1 Hestenes Operator

1.1 Construction

We construct the Hestenes operator for domains $\Omega \subset \mathbb{R}^n$ with C^m boundary mainly following paragraphs 6.2,6.3 of [2]. First we consider a simple case where Ω is a C^m half strip.

Lemma 1. Let $l, n, m \in \mathbb{N}, m \geq l, 1 \leq p \leq \infty$ and $W = \prod_{i=1}^{n-1}]a_i, b_i[$ be an open cuboid of $\mathbb{R}^{n-1}$. Moreover define

$$S = W \times \mathbb{R}$$

$$\Omega = \{(\overline{x}, x_n) | \overline{x} \in W, x_n < \phi(\overline{x})\}$$

where $\phi \in C^m(\overline{W})$,$m \geq l$, and $\|D^\alpha \phi\| \leq M < \infty$ for every $1 \leq |\alpha| \leq l$. Then there exists a bounded extension operator T from $W^{l,p}(\Omega)$ to $W^{l,p}(S)$.

To prove Lemma 1 we prove first the case $\phi \equiv 0$ in the following result, that is a generalization of Lemma 9.2 in [1].

Lemma 2. Let $l, n \in \mathbb{N}, 1 \leq p \leq \infty$ and $W = \prod_{i=1}^{n-1}]a_i, b_i[$ be an open cuboid of $\mathbb{R}^{n-1}$. There exists a bounded extension operator

$$T : W^{l,p}(S^-) \to W^{l,p}(S)$$

where

$$S = W \times \mathbb{R}$$

$$S^- = W \times \mathbb{R}^-.$$

Proof. Let $f \in W^{l,p}(S^-)$. We define

$$Tf(\overline{x}, x_n) = \begin{cases} f(x), & \text{if } x_n < 0, \\ \sum_{k=1}^{l} \alpha_k f(\overline{x}, -\beta_k x_n), & \text{if } x_n > 0, \end{cases}$$

where α_k, β_k are real numbers that satisfy $\beta_k > 0$ and

$$\sum_{k=1}^{l} \alpha_k (-\beta_k)^s = 1 \tag{1}$$

for every $s = 0, ..., l-1$. Notice that given $\beta_1, ..., \beta_l > 0$ pairwise distinct, we can always find $\alpha_1, ..., \alpha_l$ that satisfy the condition by solving a Vandermonde square system of linear equations. First we prove that $Tf \in W^{l,p}(S)$. We take any $\phi \in C_c^\infty(S)$ and consider the integral

$$\int_S Tf(x)D^\alpha\phi(x)dx = \int_{S^+} Tf(x)D^\alpha\phi(x)dx + \int_{S^-} Tf(x)D^\alpha\phi(x)dx$$

where $S^+ = \{(\overline{x}, x_n) \mid \overline{x} \in W, x_n > 0\}$ and $\alpha \in \mathbb{N}_0^n, 1 \le |\alpha| \le l$. Let's write $\alpha = (\overline{\alpha}, \alpha_n)$, with $\overline{\alpha} \in \mathbb{N}_0^{n-1}$ and $\alpha_n \in \mathbb{N}_0$. By changing variables in the integrals we get

$$\begin{aligned}\int_S Tf(x)D^\alpha\phi(x)dx &= \int_{S^+} \sum_{k=1}^{l} \alpha_k f(\overline{x}, -\beta_k x_n)D^\alpha\phi(x)dx + \int_{S^-} f(x)D^\alpha\phi(x)dx \\ &= \int_{S^-} f(\overline{y}, y_n)D^\alpha\psi(\overline{y}, y_n)dy \qquad (*)\end{aligned}$$

where $\psi(\overline{x}, x_n) = \sum_{k=1}^{l} -\alpha_k(-\beta_k)^{\alpha_n - 1}\phi\left(\overline{x}, -x_n/\beta_k\right) + \phi(\overline{x}, x_n)$.Note that ψ belongs to $\in C^\infty(S^-)$ but does not have compact support in S^-. To bypass this problem we use an auxiliary function $\nu \in C^\infty(\mathbb{R})$ that satisfies

$$\begin{cases}\nu(x) = 0, & \text{if } x > -1/2,\\ \nu(x) = 1, & \text{if } x < -1,\end{cases}$$

and we define the functions $\nu_k(t) = \nu(kt)$ for $k \in \mathbb{N}$. It's clear that $\psi(x)\nu_k(x_n) \in C_c^\infty(S^-)$, hence we can integrate by parts

$$\int_{S^-} f(x)D^\alpha(\psi(x)\nu_k(x_n))dx = (-1)^{|\alpha|}\int_{S^-} D_w^\alpha f(x)\psi(x)\nu_k(x_n)dx \qquad (2)$$

By the Leibniz rule

$$\begin{aligned}D^\alpha(\psi(x)\nu_k(x_n)) &= \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}D^{\overline{\alpha}}(\psi(x)\nu_k(x_n)) \\ &= \nu(kx_n)D^\alpha\psi(x) + \sum_{i=1}^{\alpha_n}\binom{\alpha_n}{i}k^i\nu^{(i)}(kx_n)\frac{\partial^{\alpha_n - i}}{\partial x_n^{\alpha_n - i}}D^{\overline{\alpha}}\psi(x).\end{aligned}$$

By the Dominated Convergence Theorem

$$\int_{S^-} f(x)\nu(kx_n)D^\alpha\psi(x)dx \to \int_{S^-} f(x)D^\alpha\psi(x)dx \text{ as } k \to \infty,$$

because $f \in L^1(S^- \cap \operatorname{supp}\psi)$ since $\operatorname{supp}\psi$ is bounded. Next, we claim that for every $i = 1, .., \alpha_n$

$$\int_{S^-} f(x) k^i \nu^{(i)}(kx_n) \frac{\partial^{\alpha_n - i}}{\partial x_n^{\alpha_n - i}} D^{\overline{\alpha}} \psi(x) dx \to 0 \tag{3}$$

as $k \to \infty$. To prove this first we notice that since α_k, β_k satisfies (1) we have that

$$\frac{\partial^j}{\partial x_n^j} D^{\overline{\alpha}} \psi(\overline{x}, 0) = 0 \ ; \ j = 0, ..., \alpha_n - 1,$$

hence by Taylor formula

$$\left| \frac{\partial^{\alpha_n - i}}{\partial x_n^{\alpha_n - i}} D^{\overline{\alpha}} \psi(\overline{x}, x_n) \right| \leq \frac{C|x_n|^i}{i!},$$

for all $i = 1, ..., \alpha_n$, where $C = \sup_{x \in S^-} |D^\alpha \psi(x)|$. Therefore we get the following estimate

$$\begin{aligned}
\int_{S^-} \left| f(x) k^i \nu^{(i)}(kx_n) \frac{\partial^{\alpha_n - i}}{\partial x_n^{\alpha_n - i}} D^{\overline{\alpha}} \psi(x) \right| dx &\leq \frac{\widetilde{C}C}{i!} \int_{\{x \in S^- \cap \operatorname{supp} f \ , \ -1/k < x_n < 0\}} |f(x)| k^i |x_n|^i dx \\
&\leq \frac{\widetilde{C}C}{i!} \int_{\{x \in S^- \cap \operatorname{supp} f \ , \ -1 < x_n < 0\}} |f(x)| dx
\end{aligned}$$

where $\widetilde{C} = \sup_{\mathbb{R}} |\nu^{(i)}|$. The second inequality comes from the fact that $\nu^{(i)}(x) = 0$ for $x < -1$ and $i \geq 1$. Hence we get (3) by Dominated Convergence Theorem. Passing to the limit in (2) we obtain

$$\int_{S^-} f(x) D^\alpha \psi(x) dx = (-1)^{|\alpha|} \int_{S^-} D_w^\alpha f(x) \psi(x) dx.$$

which, combined with (*), implies

$$\int_S Tf(x) D^\alpha \phi(x) dx = \int_{S^-} f(x) D^\alpha \psi(x) dx = (-1)^{|\alpha|} \int_{S^-} D_w^\alpha f(x) \psi(x) dx.$$

Finally going back to the original coordinates and using the definition of ψ we get

$$\int_S Tf(x) D^\alpha \phi(x) dx = (-1)^{|\alpha|} \int_{S^-} D_w^\alpha f(x) \left[\sum_{k=1}^{l} -\alpha_k (-\beta_k)^{\alpha_n - 1} \phi\left(\overline{x}, -\frac{x_n}{\beta_k}\right) + \phi(\overline{x}, x_n) \right] dx =$$

$$= (-1)^{|\alpha|} \int_{S^+} \sum_{k=1}^{l} \alpha_k (-\beta_k)^{\alpha_n} D_w^\alpha f(\overline{y}, -\beta_k y_n) \phi(y) dy + (-1)^{|\alpha|} \int_{S^-} D_w^\alpha f(y) \phi(y) dy$$

that implies that $D^\alpha_w Tf$ exists and

$$D^\alpha_w Tf(x) = \begin{cases} D^\alpha_w f(x), & \text{if } x \in S^-, \\ \sum_{k=1}^{l} \alpha_k(-\beta_k)^{\alpha_n} D^\alpha_w f(\overline{x}, -\beta_k x_n)\phi(x), & \text{if } x \in S^+. \end{cases}$$

It remains to prove the boundedness of T. It's immediate to verify that

$$\|Tf\|_{L^p(S^+)} \le \sum_{i=1}^{l} |\alpha_k|\beta_k^{-1/p}\|f\|_{L^p(S^-)}$$

and that we have similar bounds for the norm of the weak derivatives of Tf. Hence there exists a constant C depending only on β_k, α_k, l such that $\|Tf\|_{W^{l,p}(S^+)} \le C\|f\|_{W^{l,p}(S^-)}$. Observing that $\|Tf\|^p_{W^{l,p}(S)} = \|Tf\|^p_{W^{l,p}(S^+)} + \|f\|^p_{W^{l,p}(S^-)}$ the proof is concluded. □

Lemma 3. Let $l \in \mathbb{N}$ and Ω be a domain in $\mathbb{R}^n$. Suppose that $f \in L^1_{loc}(\Omega)$ admits all the weak derivatives up to order l and that $g : \Omega' \to \Omega$ is a diffeomorphism of class C^l with bounded derivatives $|D^\alpha g_k| \le M$ for all $1 \le |\alpha| \le l$. Then $f \circ g$ admits weak derivative up to order l. Moreover for every $1 \le |\alpha| \le l$ we have to following bounds

$$|D^\alpha(f \circ g)(x)| \le C \sum_{1 \le |\beta| \le |\alpha|} |D^\beta f(g(x))|$$

where C depends only on M and l.

