APPENDIX B

Numerical Examples for BEM

Two-Dimensional Flow Around a Body in Infinite Fluid

One of the most simple applications of boundary element methods is the computation of the potential flow around a body in an infinite fluid. The inclusion of a rigid surface is straightforward in this case and leads to the double-body flow problem which will be discussed at the end of this chapter.

Theory

We consider a submerged body of arbitrary (but smooth) shape moving with constant speed V in an infinite fluid domain. For inviscid and irrotational flow, this problem is equivalent to a body being fixed in an inflow of constant speed. For testing purposes, we may select a simple geometry like a circle (cylinder of infinite length) as a body.

For the assumed ideal fluid, there exists a velocity potential ϕ such that $\vec{v} = \nabla\phi$. For the considered ideal fluid, continuity gives Laplace's equation, which holds in the whole fluid domain:

$$\Delta\phi = \phi_{xx} + \phi_{zz} = 0$$

In addition, we require the boundary condition that water does not penetrate the body's surface (hull condition). For an inviscid fluid, this condition can be reformulated requiring just vanishing normal velocity on the body:

$$\vec{n} \cdot \nabla\phi = 0$$

$\vec{n}$ is the inward unit normal vector on the body hull. This condition is mathematically a Neumann condition as it involves only derivatives of the unknown potential.

Once a potential and its derivatives have been determined, the forces on the body can be determined by direct pressure integration:

$$f_1 = \int_s p n_1 \, \mathrm{d}S$$

$$f_2 = \int_S p n_2 \, \mathrm{d}S$$

S is the wetted surface. p is the pressure determined from Bernoulli's equation:

$$p = \frac{\rho}{2}\left(V^2 - (\nabla\phi)^2\right)$$

The force coefficients are then:

$$C_x = \frac{f_1}{\frac{\rho}{2} V^2 S}$$

$$C_z = \frac{f_2}{\frac{\rho}{2} V^2 S}$$

Numerical Implementation

The velocity potential ϕ is approximated by uniform flow superimposed by a finite number N of elements. These elements are in the sample program DOUBL2D desingularized point sources inside the body (Fig. A.10). The choice of elements is rather arbitrary, but the most simple elements are selected here for teaching purposes.

We formulate the potential ϕ as the sum of parallel uniform flow (of speed V) and a residual potential which is represented by the elements:

$$\phi = -Vx + \sum \sigma_i \varphi$$

σ_i is the strength of the ith element, φ the potential of an element of unit strength. The index i for φ is omitted for convenience but it should be understood in the equations below that φ refers to the potential of only the ith element.

Then the Neumann condition on the hull becomes:

$$\sum_{i=1}^{N} \sigma_i (\vec{n} \cdot \nabla\varphi) = V n_1$$

This equation is fulfilled on N collocation points on the body, thus forming a linear system of equations in the unknown element strengths σ_i. Once the system is solved, the velocities and pressures are determined on the body.

The pressure integral for the x force is evaluated approximately by:

$$\int_s pn_1 \, dS \approx \sum_{i=1}^{N} p_i n_{1,i} s_i$$

The pressure p_i and the inward normal on the hull n_i are taken constant over each panel. s_i is the area of one segment.

For double-body flow, an ‘element’ consists of a source at $z = z_q$ and its mirror image at $z = -z_q$. Otherwise, there is no change in the program.

Two-Dimensional Wave Resistance Problem

The extension of the theory for a two-dimensional double-body flow problem to a two-dimensional free-surface problem with optional shallow-water effect introduces these main new features:

- ‘fully non-linear’ free-surface treatment
- shallow-water treatment
- treatment of various element types in one program.

While the problem is purely academical as free-surface steady flows for ships in reality are always strongly three-dimensional, the two-dimensional problem is an important step in understanding the three-dimensional problem. Various techniques have in the history of development always been tested and refined first in the much faster and easier two-dimensional problem, before being implemented in three-dimensional codes. The two-dimensional problem is thus an important stepping stone for researchers and a useful teaching example for students.

Theory

We consider a submerged body of arbitrary (but smooth) shape moving with constant speed V under the free surface in water of constant depth. The depth may be infinite or finite. For inviscid and irrotational flow, this problem is equivalent to a body being fixed in an inflow of constant speed.

We extend the theory simply repeating the previously discussed conditions and focusing on the new conditions. Laplace’s equation holds in the whole fluid domain. The boundary conditions are:

- Hull condition: water does not penetrate the body’s surface.
- Kinematic condition: water does not penetrate the water surface.

- Dynamic condition: there is atmospheric pressure at the water surface.
- Radiation condition: waves created by the body do not propagate ahead.
- Decay condition: far ahead of and below the body, the flow is undisturbed.
- Open-boundary condition: waves generated by the body pass unreflected any artificial boundary of the computational domain.
- Bottom condition (shallow-water case): no water flows through the sea bottom.

The decay condition replaces the bottom condition if the bottom is at infinity, i.e. in the usual infinite fluid domain case.

The wave resistance problem features two special problems requiring an iterative solution:

1. A non-linear boundary condition appears on the free surface.
2. The boundaries of water (waves) are not a priori known.

The iteration starts by approximating:

- the unknown wave elevation by a flat surface
- the unknown potential by the potential of uniform parallel flow.

In each iterative step, wave elevation and potential are updated yielding successively better approximations for the solution of the non-linear problem.

The equations are formulated here in a right-handed Cartesian coordinate system with x pointing forward towards the 'bow' and z pointing upward. For the assumed ideal fluid, there exists a velocity potential ϕ such that $\vec{v} = \nabla\phi$. The velocity potential ϕ fulfills Laplace's equation in the whole fluid domain:

$$\Delta\phi = \phi_{xx} + \phi_{zz} = 0$$

The hull condition requires vanishing normal velocity on the body:

$$\vec{n}\cdot\nabla\phi = 0$$

$\vec{n}$ is the inward unit normal vector on the body hull.

The kinematic condition (no penetration of water surface) gives at $z = \zeta$:

$$\nabla\phi\cdot\nabla\zeta = \phi_z$$

For simplification, we write $\zeta(x, z)$ with $\zeta_z = \partial\zeta/\partial z = 0$.

The dynamic condition (atmospheric pressure at water surface) gives at $z = \zeta$:

$$\frac{1}{2}(\nabla\phi)^2 + gz = \frac{1}{2}V^2$$

with $g = 9.81\ \text{m/s}^2$. Combining the dynamic and kinematic boundary conditions eliminates the unknown wave elevation $z = \zeta$:

$$\frac{1}{2}\nabla\phi\cdot\nabla(\nabla\phi)^2 + g\phi_z = 0$$

This equation must still be fulfilled at $z = \zeta$. If we approximate the potential ϕ and the wave elevation ζ by arbitrary approximations Φ and $\overline{\zeta}$, linearization about the approximated potential gives at $z = \zeta$:

$$\nabla\Phi\cdot\nabla\left(\frac{1}{2}(\nabla\Phi)^2 + \nabla\Phi\cdot\nabla(\phi - \Phi)\right) + \nabla(\phi - \Phi)\cdot\nabla\left(\frac{1}{2}(\nabla\Phi)^2\right) + g\phi_z = 0$$

Φ and $\phi - \Phi$ are developed in a Taylor expansion about $\overline{\zeta}$. The Taylor expansion is truncated after the linear term. Products of $\zeta - \overline{\zeta}$ with derivatives of $\phi - \Phi$ are neglected. This yields at $z = \overline{\zeta}$:

$$\nabla\Phi\cdot\nabla\left(\frac{1}{2}(\nabla\Phi)^2 + \nabla\Phi\cdot\nabla(\phi - \Phi)\right) + \nabla(\phi - \Phi)\cdot\nabla\left(\frac{1}{2}(\nabla\Phi)^2\right) + g\phi_z \\ + \left[\frac{1}{2}\nabla\Phi\cdot\nabla(\nabla\Phi)^2 + g\Phi_z\right]_z (\zeta - \overline{\zeta}) = 0$$

A consistent linearization about Φ and $\overline{\zeta}$ substitutes ζ by an expression depending solely on $\overline{\zeta}$, $\Phi(\overline{\zeta})$ and $\phi(\overline{\zeta})$. For this purpose, the original expression for ζ is also developed in a truncated Taylor expansion and written at $z = \overline{\zeta}$:

$$\zeta = -\frac{1}{2g}(-(\nabla\Phi)^2 + 2\nabla\Phi\cdot\nabla\phi + 2\nabla\Phi\cdot\nabla\Phi_z(\zeta - \overline{\zeta}) - V^2)$$

$$\zeta - \overline{\zeta} = \frac{-\frac{1}{2}(2\nabla\Phi\cdot\nabla\phi - (\nabla\Phi)^2 - V^2) - g\overline{\zeta}}{g + \nabla\Phi\cdot\nabla\Phi_z}$$

Substituting this expression in our equation for the free-surface condition gives the consistently linearized boundary condition at $z = \overline{\zeta}$:

$$\nabla\Phi\nabla[-(\nabla\Phi)^2 + \nabla\Phi\cdot\nabla\phi] + \frac{1}{2}\nabla\phi\nabla(\nabla\Phi)^2 + g\phi_z + \frac{\left[\frac{1}{2}\nabla\Phi\nabla(\nabla\Phi)^2 + g\Phi_z\right]_z}{g + \nabla\Phi\cdot\nabla\Phi_z} \\ \times\left(-\frac{1}{2}[-(\nabla\Phi)^2 + 2\nabla\Phi\cdot\nabla\phi - V^2] - g\overline{\zeta}\right) = 0$$

The denominator in the last term becomes zero when the vertical particle acceleration is equal to gravity. In fact, the flow becomes unstable already at 0.6 to $0.7g$ both in reality and in numerical computations.

