1 Hestenes Operator

1.1 Construction

We construct the Hestenes operator for domains $\Omega \subset \mathbb{R}^n$ with C^m boundary mainly following paragraphs 6.2,6.3 of [2]. First we consider a simple case where Ω is a C^m half strip.

Lemma 1. Let $l, n, m \in \mathbb{N}, m \geq l, 1 \leq p \leq \infty$ and $W = \prod_{i=1}^{n-1}]a_i, b_i[$ be an open cuboid of $\mathbb{R}^{n-1}$. Moreover define

$$S = W \times \mathbb{R}$$

$$\Omega = \{(\overline{x}, x_n) | \overline{x} \in W, x_n < \phi(\overline{x})\}$$

where $\phi \in C^m(\overline{W})$,$m \geq l$, and $\|D^\alpha \phi\| \leq M < \infty$ for every $1 \leq |\alpha| \leq l$. Then there exists a bounded extension operator T from $W^{l,p}(\Omega)$ to $W^{l,p}(S)$.

To prove Lemma 1 we prove first the case $\phi \equiv 0$ in the following result, that is a generalization of Lemma 9.2 in [1].

Lemma 2. Let $l, n \in \mathbb{N}, 1 \leq p \leq \infty$ and $W = \prod_{i=1}^{n-1}]a_i, b_i[$ be an open cuboid of $\mathbb{R}^{n-1}$. There exists a bounded extension operator

$$T : W^{l,p}(S^-) \to W^{l,p}(S)$$

where

$$S = W \times \mathbb{R}$$

$$S^- = W \times \mathbb{R}^-.$$

Proof. Let $f \in W^{l,p}(S^-)$. We define

$$Tf(\overline{x}, x_n) = \begin{cases} f(x), & \text{if } x_n < 0, \\ \sum_{k=1}^{l} \alpha_k f(\overline{x}, -\beta_k x_n), & \text{if } x_n > 0, \end{cases}$$

where α_k, β_k are real numbers that satisfy $\beta_k > 0$ and

$$\sum_{k=1}^{l} \alpha_k (-\beta_k)^s = 1 \tag{1}$$

for every $s = 0, ..., l-1$. Notice that given $\beta_1, ..., \beta_l > 0$ pairwise distinct, we can always find $\alpha_1, ..., \alpha_l$ that satisfy the condition by solving a Vandermonde square system of linear equations. First we prove that $Tf \in W^{l,p}(S)$. We take any $\phi \in C_c^\infty(S)$ and consider the integral

$$\int_S Tf(x) D^\alpha \phi(x) dx = \int_{S^+} Tf(x) D^\alpha \phi(x) dx + \int_{S^-} Tf(x) D^\alpha \phi(x) dx$$

where $S^+ = \{(\overline{x}, x_n) \mid \overline{x} \in W, x_n > 0\}$ and $\alpha \in \mathbb{N}_0^n, 1 \le |\alpha| \le l$. Let's write $\alpha = (\overline{\alpha}, \alpha_n)$, with $\overline{\alpha} \in \mathbb{N}_0^{n-1}$ and $\alpha_n \in \mathbb{N}_0$. By changing variables in the integrals we get

$$\begin{aligned} \int_S Tf(x) D^\alpha \phi(x) dx &= \int_{S^+} \sum_{k=1}^{l} \alpha_k f(\overline{x}, -\beta_k x_n) D^\alpha \phi(x) dx + \int_{S^-} f(x) D^\alpha \phi(x) dx \\ &= \int_{S^-} f(\overline{y}, y_n) D^\alpha \psi(\overline{y}, y_n) dy \qquad (*) \end{aligned}$$

where $\psi(\overline{x}, x_n) = \sum_{k=1}^{l} -\alpha_k (-\beta_k)^{\alpha_n - 1} \phi\left(\overline{x}, -x_n/\beta_k\right) + \phi(\overline{x}, x_n)$.Note that ψ belongs to $\in C^\infty(S^-)$ but does not have compact support in S^-. To bypass this problem we use an auxiliary function $\nu \in C^\infty(\mathbb{R})$ that satisfies

$$\begin{cases} \nu(x) = 0, & \text{if } x > -1/2, \\ \nu(x) = 1, & \text{if } x < -1, \end{cases}$$

and we define the functions $\nu_k(t) = \nu(kt)$ for $k \in \mathbb{N}$. It's clear that $\psi(x)\nu_k(x_n) \in C_c^\infty(S^-)$, hence we can integrate by parts

$$\int_{S^-} f(x) D^\alpha(\psi(x)\nu_k(x_n)) dx = (-1)^{|\alpha|} \int_{S^-} D_w^\alpha f(x) \psi(x) \nu_k(x_n) dx \qquad (2)$$

By the Leibniz rule

$$\begin{aligned} D^\alpha(\psi(x)\nu_k(x_n)) &= \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} D^{\overline{\alpha}}(\psi(x)\nu_k(x_n)) \\ &= \nu(kx_n) D^\alpha \psi(x) + \sum_{i=1}^{\alpha_n} \binom{\alpha_n}{i} k^i \nu^{(i)}(kx_n) \frac{\partial^{\alpha_n - i}}{\partial x_n^{\alpha_n - i}} D^{\overline{\alpha}} \psi(x). \end{aligned}$$

By the Dominated Convergence Theorem

$$\int_{S^-} f(x) \nu(kx_n) D^\alpha \psi(x) dx \to \int_{S^-} f(x) D^\alpha \psi(x) dx \text{ as } k \to \infty,$$

because $f \in L^1(S^- \cap \operatorname{supp}\psi)$ since $\operatorname{supp}\psi$ is bounded. Next, we claim that for every $i = 1, .., \alpha_n$

$$\int_{S^-} f(x)k^i \nu^{(i)}(kx_n) \frac{\partial^{\alpha_n - i}}{\partial x_n^{\alpha_n - i}} D^{\overline{\alpha}}\psi(x)dx \to 0 \tag{3}$$

as $k \to \infty$. To prove this first we notice that since α_k, β_k satisfies (1) we have that

$$\frac{\partial^j}{\partial x_n^j} D^{\overline{\alpha}}\psi(\overline{x}, 0) = 0 \ ; \ j = 0, ..., \alpha_n - 1,$$

hence by Taylor formula

$$\left| \frac{\partial^{\alpha_n - i}}{\partial x_n^{\alpha_n - i}} D^{\overline{\alpha}}\psi(\overline{x}, x_n) \right| \leq \frac{C|x_n|^i}{i!},$$

for all $i = 1, ..., \alpha_n$, where $C = \sup_{x \in S^-} |D^\alpha \psi(x)|$. Therefore we get the following estimate

$$\begin{aligned} \int_{S^-} \left| f(x)k^i \nu^{(i)}(kx_n) \frac{\partial^{\alpha_n - i}}{\partial x_n^{\alpha_n - i}} D^{\overline{\alpha}}\psi(x) \right| dx &\leq \frac{\widetilde{C}C}{i!} \int_{\{x \in S^- \cap \operatorname{supp} f \ , \ -1/k < x_n < 0\}} |f(x)|k^i |x_n|^i dx \\ &\leq \frac{\widetilde{C}C}{i!} \int_{\{x \in S^- \cap \operatorname{supp} f \ , \ -1 < x_n < 0\}} |f(x)| dx \end{aligned}$$

where $\widetilde{C} = \sup_{\mathbb{R}} |\nu^{(i)}|$. The second inequality comes from the fact that $\nu^{(i)}(x) = 0$ for $x < -1$ and $i \geq 1$. Hence we get (3) by Dominated Convergence Theorem. Passing to the limit in (2) we obtain

$$\int_{S^-} f(x) D^\alpha \psi(x) dx = (-1)^{|\alpha|} \int_{S^-} D_w^\alpha f(x) \psi(x) dx.$$

which, combined with (*), implies

$$\int_S Tf(x) D^\alpha \phi(x) dx = \int_{S^-} f(x) D^\alpha \psi(x) dx = (-1)^{|\alpha|} \int_{S^-} D_w^\alpha f(x) \psi(x) dx.$$

Finally going back to the original coordinates and using the definition of ψ we get

$$\int_S Tf(x) D^\alpha \phi(x) dx = (-1)^{|\alpha|} \int_{S^-} D_w^\alpha f(x) \left[\sum_{k=1}^l -\alpha_k (-\beta_k)^{\alpha_n - 1} \phi\left(\overline{x}, -\frac{x_n}{\beta_k}\right) + \phi(\overline{x}, x_n) \right] dx =$$

$$= (-1)^{|\alpha|} \int_{S^+} \sum_{k=1}^l \alpha_k (-\beta_k)^{\alpha_n} D_w^\alpha f(\overline{y}, -\beta_k y_n) \phi(y) dy + (-1)^{|\alpha|} \int_{S^-} D_w^\alpha f(y) \phi(y) dy$$

that implies that $D^\alpha_w Tf$ exists and

$$D^\alpha_w Tf(x) = \begin{cases} D^\alpha_w f(x), & \text{if } x \in S^-, \\ \sum_{k=1}^l \alpha_k(-\beta_k)^{\alpha_n} D^\alpha_w f(\overline{x}, -\beta_k x_n)\phi(x), & \text{if } x \in S^+. \end{cases}$$

It remains to prove the boundedness of T. It's immediate to verify that

$$\|Tf\|_{L^p(S^+)} \leq \sum_{i=1}^{l} |\alpha_k| \beta_k^{-1/p} \|f\|_{L^p(S^-)}$$

and that we have similar bounds for the norm of the weak derivatives of Tf. Hence there exists a constant C depending only on β_k, α_k, l such that $\|Tf\|_{W^{l,p}(S^+)} \leq C\|f\|_{W^{l,p}(S^-)}$. Observing that $\|Tf\|^p_{W^{l,p}(S)} = \|Tf\|^p_{W^{l,p}(S^+)} + \|f\|^p_{W^{l,p}(S^-)}$ the proof is concluded. □

Lemma 3. Let $l \in \mathbb{N}$ and Ω be a domain in $\mathbb{R}^n$. Suppose that $f \in L^1_{loc}(\Omega)$ admits all the weak derivatives up to order l and that $g : \Omega' \to \Omega$ is a diffeomorphism of class C^l with bounded derivatives $|D^\alpha g_k| \leq M$ for all $1 \leq |\alpha| \leq l$. Then $f \circ g$ admits weak derivative up to order l. Moreover for every $1 \leq |\alpha| \leq l$ we have to following bounds

$$|D^\alpha (f \circ g)(x)| \leq C \sum_{1 \leq |\beta| \leq |\alpha|} |D^\beta f(g(x))| \tag{4}$$

where C depends only on M and l.

Proof. We prove the statement by induction on l. For $l = 1$ we know that exists a sequence of functions $\{f_k\}_k \in C^\infty(\Omega)$ such that

$$f_k \to f \qquad \text{in } L^1_{loc}(\Omega)$$

$$\frac{\partial f_k}{\partial x_i} \to \frac{\partial f}{\partial x_i} \qquad \text{in } L^1_{loc}(\Omega).$$

Take $\phi \in C^\infty_c(\Omega')$ and integrate by parts

$$\int_{\Omega'} f_k(g(x)) \frac{\partial \phi}{\partial x_i}(x) dx = -\int_{\Omega'} \left(\sum_{j=1}^{n} \frac{\partial f_k}{\partial x_j}(g(x)) \frac{\partial g_j}{\partial x_i}(x) \right) \phi(x) dx.$$

Since $\phi(g^{-1}) \in C_c^l(\Omega)$ and the derivatives of g and g^{-1} are bounded, we can pass to the limit in the above equation

$$\int_{\Omega'} f(g(x))\frac{\partial \phi}{\partial x_i}(x)dx = -\int_{\Omega'} \left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(g(x))\frac{\partial g_j}{\partial x_i}(x)\right)\phi(x)dx.$$

Hence the case $l = 1$ is proved. Now suppose that the statement is true for l. We prove the case $l+1$, so we suppose that f admits weak derivatives up to order $l+1$ and that g is of class C^{l+1}. From the case $l = 1$ we know that $\frac{\partial(f\circ g)}{\partial x_i}$ exists and that

$$\frac{\partial(f\circ g)}{\partial x_i} = \sum_{j=1}^{n}(\frac{\partial f}{\partial x_j}\circ g)\frac{\partial g_j}{\partial x_i}$$

Since $\frac{\partial f}{\partial x_j}$ admits weak derivatives up to order l, by induction hypothesis the functions $\frac{\partial f}{\partial x_j}\circ g$ admit weak derivatives up to order l. Moreover $\frac{\partial g_j}{\partial x_i}$ is of class C^l, thus by the Leibniz rule the functions $(\frac{\partial f}{\partial x_j}\circ g)\frac{\partial g_j}{\partial x_i}$ admits weak derivatives of order l. In conclusion $\frac{\partial(f\circ g)}{\partial x_i}$ admits derivatives up to order l and this conclude the proof of the case $l+1$.