Proof. We prove the statement by induction on l. For $l = 1$ we know that exists a sequence of functions $\{f_k\}_k \in C^\infty(\Omega)$ such that

$$f_k \to f \qquad \text{in } L^1_{loc}(\Omega)$$

$$\frac{\partial f_k}{\partial x_i} \to \frac{\partial f}{\partial x_i} \qquad \text{in } L^1_{loc}(\Omega).$$

Take $\phi \in C^\infty_c(\Omega')$ and integrate by parts

$$\int_{\Omega'} f_k(g(x))\frac{\partial \phi}{\partial x_i}(x)dx = -\int_{\Omega'} \left(\sum_{j=1}^{n} \frac{\partial f_k}{\partial x_j}(g(x))\frac{\partial g_j}{\partial x_i}(x)\right)\phi(x)dx.$$

Since $\phi(g^{-1}) \in C_c^l(\Omega)$ and the derivatives of g and g^{-1} are bounded, we can pass to the limit in the above equation

$$\int_{\Omega'} f(g(x)) \frac{\partial \phi}{\partial x_i}(x) dx = -\int_{\Omega'} \left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(g(x)) \frac{\partial g_j}{\partial x_i}(x) \right) \phi(x) dx.$$

Hence the case $l = 1$ is proved. Now suppose that the statement is true for l. We prove the case $l+1$, so we suppose that f admits weak derivatives up to order $l+1$ and that g is of class C^{l+1}. From the case $l = 1$ we know that $\frac{\partial (f \circ g)}{\partial x_i}$ exists and that

$$\frac{\partial (f \circ g)}{\partial x_i} = \sum_{j=1}^{n} (\frac{\partial f}{\partial x_j} \circ g) \frac{\partial g_j}{\partial x_i}$$

Since $\frac{\partial f}{\partial x_j}$ admits weak derivatives up to order l, by induction hypothesis the functions $\frac{\partial f}{\partial x_j} \circ g$ admit weak derivatives up to order l. Moreover $\frac{\partial g_j}{\partial x_i}$ is of class C^l, thus by the Leibniz rule the functions $(\frac{\partial f}{\partial x_j} \circ g)\frac{\partial g_j}{\partial x_i}$ admits weak derivatives of order l. In conclusion $\frac{\partial (f \circ g)}{\partial x_i}$ admits derivatives up to order l and this conclude the proof of the case $l+1$.

To prove the bounds we notice that the weak derivatives $D^\alpha(f \circ g)$ can be computed using the chain rule for usual derivatives. Such formula can be found in [3, formula B]:

$$D_w^\alpha(f(g))(x) = \sum_{1 \le |\beta| \le |\alpha|} D_w^\beta(f(g(x)) Q_{\alpha,\beta}(g, x)$$

In this formula $Q_{\alpha,\beta}(g,x)$ are homogeneous polynomials of degree $|\beta| \le l$ in the derivatives of order less than l of the components of g. Moreover the coefficients of these polynomials depend only on α, l, n. Hence there exists a constant C depending only on l, n, M such that $|Q_{\alpha,\beta}(g,x)| \le C$ uniformly on x. This concludes the proof. □

Proof of Lemma 1 . Let $f \in W^{l,p}(\Omega)$. Consider the function g from S^- onto Ω defined by

$$g(\overline{x}, x_n) = (\overline{x}, x_n + \phi(\overline{x}))$$

for all $(\overline{x}, x_n) \in S^-$ and its inverse g^{-1}

$$g^{-1}(\overline{x}, x_n) = (\overline{x}, x_n - \phi(\overline{x}))$$

where $S^- = W \times \mathbb{R}^-$. For all $f \in W^{l,p}(\Omega)$ we set

$$Gf = f \circ g$$

Since g is a diffeomorphism between S^- and Ω of class C^m, Lemma 3 guarantees that Gf admits weak derivatives up to order l. We claim that G defines a bounded operator from $W^{l,p}(\Omega)$ to $W^{l,p}(S^-)$, with bounded inverse. To prove this, first we compute the Jacobian matrix of g^{-1}

$$Jg^{-1}(x) = \begin{bmatrix} 1 & 0 & \dots & & 0 \\ 0 & 1 & \dots & & 0 \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \\ -\frac{\partial \phi(\overline{x})}{\partial x_1} & -\frac{\partial \phi(\overline{x})}{\partial x_2} & \dots & \dots & 1 \end{bmatrix}$$

from which $|\det(Jg^{-1}(x))| \equiv 1$. Moreover, again by Lemma 3, we have

$$|D_w^\alpha(f(g))| \leq C(l, M) \sum_{1 \leq |\beta| \leq |\alpha|} |D_w^\beta f(g)|$$

where $C(l, M)$ depends only on l and M, with $M = \sup_{1 \leq |\alpha| \leq l} \|D^\alpha \phi\|$. Next by the change of variable formula and Minkowski's inequality we get

$$\begin{aligned} \left(\int_{S^-} |D_w^\alpha(f(g))(x)|^p dx \right)^{\frac{1}{p}} &\leq \sum_{1 \leq |\beta| \leq |\alpha|} C(l, M) \left(\int_{S^-} |D_w^\beta f(g(x))|^p dx \right)^{\frac{1}{p}} \\ &= \sum_{1 \leq |\beta| \leq |\alpha|} C(l, M) \left(\int_{\Omega} |D_w^\beta f(y)|^p |\det Jg^{-1}\big|_{g(y)}| dy \right)^{\frac{1}{p}} \\ &= \sum_{1 \leq |\beta| \leq |\alpha|} C(l, M) \|D_w^\beta f\|_{L^p(\Omega)} \end{aligned}$$

Thus, using the estimates for the intermediate derivatives, that

$$\|Gf\|_{W^{l,p}(S^-)} = \|f(g)\|_{W^{l,p}(S^-)} \leq C\|f\|_{W^{l,p}(\Omega)}$$

for a constant C independent of f. In a similar way we can also prove that

$$\|G^{-1}f\|_{W^{l,p}(\Omega)} = \|f(g^{-1})\|_{W^{l,p}(\Omega)} \leq D\|f\|_{W^{l,p}(S)}.$$

Now we can just define the operator T as

$$T = G^{-1} \circ \overline{T} \circ G$$

where $\overline{T}$ is the extension operator from $W^{l,p}(S^-)$ to $W^{l,p}(S)$ defined in Lemma 2. Therefore T is bounded as composition of bounded operators. An explicit for for T is

$$Tf(x) = \begin{cases} f(x), & \text{if } x \in \Omega, \\ \sum_{i=1}^{l} \alpha_k f(\overline{x}, \phi(\overline{x}) - \beta_k(x_n - \phi(\overline{x}))), & \text{if } x \in S \setminus \overline{\Omega}. \end{cases}$$

□

We are now ready to define the Hestenes operator for a general domain Ω with C^m boundary. First we write the precise definition for this kind of domains.

Definition 1. Let $0 < d \leq D < \infty, M > 0, \varkappa > 0$ We say that an open set Ω in $\mathbb{R}^n$ has a resolved boundary with parameters $d, D, \varkappa$ if there exists a family of open cuboids $V_i, i = 1, ..., s$ (where $s \in \mathbb{N}$ if Ω is bounded and $s = \infty$ otherwise) such that

1. $(V_i)_d \cap \Omega \neq \emptyset$
2. $\Omega \subset \bigcup_{j=1}^{s} (V_i)_d$
3. The multiplicity of the cover $\{V_i\}_{i=1}^{s}$ is less than $\varkappa$.
4. There exist isometries λ_i of $\mathbb{R}^n$ such that

   $$\lambda_j(V_j) = \prod_{i=1}^{n}]a_{ij}, b_{ij}[$$

   and, if $\partial V_j \cap \Omega \neq \emptyset$,

   $$\lambda_j(V_j \cap \Omega) = \{(\overline{x}, x_n) \in \mathbb{R}^n | \overline{x} \in W_j, a_{nj} + d < x_n < \phi_j(\overline{x})\}$$

   where $W_j = \prod_{i=1}^{n-1}]a_{ij}, b_{ij}[$ and $\phi_j : W_j \to \mathbb{R}$.

   Moreover

- if $\phi_j \in C^m(\overline{W}_i)$ with $\|D^\alpha \phi_j\| \leq M < \infty$, for every $1 \leq |\alpha| \leq m$, we say that Ω has a resolved C^m boundary with parameters $d, D, \varkappa, M$.
- if $\phi_j \in \mathrm{Lip}(\overline{W}_i)$ with $\mathrm{Lip}(\phi) = M$, we say that Ω has a resolved Lipschitz boundary with parameters $d, D, \varkappa, M$.

Finally we will say that a domain Ω has a resolved C^m (or Lipschitz) boundary if there exist parameters $d, D, \varkappa, M$ for which Ω has a C^m (or Lipschitz) boundary.

Remark 1. In the notation of Lemma 1, let $a, b \in \mathbb{R}$ such that $a < \phi(\overline{x}) < b$ for every $\overline{x} \in W$. We define $S^{a,b} = W \times (a, b)$, $\Omega_a = \Omega \cap (W \times (a, \infty))$ and $\widehat{W}^{l,p}(\Omega_a) = \{f \in W^{l,p}(\Omega_a) |\, \mathrm{supp}\, f \subset S\}$. Then exists a bounded extension operator

$$T : \widehat{W}^{l,p}(\Omega_a) \to W^{l,p}(S^{a,b}).$$

To see this we can just extend $f \in \widehat{W}^{l,p}(\Omega_a)$ naturally by 0 to $f_0 \in W^{l,p}(\Omega)$ and then define

$$Tf = (\widetilde{T} f_0)\big|_{S^{a,b}}$$

where $\widetilde{T}$ is the operator of the previous Lemmma .

Theorem 1. Let $m, l \in \mathbb{N}, l \leq m$ and $1 \leq p \leq \infty$. If Ω is a domain in $\mathbb{R}^n$ has a C^m resolved boundary then there exists a bounded extension operator

$$T : W^{l,p}(\Omega) \to W^{l,p}(\mathbb{R}^n).$$

Proof Sketch. Let $f \in W^{l,p}(\Omega)$. Let $\{V_i\}_{i=1}^s$ be the covering of cuboids for Ω as in Definition 1. It's possible to construct functions $\{\psi_i\}_{i=1}^s \subset C_c^\infty(\mathbb{R}^n)$ such that the functions $\{\psi_i^2\}_{i=1}^s$ form a partition of the unity corresponding to the covering $\{V_i\}_{i=1}^s$ and satisfying $\|D^\alpha \psi_i\|_{L^\infty} \leq M_1$ with M_1 depending only on n, l, d. If $\partial\Omega \cap V_i \neq \emptyset$ by Remark 1 there exists a bounded operator

$$T_i : \widehat{W}^{l,p}(\lambda_i(\Omega \cap V_i)) \to W^{l,p}(\lambda_i(V_i))$$

where $\widehat{W}^{l,p}(\lambda_i(V_i \cap \Omega) = \{f \in W^{l,p}(V_i \cap \Omega) |\, \mathrm{supp}\, f \subset \lambda_i(V_i)\}$. If $V_i \subset \Omega$ the operator T_i is defined to be just the identity. We set

$$Tf = \sum_{i=1}^{s} \psi_i T_i(\psi_i f(\lambda_i^{-1}))(\lambda_i).$$

assuming $(\psi_i T_i(\psi_i f(\lambda_i^{-1}))(\lambda_i)) = 0$ outside V_i. The functions $\psi_i f \in W^{l,p}(V_i \cap \Omega)$ are such that $\operatorname{supp} \psi_i f \subset \overline{\Omega} \cap V_i$, hence $\psi_i f(\lambda_i) \in \widehat{W}^{l,p}(\lambda_i(V_i \cap \Omega))$ and so T is well defined. To see that T is an extension operator, take $x \in \Omega$: if $x \in \operatorname{supp} \psi_i$ then $\psi_i(x) T_i(\psi_i f(\lambda_i^{-1}))(\lambda_i(x)) = \psi_i(x)^2 f(x)$; if $x \notin \operatorname{supp} \psi_i$ then $0 = \psi_i(x) T_i(\psi_i f(\lambda_i^{-1}))(\lambda_i(x)) = \psi_i(x)^2 f(x)$. So $Tf(x) = \sum_{i=1}^{s} \psi_i^2(x) f(x) = f(x)$.