It is convenient to introduce the following abbreviations:

$$\vec{a} = \frac{1}{2}\nabla((\nabla\Phi)^2) = \begin{Bmatrix} \Phi_x\Phi_{xx} + \Phi_z\Phi_{xz} \\ \Phi_x\Phi_{xz} + \Phi_z\Phi_{zz} \end{Bmatrix}$$

$$\begin{aligned} B &= \frac{\left[\frac{1}{2}\nabla\Phi\nabla(\nabla\Phi)^2 + g\Phi_z\right]_z}{g + \nabla\Phi\cdot\nabla\Phi_z} = \frac{[\nabla\Phi\vec{a}+g\Phi_z]_z}{g + a_2} \\ &= \frac{1}{g + a_2}(\Phi_x^2\Phi_{xxz} + \Phi_z^2\Phi_{zzz} + g\Phi_{zz} + 2[\Phi_x\Phi_z\Phi_{xzz} + \Phi_{xz}\cdot a_1 + \Phi_{zz}\cdot a_2]) \end{aligned}$$

Then the boundary Φ condition at $z = \overline{\zeta}$ becomes:

$$\begin{aligned} &2(\vec{a}\,\nabla\phi + \Phi_x\Phi_z\phi_{xz}) + \Phi_x^2\phi_{xx} + \Phi_z^2\phi_{zz} + g\phi_z - B\nabla\Phi\nabla\phi \\ &= 2\vec{a}\,\nabla\Phi - B\left(\frac{1}{2}((\nabla\Phi)^2 + V^2) - g\overline{\zeta}\right) \end{aligned}$$

The non-dimensional error in the boundary condition at each iteration step is defined by:

$$\varepsilon = \max(|\vec{a}\,\nabla\Phi + g\Phi_z|)/(gV)$$

where 'max' means the maximum value of all points at the free surface.

For given velocity, Bernoulli's equation determines the wave elevation:

$$z = \frac{1}{2g}\left(V^2 - (\nabla\phi)^2\right)$$

The first step of the iterative solution is the classical linearization around uniform flow. To obtain the classical solutions for this case, the above equation should also be linearized as:

$$z = \frac{1}{2g}(V^2 + (\nabla\Phi)^2 - 2\nabla\Phi\nabla\phi)$$

However, it is computationally simpler to use the non-linear equation.

The bottom, radiation, and open-boundary conditions are fulfilled by the proper arrangement of elements as described below. The decay condition — like the Laplace equation — is automatically fulfilled by all elements.

Once a potential has been determined, the force on the body in the x direction can be determined by direct pressure integration:

$$f_1 = \int_s pn_1 \, \mathrm{d}S$$

S is the wetted surface. p is the pressure determined from Bernoulli's equation:

$$p = \frac{\rho}{2}\left(V^2 - (\nabla\phi)^2\right)$$

The force in the x direction, f_1, is the (negative) wave resistance. The non-dimensional wave resistance coefficient is:

$$C_w = -f_1 / \left(\frac{\rho}{2} V^2 S\right)$$

Numerical Implementation

The velocity potential ϕ is approximated by uniform flow superimposed by a finite number of elements. These elements are, in the sample program SHAL2D:

- desingularized point source clusters above the free surface
- desingularized point sources inside the body.

The choice of elements is rather arbitrary, but very simple elements have been selected for teaching purposes.

The height of the elements above the free surface is not corrected in SHAL2D. For usual discretizations (10 elements per wave length) and moderate speeds, this procedure should work without problems. For finer discretizations (as often found for high speeds), problems occur which require a readjustment of the panel layer. However, in most cases it is sufficient to adjust the source layer just once after the first iteration and then 'freeze' it.

We formulate the potential ϕ as the sum of parallel uniform flow (of speed V) and a residual potential which is represented by the elements:

$$\phi = -Vx + \sum \sigma_i \varphi$$

σ_i is the strength of the ith element, φ the potential of an element of unit strength. The expression 'element' refers to one source (cluster) and all its mirror images. If the collocation point and source center are sufficiently far from each other, e.g. three times the grid spacing, the source cluster may be substituted by a single point source. This accelerates the computation without undue loss of accuracy.

Then the no-penetration boundary condition on the hull becomes:

$$\sum \sigma_i (\vec{n} \cdot \nabla\varphi) = V n_1$$

The linearized free-surface condition becomes:

$$\sum \sigma_i \big(2(\vec{a}\,\nabla\varphi + \Phi_x\Phi_z\varphi_{xz}) + \Phi_x^2\varphi_{xx} + \Phi_z^2\varphi_{zz} + g\varphi_z - B\nabla\Phi\nabla\varphi\big)$$
$$= 2(\vec{a}\,\nabla\Phi + a_1 V) - B\left(\frac{1}{2}((\nabla\Phi)^2 + V^2) - g\overline{\zeta} + V\Phi_x\right)$$

These two equations form a linear system of equations in the unknown element strengths σ_i. Once the system is solved, the velocities (and higher derivatives of the potential) are determined on the water surface. Then the error ε is determined.

For shallow water, mirror images of elements at the ocean bottom are used. This technique is similar to the mirror imaging at the still waterplane used for double-body flow.

The radiation and open-boundary conditions are fulfilled using 'staggered grids'. This technique adds an extra row of panels at the downstream end of the computational domain and an extra row of collocation points at the upstream end (Fig. 3.11). For equidistant grids, this can also be interpreted as shifting or staggering the grid of collocation points vs. the grid of elements, hence the name 'staggered grid'. However, this name is misleading as for non-equidistant grids or three-dimensional grids with quasi-streamlined grid lines, adding an extra row at the ends is not the same as shifting the whole grid.

The pressure integral for the x force is evaluated approximately by:

$$\int_S pn_1 \, \mathrm{d}S \approx \sum_{i=1}^{N_B} p_i n_{1,i} s_i$$

N_B is the number of elements on the hull. The pressure p_i, and the inward normal on the hull, n_i, are taken constant over each panel. s_i is the area of one segment.

Three-Dimensional Wave Resistance Problem

The extension of the theory for a two-dimensional submerged body to a three-dimensional surface-piercing ship free to trim and sink introduces these main new features:

- surface-piercing hulls
- dynamic trim and sinkage
- transom stern
- Kutta condition for multihulls.

The theory outlined here is the theory behind the STEADY code (Hughes and Bertram 1995). The code is a typical representative of a state-of-the-art 'fully non-linear' wave resistance code of the 1990s.

Theory

We consider a ship moving with constant speed V in water of constant depth. The depth and width may be infinite and are in fact assumed to be so in most cases. For inviscid and irrotational flow, this problem is equivalent to a ship being fixed in an inflow of constant speed.

For the considered ideal fluid, continuity gives Laplace's equation, which holds in the whole fluid domain. A unique description of the problem requires further conditions on all boundaries of the fluid resp. the modeled fluid domain:

- Hull condition: water does not penetrate the ship's surface.
- Transom stern condition: for ships with a transom stern, we assume that the flow separates and the transom stern is dry. Atmospheric pressure is then enforced at the edge of the transom stern.
- Kinematic condition: water does not penetrate the water surface.
- Dynamic condition: there is atmospheric pressure at the water surface.
- Radiation condition: waves created by the ship do not propagate ahead. (This condition is not valid for transcritical depth Froude numbers when the flow becomes unsteady and soliton waves are pulsed ahead. But ships are never designed for such speeds.)
- Decay condition: far away from the ship, the flow is undisturbed.
- Open-boundary condition: waves generated by the ship pass unreflected any artificial boundary of the computational domain.
- Equilibrium: the ship is in equilibrium, i.e. trim and sinkage are changed such that the dynamic vertical force and the trim moment are counteracted.
- Bottom condition (shallow-water case): no water flows through the sea bottom.
- Kutta condition (for multihulls): at the end of each side floater the flow separates smoothly. This is approximated by setting the y velocity to zero.

The decay condition replaces the bottom condition if the bottom is at infinity, i.e. in the usual infinite fluid domain case.

The problem is solved using boundary elements (in the case of STEADY higher-order panels on the ship hull, point source clusters above the free surface). The wave resistance problem features two special problems requiring an iterative solution approach:

1. A non-linear boundary condition appears on the free surface.
2. The boundaries of water (waves) and ship (trim and sinkage) are not a priori known.

The iteration starts by approximating:

- the unknown wave elevation by a flat surface
- the unknown potential by the potential of uniform parallel flow
- the unknown position of the ship by the position of the ship at rest.

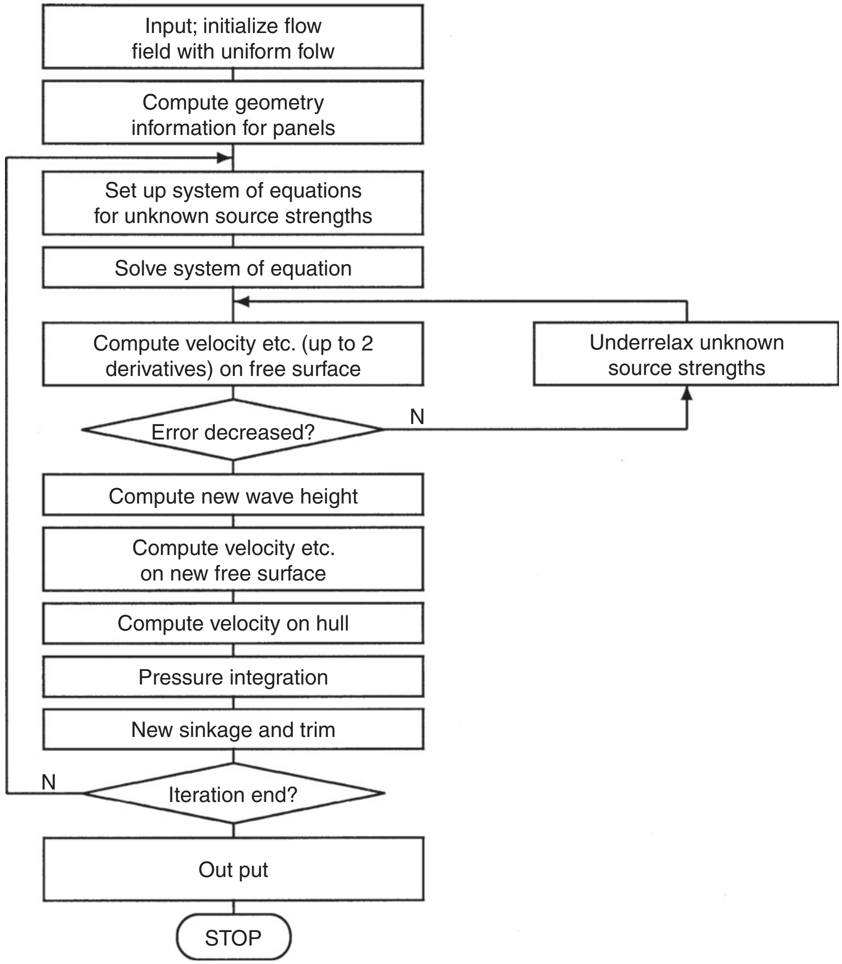

Figure B.1:
Flow chart of iterative solution

In each iterative step, wave elevation, potential, and position are updated, yielding successively better approximations for the solution of the non-linear problem (Fig. B.1).

The equations are formulated here in a right-handed Cartesian coordinate system with x pointing forward towards the bow and z pointing upward. The moment about the y-axis (and the trim angle) are positive clockwise (bow immerses for positive trim angle).