To prove the bounds we notice that the weak derivatives $D^\alpha(f\circ g)$ can be computed using the chain rule for usual derivatives. Such formula can be found in [3, formula B]:

$$D_w^\alpha(f(g))(x) = \sum_{1\le|\beta|\le|\alpha|} D_w^\beta(f(g(x))Q_{\alpha,\beta}(g,x)$$

In this formula $Q_{\alpha,\beta}(g,x)$ are homogeneous polynomials of degree $|\beta| \le l$ in the derivatives of order less than l of the components of g. Moreover the coefficients of these polynomials depend only on α, l, n. Hence there exists a constant C depending only on l, n, M such that $|Q_{\alpha,\beta}(g,x)| \le C$ uniformly on x. This concludes the proof. $\square$

Proof of Lemma 1 . Let $f \in W^{l,p}(\Omega)$. Consider the function g from S^- onto Ω defined by

$$g(\overline{x}, x_n) = (\overline{x}, x_n + \phi(\overline{x}))$$

for all $(\overline{x}, x_n) \in S^-$ and its inverse g^{-1}

$$g^{-1}(\overline{x}, x_n) = (\overline{x}, x_n - \phi(\overline{x}))$$

where $S^- = W \times \mathbb{R}^-$. For all $f \in W^{l,p}(\Omega)$ we set

$$Gf = f \circ g$$

Since g is a diffeomorphism between S^- and Ω of class C^m, Lemma 3 guarantees that Gf admits weak derivatives up to order l. We claim that G defines a bounded operator from $W^{l,p}(\Omega)$ to $W^{l,p}(S^-)$, with bounded inverse. To prove this, first we compute the Jacobian matrix of g^{-1}

$$Jg^{-1}(x) = \begin{bmatrix} 1 & 0 & \dots & & 0 \\ 0 & 1 & \dots & & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \\ -\frac{\partial\phi(\overline{x})}{\partial x_1} & -\frac{\partial\phi(\overline{x})}{\partial x_2} & \dots & \dots & 1 \end{bmatrix}$$

from which $|\det(Jg^{-1}(x))| \equiv 1$. Moreover, again by Lemma 3, we have

$$|D_w^\alpha(f(g))| \leq C(l,M) \sum_{1\leq|\beta|\leq|\alpha|} |D_w^\beta f(g)|$$

where $C(l,M)$ depends only on l and M, with $M = \sup_{1\leq|\alpha|\leq l} \|D^\alpha \phi\|$. Next by the change of variable formula and Minkowski's inequality we get

$$\begin{aligned} \left(\int_{S^-} |D_w^\alpha(f(g))(x)|^p dx\right)^{\frac{1}{p}} &\leq \sum_{1\leq|\beta|\leq|\alpha|} C(l,M) \left(\int_{S^-} |D_w^\beta f(g(x))|^p dx\right)^{\frac{1}{p}} \\ &= \sum_{1\leq|\beta|\leq|\alpha|} C(l,M) \left(\int_\Omega |D_w^\beta f(y)|^p |\det Jg^{-1}\big|_{g(y)}| dy\right)^{\frac{1}{p}} \\ &= \sum_{1\leq|\beta|\leq|\alpha|} C(l,M) \|D_w^\beta f\|_{L^p(\Omega)} \end{aligned}$$

Thus, using the estimates for the intermediate derivatives, that

$$\|Gf\|_{W^{l,p}(S^-)} = \|f(g)\|_{W^{l,p}(S^-)} \leq C\|f\|_{W^{l,p}(\Omega)}$$

for a constant C independent of f. In a similar way we can also prove that

$$\|G^{-1}f\|_{W^{l,p}(\Omega)} = \|f(g^{-1})\|_{W^{l,p}(\Omega)} \leq D\|f\|_{W^{l,p}(S)}.$$

Now we can just define the operator T as

$$T = G^{-1} \circ \overline{T} \circ G$$

where $\overline{T}$ is the extension operator from $W^{l,p}(S^-)$ to $W^{l,p}(S)$ defined in Lemma 2. Therefore T is bounded as composition of bounded operators. An explicit for for T is

$$Tf(x) = \begin{cases} f(x), & \text{if } x \in \Omega, \\ \sum_{i=1}^{l} \alpha_k f(\overline{x}, \phi(\overline{x}) - \beta_k(x_n - \phi(\overline{x}))), & \text{if } x \in S \setminus \overline{\Omega}. \end{cases}$$

□

We are now ready to define the Hestenes operator for a general domain Ω with C^m boundary. First we write the precise definition for this kind of domains.

Definition 1. Let $0 < d \leq D < \infty, M > 0, \varkappa > 0$ We say that an open set Ω in $\mathbb{R}^n$ has a resolved boundary with parameters $d, D, \varkappa$ if there exists a family of open cuboids $V_i, i = 1, ..., s$ (where $s \in \mathbb{N}$ if Ω is bounded and $s = \infty$ otherwise) such that

1. $(V_i)_d \cap \Omega \neq \emptyset$
2. $\Omega \subset \bigcup_{j=1}^{s} (V_i)_d$
3. The multiplicity of the cover $\{V_i\}_{i=1}^{s}$ is less than $\varkappa$.
4. There exist isometries λ_i of $\mathbb{R}^n$ such that

$$\lambda_j(V_j) = \prod_{i=1}^{n}]a_{ij}, b_{ij}[$$

and, if $\partial V_j \cap \Omega \neq \emptyset$,

$$\lambda_j(V_j \cap \Omega) = \{(\overline{x}, x_n) \in \mathbb{R}^n | \overline{x} \in W_j, a_{nj} + d < x_n < \phi_j(\overline{x})\}$$

where $W_j = \prod_{i=1}^{n-1}]a_{ij}, b_{ij}[$ and $\phi_j : W_j \to \mathbb{R}$.

Moreover

- if $\phi_j \in C^m(\overline{W}_i)$ with $\|D^\alpha \phi_j\| \leq M < \infty$, for every $1 \leq |\alpha| \leq m$, we say that Ω has a resolved C^m boundary with parameters $d, D, \varkappa, M$.
- if $\phi_j \in \mathrm{Lip}(\overline{W}_i)$ with $\mathrm{Lip}(\phi) = M$, we say that Ω has a resolved Lipschitz boundary with parameters $d, D, \varkappa, M$.

Finally we will say that a domain Ω has a resolved C^m (or Lipschitz) boundary if there exist parameters $d, D, \varkappa, M$ for which Ω has a C^m (or Lipschitz) boundary.

Remark 1. In the notation of Lemma 1, let $a, b \in \mathbb{R}$ such that $a < \phi(\overline{x}) < b$ for every $\overline{x} \in W$. We define $S^{a,b} = W \times (a, b)$, $\Omega_a = \Omega \cap (W \times (a, \infty))$ and $\widehat{W}^{l,p}(\Omega_a) = \{f \in W^{l,p}(\Omega_a) |\, \mathrm{supp}\, f \subset S\}$. Then exists a bounded extension operator

$$T : \widehat{W}^{l,p}(\Omega_a) \to W^{l,p}(S^{a,b}).$$

To see this we can just extend $f \in \widehat{W}^{l,p}(\Omega_a)$ naturally by 0 to $f_0 \in W^{l,p}(\Omega)$ and then define

$$Tf = (\widetilde{T} f_0)\big|_{S^{a,b}}$$

where $\widetilde{T}$ is the operator of the previous Lemmma .

Theorem 1. Let $m, l \in \mathbb{N}, l \leq m$ and $1 \leq p \leq \infty$. If Ω is a domain in $\mathbb{R}^n$ has a C^m resolved boundary then there exists a bounded extension operator

$$T : W^{l,p}(\Omega) \to W^{l,p}(\mathbb{R}^n).$$

Proof Sketch. Let $f \in W^{l,p}(\Omega)$. Let $\{V_i\}_{i=1}^s$ be the covering of cuboids for Ω as in Definition 1. It's possible to construct functions $\{\psi_i\}_{i=1}^s \subset C_c^\infty(\mathbb{R}^n)$ such that the functions $\{\psi_i^2\}_{i=1}^s$ form a partition of the unity corresponding to the covering $\{V_i\}_{i=1}^s$ and satisfying $\|D^\alpha \psi_i\|_{L^\infty} \leq M_1$ with M_1 depending only on n, l, d. If $\partial\Omega \cap V_i \neq \emptyset$ by Remark 1 there exists a bounded operator

$$T_i : \widehat{W}^{l,p}(\lambda_i(\Omega \cap V_i)) \to W^{l,p}(\lambda_i(V_i))$$

where $\widehat{W}^{l,p}(\lambda_i(V_i \cap \Omega) = \{f \in W^{l,p}(V_i \cap \Omega) |\, \mathrm{supp}\, f \subset \lambda_i(V_i)\}$. If $V_i \subset \Omega$ the operator T_i is defined to be just the identity. We set

$$Tf = \sum_{i=1}^{s} \psi_i T_i(\psi_i f(\lambda_i^{-1}))(\lambda_i).$$

assuming $(\psi_i T_i(\psi_i f(\lambda_i^{-1}))(\lambda_i)) = 0$ outside V_i. The functions $\psi_i f \in W^{l,p}(V_i \cap \Omega)$ are such that $\operatorname{supp}\psi_i f \subset \overline{\Omega} \cap V_i$, hence $\psi_i f(\lambda_i) \in \widehat{W}^{l,p}(\lambda_i(V_i \cap \Omega))$ and so T is well defined. To see that T is an extension operator, take $x \in \Omega$: if $x \in \operatorname{supp}\psi_i$ then $\psi_i(x)T_i(\psi_i f(\lambda_i^{-1}))(\lambda_i(x)) = \psi_i(x)^2 f(x)$; if $x \notin \operatorname{supp}\psi_i$ then $0 = \psi_i(x)T_i(\psi_i f(\lambda_i^{-1}))(\lambda_i(x)) = \psi_i(x)^2 f(x)$. So $Tf(x) = \sum_{i=1}^{s} \psi_i^2(x)f(x) = f(x)$.

We omit the proof of the boundedness of T, the details of which can be found in the proofs of Lemma 13-14 in [2]. □

1.2 Hestenes operator on Morrey spaces

Definition 2. Let $1 \leq p < \infty$, ϕ a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ and Ω be a domain in $\mathbb{R}^n$. For a function $f \in L^p_{loc}(\Omega)$ we define the Morrey space as

$$M_p^\phi(\Omega) = \{f \in L^p_{loc}(\Omega) \mid \|f\|_{M_p^\phi(\Omega)} < \infty\}$$

where

$$\|f\|_{M_p^\phi(\Omega)} := \sup_{B_r(x), x\in\Omega, r>0} \left(\frac{1}{\phi(r)} \int_{B_r(x)\cap\Omega} |f(y)|^p dy \right)^{\frac{1}{p}}.$$

Lemma 4. Let $k \geq 1$ and Ω be set in $\mathbb{R}^n$ with diameter $D > 0$. Then there exists an integer $C_{n,k}$ depending only on k and n such that Ω can be covered by a collection of open balls $B_1, ..., B_h$ centered in Ω with radius D/k and $h \leq C_{k,n}$.

Proof. We start by claiming that if S is a set of points in $\mathbb{R}^n$ satisfying

i) $S \subset \Omega$,

ii) $\|z_1 - z_2\| \geq D/k$ for every $z_1, z_2 \in \Omega$ with $z_1 \neq z_2$,

then $\#S \leq C_{n,k}$ where $C_{k,n}$ is an integer depending only on k and n. To see this, first note that Ω is contained in some closed cube Q of side $2D$. Then we choose $m \in \mathbb{N}$ such that $2^{m-1} > \sqrt{n}k$. Next we cover Q with $(2^m)^n$ smaller closed cubes of side $2D/2^m$. The diagonal of a smaller cube measures $2D/2^m \cdot \sqrt{n} < D/k$. Thus each of these cubes can contain at most one point of S ,so $\#S \leq (2^m)^n$. Therefore it's enough to choose $C_{n,k} = 2^{mn}$. Set $r := D/k$, we'll prove that we can cover Ω with a collection of balls $B_1, ..., B_h$ centered in Ω of radius r and such that $k \leq C_{n,k}$. Choose $x_1 \in \Omega$ and take $B_1 = B_r(x_1)$, the ball centered in x_1 of radius r. If $\Omega \subset B_1$ we are done, if not there exists $x_2 \in \Omega \setminus B_1$ and we take $B_2 = B_r(x_2)$. Again, if $\Omega \subset (B_1 \cup B_2)$ we stop, otherwise we can pick $x_3 \in \Omega \setminus (B_1 \cup B_2)$ and take $B_3 = B_r(x_3)$. We iterate this procedure : given $B_1, ..., B_i$ balls, if $\Omega \subset (B_1 \cup ... \cup B_i)$ we stop, otherwise we can choose $x_{i+1} \in \Omega \setminus (B_1 \cup ... \cup B_i)$ and take $B_{i+1} = B_r(x_{i+1})$. We claim that this procedure stops with $i \leq C_{n,k}$. Suppose it doesn't, then we can find $B_1, ..., B_{C_{n,k}+1}$ balls centered respectively at $x_1, ..., x_{C_{n,k}+1}$. Setting $S = \{x_1, ..., x_{C_{n,k}+1}\}$, it's immediate to see that S satisfies i) and ii), but $\#S = C_{n,k} + 1$, that is a contradiction. □

Lemma 5. Let $W \subset \mathbb{R}^{n-1}$ be open connected and define

$$\Omega = \{(\overline{x}, x_n) \mid \overline{x} \in W, x_n \leq \psi(\overline{x})\}$$

$$\Omega^+ = \{(\overline{x}, x_n) \mid \overline{x} \in W, x_n > \psi(\overline{x})\}$$

where $\psi \in \mathrm{Lip}(\overline{W})$. Let $\beta > 0$ and consider the function A_β from $W \times \mathbb{R}$ to Ω defined by

$$A_\beta(\overline{x}, x_n) = \begin{cases} (\overline{x}, \psi(\overline{x}) - \beta(x_n - \psi(\overline{x}))), & \text{if } (\overline{x}, x_n) \in \Omega^+, \\ (\overline{x}, x_n), & \text{if } (\overline{x}, x_n) \in \Omega. \end{cases}$$

Then for every $x_0 \in W \times \mathbb{R}$ and $r > 0$

$$A(B_r(x_0) \cap \Omega^+) \subset B_{cr}(A(x_0)) \cap \Omega$$

where $c \geq 1$ is a constant depending only on $\mathrm{Lip}\,\psi$ and β.

Proof. Notice that it is sufficient to prove that for every $x, y \in W \times \mathbb{R}$ we have

$$\|A(x) - A(y)\| \leq c\|x - y\|. \tag{5}$$

Set $M = \mathrm{Lip}\,\psi$. We distinguish three cases: 1. $x, y \in \Omega$: in this case $A(x) = x$ and $A(y) = y$, so $\|x - y\| = \|A(x) - A(y)\|$ and there is nothing to prove.

2. $x, y \in \Omega^+$: we have

$$\begin{aligned} |A(x)_n - A(y)_n| &= |\psi(\overline{x}) - \beta(x_n - \psi(\overline{x})) - \psi(\overline{y}) + \beta(y_n - \psi(\overline{y}))| \\ &\leq (1 + \beta)|\psi(\overline{x}) - \psi(\overline{y})| + \beta|x_n - y_n| \\ &\leq M(1 + \beta)\|\overline{x} - \overline{y}\| + \beta|x_n - y_n| \end{aligned}$$

Hence

$$\begin{aligned}
\|A(x) - A(y)\|^2 &= \|\overline{A(x)} - \overline{A(y)}\|^2 + |A(x)_n - A(y)_n|^2 \\
&\leq \|\overline{x} - \overline{y}\|^2 + [M(1+\beta)\|\overline{x} - \overline{y}\| + \beta|x_n - y_n|]^2 \\
&\leq (1 + 2M^2(1+\beta)^2)\|\overline{x} - \overline{y}\|^2 + 2\beta^2|x_n - y_n|^2 \\
&\leq c_1^2(M,\beta)\|x - y\|^2
\end{aligned}$$

for some constant $c_1(M,\beta)$.

3. $x \in \Omega^+, y \in \Omega$: first notice that, since $\psi(\overline{x}) < x_n$, then $x_n - y_n > \psi(\overline{x}) - y_n$. Moreover $\psi(\overline{y}) > y_n$, hence $M\|\overline{x} - \overline{y}\| \geq \psi(\overline{y}) - \psi(\overline{x}) > y_n - \psi(\overline{x})$. This implies

$$|\psi(\overline{x}) - y_n| < |x_n - y_n| + M\|\overline{x} - \overline{y}\|.$$

Now

$$\begin{aligned}
|A(x)_n - A(y)_n| &= |\psi(\overline{x}) - \beta(x_n - \psi(\overline{x})) - y_n| \\
&= |(1+\beta)(\psi(\overline{x}) - y_n) + \beta(y_n - x_n)| \\
&\leq M(1+\beta)\|\overline{x} - \overline{y}\| + (1+2\beta)|x_n - y_n|
\end{aligned}$$

and

$$\begin{aligned}
\|A(x) - A(y)\|^2 &= \|\overline{A(x)} - \overline{A(y)}\|^2 + |A(x)_n - A(y)_n|^2 \\
&\leq \|\overline{x} - \overline{y}\|^2 + [M(1+\beta)\|\overline{x} - \overline{y}\| + (1+2\beta)|x_n - y_n|]^2 \\
&\leq (1 + 2M^2(1+\beta)^2)\|\overline{x} - \overline{y}\|^2 + 2(1+2\beta)^2|x_n - y_n|^2 \\
&\leq c_2^2(M,\beta)\|x - y\|^2.
\end{aligned}$$

for some constant $c_2(M,\beta)$. Then (5) by taking $c = max(\sqrt{c}_1, \sqrt{c}_2, 1)$.