We omit the proof of the boundedness of T, the details of which can be found in the proofs of Lemma 13-14 in [2]. □

1.2 Hestenes operator on Morrey spaces

Definition 2. Let $1 \leq p < \infty$, ϕ a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ and Ω be a domain in $\mathbb{R}^n$. For a function $f \in L^p_{loc}(\Omega)$ we define the Morrey space as

$$M_p^\phi(\Omega) = \{f \in L^p_{loc}(\Omega) \mid \|f\|_{M_p^\phi(\Omega)} < \infty\}$$

where

$$\|f\|_{M_p^\phi(\Omega)} := \sup_{B_r(x), x \in \Omega, r > 0} \left(\frac{1}{\phi(r)} \int_{B_r(x) \cap \Omega} |f(y)|^p dy \right)^{\frac{1}{p}}.$$

Lemma 4. Let $k \geq 1$ and Ω be set in $\mathbb{R}^n$ with diameter $D > 0$. Then there exists an integer $C_{n,k}$ depending only on k and n such that Ω can be covered by a collection of open balls $B_1, ..., B_h$ centered in Ω with radius D/k and $h \leq C_{k,n}$.

Proof. We start by claiming that if S is a set of points in $\mathbb{R}^n$ satisfying

i) $S \subset \Omega$,

ii) $\|z_1 - z_2\| \geq D/k$ for every $z_1, z_2 \in \Omega$ with $z_1 \neq z_2$,

then $\#S \leq C_{n,k}$ where $C_{k,n}$ is an integer depending only on k and n. To see this, first note that Ω is contained in some closed cube Q of side $2D$. Then we choose $m \in \mathbb{N}$ such that $2^{m-1} > \sqrt{n}k$. Next we cover Q with $(2^m)^n$ smaller closed cubes of side $2D/2^m$. The diagonal of a smaller cube measures $2D/2^m \cdot \sqrt{n} < D/k$. Thus each of these cubes can contain at most one point of S ,so $\#S \leq (2^m)^n$. Therefore it's enough to choose $C_{n,k} = 2^{mn}$. Set $r := D/k$, we'll prove that we can cover Ω with a collection of balls $B_1, ..., B_h$ centered in Ω of radius r and such that $k \leq C_{n,k}$. Choose $x_1 \in \Omega$ and take $B_1 = B_r(x_1)$, the ball centered in x_1 of radius r. If $\Omega \subset B_1$ we are done, if not there exists $x_2 \in \Omega \setminus B_1$ and we take $B_2 = B_r(x_2)$. Again, if $\Omega \subset (B_1 \cup B_2)$ we stop, otherwise we can pick $x_3 \in \Omega \setminus (B_1 \cup B_2)$ and take $B_3 = B_r(x_3)$. We iterate this procedure : given $B_1, ..., B_i$ balls, if $\Omega \subset (B_1 \cup ... \cup B_i)$ we stop, otherwise we can choose $x_{i+1} \in \Omega \setminus (B_1 \cup ... \cup B_i)$ and take $B_{i+1} = B_r(x_{i+1})$. We claim that this procedure stops with $i \leq C_{n,k}$. Suppose it doesn't, then we can find $B_1, ..., B_{C_{n,k}+1}$ balls centered respectively at $x_1, ..., x_{C_{n,k}+1}$. Setting $S = \{x_1, ..., x_{C_{n,k}+1}\}$, it's immediate to see that S satisfies i) and ii), but $\#S = C_{n,k} + 1$, that is a contradiction. $\square$

Lemma 5. Let $W \subset \mathbb{R}^{n-1}$ be open connected and define

$$\Omega = \{(\overline{x}, x_n) \mid \overline{x} \in W, x_n \leq \psi(\overline{x})\}$$

$$\Omega^+ = \{(\overline{x}, x_n) \mid \overline{x} \in W, x_n > \psi(\overline{x})\}$$

where $\psi \in \mathrm{Lip}(\overline{W})$. Let $\beta > 0$ and consider the function A_β from $W \times \mathbb{R}$ to Ω defined by

$$A_\beta(\overline{x}, x_n) = \begin{cases} (\overline{x}, \psi(\overline{x}) - \beta(x_n - \psi(\overline{x}))), & \text{if } (\overline{x}, x_n) \in \Omega^+, \\ (\overline{x}, x_n), & \text{if } (\overline{x}, x_n) \in \Omega. \end{cases}$$

Then for every $x_0 \in W \times \mathbb{R}$ and $r > 0$

$$A(B_r(x_0) \cap \Omega^+) \subset B_{cr}(A(x_0)) \cap \Omega$$

where $c \geq 1$ is a constant depending only on $\mathrm{Lip}\,\psi$ and β.

Proof. Notice that it is sufficient to prove that for every $x, y \in W \times \mathbb{R}$ we have

$$\|A(x) - A(y)\| \leq c\|x - y\|. \tag{4}$$

Set $M = \mathrm{Lip}\,\psi$. We distinguish three cases: 1. $x, y \in \Omega$: in this case $A(x) = x$ and $A(y) = y$, so $\|x - y\| = \|A(x) - A(y)\|$ and there is nothing to prove.

2. $x, y \in \Omega^+$: we have

$$\begin{aligned} |A(x)_n - A(y)_n| &= |\psi(\overline{x}) - \beta(x_n - \psi(\overline{x})) - \psi(\overline{y}) + \beta(y_n - \psi(\overline{y}))| \\ &\leq (1 + \beta)|\psi(\overline{x}) - \psi(\overline{y})| + \beta|x_n - y_n| \\ &\leq M(1 + \beta)\|\overline{x} - \overline{y}\| + \beta|x_n - y_n| \end{aligned}$$

Hence

$$\begin{aligned}\|A(x)-A(y)\|^2 &= \|\overline{A(x)}-\overline{A(y)}\|^2+|A(x)_n-A(y)_n|^2\\ &\le \|\overline{x}-\overline{y}\|^2+[M(1+\beta)\|\overline{x}-\overline{y}\|+\beta|x_n-y_n|]^2\\ &\le (1+2M^2(1+\beta)^2)\|\overline{x}-\overline{y}\|^2+2\beta^2|x_n-y_n|^2\\ &\le c_1^2(M,\beta)\|x-y\|^2\end{aligned}$$

for some constant $c_1(M,\beta)$.

3. $x\in\Omega^+, y\in\Omega$: first notice that, since $\psi(\overline{x})<x_n$, then $x_n-y_n>\psi(\overline{x})-y_n$. Moreover $\psi(\overline{y})>y_n$, hence $M\|\overline{x}-\overline{y}\|\ge\psi(\overline{y})-\psi(\overline{x})>y_n-\psi(\overline{x})$. This implies

$$|\psi(\overline{x})-y_n|<|x_n-y_n|+M\|\overline{x}-\overline{y}\|.$$

Now

$$\begin{aligned}|A(x)_n-A(y)_n| &= |\psi(\overline{x})-\beta(x_n-\psi(\overline{x}))-y_n|\\ &= |(1+\beta)(\psi(\overline{x})-y_n)+\beta(y_n-x_n)|\\ &\le M(1+\beta)\|\overline{x}-\overline{y}\|+(1+2\beta)|x_n-y_n|\end{aligned}$$

and

$$\begin{aligned}\|A(x)-A(y)\|^2 &= \|\overline{A(x)}-\overline{A(y)}\|^2+|A(x)_n-A(y)_n|^2\\ &\le \|\overline{x}-\overline{y}\|^2+[M(1+\beta)\|\overline{x}-\overline{y}\|+(1+2\beta)|x_n-y_n|]^2\\ &\le (1+2M^2(1+\beta)^2)\|\overline{x}-\overline{y}\|^2+2(1+2\beta)^2|x_n-y_n|^2\\ &\le c_2^2(M,\beta)\|x-y\|^2.\end{aligned}$$

for some constant $c_2(M,\beta)$. Then (4) by taking $c=max(\sqrt{c}_1,\sqrt{c}_2,1)$.

$\square$

Lemma 6. Let $l,n,m\in\mathbb{N}, m\ge l, 1\le p\le\infty$, $W=\prod_{i=1}^{n-1}]a_i,b_i[$ be an open cuboid of $\mathbb{R}^{n-1}$ and ϕ a function from $\mathbb{R}^+$ to $\mathbb{R}^+$. Moreover define

$$S=W\times\mathbb{R}$$

$$\Omega=\{(\overline{x},x_n)|\overline{x}\in W, x_n<\psi(\overline{x})\}$$

where $\psi\in C^m(\overline{W})$ and $\|D^\alpha\psi\|\le M<\infty$ for every $1\le|\alpha|\le l$. Then for every $f\in W^{l,p}(\Omega)$ and $1\le|\alpha|\le l$

$$\|Tf\|_{M_p^\phi(S)}\le C\|f\|_{M_p^\phi(\Omega)}, \tag{5}$$

$$\|D_w^\alpha Tf\|_{M_p^\phi(S)}\le C\sum_{1\le|\beta|\le|\alpha|}\|D_w^\beta f\|_{M_p^\phi(\Omega)}, \tag{6}$$

where T is the Hestenes operator defined in Lemma 1 and C is a constant independent of f.

Proof. Define $\Omega^+ = \{(\overline{x}, x_n) \mid \overline{x} \in W, x_n > \psi(\overline{x})\}$. We recall the definition of T

$$Tf(x) = \begin{cases} f(x) & x \in \Omega \\ \sum_{i=1}^{l} \alpha_k f(\overline{x}, \psi(\overline{x}) - \beta_k(x_n - \psi(\overline{x}))) & x \in \Omega^+ \end{cases}$$

and observe that we can rewrite it as

$$Tf(x) = \begin{cases} f(x), & \text{if } x \in \Omega, \\ \sum_{i=1}^{l} \alpha_k f(G_k(x)), & \text{if } x \in \Omega^+, \end{cases}$$

where $G_k(\overline{x}, x_n) = (\overline{x}, \psi(\overline{x}) - \beta_k(x_n - \psi(\overline{x})))$. Note that $G_k : \Omega^+ \to \Omega$ defines a diffeomorphism from Ω^+ to Ω of class C^m and satisfying $|\det JG_k^{-1}| \equiv 1/\beta_k$. First we prove ii). Let's fix $x_0 \in S$ and a radius $r > 0$. We want to estimate the quantity

$$I = \left(\frac{1}{\psi(r)} \int_{B_r(x_0) \cap S} |D_w^\alpha Tf(x)|^p dx \right)^{\frac{1}{p}}$$

for $1 \leq |\alpha| \leq l$. To do this we estimate the integral as follows

$$I \leq \underbrace{\left(\frac{1}{\phi(r)} \int_{B_r(x_0) \cap \Omega^+} |D_w^\alpha Tf(x)|^p dx \right)^{\frac{1}{p}}}_{I_1} + \underbrace{\left(\frac{1}{\phi(r)} \int_{B_r(x_0) \cap \Omega} |D_w^\alpha Tf(x)|^p dx \right)^{\frac{1}{p}}}_{I_2}.$$

Since $Tf(x) = f(x)$ when $x \in \Omega$, we have immediately

$$I_2 \leq \|D_w^\alpha f\|_{M_p^\phi(\Omega)}.$$

It remains to estimate I_1. We start by observing that from Lemma 3 there exists a constant C_k depending only on G_k and l such that

$$|D_w^\alpha (f \circ G_k)| \leq C_k \sum_{1 \leq |\beta| \leq |\alpha|} |D_w^\beta f(G_k)|.$$