For the assumed ideal fluid, there exists a velocity potential ϕ such that $\vec{v} = \nabla\phi$. The velocity potential ϕ fulfills Laplace's equation in the whole fluid domain:

$$\Delta\phi = \phi_{xx} + \phi_{yy} + \phi_{zz} = 0$$

A unique solution requires the formulation of boundary conditions on all boundaries of the modeled fluid domain.

The hull condition (no penetration of ship hull) requires that the normal velocity on the hull vanishes:

$$\vec{n} \cdot \nabla\phi = 0$$

$\vec{n}$ is the inward unit normal vector on the ship hull.

The transom stern condition (atmospheric pressure at the edge of the transom stern $z = z_T$) is derived from Bernoulli's equation:

$$\frac{1}{2}(\nabla\phi)^2 + gz_T = \frac{1}{2}V^2$$

with $g = 9.81$ m/s^2. This condition is non-linear in the unknown potential. We assume that the water flows at the stern predominantly in the x direction, such that the y and z components are negligible. This leads to the linear condition:

$$\phi_x + \sqrt{V^2 - 2gz_T} = 0$$

For points above the height of stagnation $V^2/2g$, this condition leads to a negative term in the square root. For these points, stagnation of horizontal flow is enforced instead. Both cases can be combined as:

$$\phi_x + \sqrt{\max(0, V^2 - 2gz_T)} = 0$$

The Kutta condition is originally a pressure condition, thus also non-linear. However, the obliqueness of the flow induced at the end of each side floater is so small that a simplification can be well justified. We then enforce just zero y velocity (Joukowski condition):

$$\phi_y = 0$$

The kinematic condition (no penetration of water surface) gives at $z = \zeta$:

$$\nabla\phi \cdot \nabla\zeta = \phi_z$$

For simplification, we write $\zeta(x, y, z)$ with $\zeta_z = \partial\zeta/\partial z = 0$.

The dynamic condition (atmospheric pressure at water surface) gives at $z = \zeta$:

$$\frac{1}{2}(\nabla\phi)^2 + gz = \frac{1}{2}V^2$$

Combining the dynamic and kinematic boundary conditions and linearizing consistently yields again at $z = \overline{\zeta}$:

$$\begin{aligned}
&2(\vec{a}\,\nabla\phi + \Phi_x\Phi_y\phi_{xy} + \Phi_x\Phi_z\phi_{xz} + \Phi_y\Phi_z\phi_{yz}) + \Phi_x^2\phi_{xx} \\
&\quad + \Phi_y^2\phi_{yy} + \Phi_z^2\phi_{zz} + g\phi_z - B\nabla\Phi\nabla\phi = 2\vec{a}\,\nabla\Phi - B\left(\frac{1}{2}((\nabla\Phi)^2 + V^2) - g\overline{\zeta}\right)
\end{aligned}$$

with

$$\vec{a} = \frac{1}{2}\nabla((\nabla\Phi)^2) = \begin{Bmatrix} \Phi_x\Phi_{xx} + \Phi_y\Phi_{xy} + \Phi_z\Phi_{xz} \\ \Phi_x\Phi_{xy} + \Phi_y\Phi_{yy} + \Phi_z\Phi_{yz} \\ \Phi_x\Phi_{xz} + \Phi_y\Phi_{yz} + \Phi_z\Phi_{zz} \end{Bmatrix}$$

$$\begin{aligned} B &= \frac{\left[\frac{1}{2}\nabla\Phi\nabla(\nabla\Phi)^2 + g\Phi_z\right]_z}{g + \nabla\Phi\cdot\nabla\Phi_z} = \frac{[\nabla\Phi\vec{a} + g\Phi_z]_z}{\mathrm{g} + a_3} \\ &= \frac{1}{g + a_3}\Big(\Phi_x^2\Phi_{xxz} + \Phi_y^2\Phi_{yyz} + \Phi_z^2\Phi_{zzz} + g\Phi_{zz} + 2[\Phi_x\Phi_y\Phi_{xyz} \\ &\quad + \Phi_x\Phi_z\Phi_{xzz} + \Phi_y\Phi_z\Phi_{yzz} + \Phi_{xz}\cdot a_1 + \Phi_{yz}\cdot a_2 + \Phi_{zz}\cdot a_3]\Big) \end{aligned}$$

The bottom, radiation, and open-boundary conditions are fulfilled by the proper arrangement of elements as described below. The decay condition − like the Laplace equation − is automatically fulfilled by all elements.

Once a potential has been determined, the forces can be determined by direct pressure integration on the wetted hull. The forces are corrected by the hydrostatic forces at rest. (The hydrostatic x force and y moment should be zero, but are non-zero due to discretization errors. The discretization error is hoped to be reduced by subtracting the value for the hydrostatic force):

$$f_1 = \int_S pn_1 \, \mathrm{d}S - \int_{S_0} p_s n_1 \, \mathrm{d}S$$

$$f_3 = \int_S pn_3 \, \mathrm{d}S - \int_{S_0} p_s n_3 \, \mathrm{d}S$$

$$f_5 = \int_S p(zn_1 - xn_3) \, \mathrm{d}S - \int_{S_0} p_s(zn_1 - xn_3) \, \mathrm{d}S$$

S is the actually wetted surface. S_0 is the wetted surface of the ship at rest. $p_s = -\rho gz$ is the hydrostatic pressure, where ρ is the density of water. p is the pressure determined from Bernoulli's equation:

$$p = \frac{\rho}{2}\left(V^2 - (\nabla\phi)^2\right) - \rho gz$$

The force in the x direction, f_1, is the (negative) wave resistance. The non-dimensional wave resistance coefficient is:

$$C_w = -f_1 \Big/ \left(\frac{\rho}{2}V^2 S\right)$$

The z force and y moments are used to adjust the position of the ship. We assume small changes of the position of the ship. Δz is the deflection of the ship (positive, if the ship surfaces) and $\Delta\theta$ is the trim angle (positive if bow immerses) (Fig. B.2).

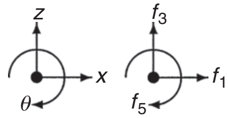

Figure B.2:
Coordinate system; x points towards bow, origin is usually amidships in still waterline; relevant forces and moment

For given Δz and $\Delta\theta$, the corresponding z force and y moment (necessary to enforce this change of position) are:

$$\begin{Bmatrix} f_3 \\ f_5 \end{Bmatrix} = \begin{bmatrix} A_{WL}\cdot\rho\cdot g & -A_{WL}\cdot\rho\cdot g\cdot x_{WL} \\ -A_{WL}\cdot\rho\cdot g\cdot x_{WL} & I_{WL}\cdot\rho\cdot g \end{bmatrix} \begin{Bmatrix} \Delta z \\ \Delta\theta \end{Bmatrix}$$

A_{WL} is the area, I_{WL} the moment of inertia, and x_{WL} the center of the still water-plane. I_{WL} and x_{WL} are taken relative to the origin, which we put amidships. Inversion of this matrix gives an equation of the form:

$$\begin{Bmatrix} \Delta z \\ \Delta\theta \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} f_3 \\ f_5 \end{Bmatrix}$$

The coefficients a_{ij} are determined once in the beginning by inverting the matrix for the still waterline. Then during each iteration the position of the ship is changed by Δz and $\Delta\theta$ giving the final sinkage and trim when converged. The coefficients should actually change as the ship trims and sinks and thus its actual waterline changes from the still waterline. However, this error just slows down the convergence, but (for convergence) does not change the final result for trim and sinkage.

Numerical Implementation

The velocity potential ϕ is approximated by parallel flow superimposed by a finite number of elements. These elements are, for STEADY higher-order panels lying on the ship surface, linear panels (constant strength) in a layer above part of the free surface, and vortex elements lying on the local center plane of any side floater. However, the choice of elements is rather arbitrary. If just wave resistance computations are performed, first-order elements are sufficient and actually preferable due to their greater robustness.

The free-surface elements are again usually ‘desingularized‘. We place them approximately one panel length above the still-water plane ($z = 0$).

We formulate the potential ϕ as the sum of parallel uniform flow (of speed V) and a residual potential which is represented by the elements:

$$\phi = -Vx + \sum \sigma_i \varphi$$

σ_i is the strength of the ith element, φ the potential of an element of unit strength. The index i for φ is omitted for convenience but it should be understood in the equations below that φ refers to the potential of only the ith element. The expression 'element' refers to one panel or vortex and all its mirror images.

Then the no-penetration boundary condition on the hull becomes:

$$\sum \sigma_i(\vec{n} \cdot \nabla\varphi) = Vn_1$$

The Kutta condition becomes:

$$\sum \sigma_i \varphi_y = 0$$

The transom stern condition becomes:

$$\sum \sigma_i \varphi_x = V - \sqrt{\max(0, V^2 - 2gz_T)}$$

The linearized free surface condition then becomes:

$$\sum \sigma_i(2(\vec{a}\nabla\varphi + \Phi_x\Phi_y\varphi_{xy} + \Phi_x\Phi_z\varphi_{xz} + \Phi_y\Phi_z\varphi_{yz}) + \Phi_x^2\varphi_{xx} + \Phi_y^2\varphi_{yy}$$
$$+ \Phi_z^2\varphi_{zz} + g\varphi_z - B\nabla\Phi\nabla\varphi) = 2(\vec{a}\nabla\Phi + a_1 V) - B\left(\frac{1}{2}((\nabla\Phi)^2 + V^2) - g\overline{\zeta} + V\Phi_x\right)$$

These four equations form a linear system of equations in the unknown element strengths σ_i. Once the system is solved, the velocities (and higher derivatives of the potential) are determined on the water surface and the error ε is determined. A special refinement accelerates and stabilizes to some extent the iteration process: if the error ε_{i+1} in iteration step i+1 is larger than the error ε_i in the previous ith step the source strengths are underrelaxed:

$$\sigma_{i+1,\text{new}} = \frac{\sigma_{i+1,\text{old}} \cdot \varepsilon_i + \sigma_i \cdot \varepsilon_{i+1}}{\varepsilon_i + \varepsilon_{i+1}}$$

Velocities and errors are evaluated again with the new source strengths. If the error is decreased the computation proceeds, otherwise the underrelaxation is repeated. If four repetitions still do not improve the error compared to the previous step, the computation is stopped. In this case, no converged non-linear solution can be found. This is usually the case if breaking waves appear in the real flow at a location of a collocation point.

Mirror images of panels are used (Fig. B.3):

1. In the y direction with respect to the center plane $y = 0$.
2. For shallow water in the z direction with respect to the water bottom $z = z_{\text{bottom}} : z' = -2|z_{\text{bottom}}| - z$.