□

Definition 3. Let $1 \leq p < \infty$, ϕ a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ and Ω be a domain in $\mathbb{R}^n$. For every $\infty \geq \delta > 0$ and every function $f \in L^p_{loc}(\Omega)$ we define the norm $\|f\|_{M_p^{\delta,\phi}}$ as

$$\|f\|_{M_p^{\delta,\phi}(\Omega)} := \sup_{B_r(x), x\in\Omega, 0<r<\delta} \left(\frac{1}{\phi(r)} \int_{B_r(x)\cap\Omega} |f(y)|^p dy \right)^{\frac{1}{p}}.$$

Remark 2. For $\delta = \infty$ we have that $\|.\|_{M_p^{\delta,\phi}(\Omega)} = \|.\|_{M_p^{\phi}(\Omega)}$.

Lemma 6. Let $l, n, m \in \mathbb{N}, m \geq l, 1 \leq p \leq \infty$, $W = \prod_{i=1}^{n-1}]a_i, b_i[$ be an open cuboid of $\mathbb{R}^{n-1}$ and ϕ a function from $\mathbb{R}^+$ to $\mathbb{R}^+$. Moreover define

$$S = W \times \mathbb{R}$$

$$\Omega = \{(\overline{x}, x_n) | \overline{x} \in W, x_n < \psi(\overline{x})\}$$

where $\psi \in C^m(\overline{W})$ and $\|D^\alpha \psi\| \leq M < \infty$ for every $1 \leq |\alpha| \leq l$. Then for every $f \in W^{l,p}(\Omega)$, $\delta > 0$ and $1 \leq |\alpha| \leq l$

$$\|Tf\|_{M_p^{\phi,\delta}(S)} \leq C\|f\|_{M_p^{\phi,\delta}(\Omega)}, \tag{6}$$

$$\|D_w^\alpha Tf\|_{M_p^{\phi,\delta}(S)} \leq C \sum_{1 \leq |\beta| \leq |\alpha|} \|D_w^\beta f\|_{M_p^{\phi,\delta}(\Omega)}, \tag{7}$$

where T is the Hestenes operator defined in Lemma 1 and C is a constant independent of f.

Proof. Define $\Omega^+ = \{(\overline{x}, x_n) \mid \overline{x} \in W, x_n > \psi(\overline{x})\}$. We recall the definition of T

$$Tf(x) = \begin{cases} f(x) & x \in \Omega \\ \sum_{i=1}^{l} \alpha_k f(\overline{x}, \psi(\overline{x}) - \beta_k(x_n - \psi(\overline{x}))) & x \in \Omega^+ \end{cases}$$

and observe that we can rewrite it as

$$Tf(x) = \begin{cases} f(x), & \text{if } x \in \Omega, \\ \sum_{i=1}^{l} \alpha_k f(G_k(x)), & \text{if } x \in \Omega^+, \end{cases}$$

where $G_k(\overline{x}, x_n) = (\overline{x}, \psi(\overline{x}) - \beta_k(x_n - \psi(\overline{x})))$. Note that $G_k : \Omega^+ \to \Omega$ defines a diffeomorphism from Ω^+ to Ω of class C^m and satisfying $|\det JG_k^{-1}| \equiv 1/\beta_k$. First we prove ii). Let's fix $x_0 \in S$ and a radius $\delta > r > 0$. We want to estimate the quantity

$$I = \left(\frac{1}{\psi(r)} \int_{B_r(x_0) \cap S} |D_w^\alpha Tf(x)|^p dx \right)^{\frac{1}{p}}$$

for $1 \leq |\alpha| \leq l$. To do this we estimate the integral as follows

$$I \leq \underbrace{\left(\frac{1}{\phi(r)} \int_{B_r(x_0) \cap \Omega^+} |D_w^\alpha Tf(x)|^p dx \right)^{\frac{1}{p}}}_{I_1} + \underbrace{\left(\frac{1}{\phi(r)} \int_{B_r(x_0) \cap \Omega} |D_w^\alpha Tf(x)|^p dx \right)^{\frac{1}{p}}}_{I_2}.$$

Since $Tf(x) = f(x)$ when $x \in \Omega$, we have immediately

$$I_2 \le \|D_w^\alpha f\|_{M_p^{\phi,\delta}(\Omega)}.$$

It remains to estimate I_1. We start by observing that from Lemma 3 there exists a constant C_k depending only on G_k and l such that

$$|D_w^\alpha (f \circ G_k)| \le C_k \sum_{1 \le |\beta| \le |\alpha|} |D_w^\beta f(G_k)|.$$

By the previous inequality and Lemma 5 we are able to produce the following bound

$$\begin{aligned}
\frac{\|D_w^\alpha (f \circ G_k)\|_{L^p(B_r(x_0)\cap\Omega^+)}}{\phi(r)^{\frac{1}{p}}} &\le C_k \sum_{1 \le |\beta| \le |\alpha|} \left(\phi(r)^{-1} \int_{G_k(B_r(x_0)\cap\Omega^+)} |D_w^\beta f(y)|^p |\det JG_k^{-1}|_{G_k(y)} |dy \right)^{\frac{1}{p}} \\
&\le C_k \beta_k^{-\frac{1}{p}} \sum_{1 \le |\beta| \le |\alpha|} \left(\phi(r)^{-1} \int_{B_{c_k r}(A_{\beta_k}(x_0))\cap\Omega} |D_w^\beta f(y)|^p dy \right)^{\frac{1}{p}}
\end{aligned}$$

where A_{α_k} is defined as in Lemma 5 and c_k depends only on β_k and M. By Lemma 4 the set $B_{c_k r}(A_{\beta_k}(x_0)) \cap \Omega$ can be covered with a collection of open balls $B_1, ..., B_h$ centered in Ω with radius r and $h \le m_k$, where m_k depends only on c_k. Hence we get

$$\frac{\|D_w^\alpha (f \circ G_k)\|_{L^p(B_r(x_0)\cap\Omega^+)}}{\phi(r)^{\frac{1}{p}}} \le C_k \beta_k^{-\frac{1}{p}} m_k \sum_{1 \le |\beta| \le |\alpha|} \|D^\beta f\|_{M_p^{\delta,\phi}(\Omega)}$$

Next we estimate I_1:

$$\begin{aligned}
I_1 = \phi(r)^{-\frac{1}{p}} |D_w^\alpha Tf\|_{L^p(B_r(x_0)\cap\Omega^+)} &\le \phi(r)^{-\frac{1}{p}} \sum_{k=1}^{l} \alpha_k \|D_w^\alpha f(G_k)\|_{L^p(B_r(x_0)\cap\Omega^+)} \\
&\le \sum_{k=1}^{l} \alpha_k C_k \beta_k^{-\frac{1}{p}} m_k \left(\sum_{1 \le |\beta| \le |\alpha|} \|D_w^\beta f\|_{M_p^{\phi,\delta}(\Omega)} \right).
\end{aligned}$$

Finally putting the estimates of I_1, I_2 together

$$\begin{aligned}
\|D^\alpha_w Tf\|_{M^\phi_p(S)} &= \sup_{x_0\in S, r>0}\left(\frac{1}{\phi(r)}\int_{B_r(x_0)\cap S}|D^\alpha_w Tf(x)|^p dx\right)^{\frac{1}{p}} \\
&\le \|D^\alpha_w f\|_{M^\phi_p(\Omega)} + \sum_{k=1}^{l}\alpha_k C_k \beta_k^{-\frac{1}{p}} m_k \left(\sum_{1\le|\beta|\le|\alpha|}\|D^\alpha_w f\|_{M^{\phi,\delta}_p(\Omega)}\right) \\
&\le \widetilde{C}\sum_{1\le|\beta|\le|\alpha|}\|D^\alpha_w f\|_{M^{\phi,\delta}_p(\Omega)}
\end{aligned}$$

where $\widetilde{C}$ depends only on $\{b_k\}_k, \{\alpha_k\}_k, l, M, p$. This proves ii). The proof of i) is exactly analogous to the proof of ii). □

Theorem 2. Let $m, l \in \mathbb{N}, l \le m$, $1 \le p \le \infty$, ϕ a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ and Ω a domain in $\mathbb{R}^n$ with C^m resolved boundary. Let also T be the Hestenes operator defined in Theorem 1. Then if Ω is bounded, for every $f \in W^{l,p}(\Omega)$, $\delta > 0$ and $1 \le |\alpha| \le l$ we have

$$\|Tf\|_{M^\phi_p(\mathbb{R}^n)} \le C\|f\|_{M^\phi_p(\Omega)}, \tag{8}$$

$$\|D^\alpha_w Tf\|_{M^{\phi,\delta}_p(\mathbb{R}^n)} \le C\sum_{1\le|\beta|\le|\alpha|}\|D^\beta_w f\|_{M^{\phi,\delta}_p(\Omega)}, \tag{9}$$

where C doesn't depend on f. If instead Ω is unbounded, for every $f \in W^{l,p}(\Omega)$ and $\delta > 0$ we have

$$\|Tf\|_{M^{\phi,\delta}_p(\mathbb{R}^n)} \le C_\delta\|f\|_{M^\phi_p(\Omega)}, \tag{10}$$

$$\|D^\alpha_w Tf\|_{M^{\phi,\delta}_p(\mathbb{R}^n)} \le C_\delta\sum_{1\le|\beta|\le|\alpha|}\|D^\beta_w f\|_{M^\phi_p(\Omega)}, \tag{11}$$

where C_δ depends on δ but not on f.

Proof. Let $f \in W^{l,p}(\Omega)$ and $\{V_i\}_{i=1}^s$ be the covering of cuboids for Ω as in the definition of set with resolved boundary. We recall the definition of T :

$$Tf = \sum_{i=1}^{s}\psi_i T_i(\psi_i f(\lambda_i^{-1}))(\lambda_i)$$

where $\{\psi_i^2\}_{i=1}^s$ form a partition of the unity corresponding to the covering $\{V_i\}_{i=1}^s$ and satisfying $\|D^\alpha \psi_i\|_{L^\infty} \le M_1$, with $|\alpha| \le l$ and M_1 depending only on n, l, d. To make the notation simpler we will rewrite T as

$$Tf = \sum_{i=1}^{s} \psi_i \widetilde{T}_i(\psi_i f)$$

where the operator $\widetilde{T}_i$ is defined as $\widetilde{T}_i f = T_i(f(\lambda_i^{-1}))(\lambda_i)$. Before starting the proof we remark some facts that will be justified at the end:

a) Let C_i the constant such that

$$\|T_i g\|_{M_p^{\phi,\delta}(\lambda_i(V_i))} \le C_i \|g\|_{M_p^{\phi,\delta}(\lambda_i(\Omega\cap V_i))},$$

$$\|D_w^\alpha T_i g\|_{M_p^{\phi,\delta}(\lambda_i(V_i)))} \le C_i \sum_{1\le|\beta|\le|\alpha|} \|D_w^\alpha g\|_{M_p^{\phi,\delta}(\lambda_i(\Omega\cap V_i)))},$$

for $1 \le |\alpha| \le l$, $g \in \widehat{W}^{l,p}(\lambda_i(\Omega \cap V_i))$ and $\delta > 0$. Then $\sup_{i=1,\dots,s} C_i \le M_2$, where M_2 depends only on Ω, l, n.

b) We have

$$\|\widetilde{T}_i g\|_{M_p^{\phi,\delta}(V_i)} \le M_2 \|g\|_{M_p^{\phi,\delta}(\Omega\cap V_i)},$$

$$\|D_w^\alpha \widetilde{T}_i g\|_{M_p^{\phi}(V_i)} \le M_3 M_2 \sum_{1\le|\beta|\le|\alpha|} \|D_w^\alpha g\|_{M_p^{\phi}(\Omega\cap V_i)},$$

for $1 \le |\alpha| \le l$,$g \in \widehat{W}^{l,p}(\Omega \cap V_i)$, $\delta > 0$ and where M_3 doesn't depend on i.

Let now $x_0 \in \mathbb{R}^n$, $0 < r < \delta$ and $B_r(x_0)$ the ball centered in x_0 of radius r. Let's consider the set $J = \{i = 1, ..., s \mid V_i \cap B_r(x_0) \neq \emptyset\}$. We notice that there exists an integer $\widetilde{s}$ depending only on the covering $(V_i)_{i=1}^s$ and on δ such that $\#J \le \widetilde{s}$. We also recall that if Ω is bounded then $\widetilde{s} \le s < \infty$. We have

$$\begin{aligned}
\left(\frac{1}{\phi(r)} \int_{B_r(x_0)} |Tf(x)|^p dx\right)^{\frac{1}{p}} &= \left(\frac{1}{\phi(r)} \int_{B_r(x_0)} |\sum_{i=1}^{s} \psi_i(x) \widetilde{T}_i(\psi_i f))(x)|^p dx\right)^{\frac{1}{p}} \\
&\le \sum_{i\in J} \left(\frac{1}{\phi(r)} \int_{B_r(x_0)\cap V_i} |\widetilde{T}_i(\psi_i f)(x)|^p| dx\right)^{\frac{1}{p}} \\
&\overset{b)}{\le} \widetilde{s} M_2 \|\psi_i f\|_{M_p^{\phi,\delta}(V_i\cap\Omega)} \le M_2 \widetilde{s} \|f\|_{M_p^{\phi,\delta}(\Omega)}.
\end{aligned}$$

This proves (8) and (10). Let now $\alpha \in \mathbb{N}_0^n$ with $1 \le |\alpha| \le l$. We have

$$
\begin{aligned}
\left(\frac{1}{\phi(r)}\int_{B_r(x_0)}|D_w^\alpha Tf(x)|^p dx\right)^{\frac{1}{p}} &= \left(\frac{1}{\phi(r)}\int_{B_r(x_0)}|D_w^\alpha \sum_{i=1}^{s}\psi_i(x)\widetilde{T}_i(\psi_i f))(x)|^p dx\right)^{\frac{1}{p}} \\
&\le C_\alpha \sum_{i\in J}\left(\frac{1}{\phi(r)}\int_{B_r(x_0)\cap V_i}\sum_{\beta\le\alpha}|D^{\alpha-\beta}\psi_i(x)D_w^\beta \widetilde{T}_i(\psi_i f)(x)|^p dx\right)^{\frac{1}{p}} \\
&\le C_\alpha M_1 \widetilde{s}\sum_{i\in J}\left(\frac{1}{\phi(r)}\int_{B_r(x_0)\cap V_i}\sum_{\beta\le\alpha}|D_w^\beta \widetilde{T}_i(\psi_i f)(x)|^p dx\right)^{\frac{1}{p}} \\
&\overset{b)}{\le} C_\alpha M_1 \widetilde{s}\sum_{\beta\le\alpha} M_2 M_3 \sum_{|\gamma|\le|\beta|}\|D_w^\gamma f\|_{M_p^{\phi,\delta}(V_i)} \\
&\le \widetilde{C}_\alpha M_1 M_2 M_3 \widetilde{s}\sum_{|\beta|\le|\alpha|}\|D_w^\beta f\|_{M_p^{\phi,\delta}(V_i)}
\end{aligned}
$$

This proves (9) and (11). Let's now prove a) and b). a) Ω has a resolved C^m boundary with parameters $\varkappa, d, D, M$. Hence, if ϕ_i are the C^m functions of Definition 1, we have $\|D^\alpha \phi_i\|_{L^\infty} \le M$ for every i and for every $1 \le |\alpha| \le l$. Therefore by the proof of Lemma 6 we deduce that C_i depends only on l, n, M and on the choice of the constants α_k, β_k, which can be chosen to be the same for every T_i. b) We notice that since λ_i are isometries, they are smooth and their derivatives are uniformly bounded with a bound depending only on n. Then the result follows from a) and from a straightforward computation using a change of variable and Lemma 3. □

2 Stein operator

2.1 Construction

In this section we will define the Stein extension operator for Lipschitz domains in $\mathbb{R}^n$. The details of the construction and the proofs of all the results in this subsection can be found in [4, Section 2-3, Ch. VI]. We start by introducing the notion of regularized distance with the following theorem. Here by $d(x, F)$ we denote the distance of a point $x \in \mathbb{R}^n$ from the set $F \subset \mathbb{R}^n$.

