Definition 1. Let $1 \leq p < \infty$, ϕ a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ and Ω be a domain in $\mathbb{R}^n$. For a function $f \in L^p_{loc}(\Omega)$ we define the cubic-Morrey norm $\|.\|_{M^{\phi}_{p,Q}(\Omega)}$ as

$$\|f\|_{M^{\phi}_{p,Q}(\Omega)} := \sup_{Q_r(x), x\in\Omega, r>0} \left(\frac{1}{\phi(r)} \int_{Q_r(x)\cap\Omega} |f(y)|^p dy \right)^{\frac{1}{p}}$$

where $Q_r(x)$ is the open cube centered in x of side $2r$.

Lemma 1. Let $1 \leq p \leq \infty$, ϕ a function from $\mathbb{R}^+$ to $\mathbb{R}^+$ and Ω be a domain in $\mathbb{R}^n$. Then then cubic-Morrey norm $\|.\|_{M^{\phi}_{p,Q}(\Omega)}$ is equivalent to the classical Morrey norm $\|.\|_{M^{\phi}_{p}(\Omega)}$. In particular

$$\|.\|_{M^{\phi}_{p}(\Omega)} \leq \|.\|_{M^{\phi}_{p,Q}(\Omega)} \leq 2^{n^2} \|.\|_{M^{\phi}_{p}(\Omega)}.$$

Proof. We start by proving some geometrical facts. Let Q be a cube in $\mathbb{R}^n$ of side $2r$. We claim that if S is a set of points in $\mathbb{R}^n$ satisfying

i) $S \subset Q$,

ii) $\|z_1 - z_2\| \geq r$ for every $z_1, z_2 \in Q$ with $z_1 \neq z_2$,

then $|S| \leq 2^{n^2}$. To see this let's cover Q with $(2^n)^n$ small closed cubes of side $2r/2^n$. The diagonal of a small cube measures $2r/2^n \cdot \sqrt{n} < r$. Thus each of these cubes can contain at most one point of S, so $|S| \leq 2^{n^2}$.

Now let $x \in \Omega$, $r > 0$ and Q be the cube centered in x of side $2r$. Consider $Q \cap \Omega$, we'll prove that we can cover this set with a collection of balls $B_1, ..., B_k$ centered in Ω of radius r and such that $k \leq 2^{n^2}$. Let's start by taking $B_1 = B_r(x)$, the ball centered in x of radius r and calling $x_1 = x$. If $(Q \cap \Omega) \subset B_1$ we are done, if not there exists $x_2 \in (Q \cap \Omega) \setminus B_1$ and we take $B_2 = B_r(x_2)$. Again, if $(Q \cap \Omega) \subset (B_1 \cup B_2)$ we stop, else we can pick $x_3 \in (Q \cap \Omega) \setminus (B_1 \cup B_2)$ and take $B_3 = B_r(x_3)$. We iterate this procedure : given $B_1, ..., B_h$ balls, if $(Q \cap \Omega) \subset (B_1 \cup ... \cup B_h)$ we stop, else we can choose $x_{h+1} \in (Q \cap \Omega) \setminus (B_1 \cup ... \cup B_h)$ and take $B_{h+1} = B_r(x_{h+1})$. We claim that this procedure stops with $h \leq 2^{n^2}$. Suppose it doesn't, then we can find $B_1, ..., B_{2^{n^2}+1}$ balls centered respectively at $x_1, ..., x_{2^{n^2}+1}$. Setting $S = \{x_1, ..., x_{2^{n^2}+1}\}$, it's immediate to see that S satisfies i) and ii), but $|S| = 2^{n^2} + 1$.