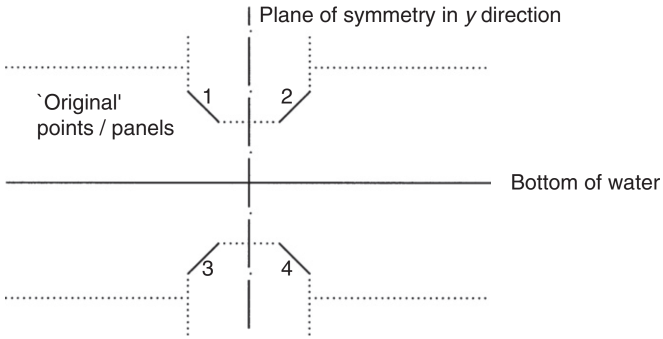

Figure B.3:
Mirror images of panels are used

The computation of the influence of one element on one collocation point uses the fact that the influence of a panel at A on a point at B has the same absolute value and opposite sign as a panel at B on a point at A. Actually, mirror images of the collocation point are produced and the influence of the original panel is computed for all mirror points. Then the sign of each influence is changed according to Table B.1.

The radiation and open-boundary conditions are fulfilled using 'staggered grids' as for the two-dimensional case. No staggering in the y direction is necessary.

For equidistant grids and collocation points along lines of $y =$ const., this can also be interpreted as shifting or staggering the grid of collocation points vs. the grid of elements, hence the name 'staggered grid'. However, for three-dimensional grids around surface-piercing

Table B.1: Sign for derivatives of potential due to interchanging source and collocation point; mirror image number as in Fig. B.4

	1	2	3	4
ϕ_x	+	+	+	+
ϕ_y	+	–	+	–
ϕ_z	+	+	+	+
ϕ_{xx}	+	+	+	+
ϕ_{xy}	+	–	+	–
ϕ_{xz}	+	+	+	+
ϕ_{yy}	+	+	+	+
ϕ_{yz}	+	–	+	–
ϕ_{xxz}	+	+	+	+
ϕ_{xyz}	+	–	+	–
ϕ_{xzz}	+	+	+	+
ϕ_{yyz}	+	+	+	+
ϕ_{yzz}	+	–	+	–

ships the grids are not staggered in a strict sense as, with the exception of the very ends, collocation points always lie directly under panel centers.

The pressure integral for the x force – the procedure for the z force and the y moment are corresponding – is evaluated approximately by:

$$\int_S pn_1 \, dS \approx 2 \sum_{i=1}^{N_B} p_i n_{1,i} s_i$$

N_B is the number of elements on the hull. The pressure p_i and the inward normal on the hull n_i are taken constant over each panel. The factor 2 is due to the port/starboard symmetry.

The non-linear solution makes it necessary to discretize the ship above the still waterline. The grid can then be transformed (regenerated) such that it always follows the actually wetted surface of the ship. However, this requires fully automatic grid generation, which is difficult on complex ship geometries prefering to discretize a ship initially to a line $z =$ const. above the free waterline. Then the whole grid can trim and sink relative to the free surface, as the grids on free surface and ship do not have to match. Then in each step, the actually wetted part of the ship grid has to be determined. The wetted area of each panel can be determined as follows.

A panel is subdivided into triangles. Each triangle is formed by one side of the panel and the panel center. Bernoulli's equation correlates the velocity in a panel to a height z_w where the pressure would equal atmospheric pressure:

$$z_w = \frac{1}{2g}(V^2 - (\nabla\phi)^2)$$

If z_w lies above the highest point of the triangle, s_i is taken as the triangle area. If z_w lies below the lowest point of the triangle, $s_i = 0$. If z_w lies between the highest and the lowest point of the triangle, the triangle is partially submerged and pierces the water surface (Fig. B.4).

In this case, the line z_w divides the triangle into a subtriangle ABC and the remaining trapezoid. If the triangle ABC is submerged (left case) s_i is taken to the area of ABC, otherwise to the triangle area minus ABC. The value of z in the pressure integral (e.g. for the hydrostatic contribution) is taken from the center of the submerged partial area.

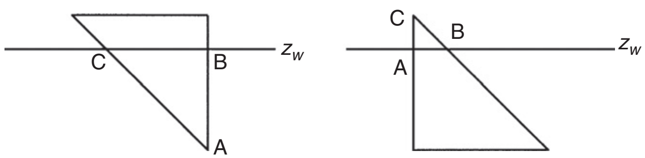

Figure B.4:
Partially submerged triangle with subtriangle ABC submerged (left) or surfaced (right)

If a panel at the upper limit of discretization is completely submerged, the discretization was chosen too low. The limit of upper discretization is given for the trimmed ship by:

$$z = m_{\text{sym}}x + n_{\text{sym}}$$

Strip Method Module (Two-Dimensional)

Strip methods as discussed in Section 4.4.2, Chapter 4, are the standard tool in evaluating ship seakeeping. An essential part of each strip method is the computation of hydrodynamic masses, damping, and exciting forces for each strip. This computation was traditionally based on conformal mapping techniques, where an analytical solution for a semicircle was transformed to a shape resembling a ship section. This technique is not capable of reproducing complex shapes as found in the forebody of modern ships, where possibly cross-section may consist of unconnected parts for bulbous bow and upper stem. Numerical 'close-fit' methods became available with the advent of computers in naval architecture and are now widely used in practice. In the following, one example of such a close-fit method to solve the two-dimensional strip problem is presented. The Fortran source code for the method is available on the internet (www.bh.com/companions/0750648511).

We compute the radiation and diffraction problems for a two-dimensional cross-section of arbitrary shape in harmonic, elementary waves. As usual, we assume an ideal fluid. Then there exists a velocity potential ϕ such that the partial derivatives of this potential in an arbitrary direction yield the velocity component of the flow in that direction. We neglect all non-linear effects in our computations. The problem is formulated in a coordinate system as shown in Fig. B.5. Indices y, z, and t denote partial derivatives with respect to these variables.

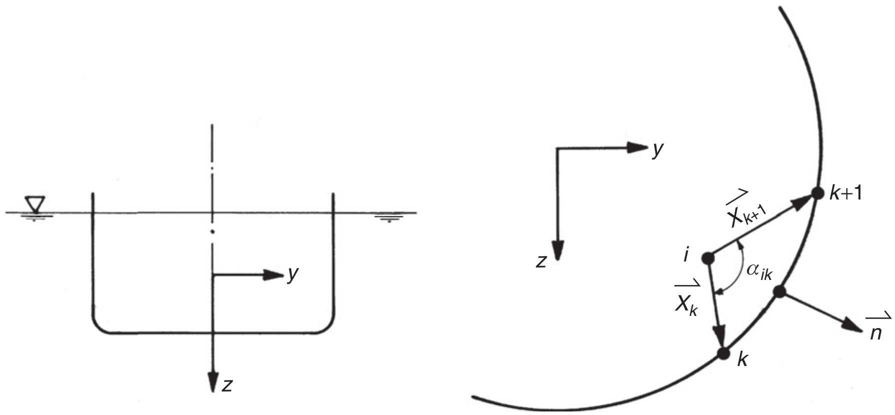

Figure B.5:
Flow chart of iterative solution

We solve the problem in the frequency domain. The two-dimensional seakeeping potentials will then be harmonic functions oscillating with encounter frequency ω_e:

$$\phi(y,z,t) = \mathrm{Re}(\hat{\phi}(y,z)\mathrm{e}^{i\omega_e t})$$

The potential must fulfill the Laplace equation:

$$\phi_{yy} + \phi_{zz} = 0$$

in the whole fluid domain ($z < 0$) subject to the following boundary conditions:

1. Decaying velocity with water depth:

$$\lim_{z \to \infty} \nabla\phi = 0$$

2. There is atmospheric pressure everywhere on the free surface $z = \zeta$ (dynamic condition). Then Bernoulli's equation yields:

$$\phi_t + \frac{1}{2}(\nabla\phi)^2 - g\zeta = 0$$

3. There is no flow through the free surface (kinematic condition), i.e. the local vertical velocity of a particle coincides with the rate of change of the surface elevation in time:

$$\phi_z = \zeta_t$$

4. Differentiation of the dynamic condition with respect to time and combination with the kinematic condition yields:

$$\phi_{tt} + \phi_y\phi_{yt} + \phi_z\phi_{zt} - g\phi_z = 0$$

This expression can be developed in a Taylor expansion around $z = 0$. Omitting all non-linear terms then yields:

$$\phi_{tt} - g\phi_z = 0$$

5. There is no flow through the body contour, i.e. the normal velocity of the water on the body contour coincides with the normal velocity of the hull (or, respectively, the relative normal velocity between body and water is zero):

$$\vec{n}\cdot\nabla\phi = \vec{n}\cdot\vec{v}$$

Here $\vec{v}$ is the velocity of the body, $\vec{n}$ is the outward unit normal vector.

6. Waves created by the body must radiate away from the body:

$$\lim_{|y|\to\infty} \phi = \mathrm{Re}(\hat{\varphi} \mathrm{e}^{-kz} \mathrm{e}^{i(\omega_e t - k|y|)})$$

$\hat{\varphi}$ here is a yet undetermined, but constant, amplitude.

Using the harmonic time dependency of the potential, we can reformulate the Laplace equation and all relevant boundary conditions such that only the time-independent complex amplitude of the potential $\hat{\phi}$ appears:

Laplace equation:

$$\hat{\phi}_{yy} + \hat{\phi}_{zz} = 0 \quad \text{for } z < 0$$

Decay condition:

$$\lim_{z\to\infty} \nabla\hat{\phi} = 0$$

Combined free-surface condition:

$$\frac{\omega_e^2}{g}\hat{\phi} + \hat{\phi}_z = 0 \quad \text{at } z = 0$$

The body boundary condition here is explicitly given for the radiation problem of the body in heave motion. This will serve as an example. The other motions (sway, roll) and the diffraction problem are treated in a very similar fashion. The body boundary condition for heave is then:

$$\vec{n}\nabla\hat{\phi} = i\omega_e n_2$$

n_2 is the z component of the (two-dimensional) normal vector $\vec{n}$.