Theorem 3. Let F be a closed set in $\mathbb{R}^n$. Then there exists a real-valued function $\Delta(.) = \Delta(., F)$ defined in F^c such that

a) $c_1 d(x,F) \leq \Delta(x) \leq c_2 d(x,F)$, for every $x \in F^c$,

b) Δ is C^∞ in F^c and

$$|D^\alpha \Delta(x)| \leq B_\alpha d(x,F)^{1-|\alpha|},$$

for every $x \in F^c$, where B_α, c_1,c_2 are constants independent of x and F.

Next we give the definition of a special Lipschitz domain.

Definition 4. A domain Ω of $\mathbb{R}^n$ is said to be a special Lipschitz domain if there exists a Lipschitz function ψ defined from $\mathbb{R}^{n-1}$ to $\mathbb{R}$ such that

$$\Omega = \{(\overline{x}, y) \in \mathbb{R}^n \ \mid \psi(\overline{x}) < y\}.$$

Moreover the Lipschitz constant $\mathrm{Lip}\,\psi$ is said to be the Lipschitz bound of Ω.

It is convenient to define first the Stein extension operator in the case of a special Lipschitz domain. To do so we need the following two lemmas.

Lemma 7. Let Ω be a special Lipschitz domain of $\mathbb{R}^n$ and set $F = \overline{\Omega}$. Let Δ be the regularized distance from F as given in Theorem 3. Then there exists a positive constant a, which depends only on the Lipschitz bound of Ω, such that if $(\overline{x}, y) \in F^c$, then $a\Delta(\overline{x}, y) \geq \psi(\overline{x}) - y$.

Lemma 8. There exists a continuous real-valued function τ defined in $[1, \infty)$ satisfying

i) $\tau(\lambda) = O(\lambda^N)$, as $\lambda \to \infty$ for every N,

ii) $\int_1^\infty \tau(\lambda) d\lambda = 1$, $\int_1^\infty \lambda^k \tau(\lambda) d\lambda = 0$, for every $k = 1, 2, ...$

Theorem 4. Let Ω be a special Lipschitz domain of $\mathbb{R}^n$ with Lipschitz bound M. Moreover let τ be the function in Lemma 8 and a the constant of Lemma 7. For every function f that is C^∞ in $\overline{\Omega}$ and bounded in $\overline{\Omega}$ together with all its partial derivatives, define

$$Tf(\overline{x}, y) = \begin{cases} f(\overline{x}, y), & \text{if } y \geq \psi(\overline{x}) \\ \int_1^\infty f(\overline{x}, y + \lambda\delta^*(\overline{x}, y))\tau(\lambda) d\lambda, & \text{if } y < \psi(\overline{x}), \end{cases} \tag{12}$$

where $\delta^*(\overline{x}, y) = 2a\Delta(\overline{x}, y)$. Then $Tf \in C^\infty(\mathbb{R}^n)$ and

$$\|Tf\|_{W^{l,p}(\mathbb{R}^n)} \leq C_{n,l}(M) \|f\|_{W^{l,p}(\Omega)},$$

where $C_{l,n}(M)$ is a constant depending only on n, l and M.

We are now ready to define the Stein extension operator in the case of special Lipschitz domains. The construction is the following. Let Ω be a special Lipschitz domain in $\mathbb{R}^n$ with Lipschitz bound M. We denote by Γ the cone with vertex at the origin given by $\Gamma = \{(\overline{x}, y) \in \mathbb{R}^n \mid M|\overline{x}| < |y|, y < 0\}$. Suppose now that $\eta \in C_c^\infty(\mathbb{R}^n)$ is a non-negative function with integral 1 and which support is contained in Γ. For every $f \in W^{l,p}(\Omega)$ and every $\varepsilon > 0$ we define

$$f_\varepsilon(x) = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} f(x-y)\eta(y/\varepsilon)dy = \int_{\mathbb{R}^n} f(x-\varepsilon y)\eta(y)dy.$$

Notice that, since the support of η is strictly inside Γ, the above integral is well defined for every x in some neighborhood of $\overline{\Omega}$ depending on ε. Hence $f_\varepsilon \in C^\infty(\overline{\Omega})$ and it is bounded with all its partial derivatives, thus Tf_ε is well defined. The Stein operator is then taken to be the limit of Tf_ε as $\varepsilon \to 0$. This limit procedure is formalized in the following result.

Theorem 5. Let $l \in \mathbb{N}, 1 \leq p \leq \infty$ and Ω be a special Lipschitz domain of $\mathbb{R}^n$ with Lipschitz bound M. For every $f \in W^{l,p}(\Omega)$ define Tf_ε as in (12). Then Tf_ε converges in $W^{l,p}(\mathbb{R}^n)$ if $p < \infty$ and in $W^{l-1,p}(\mathbb{R}^n)$ if $p = \infty$, as $\varepsilon \to 0$. Moreover setting

$$Sf = \lim_{\varepsilon \to 0} Tf_\varepsilon$$

we have that Sf extend f to $\mathbb{R}^n$ and

$$\|Sf\|_{W^{l,p}(\mathbb{R}^n)} \leq C_{l,n}(M)\|f\|_{W^{l,p}(\Omega)},$$

where $C_{l,n}(M)$ is a constant depending only on n, l and M.

Remark 3. Let Ω be a domain in $\mathbb{R}^n$ and suppose that there exists a rotation R of $\mathbb{R}^n$ such that $R(\Omega)$ is a special Lipschitz domain with Lipschitz bound M. We observe that we can use the operator S to extend the space $W^{l,p}(\Omega)$ to $W^{l,p}(\mathbb{R}^n)$ continuously. Indeed, given $f \in W^{l,p}(\Omega)$, by Lemma 3 we have $f \circ R^{-1} \in W^{l,p}(R(\Omega))$. Hence we can use Theorem 5 to extend $f \circ R^{-1}$ to $\mathbb{R}^n$ with $S(f \circ R^{-1}) \in W^{l,p}(\mathbb{R}^n)$. Then $S(f \circ R^{-1}) \circ R$ clearly extends f and $S(f \circ R^{-1}) \circ R \in W^{l,p}(\mathbb{R}^n)$ by Lemma 3. Now given $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq l$ we argue as follows. Applying repeatedly (4) we have

$$\left(\int_{\mathbb{R}^n} |D^\alpha_w(S(f\circ R^{-1})\circ R)(x)|^p dx\right)^{\frac{1}{p}} \le C\sum_{|\beta|\le|\alpha|}\left(\int_{\mathbb{R}^n} |D^\beta_w(S(f\circ R^{-1}))(R)|^p dx\right)^{\frac{1}{p}} =$$

$$= C\sum_{|\beta|\le|\alpha|}\left(\int_{\mathbb{R}^n} |D^\beta_w(S(f\circ R^{-1}))|^p|\det JR^{-1}(x)|dx\right)^{\frac{1}{p}}$$

$$= C\sum_{|\beta|\le|\alpha|}\left(\int_{\mathbb{R}^n} |D^\beta_w(S(f\circ R^{-1}))|^p dx\right)^{\frac{1}{p}}$$

$$\le CC_{l,n}(M)\sum_{|\beta|\le|\alpha|}\sum_{|\gamma|\le|\beta|}\left(\int_{\mathbb{R}^n} |D^\gamma_w(f\circ R^{-1})|^p dx\right)^{\frac{1}{p}}$$

$$\le C^2C_{l,n}(M)\sum_{|\beta|\le|\alpha|}\sum_{|\gamma|\le|\beta|}\sum_{|\eta|\le|\gamma|}\left(\int_{\mathbb{R}^n} |D^\eta_w f(R^{-1})|^p dx\right)^{\frac{1}{p}} =$$

$$= C^2C_{l,n}(M)\sum_{|\beta|\le|\alpha|}\sum_{|\gamma|\le|\beta|}\sum_{|\eta|\le|\gamma|}\left(\int_{\mathbb{R}^n} |D^\eta_w f|^p dx\right)^{\frac{1}{p}},$$

where C depends only on the bound of the derivatives of R, hence only on n. This proves the continuity of the extension. In what follows we will denote the extension operator for a rotated special Lipschitz domain, that is $S(f\circ R^{-1})\circ R$, just by Sf.

Definition 5. Let Ω be an open set in $\mathbb{R}^n$ and let $\partial\Omega$ be its boundary. We say that $\partial\Omega$ is minimally smooth if there exists an $\varepsilon>0$, $N\in\mathbb{N}$, $M>0$ and a sequence $\{U_i\}_{i=1}^s$ (where s can be $+\infty$) of open sets such that:

i) if $x\in\partial\Omega$, then $B_\varepsilon(x)\subset U_i$, for some i, where $B_\varepsilon(x)$ is the open ball centered in x of radius ε.

ii) No point of $\mathbb{R}^n$ is contained in more than N elements of the family $\{U_i\}_{i=1}^s$.

iii) For every $i=1,...,s$ there exist a special Lipschitz domain D_i and a rotation R_i of $\mathbb{R}^n$ such that

$$U_i\cap\Omega = U_i\cap R_i(D_i).$$

iv) The Lipschitz bound of D_i does not exceed M for every i.

We now give the outline of the construction of the Stein extension operator for a set with minimally smooth boundary. The details of this construction and the proof of Theorem 6 can be found in [4].

First we introduce the following notation: given a set U in $\mathbb{R}^n$ and $\varepsilon > 0$ we set $U_\varepsilon = \{x \in U \mid B_\varepsilon(x) \subset U\}$. Now let Ω be an open set in $\mathbb{R}^n$ with minimally smooth boundary $\partial\Omega$. Consider also the constants ε, N, M and the sequence of open sets $\{U_i\}_{i=1}^s$ relative to Ω as given in Definition 5. We can construct a sequence of real-valued functions $\{\lambda_i\}_{i=1}^s$ defined in $\mathbb{R}^n$, such that

- $\operatorname{supp} \lambda_i \subset U_i$ for every $i = 1, ..., s$,
- $-1 \leq \lambda_i \leq 1$,
- $\lambda_i(x) = 1$ for every $x \in U_{\varepsilon/2}$,
- every λ_i is of class C^∞, has bounded derivatives of all orders and the bounds of the derivatives of λ_i can be taken to be independent of i.

We can also construct two real-valued functions Λ_+, Λ_- defined in $\mathbb{R}^n$, that satisfy the following conditions

- $\operatorname{supp} \Lambda_+ \subset \{x \in \Omega \mid d(x, \partial\Omega) \leq \varepsilon\} \cup \{x \in \mathbb{R}^n \mid d(x, \partial\Omega) \leq \varepsilon/2\}$,
- $\operatorname{supp} \Lambda_- \subset \Omega$,
- $|\Lambda_+|, |\Lambda_-| \leq 1$
- $\Lambda_+ + \Lambda_- = 1$ in $\overline{\Omega}$,
- Λ_+, Λ_- are of class $C^\infty(\mathbb{R}^n)$ with bounded derivatives of all orders.

Consider now the extension operators S_i for $W^{l,p}(R_i(D_i))$, defined as in Remark 3. We define the extension operator E for Ω as follows

$$Ef(x) := \Lambda_+(x) \frac{\sum_{i=1}^s \lambda_i(x) S_i(\lambda_i f)(x)}{\sum_{i=1}^s \lambda_i^2(x)} + \Lambda_-(x) f(x). \tag{13}$$

Theorem 6. Let $1 \leq p \leq \infty, l, n \in \mathbb{N}$. Let Ω be an open set in $\mathbb{R}^n$ having minimally smooth boundary. Then E is an extension operator which maps $W^{l,p}(\Omega)$ continuously into $W^{l,p}(\mathbb{R}^n)$.

2.2 Stein operator in Sobolev-Morrey spaces

Definition 6. Let x be a point in $\mathbb{R}^n$ and $r > 0$. We define the open cube centered in x of side l as the set

$$Q_l(x) = (x_1 - l/2, x_1 + l/2) \times (x_2 - l/2, x_2 + l/2) \times \cdots \times (x_n - l/2, x_n + l/2)$$

where $x = (x_1, ..., x_n)$.

Definition 7. Let $1 \le p < \infty$, ϕ a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ and Ω be a domain in $\mathbb{R}^n$. For a function $f \in L^p_{loc}(\Omega)$ and $\delta > 0$ we define the norm $\|.\|_{M^{\phi,\delta}_{p,Q}(\Omega)}$ as

$$\|f\|_{M^{\phi,\delta}_{p,Q}(\Omega)} := \sup_{\substack{Q_{2r}(x)\\ x\in\Omega\\ \delta>r>0}} \left(\frac{1}{\phi(r)} \int_{Q_{2r}(x)\cap\Omega} |f(y)|^p dy \right)^{\frac{1}{p}}$$

where $Q_{2r}(x)$ is the open cube centered in x of side $2r$.