We are now ready to prove the second inequality of the statement. Let $x \in \Omega$, $r > 0$, $Q_r(x)$ be the cube centered in x of side $2r$ and $f \in L^p_{loc}(\Omega)$. By the previous part

$$\int_{Q_r(x)\cap\Omega} |f(y)|^p dy \leq \sum_{i=1}^{k} \int_{B_i\cap\Omega} |f(y)|^p dy$$

where $k \leq 2^{n^2}$ and $B_1, ..., B_k$ are balls centered in Ω of radius r. Hence

$$\|f\|_{M^\phi_{p,Q}(\Omega)} = \sup_{Q_r(x), x\in\Omega, r>0} \left(\frac{1}{\phi(r)} \int_{Q_r(x)\cap\Omega} |f(y)|^p dy \right)^{\frac{1}{p}} \leq 2^{n^2} \|f\|_{M^\phi_p(\Omega)}.$$

To prove the first inequality we observe that for every $x \in \Omega$ and $r > 0$, $(B_r(x) \cap \Omega) \subset (Q_r(x) \cap \Omega)$, where $Q_r(x)$ is the cube centered in x with side $2r$ and $B_r(x)$ is the ball of radius r centered in x. Therefore for every $f \in L^p_{loc}(\Omega)$

$$\int_{B_r(x)\cap\Omega} |f(y)|^p dy \leq \int_{Q_r(x)\cap\Omega} |f(y)|^p dy$$

and this concludes the proof.

□

Notations :

$$\psi(t) = \frac{1}{\pi t} e^{1-\frac{\sqrt[4]{t-1}}{\sqrt{2}}} \sin \frac{\sqrt[4]{t-1}}{\sqrt{2}}$$

for $t \geq 1$, so

$$|\psi(t)| \leq e^{-\frac{\sqrt[4]{t-1}}{\sqrt{2}}} \frac{e}{\pi t} \leq \frac{A}{t^3}$$

for every $t \geq 1$ and some constant A (for example $A = 10^5$).

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_n > 0\}$$

$$\mathbb{R}^n_- = \{x \in \mathbb{R}^n \mid x_n < 0\}$$

Let $f \in L^p_{loc}(\mathbb{R}^n_+)$, we define

$$Tf(\overline{x}, y) = \begin{cases} \int_1^\infty f(\overline{x}, y + \lambda\delta^*(\overline{x}, y))\psi(\lambda)d\lambda, & \text{if } y < 0, \\ f(\overline{x}, y), & \text{if } y > 0, \end{cases}$$

where $\overline{x} \in \mathbb{R}^{n-1}$. We only need to know for now that δ^* is some function defined in $\mathbb{R}^n_-$ such that $c|y| \geq \delta^*(\overline{x}, y) \geq 2|y|$ for some constant c.

Lemma 2. Let $1 \leq p < \infty, n \geq 2$ and ϕ a function from $\mathbb{R}^+$ to $\mathbb{R}^+$. Then T defines a bounded extension operator from $M_p^\phi(\mathbb{R}^n_+)$ to $M_p^\phi(\mathbb{R}^n)$.

Proof. We will prove that for an arbitrary open cube Q of side r contained in $\mathbb{R}^n$ we have

$$\left(\frac{1}{\phi(r/2)}\int_Q |Tf(x)|^p dx\right)^{\frac{1}{p}} \leq C\|f\|_{M^\phi_{p,Q}(\mathbb{R}^n_+)} \tag{1}$$

for a constant C independent of f, then the main statement follows from Lemma 1. There are three cases: 1. $Q \subset \mathbb{R}^n_+$ 2. $Q \subset \mathbb{R}^n_-$ 3. $Q \cap \{x_n = 0\} \neq \emptyset$.

1. Since $Tf = f$ in $\mathbb{R}^n_+$

$$\left(\frac{1}{\phi(r/2)}\int_Q |Tf(x)|^p dx\right)^{\frac{1}{p}} = \left(\frac{1}{\phi(r/2)}\int_Q |f(x)|^p dx\right)^{\frac{1}{p}} \leq \|f\|_{M^\phi_{p,Q}(\mathbb{R}^n_+)}$$

and we are done.