The radiation condition for $\hat{\phi}$ is derived by differentiation of the initial radiation condition for ϕ with respect to y and z, respectively. The resulting two equations allow the elimination of the unknown constant amplitude $\hat{\varphi}$, yielding:

$$i\hat{\phi}_z = \mathrm{sign}\,(y)\cdot\hat{\phi}_y$$

$\hat{\phi}$ can be expressed as the superposition of a finite number n of point source potentials (see Appendix A). The method described here uses desingularized sources located (a small distance) inside the body and above the free surface. The grid on the free surface extends to a sufficient distance to both sides depending on the wavelength of the created wave. Due to symmetry, sources at y_i, z_i should have the same strength as sources at $-y_i$, z_i. (For sway and roll motion, we have antisymmetrical source strength.) We then exploit symmetry and use source pairs as elements to represent the total potential:

$$\hat{\phi}(y,z) = \sum_{i=1}^{n} \sigma_i \varphi_i$$

$$\varphi_i = \frac{1}{4\pi}\ln\left[(y-y_i)^2 + (z-z_i)^2\right] \; + \frac{1}{4\pi}\ln\left[(y+y_i)^2 + (z-z_i)^2\right]$$

This formulation automatically fulfills the Laplace equation and the decay condition. The body, free surface, and radiation conditions are fulfilled numerically by adjusting the element strengths σ_i appropriately. We enforce these conditions only on points $y_i > 0$. Due to symmetry, they will then also be fulfilled automatically for $y_i < 0$.

The method described here uses a patch method to numerically enforce the boundary conditions (see Appendix A). The body boundary condition is then integrated over one patch, e.g. between the points k and $k + 1$ on the contour (Fig. B.5):

$$\sum_{i=1}^{n} \sigma_i \int_{p_k}^{p_{k+1}} \nabla\varphi_i \vec{n}_k \, \mathrm{d}S = i\omega_e \int_{p_k}^{p_{k+1}} n_2 \, \mathrm{d}S$$

As n_2 can be expressed as $n_2 = \mathrm{d}y/\mathrm{d}s$, this yields:

$$\sum_{i=1}^{n} \sigma_i \int_{p_k}^{p_{k+1}} \nabla\varphi_i \vec{n}_k \, \mathrm{d}S = i\omega_e (y_{k+1} - y_k)$$

The integral on the l.h.s. describes the flow per time (flux) through the patch (contour section) under consideration due to a unit source at y_i, z_i and its mirror image. The flux for just the source without its image corresponds to the portion of the angle α_{ik} (Fig. B.5):

$$\int_{p_k}^{p_{k+1}} \nabla\varphi_i \vec{n}_k \, \mathrm{d}S = \frac{\alpha_{ik}}{2\pi}$$

Correspondingly we write for the elements formed by a pair of sources:

$$\int_{p_k}^{p_{k+1}} \nabla\varphi_i \vec{n}_k \, \mathrm{d}S = \frac{\alpha_{ik}^{+}}{2\pi} + \frac{\alpha_{ik}^{-}}{2\pi}$$

The angle α_{ik}^{+} is determined by:

$$\alpha_{ik} = \arctan\left(\frac{\vec{x}_{k+1} \times \vec{x}_k}{\vec{x}_{k+1} \cdot \vec{x}_{\;k}}\right)_1$$

The index 1 here denotes the x component of the vector.

The other numerical conditions can be formulated in an analogous way. The number of patches corresponds to the number of elements. The patch conditions then form a system of linear equations for the unknown element strengths σ_i, which can be solved straightforwardly. Once the element strengths are known, the velocity can be computed everywhere. The pressure integration for the patch method, then yields the forces on the section. The forces can then again be decomposed into exciting forces (for diffraction) and radiation forces expressed as added mass and damping coefficients analogous to the decomposition described in Section 4.4, Chapter 4. The method has been encoded in the Fortran routines HMASSE and WERREG (see www.bh.com/companions/0750648511).

Rankine Panel Method in the Frequency Domain

Theory

The seakeeping method is limited theoretically to $\tau > 0.25$. In practice, accuracy problems may occur for $\tau < 0.4$. The method does not treat transom sterns. The theory given is that behind the FREDDY code (Bertram 1998).

We consider a ship moving with mean speed V in a harmonic wave of small amplitude h with $\tau = V\omega_e/g > 0.25$. ω_e is the encounter frequency, $g = 9.81$ m/s^2. The resulting (linearized) seakeeping problems are similar to the steady wave resistance problem described previously and can be solved using similar techniques.

The fundamental field equation for the assumed potential flow is again Laplace's equation. In addition, boundary conditions are postulated:

1. No water flows through the ship's surface.
2. At the trailing edge of the ship, the pressures are equal on both sides. (Kutta condition.)
3. A transom stern is assumed to remain dry. (Transom condition.)
4. No water flows through the free surface. (Kinematic free surface condition.)
5. There is atmospheric pressure at the free surface. (Dynamic free surface condition.)
6. Far away from the ship, the disturbance caused by the ship vanishes.
7. Waves created by the ship move away from the ship. For $\tau > 0.25$, waves created by the ship propagate only downstream. (Radiation condition.)
8. Waves created by the ship should leave artificial boundaries of the computational domain without reflection. They may not reach the ship again. (Open-boundary condition.)
9. Forces on the ship result in motions. (Average longitudinal forces are assumed to be counteracted by corresponding propulsive forces, i.e. the average speed V remains constant.)

Note that this verbal formulation of the boundary conditions coincides virtually with the formulation for the steady wave resistance problem.

All coordinate systems here are right-handed Cartesian systems. The inertial $Oxyz$ system moves uniformly with velocity V. x points in the direction of the body's mean velocity V, z points vertically upwards. The $\underline{Oxyz}$ system is fixed at the body and follows its motions. When the body is at rest position, $\underline{x}$, $\underline{y}$, $\underline{z}$ coincide with x, y, z. The angle of encounter μ between body and incident wave is defined such that $\mu = 180°$ denotes head sea and $\mu = 90°$ beam sea.

The body has six degrees of freedom for rigid body motion. We denote corresponding to the degrees of freedom:

u_1 surge motion of $\underline{O}$ in the x direction, relative to O

u_2 heave motion of $\underline{O}$ in the y direction, relative to O

u_3 heave motion of $\underline{O}$ in the z direction, relative to O

u_4 angle of roll = angle of rotation around the x-axis

u_5 angle of pitch = angle of rotation around the y-axis

u_6 angle of yaw = angle of rotation around the z-axis

The motion vector $\vec{u}$ and the rotational motion vector $\vec{\alpha}$ are given by:

$$\vec{u} = \{u_1, u_2, u_3\}^T \quad \text{and} \quad \vec{\alpha} = \{u_4, u_5, u_6\}^T = \{\alpha_1, \alpha_2, \alpha_3\}^T$$

All motions are assumed to be small, of order $O(h)$. Then for the three angles α_i, the following approximations are valid: $\sin(\alpha_i) = \tan(\alpha_i) = \alpha_i$, $\cos(\alpha_i) = 1$.

The relation between the inertial and the hull-bound coordinate system is given by the linearized transformation equations:

$$\vec{x} = \underline{\vec{x}} + \vec{\alpha} \times \underline{\vec{x}} + \vec{u}$$

$$\underline{\vec{x}} = \vec{x} - \vec{\alpha} \times \vec{x} - \vec{u}$$

Let $\vec{v} = \vec{v}(\vec{x})$ be any velocity relative to the $Oxyz$ system and $\underline{\vec{v}} = \underline{\vec{v}}(\underline{x})$ the velocity relative to the $\underline{Oxyz}$ system where $\vec{x}$ and $\underline{\vec{x}}$ describe the same point.

Then the velocities transform:

$$\vec{v} = \underline{\vec{v}} + \vec{\alpha} \times \underline{\vec{v}} + (\vec{\alpha}_t \times \underline{\vec{x}} + \vec{u}_t)$$

$$\underline{\vec{v}} = \vec{v} - \vec{\alpha} \times \vec{v} + (\vec{\alpha}_t \times \vec{x} + \vec{u}_t)$$

The differential operators ∇_x and $\nabla_{\underline{x}}$ transform:

$$\nabla_x = \{\partial/\partial x, \partial/\partial y, \partial/\partial z\}^T = \nabla_{\underline{x}} + \vec{\alpha} \times \nabla_{\underline{x}}$$

$$\nabla_{\underline{x}} = \{\partial/\partial \underline{x}, \partial/\partial \underline{y}, \partial/\partial \underline{z}\}^T = \nabla_x - \vec{\alpha} \times \nabla_x$$

Using a three-dimensional truncated Taylor expansion, a scalar function transforms from one coordinate system into the other:

$$f(\vec{x}) = f(\underline{\vec{x}}) + (\vec{\alpha} \times \underline{\vec{x}} + \vec{u})\nabla_{\underline{x}} f(\underline{\vec{x}})$$

$$f(\underline{\vec{x}}) = f(\vec{x}) - (\vec{\alpha} \times \vec{x} + \vec{u})\nabla_x f(\vec{x})$$

Correspondingly we write:

$$\nabla_x f(\vec{x}) = \nabla_{\underline{x}} f(\underline{\vec{x}}) + ((\vec{\alpha} \times \underline{\vec{x}} + \vec{u})\nabla_{\underline{x}})\nabla_{\underline{x}} f(\underline{\vec{x}})$$

$$\nabla_{\underline{x}} f(\underline{\vec{x}}) = \nabla_x f(\vec{x}) + ((\vec{\alpha} \times \vec{x} + \vec{u})\nabla_x)\nabla_x f(\vec{x})$$

A perturbation formulation for the potential is used:

$$\phi^{\text{total}} = \phi^{(0)} + \phi^{(1)} + \phi^{(2)} + \cdots$$

$\phi^{(0)}$ is the part of the potential which is independent of the wave amplitude h. It is the solution of the steady wave resistance problem described in the previous section (where it was denoted by just ϕ). $\phi^{(1)}$ is proportional to h, $\phi^{(2)}$ proportional to h^2, etc. Within a theory of first order (linearized theory), terms proportional to h^2 or higher powers of h are neglected. For reasons of simplicity, the equality sign is used here to denote equality of low-order terms only, i.e. $A = B$ means $A = B + O(h^2)$.

We describe both the z-component of the free surface ζ and the potential in a first-order formulation. $\phi^{(1)}$ and $\zeta^{(1)}$ are time harmonic with ω_e, the frequency of encounter:

$$\begin{aligned}\phi^{\text{total}}(x,y,z;t) &= \phi^{(0)}(x,y,z) + \phi^{(1)}(x,y,z;t)\\ &= \phi^{(0)}(x,y,z) + \mathrm{Re}(\hat{\phi}^{(1)}(x,y,z)\mathrm{e}^{i\omega_e t})\end{aligned}$$

$$\begin{aligned}\zeta^{\text{total}}(x,y;t) &= \zeta^{(0)}(x,y) + \zeta^{(1)}(x,y;t)\\ &= \zeta^{(0)}(x,y) + \mathrm{Re}(\hat{\zeta}^{(1)}(x,y)\mathrm{e}^{i\omega_e t})\end{aligned}$$

Correspondingly the symbol ^ is used for the complex amplitudes of all other first-order quantities, such as motions, forces, pressures, etc.