By the previous inequality and Lemma 5 we are able to produce the following bound

$$\frac{\|D_w^\alpha(f\circ G_k)\|_{L^p(B_r(x_0)\cap\Omega^+)}}{\phi(r)^{\frac{1}{p}}} \le C_k \sum_{1\le|\beta|\le|\alpha|}\left(\phi(r)^{-1}\int_{G_k(B_r(x_0)\cap\Omega^+)}|D_w^\beta f(y)|^p|\det JG_k^{-1}|_{G_k(y)}|dy\right)^{\frac{1}{p}}$$
$$\le C_k\beta_k^{-\frac{1}{p}}\sum_{1\le|\beta|\le|\alpha|}\left(\phi(r)^{-1}\int_{B_{c_kr}(A_{\beta_k}(x_0))\cap\Omega}|D_w^\beta f(y)|^p dy\right)^{\frac{1}{p}}$$

where A_{α_k} is defined as in Lemma 5 and c_k depends only on β_k and M. By Lemma 4 the set $B_{c_kr}(A_{\beta_k}(x_0))\cap\Omega$ can be covered with a collection of open balls $B_1,...,B_h$ centered in Ω with radius r and $h\le m_k$, where m_k depends only on c_k. Hence we get

$$\frac{\|D_w^\alpha(f\circ G_k)\|_{L^p(B_r(x_0)\cap\Omega^+)}}{\phi(r)^{\frac{1}{p}}} \le C_k\beta_k^{-\frac{1}{p}}m_k\sum_{1\le|\beta|\le|\alpha|}\|D^\beta f\|_{M_p^\phi(\Omega)}$$

Next we estimate I_1:

$$I_1=\phi(r)^{-\frac{1}{p}}|D_w^\alpha Tf\|_{L^p(B_r(x_0)\cap\Omega^+)}\le\phi(r)^{-\frac{1}{p}}\sum_{k=1}^{l}\alpha_k\|D_w^\alpha f(G_k)\|_{L^p(B_r(x_0)\cap\Omega^+)}$$
$$\le\sum_{k=1}^{l}\alpha_kC_k\beta_k^{-\frac{1}{p}}m_k\left(\sum_{1\le|\beta|\le|\alpha|}\|D_w^\beta f\|_{M_p^\phi(\Omega)}\right).$$

Finally putting the estimates of I_1,I_2 together

$$\|D_w^\alpha Tf\|_{M_p^\phi(S)}=\sup_{x_0\in S,r>0}\left(\frac{1}{\phi(r)}\int_{B_r(x_0)\cap S}|D_w^\alpha Tf(x)|^pdx\right)^{\frac{1}{p}}$$
$$\le\|D_w^\alpha f\|_{M_p^\phi(\Omega)}+\sum_{k=1}^{l}\alpha_kC_k\beta_k^{-\frac{1}{p}}m_k\left(\sum_{1\le|\beta|\le|\alpha|}\|D_w^\alpha f\|_{M_p^\phi(\Omega)}\right)$$
$$\le\widetilde{C}\sum_{1\le|\beta|\le|\alpha|}\|D_w^\alpha f\|_{M_p^\phi(\Omega)}$$

where $\widetilde{C}$ depends only on $\{b_k\}_k,\{\alpha_k\}_k,l,M,p$. This proves ii). The proof of i) is exactly analogous to the proof of ii). $\square$

Definition 3. Let $1 \leq p < \infty$, ϕ a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ and Ω be a domain in $\mathbb{R}^n$. For every $\delta > 0$ and every function $f \in L^p_{loc}(\Omega)$ we define the norm $\|f\|_{M_p^{\delta,\phi}}$ as

$$\|f\|_{M_p^{\delta,\phi}} := \sup_{B_r(x), x\in\Omega, 0<r<\delta} \left(\frac{1}{\phi(r)} \int_{B_r(x)\cap\Omega} |f(y)|^p dy \right)^{\frac{1}{p}}.$$

Theorem 2. Let $m, l \in \mathbb{N}, l \leq m$, $1 \leq p \leq \infty$, ϕ a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ and Ω a domain in $\mathbb{R}^n$ with C^m resolved boundary. Let also T be the Hestenes operator defined in Theorem 1. Then if Ω is bounded, for every $f \in W^{l,p}(\Omega)$ and $1 \leq |\alpha| \leq l$ we have

$$\|Tf\|_{M_p^{\phi}(\mathbb{R}^n)} \leq C\|f\|_{M_p^{\phi}(\Omega)}, \tag{7}$$

$$\|D_w^\alpha Tf\|_{M_p^{\phi}(\mathbb{R}^n)} \leq C \sum_{1\leq|\beta|\leq|\alpha|} \|D_w^\beta f\|_{M_p^{\phi}(\Omega)}, \tag{8}$$

where C doesn't depend on f. If instead Ω is unbounded, for every $f \in W^{l,p}(\Omega)$ and $\delta > 0$ we have

$$\|Tf\|_{M_p^{\phi,\delta}(\mathbb{R}^n)} \leq C_\delta\|f\|_{M_p^{\phi}(\Omega)}, \tag{9}$$

$$\|D_w^\alpha Tf\|_{M_p^{\phi,\delta}(\mathbb{R}^n)} \leq C_\delta \sum_{1\leq|\beta|\leq|\alpha|} \|D_w^\beta f\|_{M_p^{\phi}(\Omega)}, \tag{10}$$

where C_δ depends on δ but not on f.

Proof. Let $f \in W^{l,p}(\Omega)$ and $\{V_i\}_{i=1}^s$ be the covering of cuboids for Ω as in the definition of set with resolved boundary. We recall the definition of T :

$$Tf = \sum_{i=1}^{s} \psi_i T_i(\psi_i f(\lambda_i^{-1}))(\lambda_i)$$

where $\{\psi_i^2\}_{i=1}^s$ form a partition of the unity corresponding to the covering $\{V_i\}_{i=1}^s$ and satisfying $\|D^\alpha\psi_i\|_{L^\infty} \leq M_1$, with $|\alpha| \leq l$ and M_1 depending only on n, l, d. To make the notation simpler we will rewrite T as

$$Tf = \sum_{i=1}^{s} \psi_i \widetilde{T}_i(\psi_i f)$$

where the operator $\widetilde{T}_i$ is defined as $\widetilde{T}_i f = T_i(f(\lambda_i^{-1}))(\lambda_i)$. Before starting the proof we remark some facts that will be justified at the end:

a) Let C_i the constant such that

$$\|T_i g\|_{M_p^\phi(\lambda_i(V_i))} \leq C_i \|g\|_{M_p^\phi(\lambda_i(\Omega\cap V_i))},$$

$$\|D_w^\alpha T_i g\|_{M_p^\phi(\lambda_i(V_i)))} \leq C_i \sum_{1\leq|\beta|\leq|\alpha|} \|D_w^\alpha g\|_{M_p^\phi(\lambda_i(\Omega\cap V_i)))},$$

for $1 \leq |\alpha| \leq l$ and $g \in \widehat{W}^{l,p}(\lambda_i(\Omega \cap V_i))$. Then $\sup_{i=1,\dots,s} C_i \leq M_2$, where M_2 depends only on Ω, l, n.

b) We have

$$\|\widetilde{T}_i g\|_{M_p^\phi(V_i)} \leq M_2 \|g\|_{M_p^\phi(\Omega\cap V_i)},$$

$$\|D_w^\alpha \widetilde{T}_i g\|_{M_p^\phi(V_i)} \leq M_3 M_2 \sum_{1\leq|\beta|\leq|\alpha|} \|D_w^\alpha g\|_{M_p^\phi(\Omega\cap V_i)},$$

for $1 \leq |\alpha| \leq l$ and $g \in \widehat{W}^{l,p}(\Omega \cap V_i)$ and where M_3 doesn't depend on i.

Let now $x_0 \in \mathbb{R}^n$, $0 < r < \delta$ and $B_r(x_0)$ the ball centered in x_0 of radius r. Let's consider the set $J = \{i = 1, ..., s \mid V_i \cap B_r(x_0) \neq \emptyset\}$. We notice that there exists an integer $\widetilde{s}$ depending only on the covering $(V_i)_{i=1}^s$ and on δ such that $\#J \leq \widetilde{s}$. We also recall that if Ω is bounded then $\widetilde{s} \leq s < \infty$. We have

$$\begin{aligned}
\left(\frac{1}{\phi(r)} \int_{B_r(x_0)} |Tf(x)|^p dx\right)^{\frac{1}{p}} &= \left(\frac{1}{\phi(r)} \int_{B_r(x_0)} |\sum_{i=1}^{s} \psi_i(x) \widetilde{T}_i(\psi_i f))(x)|^p dx\right)^{\frac{1}{p}} \\
&\leq \sum_{i\in J} \left(\frac{1}{\phi(r)} \int_{B_r(x_0)\cap V_i} |\widetilde{T}_i(\psi_i f)(x)|^p| dx\right)^{\frac{1}{p}} \\
&\overset{b)}{\leq} \widetilde{s} M_2 \|\psi_i f\|_{M_p^\phi(V_i\cap\Omega)} \leq M_2 \widetilde{s} \|f\|_{M_p^\phi(\Omega)}.
\end{aligned}$$

This proves (7) and (9). Let now $\alpha \in \mathbb{N}_0^n$ with $1 \le |\alpha| \le l$. We have

$$\begin{aligned}
\left(\frac{1}{\phi(r)}\int_{B_r(x_0)} |D_w^\alpha Tf(x)|^p dx\right)^{\frac{1}{p}} &= \left(\frac{1}{\phi(r)}\int_{B_r(x_0)} |D_w^\alpha \sum_{i=1}^s \psi_i(x)\widetilde{T}_i(\psi_i f))(x)|^p dx\right)^{\frac{1}{p}} \\
&\le C_\alpha \sum_{i\in J}\left(\frac{1}{\phi(r)}\int_{B_r(x_0)\cap V_i} \sum_{\beta\le\alpha} |D^{\alpha-\beta}\psi_i(x) D_w^\beta \widetilde{T}_i(\psi_i f)(x)|^p dx\right)^{\frac{1}{p}} \\
&\le C_\alpha M_1 \widetilde{s} \sum_{i\in J}\left(\frac{1}{\phi(r)}\int_{B_r(x_0)\cap V_i} \sum_{\beta\le\alpha} |D_w^\beta \widetilde{T}_i(\psi_i f)(x)|^p dx\right)^{\frac{1}{p}} \\
&\overset{b)}{\le} C_\alpha M_1 \widetilde{s} \sum_{\beta\le\alpha} M_2 M_3 \sum_{|\gamma|\le|\beta|} \|D_w^\gamma f\|_{M_p^\phi(V_i)} \\
&\le \widetilde{C}_\alpha M_1 M_2 M_3 \widetilde{s} \sum_{|\beta|\le|\alpha|} \|D_w^\beta f\|_{M_p^\phi(V_i)}
\end{aligned}$$

This proves (8) and (10). Let's now prove a) and b). a) Ω has a resolved C^m boundary with parameters $\varkappa, d, D, M$. Hence, if ϕ_i are the C^m functions of Definition 1, we have $\|D^\alpha \phi_i\|_{L^\infty} \le M$ for every i and for every $1 \le |\alpha| \le l$. Therefore by the proof of Lemma 6 we deduce that C_i depends only on l, n, M and on the choice of the constants α_k, β_k, which can be chosen to be the same for every T_i. b) We notice that since λ_i are isometries, they are smooth and their derivatives are uniformly bounded with a bound depending only on n. Then the result follows from a) and from a straightforward computation using a change of variable and Lemma 3. □

2 Stein operator

2.1 Construction

In this section we will define the Stein extension operator for Lipschitz domains in $\mathbb{R}^n$. The details of the construction and the proofs of all the results in this subsection can be found in [4, Section 2-3, Ch. VI]. We start by introducing the notion of regularized distance with the following theorem.