Lemma 9. Let $1 \le p \le \infty$, ϕ a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ and Ω be a domain in $\mathbb{R}^n$. Then then norm $\|.\|_{M^{\phi}_{p,Q}(\Omega)}$ is equivalent to the classical Morrey norm $\|.\|_{M^{\phi}_{p}(\Omega)}$. In particular

$$\|.\|_{M^{\phi,\delta}_{p}(\Omega)} \le \|.\|_{M^{\phi,\delta}_{p,Q}(\Omega)} \le C_n \|.\|_{M^{\phi,\delta}_{p}(\Omega)}$$

where C_n is a constant depending only on n.

Proof. We prove first the second inequality of the statement. Let $x \in \Omega$, $\delta > r > 0$, $Q_{2r}(x)$ be the cube centered in x of side $2r$ and $f \in L^p_{loc}(\Omega)$. Since the set $Q_{2r}(x) \cap \Omega$ has diameter less than $2r\sqrt{n}$ by Lemma 4 there exists a collection of balls $B_1, ..., B_k$ centered in $Q_{2r}(x) \cap \Omega$ of radius r, with $k \le C_n$ where C_n depends only on n. Hence

$$\int_{Q_{2r}(x)\cap\Omega} |f(y)|^p dy \le \sum_{i=1}^{k} \int_{B_i\cap\Omega} |f(y)|^p dy$$

and

$$\|f\|_{M^{\phi,\delta}_{p,Q}(\Omega)} = \sup_{Q_{2r}(x), x\in\Omega, r>0} \left(\frac{1}{\phi(r)} \int_{Q_{2r}(x)\cap\Omega} |f(y)|^p dy \right)^{\frac{1}{p}} \le C_n \|f\|_{M^{\phi,\delta}_{p}(\Omega)}.$$

To prove the first inequality we observe that for every $x \in \Omega$ and $r > 0$, $(B_r(x) \cap \Omega) \subset (Q_{2r}(x) \cap \Omega)$, where $Q_{2r}(x)$ is the cube centered in x with side $2r$ and $B_r(x)$ is the ball of radius r centered in x. Therefore for every $f \in L^p_{loc}(\Omega)$

$$\int_{B_r(x)\cap\Omega} |f(y)|^p dy \leq \int_{Q_{2r}(x)\cap\Omega} |f(y)|^p dy$$

and this concludes the proof.

□

Lemma 10. Let Ω be an open set in $\mathbb{R}^n$ and let $f, h \in C^\infty(\mathbb{R}^n)$. Define the function $g \in C^\infty(\mathbb{R}^n)$ by $g(x) = f(\overline{x}, x_n + \lambda h(x))$ where $\overline{x} = x_1, ..., x_{n-1}$ and $0 \neq \lambda \in \mathbb{R}$. Then, for every $\alpha \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$, $D^\alpha g(x)$ is a finite sum of terms of the following form

$$c\lambda^s D^\beta f(\overline{x}, x_n + \lambda h(x))(D^{\gamma_1} h(x))^{n_1} \cdots (D^{\gamma_k} h(x))^{n_k}$$

for some constant c, with $\beta, \gamma_i \in \mathbb{N}_0^n$, $k, s, n_i \in \mathbb{N}_0$ and $\beta, \gamma_i \neq 0$, $k, s \geq 0$, $n_i > 0$. It is meant that for $k = 0$ no term $(D^{\gamma_i} h(x))^{n_i}$ is present. Moreover every term satisfies the following conditions

a) $n_1(|\gamma_1| - 1) + n_2(|\gamma_2| - 1) + ... + n_k(|\gamma_k| - 1) = |\alpha| - |\beta|$,

b) $s = 0$ if and only if $k = 0$.

Proof. We will prove the result by induction on $l = |\alpha|$. Let's prove the case $l = 1$. For every $i = 1, ..., n$ we have

$$\frac{\partial g}{\partial x_i}(x) = \frac{\partial f}{\partial x_i}(\overline{x}, x_n + \lambda h(x)) + \lambda \frac{\partial f}{\partial x_n}(\overline{x}, x_n + \lambda h(x)) \frac{\partial h}{\partial x_i}(x)$$

that clearly satisfies the statement. We assume now that the result is true for l, and suppose $|\alpha| = l + 1$. We write $D^\alpha g(x) = \frac{\partial D^\beta g}{\partial x_i}(x)$ for some $|\beta| = l$. Hence by induction hypothesis and linearity of the derivative we have that $D^\alpha g(x)$ is a finite sum of terms of the form

$$\frac{\partial}{\partial x_i}[c\lambda^s D^\gamma f(\overline{x}, x_n + \lambda h(x))(D^{\gamma_1} h(x))^{n_1} \cdots (D^{\gamma_k} h(x))^{n_k}].$$

Suppose first that $k \geq 1$, so by induction we know that

$$n_1(|\gamma_1| - 1) + n_2(|\gamma_2| - 1) + ... + n_k(|\gamma_k| - 1) = |\beta| - |\gamma| \quad (14)$$

and that $s \geq 1$. Now using the chain rule we get

$$\frac{\partial}{\partial x_i}[c\lambda^s D^\gamma f(\overline{x}, x_n + \lambda h(x))(D^{\gamma_1}h(x))^{n_1} \cdots (D^{\gamma_k}h(x))^{n_k}] =$$
$$= c\lambda^s \frac{\partial D^\gamma f}{\partial x_i}(\overline{x}, x_n + \lambda h(x))(D^{\gamma_1}h(x))^{n_1} \cdots (D^{\gamma_k}h(x))^{n_k} +$$
$$+ c\lambda^{s+1}\frac{\partial D^\gamma f}{\partial x_n}(\overline{x}, x_n + \lambda h(x))(D^{\gamma_1}h(x))^{n_1} \cdots (D^{\gamma_k}h(x))^{n_k}\frac{\partial h}{\partial x_i}(x) +$$
$$+ \sum_{j=1}^{k} c\lambda^s n_j D^\gamma f(\overline{x}, x_n + \lambda h(x))(D^{\gamma_1}h(x))^{n_1} \cdots (D^{\gamma_k}h(x))^{n_k}\frac{\frac{\partial D^{\gamma_j}h}{\partial x_i}(x)}{D^{\gamma_j}h(x)}. \quad (15)$$

Let's see that every term in the right hand side of (15) satisfies a). By (14) we have

$$n_1(|\gamma_1| - 1) + n_2(|\gamma_2| - 1) + ... + n_k(|\gamma_k| - 1) = |\beta| - |\gamma| = |\alpha| - |\gamma + e_i|$$

where $e_i = (0, ..., 1, ..., 0)$, is the n-th element of the canonical base of $\mathbb{R}^n$. Hence that first summand satisfies a). Again by (14)

$$n_1(|\gamma_1| - 1) + n_2(|\gamma_2| - 1) + ... + n_k(|\gamma_k| - 1) + (|e_i| - 1) = |\alpha| - |\gamma + e_n|$$

and this proves a) for the second term. Now we consider the final sum, we will prove a) just for $j = 1$, since the other terms can be discussed in the same way. We need to prove that

$$n_1(|\gamma_1|-1)+...+(n_j-1)(|\gamma_j|-1)+...+n_k(|\gamma_k|-1)+(|\gamma_j+e_i|-1) = |\alpha|-|\gamma|.$$

Expanding the left-hand side we get

$$n_1(|\gamma_1| - 1) + n_2(|\gamma_2| - 1) + ... + n_k(|\gamma_k| - 1) + 1$$

and since $|\beta| = |\alpha| - 1$ we conclude using (14). We observe that, since $k, s \geq 1$, all the terms also satisfies b).
Suppose know that $k = 0$, hence we need to consider

$$\frac{\partial}{\partial x_i}[cD^\gamma f(\overline{x}, x_n + \lambda h(x))]$$

that becomes

$$c\frac{\partial D^\gamma f}{\partial x_i}(\overline{x}, x_n + \lambda h(x)) + c\lambda\frac{\partial D^\gamma f}{\partial x_n}(\overline{x}, x_n + \lambda h(x))\frac{\partial h}{\partial x_i}(x).$$

By induction and by a) we know that $|\gamma| = |\beta|$, therefore it's immediate that both the above terms satisfies a) and b).

Remark 4. Let Ω be a special Lipschitz domain and let $\delta^*(\overline{x}, y)$ be the function defined in Theorem 4. Then for every $(\overline{x}, y)$ with $\psi(\overline{x}) > y$ the following holds

$$c(\psi(\overline{x}) - y) \geq \delta^*(\overline{x}, y) \geq 2(\psi(\overline{x}) - y),$$

where c is some constant depending only on n. The second inequality follows directly from the definition of δ^* and Lemma 7. Next we notice that $(\psi(\overline{x}) - y) \geq d(x, \overline{\Omega})$, hence the first inequality follows from a) of Theorem 3.

$\square$

Lemma 11. Let $1 \leq p < \infty, n \geq 2$, ϕ a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ and Ω be a special Lipschitz domain of $\mathbb{R}^n$ with Lipschitz bound M. Moreover let T be the operator defined in Theorem 4 and $f \in C^\infty(\overline{\Omega})$ be a function bounded in $\overline{\Omega}$ together with all its partial derivatives. Then for every $\alpha \in \mathbb{N}_0^n$ and $\delta > 0$

$$\|D^\alpha Tf\|_{M_p^{\phi,\delta}(\mathbb{R}^n)} \leq C_{l,n}(M) \sum_{|\beta| \leq |\alpha|} \|D^\beta f\|_{M_p^{\phi,\delta}(\Omega)} \tag{16}$$

where $l = |\alpha|$ and $C_{l,n}(M)$ is a constant depending only on l, n and M.

Proof. Let's start by proving the case $l = 0$. By Lemma 9 it's enough to prove that for an arbitrary open cube Q of side $0 < r < \delta$ in $\mathbb{R}^n$ with sides parallel to the axis we have

$$\left(\frac{1}{\phi(r/2)} \int_Q |Tf(x)|^p dx \right)^{\frac{1}{p}} \leq C_n(M) \|f\|_{M_{p,Q}^{\phi,\delta/2}(\Omega)} \tag{17}$$

for a constant $C_n(M)$ depending only on n, M. Let's define $\Omega^- = \{(\overline{x}, y) \in \mathbb{R}^n \mid \overline{x} \in \mathbb{R}^{n-1},\ y < \psi(\overline{x})\}$. There are three cases: 1. $Q \subset \Omega$ 2. $Q \subset \Omega^-$ 3. $Q \cap \{y = \psi(\overline{x})\} \neq \emptyset$.

Case 1. Since $Tf = f$ in Ω

$$\left(\frac{1}{\phi(r/2)} \int_Q |Tf(x)|^p dx \right)^{\frac{1}{p}} = \left(\frac{1}{\phi(r/2)} \int_Q |f(x)|^p dx \right)^{\frac{1}{p}} \leq \|f\|_{M_{p,Q}^{\phi,\delta/2}(\Omega)}$$

and we are done.

Case 2. Let's write Q as $Q = \{(\overline{x}, y) \in \mathbb{R}^n \mid \overline{x} \in F, y \in (a - r, a)\}$ where F is an open cube of $\mathbb{R}^{n-1}$ of side r and $a < \psi(\overline{x})$ for every $\overline{x} \in F$. Fix now

$(\overline{x}, y) \in Q$. By Lemma 8 there exists a constant A_3 such that $|\tau(\lambda)| \le A_3/\lambda^3$ for every $\lambda \ge 1$. From the definition of Tf we have

$$|Tf(\overline{x}, y)| \le \int_1^\infty |f(\overline{x}, y+\lambda\delta^*(\overline{x}, y))||\tau(\lambda)|d\lambda \le A_3 \int_1^\infty |f(\overline{x}, y+\lambda\delta^*(\overline{x}, y))|\frac{1}{\lambda^3}d\lambda \tag{18}$$

Let's apply the change of variable $s = y + \lambda\delta^*(\overline{x}, y)$

$$|Tf(\overline{x}, y)| \le A_3 \int_{y+\delta^*}^\infty |f(\overline{x}, s)|\frac{(\delta^*)^2}{(s-y)^3}ds \le A_3 c^2 \int_{2\psi(\overline{x})-y}^\infty |f(\overline{x}, s)|\frac{(\psi(x)-y)^2}{(s-y)^3}ds \tag{19}$$

because $c(\psi(x) - y) \ge \delta^* \ge 2(\psi(x) - y)$ as seen in Remark 4. Let's now decompose the last integral as follows

$$|Tf(\overline{x}, y)| \le \sum_{k=0}^\infty A_3 c^2 \int_{2\psi(\overline{x})-y+kr}^{2\psi(\overline{x})-y+(k+1)r} |f(\overline{x}, s)|\frac{(\psi(\overline{x})-y)^2}{(s-y)^3}ds.$$

Now by applying Minkowski's inequality for an infinite sum we get

$$\begin{aligned}&\left(\int_{a-r}^a |Tf(\overline{x}, y)|^p dy\right)^{\frac{1}{p}} \\ &\le A_3 c^2 \sum_{k=0}^\infty \left(\int_{a-r}^a \left(\int_{2\psi(\overline{x})-y+kr}^{2\psi(\overline{x})-y+(k+1)r} \frac{|f(\overline{x}, s)|(\psi(x)-y)^2}{(s-y)^3}ds\right)^p dy\right)^{\frac{1}{p}}\end{aligned} \tag{20}$$

Next we plan to estimate each summand in (20). To each summand in the right-hand side of (20) we apply the change of variable $y = \psi(\overline{x}) - z$ and we get

$$\left(\int_{\psi(x)-a}^{\psi(x)-a+r} \left(\int_{\psi(x)+z+kr}^{\psi(x)+z+(k+1)r} |f(\overline{x}, s)|\frac{z^2}{(s-\psi(x)+z)^3}ds\right)^p dz\right)^{\frac{1}{p}}$$

and the change of variable $u = s - \psi(x)$

$$\left(\int_{\psi(x)-a}^{\psi(x)-a+r} \left(\int_{z+kr}^{z+(k+1)r} |f(\overline{x}, u+\psi(x))|\frac{z^2}{(u+z)^3}du\right)^p dz\right)^{\frac{1}{p}}.$$

Then we apply the change of variable $t = u/z$

$$\left(\int_{\psi(\overline{x})-a}^{\psi(\overline{x})-a+r} \left(\int_{1+kr/z}^{1+(k+1)r/z} |f(\overline{x}, tz+\psi(x))|\frac{1}{(t+1)^3}dt\right)^p dz\right)^{\frac{1}{p}}.$$

that can be rewritten as

$$\left(\int_{\psi(\overline{x})-a}^{\psi(\overline{x})-a+r}\left(\int_{1+kr/(\psi(\overline{x})-a+r)}^{1+(k+1)r/(\psi(\overline{x})-a)}|f(\overline{x},tz+\psi(x))|\mathbb{1}_{(1+kr/z,1+(k+1)r/z)}(t)\frac{1}{(t+1)^3}dt\right)^p dz\right)^{\frac{1}{p}}.$$