The superposition principle can be used within a linearized theory. Therefore the radiation problems for all six degrees of freedom of the rigid-body motions and the diffraction problem are solved separately. The total solution is a linear combination of the solutions for each independent problem.

The harmonic potential $\phi^{(1)}$ is divided into the potential of the incident wave $\phi^{(w)}$, the diffraction potential ϕ^d, and six radiation potentials:

$$\phi^{(1)} = \phi^d + \phi^w + \sum_{i=1}^{6} \phi^i u_i$$

It is convenient to decompose ϕ^w and ϕ^d into symmetrical and antisymmetrical parts to take advantage of the (usual) geometrical symmetry:

$$\phi^w(x, y, z) = \underbrace{\frac{\phi^w(x, y, z) + \phi^w(x, -y, z)}{2}}_{\phi^{w,s}} + \underbrace{\frac{\phi^w(x, y, z) - \phi^w(x, -y, z)}{2}}_{\phi^{w,a}}$$

$$\phi^d = \phi^{d,s} + \phi^{d,a} = \phi^7 + \phi^8$$

Thus:

$$\phi^{(1)} = \phi^{w,s} + \phi^{w,a} + \sum_{i=1}^{6} \phi^i u_i + \phi^7 + \phi^8$$

The conditions satisfied by the steady flow potential $\phi^{(0)}$ are repeated here without further comment.

The particle acceleration in the steady flow is:	$\vec{a}^{(0)} = (\nabla\phi^{(0)}\nabla)\nabla\phi^{(0)}$
We define an acceleration vector $\vec{a}^g$	$\vec{a}^g = \vec{a}^{(0)} + \{0, 0, g\}^T$
For convenience I introduce an abbreviation:	$B = \frac{1}{a_3^g}\frac{\partial}{\partial z}(\nabla\phi^{(0)}\vec{a}^g)$
In the whole fluid domain:	$\Delta\phi^{(0)} = 0$
At the steady free surface:	$\nabla\phi^{(0)}\vec{a}^g = 0$
	$\frac{1}{2}(\nabla\phi^{(0)})^2 + g\zeta^{(0)} = \frac{1}{2}V^2$
On the body surface:	$\underline{\vec{n}}(\underline{\vec{x}})\nabla\phi^{(0)}(\underline{\vec{x}}) = 0$

Also, suitable radiation and decay conditions are observed.

The linearized potential of the incident wave on water of infinite depth is expressed in the inertial system:

$$\phi^w = \mathrm{Re}\left(-\frac{igh}{\omega}\,\mathrm{e}^{-ik(x\cos\mu - y\sin\mu) - kz}\mathrm{e}^{i\omega_e t}\right) = \mathrm{Re}(\hat{\phi}^w \mathrm{e}^{i\omega_e t})$$

$\omega = \sqrt{gk}$ is the frequency of the incident wave, $\omega_e = |\omega - kV\cos\mu|$ the frequency of encounter, k is the wave number. The derivation of the expression for ϕ^w assumes a linearization around $z = 0$. The same formula will be used now in the seakeeping computations, although the average boundary is at the steady wave elevation, i.e. different near the ship. This may be an inconsistency, but the diffraction potential should compensate this 'error'.

We write the complex amplitude of the incident wave as:

$$\hat{\phi}^w = -\frac{igh}{\omega} e^{\vec{x}\vec{d}} \quad \text{with} \quad \vec{d} = \{-ik \cos \mu,\ ik \sin \mu,\ -kz\}^T$$

At the free surface ($z = \zeta^{\text{total}}$) the pressure is constant, namely atmospheric pressure ($p = p_0$):

$$\frac{D(p - p_0)}{Dt} = \frac{\partial(p - p_0)}{\partial t} + (\nabla\phi^{\text{total}}\nabla)(p - p_0) = 0$$

Bernoulli's equation gives at the free surface ($z = \zeta^{\text{total}}$) the dynamic boundary condition:

$$\phi_t^{\text{total}} + \frac{1}{2}(\nabla\phi^{\text{total}})^2 + g\zeta^{\text{total}} + \frac{p}{\rho} = \frac{1}{2}V^2 + \frac{p_0}{\rho}$$

The kinematic boundary condition gives at $z = \zeta^{\text{total}}$:

$$\frac{D\zeta^{\text{total}}}{Dt} = \frac{\partial}{\partial t}\zeta^{\text{total}} + (\nabla\phi^{\text{total}}\nabla)\zeta^{\text{total}} = \phi_z^{\text{total}}$$

Combining the above three equations yields at $z = \zeta^{\text{total}}$:

$$\phi_{tt}^{\text{total}} + 2\nabla\phi^{\text{total}}\nabla\phi_t^{\text{total}} + \nabla\phi^{\text{total}}\nabla\left(\frac{1}{2}\nabla\phi^{\text{total}}\right)^2 + g\phi_z^{\text{total}} = 0$$

Formulating this condition in $\phi^{(0)}$ and $\phi^{(1)}$ and linearizing with regard to instationary terms gives at $z = \zeta^{\text{total}}$:

$$\phi_{tt}^{(1)} + 2\nabla\phi^{(0)}\nabla\phi_t^{(1)} + \nabla\phi^{(0)}\nabla\left(\frac{1}{2}(\nabla\phi^{(0)})^2 + \nabla\phi^{(1)}\nabla\phi^{(0)}\right)$$
$$+\nabla\phi^{(1)}\nabla\left(\frac{1}{2}(\nabla\phi^{(0)})^2\right) + g\phi_z^{(0)} + g\phi_z^{(1)} = 0$$

We develop this equation in a linearized Taylor expansion around $\zeta^{(0)}$ using the abbreviations $\vec{a}, \vec{a}^g$, and B for steady flow contributions. This yields at $z = \zeta^0$:

$$\phi_{tt}^{(1)} + 2\nabla\phi^{(0)}\nabla\phi_t^{(1)} + \nabla\phi^{(0)}\vec{a}^{\,g} + \nabla\phi^{(0)}(\nabla\phi^{(0)}\nabla)\nabla\phi^{(1)} + \nabla\phi^{(1)}(\vec{a} + \vec{a}^g) + Ba_3^g\zeta^{(1)} = 0$$

The steady boundary condition can be subtracted, yielding:

$$\phi_{tt}^{(1)} + 2\nabla\phi^{(0)}\nabla\phi_t^{(1)} + \nabla\phi^{(0)}(\nabla\phi^{(0)}\nabla)\nabla\phi^{(1)} + \nabla\phi^{(1)}(\vec{a} + \vec{a}^g) + Ba_3^g\zeta^{(1)} = 0$$

$\zeta^{(1)}$ will now be substituted by an expression depending solely on $\zeta^{(0)}, \phi^{(0)}(\zeta^{(0)})$ and $\phi^{(1)}(\zeta^{(0)})$. To this end, Bernoulli's equation is also developed in a Taylor expansion. Bernoulli's equation yields at $z = \zeta^{(0)} + \zeta^{(1)}$:

$$\phi_t^{\text{total}} + \frac{1}{2}(\nabla\phi^{\text{total}})^2 + g\zeta^{\text{total}} = \frac{1}{2}V^2$$

A truncated Taylor expansion gives at $z = \zeta^{(0)}$:

$$\phi_t^{(1)} + \frac{1}{2}(\nabla\phi^{\text{total}})^2 + g\zeta^{(0)} - \frac{1}{2}V^2 + (\nabla\phi^{\text{total}}\phi_z^{\text{total}} + g)\zeta^{(1)} = 0$$

Formulating this condition in $\phi^{(0)}$ and $\phi^{(1)}$, linearizing with regard to instationary terms and subtracting the steady boundary condition yields:

$$\phi_t^{(1)} + \nabla\phi^{(0)}\nabla\phi^{(1)} + a_3^g\zeta^{(1)} = 0$$

This can be reformulated as:

$$\zeta^{(1)} = \frac{\phi_t^{(1)} + \nabla\phi^{(0)}\nabla\phi^{(1)}}{a_3^g}$$

By inserting this expression in the free-surface condition and performing the time derivatives leaving only complex amplitudes, the free-surface condition at $z = \zeta^{(0)}$ becomes:

$$(-\omega_e^2 + Bi\omega_e)\hat{\phi}^{(1)} + ((2i\omega_e + B)\nabla\phi^{(0)} + \vec{a}^{(0)} + \vec{a}^g)\nabla\hat{\phi}^{(1)} + \nabla\phi^{(0)}(\nabla\phi^{(0)}\nabla)\nabla\hat{\phi}^{(1)} = 0$$

The last term in this condition is explicitly written:

$$\begin{aligned}\nabla\phi^{(0)}(\nabla\phi^{(0)}\nabla)\nabla\hat{\phi}^{(1)} = {} & (\phi_x^{(0)})^2\phi_{xx}^{(1)} + (\phi_y^{(0)})^2\phi_{yy}^{(1)} + (\phi_z^{(0)})^2\phi_{zz}^{(1)} + 2\cdot(\phi_x^{(0)}\phi_y^{(0)}\phi_{xy}^{(1)} \\ & + \phi_x^{(0)}\phi_z^{(0)}\phi_{xz}^{(1)} + \phi_y^{(0)}\phi_z^{(0)}\phi_{yz}^{(1)})\end{aligned}$$

Complications in formulating the kinematic boundary condition on the body's surface arise from the fact that the unit normal vector is conveniently expressed in the body-fixed coordinate system, while the potential is usually given in the inertial system. The body surface is defined in the body-fixed system by the relation $\underline{S}(\underline{\vec{x}}) = 0$.