Theorem 3. Let F be a closed set in $\mathbb{R}^n$ and denote $d(x, F)$ the distance of x from F. Then there exists a function $\Delta(x) = \Delta(x, F)$ defined in F^c such that

a) $c_1 d(x,F) \leq \Delta(x) \leq c_2 d(x,F)$,

b) $\Delta(x)$ is C^∞ in F^c and

$$|D^\alpha \Delta(x)| \leq B_\alpha d(x,F)^{1-|\alpha|},$$

where B_α, c_1,c_2 are constants independent of F and $d(x,F)$ is the distance of x from F.

Next we give the definition of a special Lipschitz domain.

Definition 4. A domain Ω of $\mathbb{R}^n$ it's said to be a special Lipschitz domain if exists a Lipschitz function ψ defined from $\mathbb{R}^{n-1}$ to $\mathbb{R}$ such that

$$\Omega = \{(\overline{x}, y) \in \mathbb{R}^n \mid \psi(\overline{x}) < y\}.$$

Moreover the constant $\operatorname{Lip}\psi$ is said to be the Lipschitz bound of Ω.

It's convenient to define first the Stein extension operator in the case of a special Lipschitz domain, to do this we need the following two lemmas.

Lemma 7. Let Ω be a special Lipschitz domain of $\mathbb{R}^n$ and set $F = \overline{\Omega}$. Suppose $\Delta(\overline{x}, y)$ is the regularized distance from F as given in Theorem 3. Then there exists a constant a, which depends only on the Lipschitz bound of Ω, so that if $(\overline{x}, y) \in F^c$, then $a\Delta(\overline{x}, y) \geq \psi(\overline{x}) - y$.

Lemma 8. There exists a continuous function τ defined in $[1, \infty)$ satisfying

i) $\tau(\lambda) = O(\lambda^N)$, as $\lambda \to \infty$ for every N,

ii) $\int_1^\infty \tau(\lambda) d\lambda = 1$, $\int_1^\infty \lambda^k \tau(\lambda) d\lambda = 0$, for every $k = 1, 2, ...$

Theorem 4. Let Ω be a special Lipschitz domain of $\mathbb{R}^n$ with Lipschitz bound M. Moreover let τ be the function in Lemma 8 and a the constant of Lemma 7. For every function f that is C^∞ in $\overline{\Omega}$ and bounded in $\overline{\Omega}$ together with all its partial derivatives, define

$$Tf(\overline{x}, y) = \begin{cases} f(\overline{x}, y), & \text{if } y \geq \psi(\overline{x}) \\ \int_1^\infty f(\overline{x}, y + \lambda\delta^*(\overline{x}, y))\tau(\lambda) d\lambda, & \text{if } y < \psi(\overline{x}), \end{cases}$$

where $\delta^*(\overline{x}, y) = 2a\Delta(\overline{x}, y)$. Then $Tf \in C^\infty(\mathbb{R}^n)$ and

$$\|Tf\|_{W^{l,p}(\mathbb{R}^n)} \leq C_{n,l}(M) \|f\|_{W^{l,p}(\Omega)},$$

where $C_{l,n}(M)$ is a constant depending only on n, l and M.

Theorem 5. Let $l \in \mathbb{N}, 1 \leq p \leq \infty$ and Ω be a special Lipschitz domain of $\mathbb{R}^n$ with Lipschitz bound M. Denote with Γ the cone with vertex at the origin given by $\Gamma = \{(\overline{x}, y) \in \mathbb{R}^n \mid M|\overline{x}| < |y|, y < 0\}$. Suppose now that $\eta \in C_c^\infty(\mathbb{R}^n)$ is a non-negative function with total integral 1 and which support is contained in Γ. For every $f \in W^{l,p}(\Omega)$ and every $x \in \overline{\Omega}$ define $f_\varepsilon(x) = \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} f(x-y)\eta(y/\varepsilon)dy$. Then Tf_ε is well defined and the sequence $\{Tf_\varepsilon\}_{\varepsilon>0}$ converges in $W^{l,p}(\mathbb{R}^n)$ if $p < \infty$ and in $W^{l-1,p}(\mathbb{R}^n)$ if $p = \infty$, as $\varepsilon \to 0$. Moreover setting

$$Sf = \lim_{\varepsilon \to 0} Tf_\varepsilon$$

we have that Sf extend f to $\mathbb{R}^n$ and

$$\|Sf\|_{W^{l,p}(\mathbb{R}^n)} \leq C_{l,n}(M)\|f\|_{W^{l,p}(\Omega)},$$

where $C_{l,n}(M)$ is a constant depending only on n, l and M.

2.2 Stein operator in Sobolev-Morrey spaces

Definition 5. Let x be a point in $\mathbb{R}^n$ and $r > 0$. We define the open cube centered in x of side l as the set

$$Q_l(x) = (x_1 - l/2, x_1 + l/2) \times (x_2 - l/2, x_2 + l/2) \times \cdots \times (x_n - l/2, x_n + l/2)$$

where $x = (x_1, ..., x_n)$.

Definition 6. Let $1 \leq p < \infty$, ϕ a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ and Ω be a domain in $\mathbb{R}^n$. For a function $f \in L^p_{loc}(\Omega)$ we define the norm $\|.\|_{M^\phi_{p,Q}(\Omega)}$ as

$$\|f\|_{M^\phi_{p,Q}(\Omega)} := \sup_{Q_{2r}(x), x\in\Omega, r>0} \left(\frac{1}{\phi(r)}\int_{Q_r(x)\cap\Omega} |f(y)|^p dy\right)^{\frac{1}{p}}$$

where $Q_{2r}(x)$ is the open cube centered in x of side $2r$.

Lemma 9. Let $1 \leq p \leq \infty$, ϕ a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ and Ω be a domain in $\mathbb{R}^n$. Then then norm $\|.\|_{M^\phi_{p,Q}(\Omega)}$ is equivalent to the classical Morrey norm $\|.\|_{M^\phi_p(\Omega)}$. In particular

$$\|.\|_{M^\phi_p(\Omega)} \leq \|.\|_{M^\phi_{p,Q}(\Omega)} \leq C_n\|.\|_{M^\phi_p(\Omega)}$$

where C_n is a constant depending only on n.

Proof. We prove first the second inequality of the statement. Let $x \in \Omega$, $r > 0$, $Q_{2r}(x)$ be the cube centered in x of side $2r$ and $f \in L^p_{loc}(\Omega)$. Since the set $Q_{2r}(x) \cap \Omega$ has diameter less than $2r\sqrt{n}$ by Lemma 4 there exists a collection of balls $B_1, ..., B_k$ centered in $Q_{2r}(x) \cap \Omega$ of radius r, with $k \leq C_n$ where C_n depends only on n. Hence

$$\int_{Q_{2r}(x)\cap\Omega} |f(y)|^p dy \leq \sum_{i=1}^{k} \int_{B_i\cap\Omega} |f(y)|^p dy$$

and

$$\|f\|_{M^{\phi}_{p,Q}(\Omega)} = \sup_{Q_{2r}(x), x\in\Omega, r>0} \left(\frac{1}{\phi(r)} \int_{Q_{2r}(x)\cap\Omega} |f(y)|^p dy \right)^{\frac{1}{p}} \leq C_n \|f\|_{M^{\phi}_{p}(\Omega)}.$$

To prove the first inequality we observe that for every $x \in \Omega$ and $r > 0$, $(B_r(x) \cap \Omega) \subset (Q_{2r}(x) \cap \Omega)$, where $Q_{2r}(x)$ is the cube centered in x with side $2r$ and $B_r(x)$ is the ball of radius r centered in x. Therefore for every $f \in L^p_{loc}(\Omega)$

$$\int_{B_r(x)\cap\Omega} |f(y)|^p dy \leq \int_{Q_{2r}(x)\cap\Omega} |f(y)|^p dy$$

and this concludes the proof.

$\square$

Lemma 10. Let Ω be an open set in $\mathbb{R}^n$ and let $f, h \in C^\infty(\mathbb{R}^n)$. Define the function $g \in C^\infty(\mathbb{R}^n)$ as $g(x) = f(\overline{x}, x_n + \lambda h(x))$ where $\overline{x} = x_1, ..., x_{n-1}$ and $0 \neq \lambda \in \mathbb{R}$. Then for every $\alpha \in \mathbb{N}^n_0$ and $x \in \mathbb{R}^n$ the number $D^\alpha g(x)$ is a finite sum of terms of the following form

$$c\lambda^s D^\beta f(\overline{x}, x_n + \lambda h(x))(D^{\gamma_1} h(x))^{n_1} \cdots (D^{\gamma_k} h(x))^{n_k}$$

for some constant c, with $\beta, \gamma_i \in \mathbb{N}^n_0$, $k, s, n_i \in \mathbb{N}_0$ and $\beta, \gamma_i \neq 0$, $k, s \geq 0$, $n_i > 0$. Moreover every term satisfies the following conditions

a) $n_1(|\gamma_1| - 1) + n_2(|\gamma_2| - 1) + ... + n_k(|\gamma_k| - 1) = |\alpha| - |\beta|$,

b) $s = 0$ if and only if $k = 0$.

Proof. We will prove the result by induction on $l = |\alpha|$. Let's prove the case $l = 1$. For every $i = 1, ..., n$ we have

$$\frac{\partial g}{\partial x_i}(x) = \frac{\partial f}{\partial x_i}(\overline{x}, x_n + \lambda h(x)) + \lambda \frac{\partial f}{\partial x_n}(\overline{x}, x_n + \lambda h(x)) \frac{\partial h}{\partial x_i}(x)$$

that clearly satisfies the statement. We assume now that the result is true for l, and suppose $|\alpha| = l+1$. We write $D^\alpha g(x) = \frac{\partial D^\beta g}{\partial x_i}(x)$ for some $|\beta| = l$. Hence by induction hypothesis and linearity of the derivative we have that $D^\alpha g(x)$ is a finite sum of terms of the form

$$\frac{\partial}{\partial x_i}[c\lambda^s D^\gamma f(\overline{x}, x_n + \lambda h(x))(D^{\gamma_1}h(x))^{n_1} \cdots (D^{\gamma_k}h(x))^{n_k}].$$