By Minkowsi's integral inequality and setting $\alpha = r/(\psi(\overline{x})-a)$

$$\left(\int_{a\psi(\overline{x})-a}^{\psi(\overline{x})-a+r}\left(\int_{1+k\alpha/(\alpha+1)}^{1+(k+1)\alpha}|f(\overline{x},tz+\psi(x))|\mathbb{1}_{(1+kr/z,1+(k+1)r/z)}(t)\frac{1}{(t+1)^3}dt\right)^p dz\right)^{\frac{1}{p}}.$$

$$\leq \int_{1+k\alpha/(\alpha+1)}^{1+(k+1)\alpha}\left(\int_{\psi(\overline{x})-a}^{\psi(\overline{x})-a+r}|f(\overline{x},tz+\psi(x))|^p\mathbb{1}_{(1+kr/z,1+(k+1)r/z)}(t)\frac{1}{(t+1)^{3p}}dz\right)^{\frac{1}{p}}dt.$$

We notice that for every $t,z\in\mathbb{R}$ with $\psi(\overline{x})-a\leq z\leq\psi(\overline{x})-a+r$

$$\mathbb{1}_{(1+kr/z,1+(k+1)r/z)}(t)\leq\mathbb{1}_{(\psi(\overline{x})-a+kr,\psi(\overline{x})-a+(k+2)r)}(tz)$$

hence using the change of variable $w = tz$

$$\int_{1+k\alpha/(\alpha+1)}^{1+(k+1)\alpha}\left(\int_{\psi(\overline{x})-a}^{\psi(\overline{x})-a+r}|f(\overline{x},tz+\psi(x))|^p\mathbb{1}_{(1+kr/z,1+(k+1)r/z)}(t)\frac{1}{(t+1)^{3p}}dz\right)^{\frac{1}{p}}dt$$

$$\leq \int_{1+k\alpha/(\alpha+1)}^{1+(k+1)\alpha}\left(\int_{\psi(\overline{x})-a+kr}^{\psi(\overline{x})-a+(k+2)r}|f(\overline{x},w+\psi(\overline{x}))|^p\frac{1}{t(t+1)^{3p}}dw\right)^{\frac{1}{p}}dt$$

$$= \int_{1+k\alpha/(\alpha+1)}^{1+(k+1)\alpha}\frac{1}{t^{\frac{1}{p}}(t+1)^3}dt\left(\int_{\psi(\overline{x})-a+kr}^{\psi(\overline{x})-a+(k+2)r}|f(\overline{x},w+\psi(\overline{x}))|^p dw\right)^{\frac{1}{p}}$$

$$\leq \int_{1+k\alpha/(\alpha+1)}^{1+(k+1)\alpha}\frac{1}{(t+1)^3}dt\left(\int_{\psi(\overline{x})-a+kr}^{\psi(\overline{x})-a+(k+2)r}|f(\overline{x},w+\psi(\overline{x}))|^p dw\right)^{\frac{1}{p}}$$

$$\leq \int_{1+k\alpha/(\alpha+1)}^{1+(k+2)\alpha}\frac{1}{(t+1)^3}dt\left(\int_{\psi(\overline{x})-a+kr}^{\psi(\overline{x})-a+(k+2)r}|f(\overline{x},w+\psi(\overline{x}))|^p dw\right)^{\frac{1}{p}}$$

$$= \frac{1}{2}\left[\frac{1}{(2+k\alpha/(\alpha+1))^2}-\frac{1}{(2+(k+2)\alpha)^2}\right]\left(\int_{\psi(\overline{x})-a+kr}^{\psi(\overline{x})-a+(k+2)r}|f(\overline{x},w+\psi(\overline{x}))|^p dw\right)^{\frac{1}{p}}$$

$$= \frac{s_k(\alpha)}{2}\left(\int_{\psi(\overline{x})-a+kr}^{\psi(\overline{x})-a+(k+2)r}|f(\overline{x},w+\psi(\overline{x}))|^p dw\right)^{\frac{1}{p}}.$$

Where $s_k(\alpha) = \frac{1}{(2+k\alpha/(\alpha+1))^2} - \frac{1}{(2+(k+2)\alpha)^2}$. Plugging this estimate inside (20) we get

$$\begin{aligned}\left(\int_{a-r}^{a} |Tf(\overline{x},y)|^p dy\right)^{\frac{1}{p}} &\leq A_3 \frac{c^2}{2} \sum_{k=0}^{\infty} s_k(\alpha) \left(\int_{\psi(\overline{x})-a+kr}^{\psi(\overline{x})-a+(k+2)r} |f(\overline{x}, w+\psi(\overline{x}))|^p dw\right)^{\frac{1}{p}} \\ &= A_3 \frac{c^2}{2} \sum_{k=0}^{\infty} s_k(\alpha) \left(\int_{2\psi(\overline{x})-a+kr}^{2\psi(\overline{x})-a+(k+2)r} |f(\overline{x},y)|^p dy\right)^{\frac{1}{p}}.\end{aligned} \tag{21}$$

Taking the L^p norm on F on both sides and applying again Minkowski inequality we obtain

$$\begin{aligned}\left(\int_F \int_{a-r}^{a} |Tf(\overline{x},y)|^p dy d\overline{x}\right)^{\frac{1}{p}} &\leq A_3 \frac{c^2}{2} \sum_{k=0}^{\infty} s_k(\alpha) \left(\int_F \int_{2\psi(\overline{x})-a+kr}^{2\psi(\overline{x})-a+(k+2)r} |f(\overline{x},y)|^p dy d\overline{x}\right)^{\frac{1}{p}} \\ &= A_3 \frac{c^2}{2} \sum_{k=0}^{\infty} s_k(\alpha) \|f\|_{L^p(S_k)}.\end{aligned} \tag{22}$$

where $S_k = \{(\overline{x},y) \in \mathbb{R}^n \mid \overline{x} \in F,\ 2\psi(\overline{x}) - a + kr < y < 2\psi(\overline{x}) - a + (k+2)r\}$. The set S_k has the following two properties

a) S_k has diameter less than dr, where d is a constant depending only on n and M.

b) $S_k \subset \Omega$.

To prove a), let $(\overline{x}_1, y_1), (\overline{x}_2, y_2)$ be two arbitrary points in S_k. We can suppose that $y_2 \geq y_1$. Then

$$\begin{aligned}|y_1 - y_2| &= y_2 - y_1 \\ &\leq 2\psi(\overline{x}_2) - a + (k+2)r - (2\psi(\overline{x}_1) - a + kr) \\ &= 2(\psi(\overline{x}_2) - \psi(\overline{x}_1)) + 2r \leq 2M|\overline{x}_1 - \overline{x}_2| + 2r.\end{aligned}$$

Moreover

$$|\overline{x}_1 - \overline{x}_2| \leq r\sqrt{n-1}$$

because $\overline{x}_1, \overline{x}_2$ belongs to the $n-1$-dimensional cube F. This proves a). To prove b) just notice that for every $(\overline{x}, y) \in S_k$ we have $y > 2\psi(\overline{x}) - a > \psi(\overline{x})$. Property a) together with Lemma 4 implies that there exists a collection of open cubes $Q_1, ..., Q_m$ centered in S_k of side r that covers S_k, with $m \in \mathbb{N}$ depending only on M and n. Hence

$$S_k \subset \bigcup_{i=1}^{m} (Q_i \cap \Omega)$$

and property b) assures that every cube Q_i is centered in Ω. Therefore by (22)

$$\|Tf\|_{L^p(Q)} \leq \frac{A_3 c^2}{2} \sum_{k=0}^{\infty} s_k(\alpha)(\|f\|_{L^p(Q_1 \cap \Omega)} + ... + \|f\|_{L^p(Q_m \cap \Omega)}),$$

then dividing in both sides by $\phi(r/2)^{\frac{1}{p}}$ we obtain

$$\left(\frac{1}{\phi(r/2)} \int_Q |Tf(x)|^p dx \right)^{\frac{1}{p}} \leq \frac{A_3 c^2 m}{2} \sum_{k=0}^{\infty} s_k(\alpha) \|f\|_{M_{p,Q}^{\phi,\delta/2}(\Omega)}$$

We want now to estimate the series $\sum_{k=0}^{\infty} s_k(\alpha)$. First we rewrite it in the following way

$$\begin{aligned}
\sum_{k=0}^{\infty} s_k(\alpha) &= \sum_{k=0}^{\infty} \frac{1}{(2 + k\alpha/(\alpha+1))^2} - \frac{1}{(2 + (k+2)\alpha)^2} = \\
&= \sum_{k=0}^{\infty} \frac{(\alpha+1)^2}{(2 + (k+2)\alpha)^2} - \frac{1}{(2 + (k+2)\alpha)^2} = \\
&= \sum_{k=0}^{\infty} \frac{\alpha(\alpha+2)}{(2 + (k+2)\alpha)^2} = \sum_{k=2}^{\infty} \frac{\alpha(\alpha+2)}{(2 + k\alpha)^2}.
\end{aligned}$$

To bound this series we distinguish two cases, when $\alpha \leq 1$ and when $\alpha > 1$. In the first case we can bound the series using a Riemann Sum

$$\begin{aligned}
\sum_{k=2}^{\infty} \frac{\alpha(\alpha+2)}{(k\alpha+2)^2} &\leq 3 \sum_{k=2}^{\infty} \frac{\alpha}{(k\alpha+2)^2} = \\
&= 3 \sum_{k=2}^{\infty} \int_{\alpha(k-1)}^{\alpha k} \frac{1}{(\alpha k+2)^2} dt \leq 3 \int_0^{\infty} \frac{1}{(t+2)^2} dt = \frac{3}{2}.
\end{aligned}$$

In the second case

$$\sum_{k=2}^{\infty}\frac{\alpha(\alpha+2)}{(k\alpha+2)^2}\leq\sum_{k=2}^{\infty}\frac{\alpha(\alpha+2)}{k^2\alpha^2}=\sum_{k=2}^{\infty}\frac{1+\frac{2}{\alpha}}{k^2}\leq 3\big(\frac{\pi^2}{6}-1\big)<2.$$

Hence we get

$$\left(\frac{1}{\phi(r/2)}\int_Q|Tf(x)|^pdx\right)^{\frac{1}{p}}\leq\frac{3mA_3c^2}{2}\|f\|_{M^{\phi,\delta/2}_{p,Q}(\Omega)}$$

that shows (17).

Case 3. We write Q as $F\times(a-r,a)$ and and we define $Q^+=Q\cap\Omega$ and $Q^-=Q\cap\Omega^-$. Then

$$\|Tf\|_{L^p(Q)}\leq\|f\|_{L^p(Q^+)}+\|Tf\|_{L^p(Q^-)}.$$

Moreover Q^- can be furtherly decompose as $Q^-=Q_1^-\cup Q_2^-$ where $Q_1^-=\{(\overline{x},y)\in Q^-\mid\psi(\overline{x})>a\}$ and $Q_2^-=\{(\overline{x},y)\in Q^-|\psi(\overline{x})\leq a\}$. Hence

$$\begin{aligned}\int_{Q^-}|Tf(x)|^pdx&=\int_{Q_1^-}|Tf(x)|^pdx+\int_{Q_2^-}|Tf(x)|^pdx\\&=\int_{S_1}\int_{a-r}^{a}|Tf(\overline{x},y)|^pdyd\overline{x}+\int_{S_2}\int_{a-r}^{\psi(\overline{x})}|Tf(\overline{x},y)|^pdyd\overline{x}\end{aligned}$$

for two suitable measurable sets S_1 and S_2 with $S_1\cup S_2=F$. From (21) we know that if $\overline{x}\in S_1$ then

$$\left(\int_{a-r}^{a}|Tf(\overline{x},y)|^pdy\right)^{\frac{1}{p}}\leq A_3\frac{c^2}{2}\sum_{k=0}^{\infty}s_k(\alpha)\left(\int_{2\psi(\overline{x})-a+kr}^{2\psi(\overline{x})-a+(k+2)r}|f(\overline{x},y)|^pdy\right)^{\frac{1}{p}}.$$

Hence taking the L^p norm over S_1 and reasoning as in Case 2 we obtain

$$\frac{1}{\phi(r/2)^{\frac{1}{p}}}\|Tf\|_{L^p(Q_1^-)}\leq c_1\|f\|_{M^{\phi,\delta/2}_p(\Omega)}\tag{23}$$

for some constant c_1 depending only on n and M. If instead $\overline{x}\in S_2$, since $\psi(\overline{x})\leq a$, we have

$$\int_{a-r}^{\psi(\overline{x})}|Tf(\overline{x},y)|^pdy\leq\int_{\psi(\overline{x})-r}^{\psi(\overline{x})}|Tf(\overline{x},y)|^pdy.\tag{24}$$

Now from (§) with $a = \psi(\overline{x}) - \delta$ ($\delta > 0$) we obtain

$$\left(\int_{\psi(\overline{x})-\delta-r}^{\psi(\overline{x})-\delta} |Tf(\overline{x},y)|^p dy\right)^{\frac{1}{p}} \leq A_3 \frac{c^2}{2} \sum_{k=0}^{\infty} s_k(\alpha) \left(\int_{\psi(\overline{x})+\delta+kr}^{\psi(\overline{x})+\delta+(k+2)r} |f(\overline{x},y)|^p dy\right)^{\frac{1}{p}}.$$

Taking this time the L^p norm in S_2

$$\begin{aligned}\left(\int_{S_2}\int_{\psi(\overline{x})-\delta-r}^{\psi(\overline{x})-\delta} |Tf(\overline{x},y)|^p dy d\overline{x}\right)^{\frac{1}{p}} &\leq A_3 \frac{c^2}{2} \sum_{k=0}^{\infty} s_k(\alpha) \left(\int_{S_2}\int_{\psi(\overline{x})+\delta+kr}^{\psi(\overline{x})+\delta+(k+2)r} |f(\overline{x},y)|^p dy d\overline{x}\right)^{\frac{1}{p}} \\ &= A_3 \frac{c^2}{2} \sum_{k=0}^{\infty} s_k(\alpha) \|f\|_{L^p(S'_k)}.\end{aligned}$$

One can observe that the sets S'_k have the properties a) and b) like the sets S_k in Case 2, therefore

$$\left(\frac{1}{\phi(r/2)}\int_{S_2}\int_{\psi(\overline{x})-\delta-r}^{\psi(\overline{x})-\delta} |Tf(\overline{x},y)|^p dy d\overline{x}\right)^{\frac{1}{p}} \leq c_2 \|f\|_{M_p^{\phi,\delta/2}(\Omega)}$$

for some constant c_2 depending only on n and M. We now let δ go to 0

$$\left(\frac{1}{\phi(r/2)}\int_{S_2}\int_{\psi(\overline{x})-r}^{\psi(\overline{x})} |Tf(\overline{x},y)|^p dy d\overline{x}\right)^{\frac{1}{p}} \leq c_2 \|f\|_{M_p^{\phi}(\Omega)}. \tag{25}$$

Combining the above inequality with (24) we obtain

$$\left(\frac{1}{\phi(r/2)}\int_{S_2}\int_{a-r}^{\psi(\overline{x})} |Tf(\overline{x},y)|^p dy d\overline{x}\right)^{\frac{1}{p}} \leq c_2 \|f\|_{M_p^{\phi}(\Omega)}.$$

Thus from (23) and (25)

$$\frac{1}{\phi(r/2)^{\frac{1}{p}}}\|Tf\|_{L^p(Q^-)} \leq \frac{1}{\phi(r/2)^{\frac{1}{p}}}\|Tf\|_{L^p(Q_1^-)} + \frac{1}{\phi(r/2)^{\frac{1}{p}}}\|Tf\|_{L^p(Q_2^-)} \leq (c_1+c_2)\|f\|_{M_p^{\phi}(\Omega)}$$

Finally it's immediate to verify that $\|f\|_{L^p(Q^+)} \leq \phi(r/2)^{\frac{1}{p}}\|f\|_{M_p^{\phi}(\Omega)}$. This concludes the proof of Case 3.