Water does not penetrate the body's surface, i.e. relative to the body-fixed coordinate system the normal velocity is zero, at $\underline{S}(\underline{\vec{x}}) = 0$:

$$\underline{\vec{n}}(\underline{\vec{x}})\cdot\underline{\vec{v}}(\underline{\vec{x}}) = 0$$

$\underline{\vec{n}}$ is the inward unit normal vector. The velocity transforms into the inertial system as:

$$\underline{\vec{v}}(\underline{\vec{x}}) = \vec{v}(\vec{x}) - \vec{\alpha}\times\vec{v}(\vec{x}) - (\vec{\alpha}_t\times\vec{x} + \vec{u}_t)$$

where $\vec{x}$ is the inertial system description of the same point as $\underline{\vec{x}}$. $\vec{v}$ is expressed as the sum of the derivatives of the steady and the first-order potential:

$$\vec{v}(\vec{x}) = \nabla\phi^{(0)}(\vec{x}) + \nabla\phi^{(1)}(\vec{x})$$

For simplicity, the subscript x for the ∇ operator is dropped. It should be understood that from now on the argument of the ∇ operator determines its type, i.e. $\nabla\phi(\vec{x}) = \nabla_x\phi(\vec{x})$ and $\nabla\phi(\underline{\vec{x}}) = \nabla_{\underline{x}}\phi(\underline{\vec{x}})$. As $\phi^{(1)}$ is of first order small, $\phi^{(1)}(\underline{\vec{x}}) = \phi^{(1)}(\vec{x}) = \phi^{(1)}$.

The r.h.s. of the above equation for $\vec{v}(\vec{x})$ transforms back into the hull-bound system:

$$\vec{v}(\vec{x}) = \nabla\phi^{(0)}(\underline{\vec{x}}) + ((\vec{\alpha}\times\underline{\vec{x}} + \vec{u})\nabla)\nabla\phi^{(0)}(\underline{\vec{x}}) + \nabla\phi^{(1)}$$

Combining the above equations and omitting higher-order terms yields:

$$\underline{\vec{n}}(\underline{\vec{x}})(\nabla\phi^{(0)}(\underline{\vec{x}}) - \vec{\alpha}\times\nabla\phi^{(0)} + ((\vec{\alpha}\times\underline{\vec{x}} + \vec{u})\nabla)\nabla\phi^{(0)} + \nabla\phi^{(1)} - (\vec{\alpha}_t\times\vec{x} + \vec{u}_t)) = 0$$

This boundary condition must be fulfilled at any time. The steady terms give the steady body-surface condition as mentioned above. Because only terms of first order are left, we can exchange $\vec{x}$ and $\underline{\vec{x}}$ at our convenience. Using some vector identities we derive:

$$\underline{\vec{n}}\nabla\hat{\phi}^{(1)} + \hat{\vec{u}}[(\underline{\vec{n}}\nabla)\nabla\phi^{(0)} - i\omega_e\underline{\vec{n}}] + \hat{\vec{\alpha}}[\underline{\vec{n}}\times\nabla\phi^{(0)} + \underline{\vec{x}}\times((\underline{\vec{n}}\nabla)\nabla\phi^{(0)} - i\omega_e\underline{\vec{n}})] = 0$$

where all derivatives of potentials can be taken with respect to the inertial system.

With the abbreviation $\vec{m} = (\underline{\vec{n}}\nabla)\nabla\phi^{(0)}$ the boundary condition at $\underline{S}(\underline{\vec{x}}) = 0$ becomes:

$$\underline{\vec{n}}\nabla\hat{\phi}^{(1)} + \hat{\vec{u}}(\vec{m} - i\omega_e\underline{\vec{n}}) + \hat{\vec{\alpha}}(\underline{\vec{x}}\times(\vec{m} - i\omega_e\underline{\vec{n}}) + \underline{\vec{n}}\times\nabla\phi^{(0)}) = 0$$

The Kutta condition requires that at the trailing edge the pressures are equal on both sides. This is automatically fulfilled for the symmetric contributions (for monohulls). Then only the antisymmetric pressures have to vanish:

$$-\rho(\phi_t^i + \nabla\phi^{(0)}\nabla\hat{\phi}^i) = 0$$

This yields on points at the trailing edge:

$$i\omega_e\hat{\phi}^i + \nabla\phi^{(0)}\nabla\hat{\phi}^i = 0$$

Diffraction and radiation problems for unit amplitude motions are solved independently as described in the next section. After the potentials $\hat{\phi}^i$ ($i = 1 \ldots 8$) have been determined, only the motions u_i remain as unknowns.

The forces $\vec{F}$ and moments $\vec{M}$ acting on the body result from the body's weight and from integrating the pressure over the instantaneous wetted surface S. The body's weight $\vec{G}$ is:

$$\vec{G} = \{0, 0, -mg\}^T$$

m is the body's mass. (In addition, a propulsive force counteracts the resistance. This force could be included in a similar fashion as the weight. However, resistance and propulsive force are assumed to be negligibly small compared to the other forces.)

$\vec{F}$ and $\vec{M}$ are expressed in the inertial system ($\vec{n}$ is the inward unit normal vector):

$$\vec{F} = \int_S (p(\vec{x}) - p_0)\vec{n}(\vec{x})\, \mathrm{d}S + \vec{G}$$

$$\vec{M} = \int_S (p(\vec{x}) - p_0)(\vec{x} \times \vec{n}(\vec{x}))\, \mathrm{d}S + \vec{x}_g \times \vec{G}$$

$\vec{x}_g$ is the center of gravity. The pressure is given by Bernoulli's equation:

$$p(\vec{x}) - p_0 = -\rho\left(\frac{1}{2}(\nabla\phi^{\text{total}}(\vec{x}))^2 - \frac{1}{2}V^2 + gz + \phi_t^{\text{total}}(\vec{x})\right)$$

$$= \underbrace{-\rho\left(\frac{1}{2}(\nabla\phi^{\text{total}}(\vec{x}))^2 - \frac{1}{2}V^2 + gz\right)}_{p^{(0)}} - \underbrace{\rho(\nabla\phi^{(0)}\nabla\phi^{(1)} + \phi_t^{(1)})}_{p^{(1)}}$$

The r.h.s. of the expressions for $\vec{F}$ and $\vec{M}$ are now transformed from the inertial system to the body-fixed system. This includes a Taylor expansion around the steady position of the body. The normal vector $\vec{n}$ and the position $\vec{x}$ are readily transformed as usual:

$$\vec{x} = \underline{\vec{x}} + \vec{\alpha} \times \underline{\vec{x}} + \vec{u}$$

$$\vec{n}(\vec{x}) = \underline{\vec{n}}(\underline{\vec{x}}) + \vec{\alpha} \times \underline{\vec{n}}(\underline{\vec{x}})$$

The steady parts of the equations give:

$$\vec{F}^{(0)} = \int_{\underline{S}^{(0)}} p^{(0)}\, \underline{\vec{n}}\, \mathrm{d}\underline{S} + \vec{G} = 0$$

$$\vec{M}^{(0)} = \int_{\underline{S}^{(0)}} p^{(0)}(\underline{\vec{x}} \times \underline{\vec{n}})\mathrm{d}\underline{S} + \underline{\vec{x}}_g \times \vec{G} = 0$$

The ship is in equilibrium for steady flow. Therefore the steady forces and moments are all zero.

The first-order parts give (r.h.s. quantities are now all functions of $\underline{\vec{x}}$):

$$\vec{F}^{(1)} = \int_{\underline{S}^{(0)}} [(p^{(1)} + \nabla p^{(0)}(\vec{\alpha} \times \underline{\vec{x}} + \vec{u})]\,\underline{\vec{n}}\,\mathrm{d}\underline{S} - \vec{\alpha} \times \vec{G}$$

$$\vec{M}^{(1)} = \int_{\underline{S}^{(0)}} [(p^{(1)} + \nabla p^{(0)}(\vec{\alpha} \times \underline{\vec{x}} + \vec{u})](\underline{\vec{x}} \times \underline{\vec{n}})\mathrm{d}\underline{S} - \underline{\vec{x}}_g \times (\vec{\alpha} \times \vec{G})$$

where $(\vec{\alpha} \times \underline{\vec{x}}) \times \underline{\vec{n}} + \underline{\vec{x}} \times (\vec{\alpha} \times \underline{\vec{n}}) = \vec{\alpha} \times (\underline{\vec{x}} \times \underline{\vec{n}})$ and the expressions for $\vec{F}^{(0)}$ and $\vec{M}^{(0)}$ have been used. Note: $\nabla p^{(0)} = -\rho\vec{a}^{g}$. The difference between instantaneous wetted surface and average wetted surface still does not have to be considered as the steady pressure $p^{(0)}$ is small in the region of difference.

The instationary pressure is divided into parts due to the incident wave, radiation and diffraction:

$$p^{(1)} = p^w + p^d + \sum_{i=1}^{6} p^i u_i$$

Again the incident wave and diffraction contributions can be decomposed into symmetrical and antisymmetrical parts:

$$p^w = p^{w,s} + p^{w,a}$$

$$p^d = p^{d,s} + p^{d,a} = p^7 + p^8$$

Using the unit motion potentials, the pressure parts p^i are derived:

$$p^i = -p(\phi_t^i + \nabla\phi^{(0)}\nabla\phi^i)$$

$$p^w = -p(\phi_t^w + \nabla\phi^{(0)}\nabla\phi^w)$$

$$p^d = -p(\phi_t^d + \nabla\phi^{(0)}\nabla\phi^d)$$

The individual terms in the integrals for $\vec{F}^{(1)}$ and $\vec{M}^{(1)}$ are expressed in terms of the motions u_t, using the vector identity $(\vec{\alpha} \times \underline{\vec{x}})\vec{a}^g = \vec{\alpha}(\underline{\vec{x}} \times \vec{a}^g)$:

$$\begin{aligned}\vec{F}^{(1)} = &\int_{\underline{S}^{(0)}} \left(p^w + p^d\right)\underline{\vec{n}}\,\mathrm{d}\underline{S} + \sum_{i=1}^{6}\left(\int_{\underline{S}^{(0)}} p^i\,\underline{\vec{n}}\,\mathrm{d}\underline{S}\right)u_i \\ &+ \int_{\underline{S}^{(0)}} -\rho(\vec{u}\vec{a}^g + \vec{\alpha}(\underline{\vec{x}} \times \vec{a}^g))\vec{n}\,\mathrm{d}\underline{S} - \vec{\alpha} \times \vec{G}\end{aligned}$$

$$\vec{M}^{(1)} = \int_{\underline{S}^{(0)}} (p^w + p^d)(\vec{\underline{x}} \times \vec{n})\mathrm{d}\underline{S} + \sum_{i=1}^{6} \left(\int_{\underline{S}^{(0)}} p^i (\vec{x} \times \vec{\underline{n}})\mathrm{d}\underline{S} \right) u_i$$
$$- \vec{\underline{x}}_g \times (\vec{\alpha} \times \vec{G}) + \int_{\underline{S}^{(0)}} -\rho(\vec{u}\,\vec{a}^g + \vec{\alpha}(\vec{\underline{x}} \times \vec{a}^g))(\vec{x} \times \vec{n})\mathrm{d}\underline{S}$$

The relation between forces, moments and motion acceleration is:

$$\vec{F}^{(1)} = m(\vec{u}_{tt} + \vec{\alpha}_{tt} \times \vec{\underline{x}}_g)$$
$$\vec{M}^{(1)} = m(\vec{\underline{x}}_g \times \vec{u}_{tt}) + I\vec{\alpha}_{tt}$$

I is the matrix of moments of inertia:

$$I = \begin{bmatrix} \Theta_{\underline{x}} & 0 & -\Theta_{\underline{xz}} \\ 0 & \Theta_{\underline{y}} & 0 \\ -\Theta_{\underline{xz}} & 0 & \Theta_{\underline{z}} \end{bmatrix}$$

where mass distribution symmetrical in y is assumed. $\Theta_{\underline{x}}$ etc. are the moments of inertia and the centrifugal moments with respect to the origin of the body-fixed $\underline{Oxyz}$ system:

$$\Theta_{\underline{x}} = \int (\underline{y}^2 + \underline{z}^2)\,\mathrm{d}m; \quad \Theta_{\underline{x}\,\underline{y}} = \int \underline{x}\,\underline{y}\,\mathrm{d}m; \quad \text{etc.}$$

Combining the above equations for $\vec{F}^{(1)}$ and $\vec{M}^{(1)}$ yields a linear system of equations in the unknown u_i that is quickly solved using Gauss elimination.