Suppose first that $k \geq 1$, so by induction we know that

$$n_1(|\gamma_1| - 1) + n_2(|\gamma_2| - 1) + ... + n_k(|\gamma_k| - 1) = |\beta| - |\gamma| \tag{11}$$

and that $s \geq 1$. Now expanding the derivation using the chain rule we get

$$\begin{aligned}
&\frac{\partial}{\partial x_i}[c\lambda^s D^\gamma f(\overline{x}, x_n + \lambda h(x))(D^{\gamma_1}h(x))^{n_1} \cdots (D^{\gamma_k}h(x))^{n_k}] = \\
&= c\lambda^s \frac{\partial D^\gamma f}{\partial x_i}(\overline{x}, x_n + \lambda h(x))(D^{\gamma_1}h(x))^{n_1} \cdots (D^{\gamma_k}h(x))^{n_k} + \\
&+ c\lambda^{s+1} \frac{\partial D^\gamma f}{\partial x_n}(\overline{x}, x_n + \lambda h(x))(D^{\gamma_1}h(x))^{n_1} \cdots (D^{\gamma_k}h(x))^{n_k} \frac{\partial h}{\partial x_i}(x) + \\
&+ \sum_{j=1}^{k} c\lambda^s n_j D^\gamma f(\overline{x}, x_n + \lambda h(x))(D^{\gamma_1}h(x))^{n_1} \cdots (D^{\gamma_k}h(x))^{n_k} \frac{\frac{\partial D^{\gamma_j}h}{\partial x_i}(x)}{D^{\gamma_j}h(x)}.
\end{aligned}$$

Let's see that every term satisfies a). By (11) we have

$$n_1(|\gamma_1| - 1) + n_2(|\gamma_2| - 1) + ... + n_k(|\gamma_k| - 1) = |\beta| - |\gamma| = |\alpha| - |\gamma + e_i|$$

where $e_i = (0, ..., \underset{i}{1}, ..., 0)$, hence that first summand satisfies a). Again by (11)

$$n_1(|\gamma_1| - 1) + n_2(|\gamma_2| - 1) + ... + n_k(|\gamma_k| - 1) + (|e_i| - 1) = |\alpha| - |\gamma + e_n|$$

and this proves a) for the second term. Now we consider the final sum, we will prove a) just for $j = 1$, the other terms are dealt in the same way. We need to prove that

$$n_1(|\gamma_1|-1)+...+(n_j-1)(|\gamma_j|-1)+...+n_k(|\gamma_k|-1)+(|\gamma_j+e_i|-1) = |\alpha|-|\gamma|.$$

Expanding the left-hand side we get

$$n_1(|\gamma_1| - 1) + n_2(|\gamma_2| - 1) + ... + n_k(|\gamma_k| - 1) + 1$$

and since $|\beta| = |\alpha| - 1$ we conclude using (11). We observe that, since $k, s \geq 1$, all the terms also satisfies b). Suppose know that $k = 0$, hence we need to consider

$$\frac{\partial}{\partial x_i}[cD^\gamma f(\overline{x}, x_n + \lambda h(x))]$$

that becomes

$$c\frac{\partial D^\gamma f}{\partial x_i}(\overline{x}, x_n + \lambda h(x)) + c\lambda\frac{\partial D^\gamma f}{\partial x_n}(\overline{x}, x_n + \lambda h(x))\frac{\partial h}{\partial x_i}(x).$$

By induction and by a) we know that $|\gamma| = |\beta|$, therefore it's immediate that both the above terms satisfies a) and b).

Remark 2. Let Ω a special Lipschitz domain and let $\delta^*(\overline{x}, y)$ be the function defined in Theorem 4. Then for every $(\overline{x}, y)$ with $\psi(\overline{x}) > y$ the following holds

$$c(\psi(\overline{x}) - y) \geq \delta^*(\overline{x}, y) \geq 2(\psi(\overline{x}) - y),$$

where c is some constant depending only on n. The second inequality follows directly from the definition of δ^* and Lemma 7. Next we notice that $(\psi(\overline{x}) - y) \geq d(x, \overline{\Omega})$, hence the first inequality follows from a) of Theorem 3.

□

Lemma 11. Let $1 \leq p < \infty, n \geq 2$, ϕ a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ and Ω be a special Lipschitz domain of $\mathbb{R}^n$ with Lipschitz bound M. Moreover let T be the operator defined in Theorem 4 and $f \in C^\infty(\overline{\Omega})$ be a function bounded in $\overline{\Omega}$ together with all its partial derivatives. Then for every $\alpha \in \mathbb{N}_0^n$

$$\|D^\alpha Tf\|_{M_p^\phi(\mathbb{R}^n)} \leq C_{l,n}(M) \sum_{|\beta| \leq |\alpha|} \|D^\beta f\|_{M_p^\phi(\Omega)} \tag{12}$$

where $l = |\alpha|$ and $C_{l,n}(M)$ is a constant depending only on l, n and M.

Proof. Let's start by proving the case $l = 0$. By Lemma 9 it's enough to prove that for an arbitrary open cube Q of side r in $\mathbb{R}^n$ with sides parallel to the axis we have

$$\left(\frac{1}{\phi(r/2)}\int_Q |Tf(x)|^p dx\right)^{\frac{1}{p}} \leq C_n(M)\|f\|_{M_{p,Q}^\phi(\Omega)} \tag{13}$$

for a constant $C_n(M)$ depending only on n, M. Let's define $\Omega^- = \{(\overline{x}, y) \in \mathbb{R}^n \mid \overline{x} \in \mathbb{R}^{n-1},\ y < \psi(\overline{x})\}$. There are three cases: 1. $Q \subset \Omega$ 2. $Q \subset \Omega^-$ 3. $Q \cap \{y = \psi(\overline{x})\} \neq \emptyset$.

1. Since $Tf = f$ in Ω

$$\left(\frac{1}{\phi(r/2)}\int_Q |Tf(x)|^p dx\right)^{\frac{1}{p}} = \left(\frac{1}{\phi(r/2)}\int_Q |f(x)|^p dx\right)^{\frac{1}{p}} \leq \|f\|_{M^{\phi}_{p,Q}(\Omega)}$$

and we are done.

2. Let's write Q as $Q = \{(\overline{x}, y) \in \mathbb{R}^n \mid \overline{x} \in F, y \in (a-r, a)\}$ where F is an open cube of $\mathbb{R}^{n-1}$ of side r and $a < \phi(\overline{x})$ for every $\overline{x} \in F$. Fix now $(\overline{x}, y) \in Q$. By Lemma 8 there exists a constant A_3 such that $|\tau(\lambda)| \leq A_3/\lambda^3$ for every $\lambda \geq 1$. From the definition of Tf we have

$$|Tf(\overline{x}, y)| \overset{(\bullet)}{\leq} \int_1^{\infty} |f(\overline{x}, y+\lambda\delta^*(\overline{x}, y))||\tau(\lambda)|d\lambda \leq A_3 \int_1^{\infty} |f(\overline{x}, y+\lambda\delta^*(\overline{x}, y))|\frac{1}{\lambda^3}d\lambda$$

Let's apply the change of variable $s = y + \lambda\delta^*(\overline{x}, y)$

$$|Tf(\overline{x}, y)| \overset{(\bullet\bullet)}{\leq} A_3 \int_{y+\delta^*}^{\infty} |f(\overline{x}, s)|\frac{(\delta^*)^2}{(s-y)^3}ds \leq A_3 c^2 \int_{2\psi(\overline{x})-y}^{\infty} |f(\overline{x}, s)|\frac{(\psi(x)-y)^2}{(s-y)^3}ds$$

because $c(\psi(x) - y) \geq \delta^* \geq 2(\psi(x) - y)$ as seen in Remark 2. Let's now decompose the last integral as follows

$$|Tf(\overline{x}, y)| \leq \sum_{k=0}^{\infty} A_3 c^2 \int_{2\psi(\overline{x})-y+kr}^{2\psi(\overline{x})-y+(k+1)r} |f(\overline{x}, s)|\frac{(\psi(\overline{x})-y)^2}{(s-y)^3}ds.$$

Now by applying Minkowski's inequality for an infinite sum we get

$$\left(\int_{a-r}^{a} |Tf(\overline{x}, y)|^p dy\right)^{\frac{1}{p}} \leq A_3 c^2 \sum_{k=0}^{\infty} \left(\int_{a-r}^{a}\left(\int_{2\psi(\overline{x})-y+kr}^{2\psi(\overline{x})-y+(k+1)r} \frac{|f(\overline{x}, s)|(\psi(x)-y)^2}{(s-y)^3}ds\right)^p dy\right)^{\frac{1}{p}} \quad (*)$$

Next we plan to estimate each summand. In the right-hand side of (*) we apply the change of variable $y = \psi(\overline{x}) - z$

$$\left(\int_{\psi(x)-a}^{\psi(x)-a+r}\left(\int_{\psi(x)+z+kr}^{\psi(x)+z+(k+1)r} |f(\overline{x}, s)|\frac{z^2}{(s-\psi(x)+z)^3}ds\right)^p dz\right)^{\frac{1}{p}}$$

and the change of variable $u = s - \psi(x)$

$$\left(\int_{\psi(x)-a}^{\psi(x)-a+r} \left(\int_{z+kr}^{z+(k+1)r} |f(\overline{x}, u+\psi(x))| \frac{z^2}{(u+z)^3} du\right)^p dz\right)^{\frac{1}{p}}.$$

Then we apply the change of variable $t = u/z$

$$\left(\int_{\psi(\overline{x})-a}^{\psi(\overline{x})-a+r} \left(\int_{1+kr/z}^{1+(k+1)r/z} |f(\overline{x}, tz+\psi(x))| \frac{1}{(t+1)^3} dt\right)^p dz\right)^{\frac{1}{p}}.$$

that can be rewritten as

$$\left(\int_{\psi(\overline{x})-a}^{\psi(\overline{x})-a+r} \left(\int_{1+kr/(\psi(\overline{x})-a+r)}^{1+(k+1)r/(\psi(\overline{x})-a)} |f(\overline{x}, tz+\psi(x))| \mathbb{1}_{(1+kr/z, 1+(k+1)r/z)}(t) \frac{1}{(t+1)^3} dt\right)^p dz\right)^{\frac{1}{p}}.$$

By Minkowsi's integral inequality and setting $\alpha = r/(\psi(\overline{x}) - a)$

$$\left(\int_{a\psi(\overline{x})-a}^{\psi(\overline{x})-a+r} \left(\int_{1+k\alpha/(\alpha+1)}^{1+(k+1)\alpha} |f(\overline{x}, tz+\psi(x))| \mathbb{1}_{(1+kr/z, 1+(k+1)r/z)}(t) \frac{1}{(t+1)^3} dt\right)^p dz\right)^{\frac{1}{p}}.$$

$$\leq \int_{1+k\alpha/(\alpha+1)}^{1+(k+1)\alpha} \left(\int_{\psi(\overline{x})-a}^{\psi(\overline{x})-a+r} |f(\overline{x}, tz+\psi(x))|^p \mathbb{1}_{(1+kr/z, 1+(k+1)r/z)}(t) \frac{1}{(t+1)^{3p}} dz\right)^{\frac{1}{p}} dt.$$