We consider now the case $l > 0$. By Lemma 9 it's again enough to prove that for an arbitrary open cube Q of side r contained in $\mathbb{R}^n$ we have

$$\left(\frac{1}{\phi(r/2)}\int_Q |D^\alpha Tf(x)|^p dx\right)^{\frac{1}{p}} \leq C_{l,n}(M) \sum_{|\beta|\leq|\alpha|} \|D^\beta f\|_{M^\phi_{p,Q}(\Omega)} \qquad (26)$$

for a constant $C_{l,n}(M)$ depending only on l, n, M. We will consider the same three cases that appeared with $l = 0$. Since $D^\alpha Tf = D^\alpha f$ in Ω, the first case is trivial as before. We will see that the cases 2 and 3 also follow from the computations done with $l = 0$. We start observing that by the boundedness of f and all its derivatives we can differentiate under the integral sign to get

$$D^\alpha Tf(\overline{x}, y) = \int_1^\infty D^\alpha g_\lambda(\overline{x}, y)\tau(\lambda)d\lambda$$

for every $(\overline{x}, y) \in \Omega^-$, where $g_\lambda(\overline{x}, y) = f(\overline{x}, y + \lambda\delta^*(\overline{x}, y))$. By Lemma 10 $D^\alpha g_\lambda(\overline{x}, y)$ is a finite sum of terms of the type

$$\widetilde{c}\lambda^s D^\beta f(\overline{x}, y + \lambda\delta^*(\overline{x}, y)(D^{\gamma_1}\delta^*(x))^{n_1}\cdots(D^{\gamma_k}\delta^*(x))^{n_k}.$$

For each of these terms we also set

$$\begin{aligned} &T_{s,\beta,(\gamma_1,n_1),\ldots,(\gamma_k,n_k)}(x) \\ &= \int_1^\infty \lambda^s D^\beta f(\overline{x}, y + \lambda\delta^*(\overline{x}, y)(D^{\gamma_1}\delta^*(x))^{n_1}\cdots(D^{\gamma_k}\delta^*(x))^{n_k}\tau(\lambda)d\lambda. \end{aligned}$$

In this way $D^\alpha Tf(\overline{x}, y)$ is a finite sum of terms of type $\widetilde{c}T_{s,\beta,(\gamma_1,n_1),\ldots,(\gamma_k,n_k)}(x)$. Now, since the constants $\widetilde{c}$ and the number of terms of the sum depend only on l and n, we just need to estimate the quantities

$$\left(\frac{1}{\phi(r/2)}\int_Q \left|T_{s,\beta,(\gamma_1,n_1),\ldots,(\gamma_k,n_k)}(x)\right|^p dx\right)^{\frac{1}{p}}.$$

We start by assuming that $|\beta| = |\alpha|$. By the property a) in Lemma 10 and by the estimates of the derivatives of $\delta^*(= 2a\Delta)$ given in Theorem 3 we have that

$$\begin{aligned} |T_{s,\beta,(\gamma_1,n_1),\ldots,(\gamma_k,n_k)}(x)| &\leq c_3 \int_1^\infty \lambda^s |D^\beta f(\overline{x}, y + \lambda\delta^*(\overline{x}, y)||\tau(\lambda)|d\lambda \\ &\leq c_3 A_{s+3} \int_1^\infty |D^\beta f(\overline{x}, y + \lambda\delta^*(\overline{x}, y)|\frac{1}{\lambda^3}d\lambda \end{aligned}$$

where A_{s+3} is such that $|\tau(\lambda)| \leq A_{s+3}/\lambda^{s+3}$ and c_3 depends only on n and M. We are now in the same situation as in the second inequality of (18). Hence we can proceed the estimate in the same way as in case $l = 0$ to get

$$\left(\frac{1}{\phi(r/2)}\int_Q \left|T_{s,\beta,(\gamma_1,n_1),\dots,(\gamma_k,n_k)}(x)\right|^p dx\right)^{\frac{1}{p}} \leq c_4\|D^\beta f\|_{M_p^\phi(\Omega)}$$

for every Q in case 2 and

$$\left(\frac{1}{\phi(r/2)}\int_{Q\cap\Omega^-} \left|T_{s,\beta,(\gamma_1,n_1),\dots,(\gamma_k,n_k)}(x)\right|^p dx\right)^{\frac{1}{p}} \leq c_5\|D^\beta f\|_{M_p^\phi(\Omega)}$$

for every Q in Case 3, where c_4, c_5 depend only on n and M. Suppose now that $|\alpha| > |\beta|$. Arguing as above, by Theorem 3 and Lemma 10 we get

$$\begin{aligned}
&|T_{s,\beta,(\gamma_1,n_1),\dots,(\gamma_k,n_k)}(x)| \\
&\leq c_6\frac{1}{d(x,\overline{\Omega})^{|\alpha|-|\beta|}}\left|\int_1^\infty \lambda^s D^\beta f(\overline{x}, y+\lambda\delta^*(\overline{x},y)\tau(\lambda)d\lambda\right| \\
&\leq c_6\frac{1}{(\psi(\overline{x})-y)^{|\alpha|-|\beta|}}\left|\int_1^\infty \lambda^s D^\beta f(\overline{x}, y+\lambda\delta^*(\overline{x},y)\tau(\lambda)d\lambda\right|. \qquad (27)
\end{aligned}$$

Where c_6 depends only on n, l and M. We now write the Taylor expansion with integral remainder of the function $t \mapsto D^\beta f(\overline{x}, y+t)$ centered in $\delta^*(\overline{x},y)$ up to order $m = |\alpha| - |\beta|$ and evaluated at $\lambda\delta^*(\overline{x},y)$

$$D^\beta f(\overline{x}, y+\lambda\delta^*) = \sum_{i=0}^{m-1}\frac{(\lambda\delta^*-\delta^*)^i}{i!}\frac{\partial^i D^\beta f}{\partial x_n^i}(\overline{x}, y+\delta^*) + \int_{\delta^*}^{\lambda\delta^*}\frac{(\lambda\delta^*-t)^{m-1}}{m!}\frac{\partial^m D^\beta f}{\partial x_n^m}(\overline{x}, y+t)dt.$$

We observe that the terms inside the first sum in the right hand side don't give any contribution in (27), since

$$\begin{aligned}
&\int_1^\infty \frac{\lambda^s(\lambda\delta^*-\delta^*)^i}{i!}\frac{\partial^i D^\beta f}{\partial x_n^i}(\overline{x}, y+\delta^*)\tau(\lambda)d\lambda \\
&= \frac{\partial^i D^\beta f}{\partial x_n^i}(\overline{x}, y+\delta^*)\frac{(\delta^*)^i}{i!}\int_1^\infty \lambda^s(\lambda-1)^i\tau(\lambda)d\lambda = 0
\end{aligned}$$

by the properties of τ, since $s > 0$ by Lemma 10. Hence combining this with (27) we obtain

$$\begin{aligned}
&|T_{s,\beta,(\gamma_1,n_1),\dots,(\gamma_k,n_k)}(x)| \\
&\leq \frac{c_6}{(\psi(\overline{x})-y)^m}\left|\int_1^\infty\int_{\delta^*}^{\lambda\delta^*}\frac{(\lambda\delta^*-t)^{m-1}}{m!}\frac{\partial^m D\beta f}{\partial x_n^m}(\overline{x}, y+t)dt\lambda^s\tau(\lambda)d\lambda\right|.
\end{aligned}$$

Observing that $(\lambda\delta^* - t)^{m-1} \leq (\lambda\delta^*)^{m-1}$, recalling that $\psi(\overline{x}) - y \geq c\delta^*$ and using the change of variable $u = y + t$ we get

$$|T_{s,\beta,(\gamma_1,n_1),\ldots,(\gamma_k,n_k)}(x)| \leq \frac{c_6}{c^m m!\delta^*} \int_1^\infty \int_{y+\delta^*}^{y+\lambda\delta^*} \left| \frac{\partial^m D\beta f}{\partial x_n^m}(\overline{x}, u) \right| \lambda^{s+m-1} |\tau(\lambda)| du d\lambda.$$

Performing a change of order of integration we deduce

$$|T_{s,\beta,(\gamma_1,n_1),\ldots,(\gamma_k,n_k)}(x)| \leq \frac{c_6}{c^m m!\delta^*} \int_{y+\delta^*}^\infty \left| \frac{\partial^m D^\beta f}{\partial x_n^m}(\overline{x}, u) \right| \int_{(u-y)/\delta^*}^\infty |\lambda^{s+m-1}\tau(\lambda)| d\lambda du.$$

Finally recalling that that $|\tau(\lambda)| \leq A_{m+s+3}/\lambda^{s+m+3}$ for some constant A_{m+s+3} we can write

$$|T_{s,\beta,(\gamma_1,n_1),\ldots,(\gamma_k,n_k)}(x)| \leq \frac{c_6 A_{m+s+3}}{3c^m m!} \int_{y+\delta^*}^\infty \left| \frac{\partial^m D^\beta f}{\partial x_n^m}(\overline{x}, u) \right| \frac{(\delta^*)^2}{(u-y)^3} du.$$

We observe that we are now in the same situation as in the first inequality of (19) of the case $l = 0$ and the same computations lead us to

$$\left(\frac{1}{\phi(r/2)} \int_Q \left| T_{s,\beta,(\gamma_1,n_1),\ldots,(\gamma_k,n_k)}(x) \right|^p dx \right)^{\frac{1}{p}} \leq c_7 \left\| \frac{\partial^m D\beta f}{\partial x_n^m} \right\|_{M_p^\phi(\Omega)}$$

for every Q in case 2 and

$$\left(\frac{1}{\phi(r/2)} \int_{Q\cap\Omega^-} \left| T_{s,\beta,(\gamma_1,n_1),\ldots,(\gamma_k,n_k)}(x) \right|^p dx \right)^{\frac{1}{p}} \leq c_8 \left\| \frac{\partial^m D^\beta f}{\partial x_n^m} \right\|_{M_p^\phi(\Omega)}$$

for every Q in case 3, where c_7, c_8 depend only on n, l and M. This concludes also the proof of the case $l > 0$. $\square$

Theorem 7. Let $1 \leq p < \infty, n \geq 2$, ϕ a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ and Ω be a special Lipschitz domain of $\mathbb{R}^n$ with Lipschitz bound M. Moreover let S be the Stein extension operator. Then for every $f \in W^{l,p}(\Omega)$ and every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq l$

$$\|D_w^\alpha Sf\|_{M_p^\phi(\mathbb{R}^n)} \leq C_{l,n}(M) \sum_{|\beta|\leq|\alpha|} \|D_w^\beta f\|_{M_p^\phi(\Omega)} \tag{28}$$

where $C_{l,n}(\Omega)$ depends only on n, l and M.

Proof. We recall definition of the operator S. Set Γ to be the cone $\Gamma = \{(\overline{x}, y) \in \mathbb{R}^n \mid M|\overline{x}| < |y|, y < 0\}$ and let $\eta \in C_c^\infty(\mathbb{R}^n)$ be a function with total integral 1 and support is contained in Γ. Then, given $f \in W^{l,p}(\Omega)$, Sf is defined to be the limit in $W^{l,p}(\mathbb{R}^n)$ of Tf_ε as $\varepsilon \to 0$, where $f_\varepsilon(x) = 1/\varepsilon^n \int_{\mathbb{R}^n} f(x-y)\eta(y/\varepsilon)$ for every x in an appropriate neighborhood of $\overline{\Omega}$. We claim that for every $f \in W^{l,p}(\Omega)$ and $|\alpha| \le l$

$$\|D_w^\alpha f_\varepsilon\|_{M_p^\phi(\Omega)} \le \|D_w^\alpha f\|_{M_p^\phi(\Omega)}. \tag{29}$$

To see this first we notice that $D_w^\alpha f_\varepsilon(x) = 1/\varepsilon^n \int_{\mathbb{R}^n} D_w^\alpha f(x-y)\eta(y/\varepsilon)dy$ for every $x \in \Omega$. Let now $B_{x_0}(r)$ a ball centered in Ω of radius r. By Minkowski's integral inequality

$$\begin{aligned}
\left(\frac{1}{\phi(r)}\int_{B_r(x_0)\cap\Omega} |D^\alpha f_\varepsilon(x)|^p\right)^{\frac{1}{p}} &= \left(\frac{1}{\phi(r)}\int_{B_r(x_0)\cap\Omega} \left|\frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} D_w^\alpha f(x-y)\eta\left(\frac{y}{\varepsilon}\right)dy\right|^p dx\right)^{\frac{1}{p}} \\
&\le \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} \eta\left(\frac{y}{\varepsilon}\right)\left(\frac{1}{\phi(r)}\int_{B_r(x_0)\cap\Omega} |D^\alpha f(x-y)|^p dx\right)^{\frac{1}{p}} dy \\
&\le \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} \eta\left(\frac{y}{\varepsilon}\right)\left(\frac{1}{\phi(r)}\int_{B_r(x_0-y)\cap\Omega} |D^\alpha f(x)|^p dx\right)^{\frac{1}{p}} dy \\
&\le \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} \eta\left(\frac{y}{\varepsilon}\right)\|D^\alpha f\|_{M_p^\phi(\Omega)} dy = \|D^\alpha f\|_{M_p^\phi(\Omega)}
\end{aligned}$$

because $B_r(x_0) \cap \Omega - y \subset B_r(x_0 - y) \cap \Omega$ and $x_0 - y \in \Omega$ for every $x_0 \in \Omega$ and $y \in \Gamma$. This proves (29). Now combining (29) with (16) we get

$$\|D^\alpha T f_\varepsilon\|_{M_p^\phi(\mathbb{R}^n)} \le C_{l,n}(M) \sum_{|\beta|\le|\alpha|} \|D^\beta f\|_{M_p^\phi(\Omega)},$$

for every $\varepsilon > 0$ and every $|\alpha| \le l$, with $C_{l,n}(M)$ independent of ε. In particular, for every ball B in $\mathbb{R}^n$ of radius $r > 0$ we have

$$\left(\frac{1}{\phi(r)}\int_B |D^\alpha T f_\varepsilon(x)|^p dx\right)^{\frac{1}{p}} \le C_{l,n}(M) \sum_{|\beta|\le|\alpha|} \|D^\beta f\|_{M_p^\phi(\Omega)} \tag{30}$$

Since Tf_ε converges to Sf in $W^{l,p}(\mathbb{R}^n)$, then $D^\alpha T f_\varepsilon$ converges to $D_w^\alpha Sf$ in $L^p(\mathbb{R}^n)$ for every $|\alpha| \le l$ and as a consequence also in $L^p(B)$ for every ball B. Hence we can pass to the limit as $\varepsilon \to 0$ in (30) and obtain

$$\left(\frac{1}{\phi(r)}\int_B |D_w^\alpha S(x)|^p dx\right)^{\frac{1}{p}} \le C_{l,n}(M) \sum_{|\beta|\le|\alpha|} \|D_w^\beta f\|_{M_p^\phi(\Omega)}$$

for every ball B of radius r. This concludes the proof. $\square$

Remark 5. Theorem 7 holds also if Ω is a rotation of some Lipschitz domain. This can be shown using Remark 3 and similar computations.