Numerical Implementation

Systems of equations for unknown potentials

The two unknown diffraction potentials and the six unknown radiation potentials are determined by approximating the unknown potentials by a superposition of a finite number of Rankine higher-order panels on the ship and above the free surface. For the antisymmetric cases, in addition Thiart elements (Appendix A) are arranged and a Kutta condition is imposed on collocation points at the last column of collocation points on the stern. Radiation and open-boundary conditions are fulfilled by the ‘staggering’ technique (adding one row of collocation points at the upstream end of the free-surface grid and one row of source elements at the downstream end of the free-surface grid). This technique only works well for $\tau > 0.4$.

Elements use mirror images at $y = 0$ and for shallow water at $z = z_{\text{bottom}}$. For the symmetrical cases, all mirror images have the same strength. For the antisymmetrical case, the mirror images on the negative y sector(s) have negative element strength of the same absolute magnitude.

Each unknown potential is then written as:

$$\hat{\phi}^i = \sum \hat{\sigma}_i \varphi$$

σ_i is the strength of the ith element, φ the potential of an element of unit strength. φ is real for the Rankine elements and complex for the Thiart elements.

The same grid on the hull may be used as for the steady problem, but the grid on the free surface should be created new depending on the wave length of the incident wave. The quantities on the new grid can be interpolated within the new grid from the values on the old grid. Outside the old grid in the far field, all quantities are set to uniform flow on the new grid.

For the boundary condition on the free surface, we introduce the following abbreviations:

$$f_q = -\omega_e^2 + i\omega_e B$$

$$f_{qx} = (2i\omega_e + B)\phi_x^{(0)} + 2a_1$$

$$f_{qy} = (2i\omega_e + B)\phi_y^{(0)} + 2a_2$$

$$f_{qz} = (2i\omega_e + B)\phi_z^{(0)} + 2a_3$$

$$f_{qxx} = \phi_x^{(0)} \cdot \phi_x^{(0)} - \phi_z^{(0)} \cdot \phi_z^{(0)}$$

$$f_{qxy} = 2 \cdot \phi_x^{(0)} \cdot \phi_y^{(0)}$$

$$f_{qxz} = 2 \cdot \phi_x^{(0)} \cdot \phi_z^{(0)}$$

$$f_{qyy} = \phi_y^{(0)} \cdot \phi_y^{(0)} - \phi_z^{(0)} \cdot \phi_z^{(0)}$$

$$f_{qyz} = 2 \cdot \phi_y^{(0)} \cdot \phi_z^{(0)}$$

Then we can write the free-surface condition for the radiation cases ($i = 1 \ldots 6$):

$$\sum \hat{\sigma}_i (f_q\varphi + f_{qx}\varphi_x + f_{qy}\varphi_y + f_{qz}\varphi_z + f_{qxx}\varphi_{xx} + f_{qxy}\varphi_{xy} + f_{qxz}\varphi_{xz} + f_{qyy}\varphi_{yy} + f_{qyz}\varphi_{yz}) = 0$$

where it has been exploited that all potentials fulfill Laplace's equation. Similarly, we get for the symmetrical diffraction problem:

$$\begin{aligned}
&\sum \hat{\sigma}_i (f_q\varphi + f_{qx}\varphi_x + f_{qy}\varphi_y + f_{qz}\varphi_z + f_{qxx}\varphi_{xx} + f_{qxy}\varphi_{xy} + f_{qxz}\varphi_{xz} + f_{qyy}\varphi_{yy} + f_{qyz}\varphi_{yz}) \\
&\quad + f_q\hat{\phi}^{w,s} + f_{qx}\hat{\phi}_x^{w,s} + f_{qy}\hat{\phi}_y^{w,s} + f_{qz}\hat{\phi}_z^{w,s} + f_{qxx}\hat{\phi}_{xx}^{w,s} + f_{qxy}\hat{\phi}_{xy}^{w,s} + f_{qxz}\hat{\phi}_{xz}^{w,s} + f_{qyy}\hat{\phi}_{yy}^{w,s} \\
&\quad + f_{qyz}\hat{\phi}_{yz}^{w,s} = 0
\end{aligned}$$

The expression for the antisymmetrical diffraction problem is written correspondingly using $\hat{\phi}^{w,a}$ on the r.h.s.

Hull condition

For the hull conditions for the eight radiation and diffraction problems, we introduce the following abbreviations, where the auxiliary variable h is used as a local variable with different meaning than further below for the system of equations for the motions:

$$\{h_1, h_2, h_3\}^T = \vec{m} \quad - i\omega_e \underline{\vec{n}}$$

$$\{h_4, h_5, h_6\}^T = \underline{\vec{x}} \times (\vec{m} - i\omega_e \underline{\vec{n}}) + \underline{\vec{n}} \times \nabla\phi^{(0)}$$

$$h_7 = \nabla\hat{\phi}^{w,s} \cdot \underline{\vec{n}}$$

$$h_8 = \nabla\hat{\phi}^{w,a} \cdot \underline{\vec{n}}$$

Then the hull condition can be written for the jth case ($j = 1 \ldots 8$):

$$\sum \hat{\sigma}_i(\underline{\vec{n}} \cdot \varphi) + h_j = 0$$

The Kutta condition is simply written:

$$\sum \hat{\sigma}_i(i\omega_e\varphi + \nabla\phi^{(0)}\nabla\varphi) = 0 \qquad \text{for case } j = 2, 4, 6$$

$$\sum \hat{\sigma}_i(i\omega_e\varphi + \nabla\phi^{(0)}\nabla\varphi) + i\omega_e\hat{\phi}^{w,s} + \nabla\phi^{(0)}\nabla\hat{\phi}^{w,s} = 0 \qquad \text{for case } j = 8$$

The l.h.s. of the four systems of equations for the symmetrical cases and the l.h.s. for the four systems of equations for the antisymmetrical cases each share the same coefficients. Thus four systems of equations can be solved simultaneously using a Gauss elimination procedure.

System of equations for motions

We introduce the abbreviations:

$$\{h_1, h_2, h_3\}^T = -\rho\vec{a}^{\,g}$$

$$\{h_4, h_5, h_6\}^T = -\rho\underline{\vec{x}} \times \vec{a}^{\,g}$$

$$h_7 = p^{w,s} = -\rho(i\omega_e\hat{\phi}^{w,s} + \nabla\phi^{(0)}\nabla\hat{\phi}^{w,s})$$

$$h_8 = p^{w,a} = -\rho(i\omega_e\hat{\phi}^{w,a} + \nabla\phi^{(0)}\nabla\hat{\phi}^{w,a})$$

Recall that the instationary pressure contribution is:

$$p^i = -\rho(i\omega_e\hat{\phi}^i + \nabla\phi^{(0)}\nabla\hat{\phi}^i)$$

Then we can rewrite the conditions for $\vec{F}^{(1)}$ and $\vec{M}^{(1)}$:

$$-m(\vec{u}_{tt} + \vec{\alpha}_{tt} \times \vec{\underline{x}}_g) + \sum_{i=1}^{8}\left(\int_{\underline{S}^{(0)}} (p^i + h^i)\,\underline{\vec{n}}\;\mathrm{d}\underline{S}\right)u_i - \vec{\alpha} \times \vec{G} = 0$$

$$-m(\vec{\underline{x}}_g \times \vec{u}_{tt}) - I\vec{\alpha}_{tt}\sum_{i=1}^{8}\left(\int_{\underline{S}^{(0)}} (p^i + h^i)(\vec{\underline{x}} \times \underline{\vec{n}})\;\mathrm{d}\underline{S}\right)u_i - \vec{\underline{x}}_g \times (\vec{\alpha} \times \vec{G}) = 0$$

The weight terms $-\vec{\alpha} \times \vec{G}$ and $-\vec{\underline{x}}_g \times (\vec{\alpha} \times \vec{G})$ contribute with $W = mg$:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & W & 0 \\ 0 & 0 & 0 & -W & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \underline{z}_g W & 0 & 0 \\ 0 & 0 & 0 & 0 & \underline{z}_g W & 0 \\ 0 & 0 & 0 & \underline{x}_g W & & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

The mass terms $-m(\hat{u}_{tt} + \hat{\alpha}_{tt} \times \vec{\underline{x}}_g)$ and $-m(\vec{\underline{x}}_g \times \hat{u}_{tt}) - I\hat{\alpha}_{tt}$ contribute:

$$-m\,\frac{\partial^2}{\partial t^2}\begin{bmatrix} 1 & 0 & 0 & 0 & \underline{z}_g & 0 \\ 0 & 1 & 0 & -\underline{z}_g & 0 & \underline{x}_g \\ 0 & 0 & 1 & 0 & -\underline{x}_g & 0 \\ 0 & -\underline{z}_g & 0 & k_{\underline{x}}^2 & 0 & -k_{\underline{xz}}^2 \\ \underline{z}_g & 0 & -\underline{x}_g & 0 & k_{\underline{y}}^2 & 0 \\ 0 & \underline{x}_g & 0 & -k_{\underline{xz}}^2 & 0 & k_{\underline{z}}^2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{Bmatrix}$$

where the radii of inertia k have been introduced, e.g. $\Theta_{\underline{x}} = mk_{\underline{x}}^2$, etc.