We notice that for every $t, z \in \mathbb{R}$ with $\psi(\overline{x}) - a \leq z \leq \psi(\overline{x}) - a + r$

$$\mathbb{1}_{(1+kr/z, 1+(k+1)r/z)}(t) \leq \mathbb{1}_{(\psi(\overline{x})-a+kr, \psi(\overline{x})-a+(k+2)r)}(tz)$$

hence using the change of variable $w = tz$

$$\int_{1+k\alpha/(\alpha+1)}^{1+(k+1)\alpha} \left(\int_{\psi(\overline{x})-a+kr}^{\psi(\overline{x})-a+(k+2)r} |f(\overline{x}, tz+\psi(x))|^p \mathbb{1}_{(1+kr/z, 1+(k+1)r/z)}(t) \frac{1}{(t+1)^{3p}} dz \right)^{\frac{1}{p}} dt$$

$$\leq \int_{1+k\alpha/(\alpha+1)}^{1+(k+1)\alpha} \left(\int_{\psi(\overline{x})-a+kr}^{\psi(\overline{x})-a+(k+2)r} |f(\overline{x}, w+\psi(\overline{x}))|^p \frac{1}{t(t+1)^{3p}} dw \right)^{\frac{1}{p}} dt$$

$$= \int_{1+k\alpha/(\alpha+1)}^{1+(k+1)\alpha} \frac{1}{t^{\frac{1}{p}}(t+1)^3} dt \left(\int_{\psi(\overline{x})-a+kr}^{\psi(\overline{x})-a+(k+2)r} |f(\overline{x}, w+\psi(\overline{x}))|^p dw \right)^{\frac{1}{p}}$$

$$\leq \int_{1+k\alpha/(\alpha+1)}^{1+(k+1)\alpha} \frac{1}{(t+1)^3} dt \left(\int_{\psi(\overline{x})-a+kr}^{\psi(\overline{x})-a+(k+2)r} |f(\overline{x}, w+\psi(\overline{x}))|^p dw \right)$$

$$= \frac{1}{2} \left[\frac{1}{(1+(k+1)\alpha)^2} - \frac{1}{(1+k\alpha/(\alpha+1))^2} \right] \left(\int_{\psi(\overline{x})-a+kr}^{\psi(\overline{x})-a+(k+2)r} |f(\overline{x}, w+\psi(\overline{x}))|^p dw \right)^{\frac{1}{p}}$$

$$= \frac{s_k(\alpha)}{2} \left(\int_{\psi(\overline{x})-a+kr}^{\psi(\overline{x})-a+(k+2)r} |f(\overline{x}, w+\psi(\overline{x}))|^p dw \right)^{\frac{1}{p}}.$$

Plugging this estimate inside (*) we get

$$\begin{aligned}
\left(\int_{a-r}^{a} |Tf(\overline{x}, y)|^p dy \right)^{\frac{1}{p}} &\leq A_3 \frac{c^2}{2} \sum_{k=0}^{\infty} s_k(\alpha) \left(\int_{\psi(\overline{x})-a+kr}^{\psi(\overline{x})-a+(k+2)r} |f(\overline{x}, w+\psi(\overline{x}))|^p dw \right)^{\frac{1}{p}} \\
&= A_3 \frac{c^2}{2} \sum_{k=0}^{\infty} s_k(\alpha) \left(\int_{2\psi(\overline{x})-a+kr}^{2\psi(\overline{x})-a+(k+2)r} |f(\overline{x}, y)|^p dy \right)^{\frac{1}{p}}.
\end{aligned}$$

Taking the L^p norm on F on both sides and applying again Minkowski inequality we obtain

$$\begin{aligned}
\left(\int_F \int_{a-r}^{a} |Tf(\overline{x}, y)|^p dy d\overline{x} \right)^{\frac{1}{p}} &\leq A_3 \frac{c^2}{2} \sum_{k=0}^{\infty} s_k(\alpha) \left(\int_F \int_{2\psi(\overline{x})-a+kr}^{2\psi(\overline{x})-a+(k+2)r} |f(\overline{x}, y)|^p dy d\overline{x} \right)^{\frac{1}{p}} \\
&= A_3 \frac{c^2}{2} \sum_{k=0}^{\infty} s_k(\alpha) \|f\|_{L^p(S_k)}. \qquad (**)
\end{aligned}$$

where $S_k = \{(\overline{x}, y) \in \mathbb{R}^n \mid \overline{x} \in F,\ 2\psi(\overline{x}) - a + kr < y < 2\psi(\overline{x}) - a + (k+2)r\}$. Clearly the set S_k has diameter less than dr, where d is a constant depending only on n and M. Hence by Lemma 4 there exists a collection of open cubes $Q_1, ..., Q_m$ centered in S_k of side r that covers S_k, with $m \in \mathbb{N}$ depending only on M and n. Moreover for every $(\overline{x}, y) \in S_k$ we have $y > 2\psi(\overline{x}) - a > \psi(\overline{x})$, so $S_k \subset \Omega$. This implies that

$$S_k \subset \bigcup_{i=1}^{m} (Q_i \cap \Omega)$$

and that every cube Q_i is centered in Ω. Therefore by (**)

$$\|Tf\|_{L^p(Q)} \leq \frac{c^2}{2} \sum_{k=0}^{\infty} s_k(\alpha)(\|f\|_{L^p(Q_1 \cap \Omega)} + ... + \|f\|_{L^p(Q_m \cap \Omega)}),$$

then dividing in both sides by $\psi(r/2)^{\frac{1}{p}}$ we obtain

$$\left(\frac{1}{\phi(r/2)} \int_Q |Tf(x)|^p dx \right)^{\frac{1}{p}} \leq \frac{A_3 c^2 m}{2} \sum_{k=0}^{\infty} s_k(a, r) \|f\|_{M_{p,Q}(\Omega)}$$

We want now to estimate the series $\sum_{k=0}^{\infty} s_k(\alpha)$. First we notice that can be rewritten as as

$$\sum_{k=0}^{\infty} s_k(\alpha) = \sum_{k=1}^{\infty} \frac{\alpha(\alpha+2)}{(k\alpha+1)^2}.$$

To bound this series we distinguish two cases, when $\alpha \leq 1$ and when $\alpha > 1$. In the first case we can bound the series using a Riemann Sum

$$\sum_{k=1}^{\infty} \frac{\alpha(\alpha+2)}{(k\alpha+1)^2} \leq 3 \sum_{k=1}^{\infty} \frac{\alpha}{(k\alpha+1)^2} = 3 \sum_{k=1}^{\infty} \int_{\mathbb{R}} \mathbb{1}_{(\alpha(k-1), \alpha k)}(t) \frac{1}{(\alpha k+1)^2} dt \leq 3 \int_{\mathbb{R}} \frac{1}{(t+1)^2} dt = 3.$$

In the second case

$$\sum_{k=1}^{\infty} \frac{\alpha(\alpha+2)}{(k\alpha+1)^2} \leq \sum_{k=1}^{\infty} \frac{\alpha(\alpha+2)}{k^2\alpha^2} = \sum_{k=1}^{\infty} \frac{1+\frac{2}{\alpha}}{k^2} \leq 3\frac{\pi^2}{6} < 5.$$

Hence we get

$$\left(\frac{1}{\phi(r/2)} \int_Q |Tf(x)|^p dx \right)^{\frac{1}{p}} \leq A_3 5 c^2 \|f\|_{M^{\phi}_{p,Q}(\Omega)}$$

that shows (1).

3. We write Q as $F \times (a-r, a)$ and and we define $Q^+ = Q \cap \Omega$ and $Q^- = Q \cap \Omega^-$. Then

$$\|Tf\|_{L^p(Q)} \leq \|f\|_{L^p(Q^+)} + \|Tf\|_{L^p(Q^-)}.$$

Moreover Q^+ can be written as $\{(\overline{x}, y) \mid \overline{x} \in S, a-r < y < \min(\psi(\overline{x}), a)\}$ for some set $S \subset F$. Hence

$$\int_{Q^-} |Tf(x)|^p dx = \int_S \int_{a-r}^{\min(\psi(\overline{x}),a)} |Tf(\overline{x}, y)|^p dy d\overline{x}.$$

We can then proceed as in 2. to obtain

$$\begin{aligned}\left(\int_S \int_{a-r}^{a} |Tf(\overline{x}, y)|^p dy d\overline{x}\right)^{\frac{1}{p}} &\leq A_3 \frac{c^2}{2} \sum_{k=0}^{\infty} s_k(\alpha) \left(\int_S \int_{2\psi(\overline{x})-\min(a,\psi(\overline{x}))+kr}^{2\psi(\overline{x})-a+(k+2)r} |f(\overline{x}, y)|^p dy d\overline{x}\right)^{\frac{1}{p}} \\ &= A_3 \frac{c^2}{2} \sum_{k=0}^{\infty} s_k(\alpha) \|f\|_{L^p(S'_k)}.\end{aligned}$$

One can observe that the sets S'_k have the same property as the sets S_k in 2. Therefore

$$\frac{1}{\psi(r/2)^{\frac{1}{p}}} \|Tf\|_{L^p(Q^-)} \leq c_1 \|f\|_{M_p^\phi(\Omega)}$$

for some constant c_1 depending only on n and M. Finally it's immediate to verify that $\|f\|_{L^p(Q^+)} \leq \phi(r/2)^{\frac{1}{p}} \|f\|_{M_p^\phi(\Omega)}$. This concludes the proof of case 3.

We consider now the case $l > 0$. By Lemma 9 it's again enough to prove that for an arbitrary open cube Q of side r contained in $\mathbb{R}^n$ we have

$$\left(\frac{1}{\phi(r/2)} \int_Q |D^\alpha Tf(x)|^p dx\right)^{\frac{1}{p}} \leq C_{l,n}(M) \sum_{|\beta| \leq |\alpha|} \|D^\beta f\|_{M_{p,Q}^\phi(\Omega)} \qquad (14)$$

for a constant $C_{l,n}(M)$ depending only on l, n, M. We will consider the same three cases that appeared with $l = 0$. Since $D^\alpha Tf = D^\alpha f$ in Ω, the first case is trivial as before. We will see that the cases 2 and 3 also follow from the computations done with $l = 0$. We start observing that by the boundedness of f and all its derivatives we can differentiate under the integral sign to get

$$D^\alpha Tf(\overline{x}, y) = \int_1^\infty D^\alpha g_\lambda(\overline{x}, y) \tau(\lambda) d\lambda$$

for every $(\overline{x}, y) \in \Omega^-$, where $g_\lambda(\overline{x}, y) = f(\overline{x}, y + \lambda\delta^*(\overline{x}, y))$. By Lemma 10 $D^\alpha g_\lambda(\overline{x}, y)$ is a finite sum of terms of the type

$$\widetilde{c}\lambda^s D^\beta f(\overline{x}, y + \lambda\delta^*(\overline{x}, y)(D^{\gamma_1}\delta^*(x))^{n_1} \cdots (D^{\gamma_k}\delta^*(x))^{n_k}.$$

For each of these terms we also set

$$T_{s,,\beta,(\gamma_1,n_1),\ldots,(\gamma_k,n_k)}(x) = \int_1^\infty \lambda^s D^\beta f(\overline{x}, y+\lambda\delta^*(\overline{x}, y)(D^{\gamma_1}\delta^*(x))^{n_1} \cdots (D^{\gamma_k}\delta^*(x))^{n_k}\tau(\lambda)d\lambda.$$

Now, since the constants $\widetilde{c}$ and the number of terms of the sum depend only on l and n, we just need to estimate the quantities

$$\left(\frac{1}{\phi(r/2)}\int_Q \left|T_{s,\beta,(\gamma_1,n_1),\ldots,(\gamma_k,n_k)}(x)\right|^p dx\right)^{\frac{1}{p}}.$$

