:

$$\bar{\vec{v}} = \overline{\nabla\phi} = \frac{\phi_A-\phi_C}{\vec{n}_{AB}^2}\,\vec{n}_{AB} + \frac{\phi_B-\phi_A}{\vec{n}_{AC}^2}\,\vec{n}_{AC}$$

$$\vec{n}_{AB} = \vec{b} - \frac{\vec{c}\cdot\vec{b}}{\vec{c}^2}\vec{c} \quad \text{and} \quad \vec{n}_{AC} = \vec{c} - \frac{\vec{b}\cdot\vec{c}}{\vec{b}^2}\vec{b}$$

With known $\bar{\vec{v}}$ and corner velocities $\vec{v}_A$, $\vec{v}_B$, $\vec{v}_C$, the pressure force on the triangle can be determined:

$$\Delta\vec{f} = \vec{n}\int p\,\mathrm{d}A = \vec{n}\,\frac{\rho}{2}\,(V^2\cdot A - \int \vec{v}^2\mathrm{d}A)$$

where $\vec{v}$ is *not* constant! $A = 1/2|\vec{a}\times\vec{b}|$ is the patch area. To evaluate this equation, the velocity within the patch is approximated by:

$$\vec{v} = \bar{\vec{v}} + (\vec{v}_A - \bar{\vec{v}})(2r^2 - r) + (\vec{v}_B - \bar{\vec{v}})(2s^2 - s) + (\vec{v}_C - \bar{\vec{v}})(2t^2 - t)$$

r is the 'triangle coordinate' directed to patch corner A: $r = 1$ at A and $r = 0$ at the line BC. s and t are the corresponding 'triangle coordinates' directed to B resp. C. Using this quadratic v formula, the integral of $\vec{v}^2$ over the triangle area is found after some algebraic manipulations as:

$$\int \vec{v}^2 \mathrm{d}A = A \cdot \left[\bar{\vec{v}}^{\,2} + \frac{1}{30}(\vec{v}_A - \bar{\vec{v}})^2 + \frac{1}{30}(\vec{v}_B - \bar{\vec{v}})^2 + \frac{1}{30}(\vec{v}_C - \bar{\vec{v}})^2 \right.$$
$$- \frac{1}{90}(\vec{v}_A - \bar{\vec{v}})\ (\vec{v}_B - \bar{\vec{v}})\ - \frac{1}{90}(\vec{v}_B - \bar{\vec{v}})\ (\vec{v}_C - \bar{\vec{v}})$$
$$\left. - \frac{1}{90}(\vec{v}_C - \bar{\vec{v}})(\vec{v}_A - \bar{\vec{v}}) \right]$$