In Theorem 7 we proved that the Stein operator S preserves the Sobolev-Morrey spaces, in the case of a special Lipschitz domains. Our next goal is to extend this property to the more general Stein operator E, defined in (13), which acts on open set with a minimally smooth boundary. We start with a simple construction.

Let Ω be an open set in $\mathbb{R}^n$ with minimally smooth boundary with parameters ε, M, N and a covering $\{U_i\}_{i=1}^s$. Let's define

$$V_i := \bigcup_{\substack{x\in\partial\Omega,\\ B_\varepsilon(x)\subset U_i}} B_\varepsilon(x).$$

We consider the family $\{V_i\}_{i=1}^{\widetilde{s}}$ containing the sets V_i that are non-empty. We observe that the sequence $\{V_i\}_{i=1}^{\widetilde{s}}$ satisfies conditions i),ii),iii) and iv) of Definition 5 for Ω, with the same constants ε, M, N. Hence we can substitute the covering $\{U_i\}_{i=1}^s$ with the covering $\{V_i\}_{i=1}^{\widetilde{s}}$. We will call the sequence $\{V_i\}_{i=1}^{\widetilde{s}}$ a *special covering* of Ω. We remark now a crucial property of this covering.

Definition 8. Let V be an open set in $\mathbb{R}^n$ and $\varepsilon > 0$. We say that V has the ε-ball property if for every $x \in V$ exists an open ball B of radius ε contained in V such that $x \in B$.

Let Ω be an open set in $\mathbb{R}^n$ with minimally smooth boundary with parameters ε, M, N and let $\{U_i\}_{i=1}^s$ be a special covering for Ω. Then it's immediate to verify that U_i has the ε-ball property for every $i = 1, ..., s$.

Theorem 8. Let $1 \leq p < \infty, n \geq 2$ and Ω be an open set in $\mathbb{R}^n$ with minimally smooth boundary. Let $\{U_i\}_{i=1}^s$ be a special covering for Ω. Moreover let E be the operator defined in (13) using the sequence $\{U_i\}_{i=1}^s$. Then if Ω is bounded, for every $f \in W^{l,p}(\Omega)$ and every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \leq l$

$$\|D_w^\alpha Ef\|_{M_p^\phi(\mathbb{R}^n)} \leq C \sum_{|\beta|\leq|\alpha|} \|D_w^\beta f\|_{M_p^\phi(\Omega)} \tag{31}$$

where C doesn't depend on f. If instead Ω is unbounded, for every $f \in W^{l,p}(\Omega)$ and $\delta > 0$

$$\|D_w^\alpha Ef\|_{M_p^{\phi,\delta}(\mathbb{R}^n)} \leq C_\delta \sum_{|\beta|\leq|\alpha|} \|D_w^\beta f\|_{M_p^\phi(\Omega)} \tag{32}$$

where C_δ depends on δ but not on f.

Proof. Let ε, N, M be the parameters relative to the covering $\{U_i\}_{i=1}^s$ for Ω. Let B an open ball of radius δ in $\mathbb{R}^n$ and consider the set $J = \{i \in \{1, ..., s\} \mid B \cap U_i \neq \emptyset\}$. We will prove that $\#J \leq c$, where c is a constant that depends only on $\varepsilon, N, \delta, n$. We consider first the case when Ω is bounded. Then also its ε-neighborhood Ω^ε is bounded. Moreover, by definition $U_i \cap \Omega^\varepsilon$ contains a ball of radius ε, hence $|U_i \cap \Omega^\varepsilon| > \varepsilon^2\omega_n$, where ω_n is the volume of the n-dimensional unit ball. Since the covering $\{U_i\}_{i=1}^s$ has multiplicity less than N and $U_i \subset \Omega^\varepsilon$, we have that $\sum_{i=1}^s |U_i \cap \Omega^\varepsilon| \leq N|\Omega^\varepsilon|$. This implies that $s \leq N|\Omega^\varepsilon|/(\varepsilon^2\omega_n)$ and so $\#J \leq N|\Omega^\varepsilon|/(\varepsilon^2\omega_n) = c$. We observe that in this case c doesn't depend on δ. Suppose now that Ω is unbounded. Since the diameter of B is 2δ, by Lemma 4 there exists a family of m balls of radius ε that covers B, where m depends only on δ, ε and n. Suppose now that $\#J > mp$, for some integer $p \in \mathbb{N}$, then at least one of these balls intersects at least $p+1$ U_i's. Let's call this ball B_ε. We know that there exists points x_i, $i = 1, ..., p+1$, with $x_i \in B_\varepsilon \cap U_i$. Since each U_i has the ε-ball property, there are B_i, $i = 1, ..., p+1$, open balls of radius ε with $B_i \subset U_i$ and $x_i \in B_i$. We now label c_i the center of the ball B_i and we notice that the set $\{c_1, ..., c_{p+1}\}$ is contained in a ball of radius 2ε. Indeed $|x_i - c_i| \leq \varepsilon$ and $x_i \in B_\varepsilon$, for every i. Therefore by Lemma 4 we can cover the set $\{c_1, ..., c_{p+1}\}$ with q open balls of radius $\varepsilon/2$, where q depends only on n. Now suppose that $p > qN$, then at least one of these balls, that we label $B_{\varepsilon/2}$, contains at least $N+1$ points of $\{c_1, ..., c_{p+1}\}$. Without loss of generality we can suppose that they are $c_1, ..., c_{N+1}$, but then we must have that $B_1 \cap B_2 \cap ... \cap B_{N+1} \neq \emptyset$. Indeed each of these balls contains the center of $B_{\varepsilon/2}$. However, since $B_i \subset U_i$ this is in contrast with property ii) of Definition 5. Hence we proved that if $\#J \geq mp$ then $p \leq qN$, hence $\#J < m(Np+1)$. This is what we wanted to prove. Now that we proved this estimate we can proceed with the proof of the theorem in the case $|\alpha| = 0$. Let $f \in W^{l,p}(\Omega)$, by applying the definition of Ef we get

$$\left(\frac{1}{\phi(r)}\int_B |Ef(x)|^p dx\right)^{\frac{1}{p}}$$
$$\leq \left(\frac{1}{\phi(r)}\int_B \left|\Lambda_+(x)\frac{\sum_{i=1}^s \lambda_i(x)S_i(f\lambda_i)(x)}{\sum_{i=1}^s \lambda_i^2(x)}\right|^p dx\right)^{\frac{1}{p}} + \left(\frac{1}{\phi(r)}\int_B |\Lambda_-(x)f(x)|^p dx\right)^{\frac{1}{p}}.$$

The second integral can be bound as follows

$$\left(\frac{1}{\phi(r)}\int_B |\Lambda_-(x)f(x)|^p dx\right)^{\frac{1}{p}} \leq \left(\frac{1}{\phi(r)}\int_{B\cap\Omega} |f(x)|^p dx\right)^{\frac{1}{p}}$$
$$\leq \sum_{j=1}^m \left(\frac{1}{\phi(r)}\int_{B_j\cap\Omega} |f(x)|^p dx\right)^{\frac{1}{p}} \leq m\|f\|_{M_p^\phi(\Omega)} \tag{33}$$

where $B_1, ..., B_m$ is a collection of balls of radius δ centered in Ω with m depending only on n. To bound the first integral we will use that $\sum_{i=1}^s \lambda_i^2(x) \geq 1$ whenever $x \in \operatorname{supp}\Lambda_+$ and that $\operatorname{supp}\lambda_i \subset U_i$. Moreover we recall that exist rigid rotations R_i and special Lipschitz domains D_i such that $U_i \cap \Omega = U_i \cap R_i(D_i)$. We have

$$\left(\frac{1}{\phi(r)}\int_B \left|\Lambda_+(x)\frac{\sum_{i=1}^s \lambda_i(x)S_i(f\lambda_i)(x)}{\sum_{i=1}^s \lambda_i^2(x)}\right|^p dx\right)^{\frac{1}{p}} \leq \left(\frac{1}{\phi(r)}\int_B |\sum_{i=1}^s \lambda_i(x)S_i(f\lambda_i)(x)|^p dx\right)^{\frac{1}{p}}$$
$$\leq \sum_{i\in J}\left(\frac{1}{\phi(r)}\int_B |S_i(f\lambda_i)(x)|^p dx\right)^{\frac{1}{p}} \leq \sum_{i\in J}\|S_i(f\lambda_i)\|_{M_p^\phi(\mathbb{R}^n)}$$
$$\leq C_n(M)\sum_{i\in J}\|f\lambda_i\|_{M_p^\phi(R_i(D_i))} \leq C_n(M)\sum_{i\in J}\|f\|_{M_p^\phi(R_i(D_i)\cap U_i)} =$$
$$= C_n(M)\sum_{i\in J}\|f\|_{M_p^\phi(\Omega\cap U_i)} \leq C_n(M)c\|f\|_{M_p^\phi(\Omega)}.$$

Here we have used inequality (28) for S_i and $C_n(M)$ is a constant depending only on n and M. This combined with (33) proves (32) when $|\alpha| = 0$. We prove now (32) when $|\alpha| > 0$. Let's first define the functions

$$\mu_i = \frac{\Lambda_+\lambda_i}{\sum_{j=1}^s \lambda_j^2}$$

for every $i = 1, ..., s$. Then we can rewrite Ef as

$$Ef(x) = \sum_{i=1}^{s} \mu_i(x) S_i(f\lambda_i)(x) + \Lambda_-(x) f(x).$$

We recall that every λ_i has all bounded derivatives with a bound independent of i and that $\sum_{j=1}^{s} \lambda_j^2(x) \geq 1$ when $x \in \operatorname{supp} \Lambda_+$. Moreover for every $x \in \mathbb{R}^n$ the sum $\sum_{i=1}^{s} \lambda_i(x)$ has at most N terms different from 0. Using these facts and the Leibeniz rule it can be proved that also every μ_i has all bounded derivatives with a bound independent of i. Let's consider again an open ball B in $\mathbb{R}^n$ of radius δ and the set $J = \{i \in \{1, ..., s\} \mid B \cap U_i \neq \emptyset\}$. For every $x \in B$ we have

$$Ef(x) = \sum_{i \in J} \mu_i(x) S_i(f\lambda_i)(x) + \Lambda_-(x) f(x).$$

and since the set J is finite we deduce

$$D_w^\alpha Ef(x) = \sum_{i \in J} D_w^\alpha(\mu_i(x) S_i(f\lambda_i)(x)) + D_w^\alpha(\Lambda_-(x) f(x)).$$

Now using the Leibeniz rule we get

$$|D_w^\alpha Ef(x)| \leq C_\alpha \sum_{i \in J} \sum_{\beta \leq \alpha} |D_w^\beta S_i(f\lambda_i)(x)| + C_\alpha \sum_{\beta \leq \alpha} |D_w^\beta f(x)| \mathbb{1}_\Omega(x)$$

where C_α is a constant depending only on α, n and on the bound of the derivatives of μ_i from order 0 up to order $|\alpha|$, but independent of i. Hence

$$\left(\frac{1}{\phi(r)} \int_B |D_w^\alpha Ef(x)|^p dx \right)^{\frac{1}{p}}$$
$$\leq C_\alpha \sum_{i \in J} \sum_{\beta \leq \alpha} \left(\frac{1}{\phi(r)} \int_B |D_w^\beta S_i(f\lambda_i)(x)|^p dx \right)^{\frac{1}{p}} + C_\alpha \sum_{\beta \leq \alpha} \left(\frac{1}{\phi(r)} \int_{B \cap \Omega} |D_w^\beta f(x)|^p dx \right)^{\frac{1}{p}}.$$

Arguing as before we can estimate the second integral as follows

$$C_\alpha \sum_{\beta \leq \alpha} \left(\frac{1}{\phi(r)} \int_{B \cap \Omega} |D_w^\beta f(x)|^p dx \right)^{\frac{1}{p}} \leq C_\alpha m \sum_{\beta \leq \alpha} \|D^\beta w f\|_{M_p^\phi(\Omega)}. \tag{34}$$

We can estimate the first integral using inequality (28) for S_i. In particular we get

$$
\begin{aligned}
& C_\alpha \sum_{i\in J} \sum_{\beta\leq\alpha} \left(\frac{1}{\phi(r)} \int_B |D_w^\beta S_i(f\lambda_i)(x)|^p dx \right)^{\frac{1}{p}} \\
& \leq C_{l,n}(M) C_\alpha \sum_{i\in J} \sum_{\beta\leq\alpha} \sum_{|\gamma|\leq|\beta|} \|D_w^\gamma(\lambda_i f)\|_{M_p^\phi(R_i(D_i))} \\
& \leq C_\alpha C_{l,n}(M) D \sum_{i\in J} \sum_{\beta\leq\alpha} \sum_{|\gamma|\leq|\beta|} \|D_w^\gamma f\|_{M_p^\phi(R_i(D_i)\cap U_i)} = \\
& = C_{l,n}(M) C_\alpha D \sum_{i\in J} \sum_{\beta\leq\alpha} \sum_{|\gamma|\leq|\beta|} \|D_w^\gamma f\|_{M_p^\phi(\Omega\cap U_i)} \\
& \leq C_{l,n}(M) C_\alpha m \widetilde{D} \sum_{i\in J} \sum_{\beta\leq\alpha} \|D_w^\beta f\|_{M_p^\phi(\Omega)} \\
& \leq C_{l,n}(M) C_\alpha m \widetilde{D} c \sum_{\beta\leq\alpha} \|D_w^\beta f\|_{M_p^\phi(\Omega)}, \qquad (35)
\end{aligned}
$$

where $D, \widetilde{D}$ are constants depending only on n and the bound on the derivatives of λ_i. Inequality (35) together with (34) gives (32) for $|\alpha| > 0$. We finally observe that in the proof of (32) the only constant depending on δ is c, but we know that if Ω is bounded, c doesn't actually depend on δ. This proves (31).

□

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