We start by assuming that $|\beta| = |\alpha|$. By the property a) in Lemma 10 and by the estimates of the derivatives of $\delta^*(= 2a\Delta)$ given in Theorem 3 we have that

$$|T_{s,\beta,(\gamma_1,n_1),\ldots,(\gamma_k,n_k)}(x)| \leq c_2 \int_1^\infty |D^\beta f(\overline{x}, y + \lambda\delta^*(\overline{x}, y)||\tau(\lambda)|d\lambda$$

where c_2 depends only on n and M. We are now in the same situation as in the inequality ($\bullet$) of case $l = 0$. Hence we can proceed the estimate in the same way to get

$$\left(\frac{1}{\phi(r/2)}\int_Q \left|T_{s,\beta,(\gamma_1,n_1),\ldots,(\gamma_k,n_k)}(x)\right|^p dx\right)^{\frac{1}{p}} \leq c_3\|D^\beta f\|_{M_p^\phi(\Omega)}$$

for every Q in case 2 and

$$\left(\frac{1}{\phi(r/2)}\int_{Q\cap\Omega^-} \left|T_{s,\beta,(\gamma_1,n_1),\ldots,(\gamma_k,n_k)}(x)\right|^p dx\right)^{\frac{1}{p}} \leq c_4\|D^\beta f\|_{M_p^\phi(\Omega)}$$

for every Q in case 3, where c_3, c_4 depend only on n and M. Suppose now that $|\alpha| > |\beta|$. Arguing as above, by Theorem 3 and Lemma 10 we get

$$\begin{aligned}
&|T_{s,\beta,(\gamma_1,n_1),\ldots,(\gamma_k,n_k)}(x)| \\
&\leq c_5 \frac{1}{d(x,\overline{\Omega})^{|\alpha|-|\beta|}} \left|\int_1^\infty \lambda^s D^\beta f(\overline{x}, y + \lambda\delta^*(\overline{x}, y)\tau(\lambda)d\lambda\right| \\
&\leq c_5 \frac{1}{(\psi(\overline{x}) - y)^{|\alpha|-|\beta|}} \left|\int_1^\infty \lambda^s D^\beta f(\overline{x}, y + \lambda\delta^*(\overline{x}, y)\tau(\lambda)d\lambda\right|. \qquad (15)
\end{aligned}$$

Where c_5 depends only on n, l and M. We now write the Taylor expansion with integral remainder of the function $D^\beta f(\overline{x}, y+t)$ centered in $\delta^*(\overline{x}, y)$ up to order $m = |\alpha| - |\beta|$ and evaluated at $\lambda\delta^*(\overline{x}, y)$

$$D^\beta f(\overline{x}, y+\lambda\delta^*) = \sum_{i=0}^{m-1} \frac{(\lambda\delta^* - \delta^*)^i}{i!} \frac{\partial^i D\beta f}{\partial x_n^i}(\overline{x}, y+\delta^*) + \int_{\delta^*}^{\lambda\delta^*} \frac{(\lambda\delta^* - t)^{m-1}}{m!} \frac{\partial^m D\beta f}{\partial x_n^m}(\overline{x}, y+t)dt.$$

We observe that the terms inside the sum doesn't give any contribution in (8), since

$$\begin{aligned}&\int_1^\infty \frac{\lambda^s(\lambda\delta^* - \delta^*)^i}{i!} \frac{\partial^i D\beta f}{\partial x_n^i}(\overline{x}, y+\delta^*)\tau(\lambda)d\lambda \\ &= \frac{\partial^i D\beta f}{\partial x_n^i}(\overline{x}, y+\delta^*)\frac{(\delta^*)^i}{i!} \int_1^\infty \lambda^s(\lambda-1)^i\tau(\lambda)d\lambda = 0\end{aligned}$$

by the properties of τ, since $s > 0$ by Lemma 10. Hence combining this with (15) we obtain

$$|T_{s,\beta,(\gamma_1,n_1),\ldots,(\gamma_k,n_k)}(x)| \leq \frac{c_5}{(\psi(\overline{x}) - y)^m} \left| \int_1^\infty \int_{\delta^*}^{\lambda\delta^*} \frac{(\lambda\delta^* - t)^{m-1}}{m!} \frac{\partial^m D\beta f}{\partial x_n^m}(\overline{x}, y+t)dt\lambda^s\tau(\lambda)d\lambda \right|.$$

Observing that $(\lambda\delta^* - t)^{m-1} \leq (\lambda\delta^*)^{m-1}$, recalling that $\psi(\overline{x}) - y \geq c\delta^*$ and using the change of variable $u = y + t$ we get

$$|T_{s,\beta,(\gamma_1,n_1),\ldots,(\gamma_k,n_k)}(x)| \leq \frac{c_5}{c^m m!\delta^*} \int_1^\infty \int_{y+\delta^*}^{y+\lambda\delta^*} \left| \frac{\partial^m D\beta f}{\partial x_n^m}(\overline{x}, u) \right| \lambda^{s+m-1}|\tau(\lambda)|dud\lambda.$$

Perform a changing of order of integration we deduce

$$|T_{s,\beta,(\gamma_1,n_1),\ldots,(\gamma_k,n_k)}(x)| \leq \frac{c_5}{c^m m!\delta^*} \int_{y+\delta^*}^\infty \left| \frac{\partial^m D\beta f}{\partial x_n^m}(\overline{x}, u) \right| \int_{(u-y)/\delta^*}^\infty |\lambda^{s+m-1}\tau(\lambda)|d\lambda du.$$

Finally recalling that that $|\tau(\lambda)| \leq A_{m+s}/\lambda^{s+m+3}$ for some constant A_{m+s} we can write

$$|T_{s,\beta,(\gamma_1,n_1),\ldots,(\gamma_k,n_k)}(x)| \leq \frac{c_5 A_{m+s}}{3c^m m!} \int_{y+\delta^*}^\infty \left| \frac{\partial^m D\beta f}{\partial x_n^m}(\overline{x}, u) \right| \frac{(\delta^*)^2}{(u-y)^3}du.$$

We observe that we are now in the same situation as in the inequality (••) of the case $l = 0$ and the same computations lead us to

$$\left(\frac{1}{\phi(r/2)} \int_Q \left|T_{s,\beta,(\gamma_1,n_1),\ldots,(\gamma_k,n_k)}(x)\right|^p dx \right)^{\frac{1}{p}} \leq c_6 \|\frac{\partial^m D\beta f}{\partial x_n^m}\|_{M_p^\phi(\Omega)}$$

for every Q in case 2 and

$$\left(\frac{1}{\phi(r/2)}\int_{Q\cap\Omega^-}\left|T_{s,\beta,(\gamma_1,n_1),\ldots,(\gamma_k,n_k)}(x)\right|^p dx\right)^{\frac{1}{p}} \le c_7\|\frac{\partial^m D\beta f}{\partial x_n^m}\|_{M_p^\phi(\Omega)}$$

for every Q in case 3, where c_6, c_7 depend only on n, l and M. This concludes also the proof of the case $l > 0$. $\square$

Theorem 6. Let $1 \le p < \infty, n \ge 2$, ϕ a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ and Ω be a special Lipschitz domain of $\mathbb{R}^n$ with Lipschitz bound M. Moreover let S be the Stein extension operator. Then for every $f \in W^{l,p}(\Omega)$ and every $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le l$

$$\|D^\alpha Sf\|_{M_p^\phi(\mathbb{R}^n)} \le C_{l,n}(M)\sum_{|\beta|\le|\alpha|}\|D^\beta f\|_{M_p^\phi(\Omega)} \tag{16}$$

where $C_{l,n}(\Omega)$ depends only on n, l and M.

Proof. We recall definition of the operator S. Set Γ to be the cone $\Gamma = \{(\overline{x}, y) \in \mathbb{R}^n \mid M|\overline{x}| < |y|, y < 0\}$ and let $\eta \in C_c^\infty(\mathbb{R}^n)$ be a function with total integral 1 and which support is contained in Γ. Then, given $f \in W^{l,p}(\Omega)$, Sf is defined to be the limit in $W^{l,p}(\mathbb{R}^n)$ of Tf_ε as $\varepsilon \to 0$, where $f_\varepsilon(x) = 1/\varepsilon^n \int_{\mathbb{R}^n} f(x-y)\eta(y/\varepsilon)$ for every x in an appropriate neighborhood of $\overline{\Omega}$. We claim that for every $f \in W^{l,p}(\Omega)$ and $|\alpha| \le l$

$$\|D_w^\alpha f_\varepsilon\|_{M_p^\phi(\Omega)} \le \|D_w^\alpha f\|_{M_p^\phi(\Omega)}. \tag{17}$$

To see this first we notice that $D_w^\alpha f_\varepsilon(x) = 1/\varepsilon^n \int_{\mathbb{R}^n} D_w^\alpha f(x-y)\eta(y/\varepsilon)dy$ for every $x \in \Omega$. Let now $B_{x_0}(r)$ a ball centered in Ω of radius r, by Minkowski's integral inequality

$$\begin{aligned}
\left(\frac{1}{\phi(r)}\int_{B_r(x_0)\cap\Omega}|D^\alpha f_\varepsilon(x)|^p\right)^{\frac{1}{p}} &= \left(\frac{1}{\phi(r)}\int_{B_r(x_0)\cap\Omega}\left|\frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}D_w^\alpha f(x-y)\eta(\frac{y}{\varepsilon})dy\right|^p dx\right)^{\frac{1}{p}} \\
&\le \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}\eta(\frac{y}{\varepsilon})\left(\frac{1}{\phi(r)}\int_{B_r(x_0)\cap\Omega}|D^\alpha f(x-y)|^p dx\right)^{\frac{1}{p}} dy \\
&\le \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}\eta(\frac{y}{\varepsilon})\left(\frac{1}{\phi(r)}\int_{B_r(x_0-y)\cap\Omega}|D^\alpha f(x)|^p dx\right)^{\frac{1}{p}} dy \\
&\le \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}\eta(\frac{y}{\varepsilon})\|D^\alpha f\|_{M_p^\phi(\Omega)}dy = \|D^\alpha f\|_{M_p^\phi(\Omega)}
\end{aligned}$$

because $B_r(x_0) \cap \Omega - y \subset B_r(x_0 - y) \cap \Omega$ and $x_0 - y \in \Omega$ for every $x_0 \in \Omega$ and $y \in \Gamma$. This proves (17). Now combining (17) with (12) we get

$$\|D^\alpha T f_\varepsilon\|_{M_p^\phi(\mathbb{R}^n)} \leq C_{l,n}(M) \sum_{|\beta| \leq |\alpha|} \|D^\beta f\|_{M_p^\phi(\Omega)},$$

for every $\varepsilon > 0$ and every $|\alpha| \leq l$, with $C_{l,n}(M)$ independent of ε. In particular, for every ball B in $\mathbb{R}^n$ of radius $r > 0$ we have

$$\left(\frac{1}{\phi(r)} \int_B |D^\alpha T f_\varepsilon(x)|^p dx \right)^{\frac{1}{p}} \leq C_{l,n}(M) \sum_{|\beta| \leq |\alpha|} \|D^\beta f\|_{M_p^\phi(\Omega)} \tag{18}$$

Since Tf_ε converges to Sf in $W^{l,p}(\mathbb{R}^n)$, then $D^\alpha T f_\varepsilon$ converges to $D^\alpha S f$ in $L^p(\mathbb{R}^n)$ for every $|\alpha| \leq l$ and as a consequence also in $L^p(B)$ for every ball B. Hence we can pass to the limit as $\varepsilon \to 0$ in (18) and obtain

$$\left(\frac{1}{\phi(r)} \int_B |D^\alpha S(x)|^p dx \right)^{\frac{1}{p}} \leq C_{l,n}(M) \sum_{|\beta| \leq |\alpha|} \|D^\beta f\|_{M_p^\phi(\Omega)}$$

for every ball B of radius ε. This concludes the proof. $\square$

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