2. Let's write Q as $Q = \{(\overline{x}, y) \in \mathbb{R}^n \mid \overline{x} \in F, y \in (-a-r, -a)\}$ where $a > 0$ and F is an open cube of $\mathbb{R}^{n-1}$ of side r. Fix now $(\overline{x}, y) \in Q$, from the definition of Tf we have

$$|Tf(\overline{x}, y)| \leq \int_1^\infty |f(\overline{x}, y+\lambda\delta^*(\overline{x}, y))||\psi(\lambda)|d\lambda \leq A\int_1^\infty |f(\overline{x}, y+\lambda\delta^*(\overline{x}, y))|\frac{1}{\lambda^3}d\lambda$$

Let's apply the change of variable $s = y + \lambda\delta^*(\overline{x}, y)$

$$|Tf(\overline{x}, y)| \leq \int_{y+\delta^*}^\infty |f(\overline{x}, s)|\frac{(\delta^*)^2}{(s-y)^3}ds \leq c^2\int_{|y|}^\infty |f(\overline{x}, s)|\frac{|y|^2}{(s-y)^3}ds$$

because $c|y| \geq \delta^* \geq 2|y|$. Let's now decompose the last integral as follows

$$|Tf(\overline{x}, y)| \leq \sum_{k=0}^\infty c^2\int_{|y|+kr}^{|y|+(k+1)r} |f(\overline{x}, s)|\frac{|y|^2}{(s-y)^3}ds.$$

Now by applying Minkowski's inequality for an infinite sum we get

$$\left(\int_{-a-r}^{-a} |Tf(\overline{x}, y)|^p dy\right)^{\frac{1}{p}} \leq c^2\sum_{k=0}^\infty \left(\int_{-a-r}^{-a}\left(\int_{|y|+kr}^{|y|+(k+1)r} |f(\overline{x}, s)|\frac{|y|^2}{(s-y)^3}ds\right)^p dy\right)^{\frac{1}{p}}.$$

Next we plan to estimate each summand. First we apply to it the change of variable $'y = -y'$

$$\left(\int_a^{a+r}\left(\int_{y+kr}^{y+(k+1)r}|f(\overline{x},s)|\frac{y^2}{(s+y)^3}ds\right)^p dy\right)^{\frac{1}{p}}$$

then we apply the change of variable $t = s/y$

$$\left(\int_a^{a+r}\left(\int_{1+kr/y}^{1+(k+1)r/y}|f(\overline{x},ty)|\frac{1}{(t+1)^3}dt\right)^p dy\right)^{\frac{1}{p}}.$$

that can be rewritten as

$$\left(\int_a^{a+r}\left(\int_{1+kr/(a+r)}^{1+(k+1)r/a}|f(\overline{x},ty)|\mathbb{1}_{(1+kr/y,1+(k+1)r/y)}(t)\frac{1}{(t+1)^3}dt\right)^p dy\right)^{\frac{1}{p}}.$$

By Minkowsi's integral inequality

$$\left(\int_a^{a+r}\dots\right)^{\frac{1}{p}} \le \int_{1+kr/(a+r)}^{1+(k+1)r/a}\left(\int_a^{a+r}|f(\overline{x},ty)|^p\mathbb{1}_{(1+kr/y,1+(k+1)r/y)}(t)\frac{1}{(t+1)^{3p}}dy\right)^{\frac{1}{p}}dt.$$

We notice that for every $t, y \in \mathbb{R}$ with $a \le y \le a+r$

$$\mathbb{1}_{(1+kr/y,1+(k+1)r/y)}(t) \le \mathbb{1}_{(a+kr,a+(k+2)r)}(ty)$$

hence using the change of variable $z = ty$

$$\begin{aligned}\left(\int_a^{a+r}\dots\right)^{\frac{1}{p}} &\le \int_{1+kr/(a+r)}^{1+(k+1)r/a}\left(\int_{a+kr}^{a+(k+2)r}|f(\overline{x},z)|^p\frac{1}{t(t+1)^{3p}}dz\right)^{\frac{1}{p}}dt \\ &= \int_{1+kr/(a+r)}^{1+(k+1)r/a}\frac{1}{t^{\frac{1}{p}}(t+1)^3}dt\left(\int_{a+kr}^{a+(k+2)r}|f(\overline{x},z)|^p dz\right)^{\frac{1}{p}} \\ &\le \int_{1+kr/(a+r)}^{1+(k+1)r/a}\frac{1}{(t+1)^3}dt\left(\int_{a+kr}^{a+(k+2)r}|f(\overline{x},z)|^p dz\right) \\ &= \frac{1}{2}\left[\frac{1}{(1+(k+1)r/a)^2}-\frac{1}{1+kr/(a+r)^2}\right]\left(\int_{a+kr}^{a+(k+2)r}|f(\overline{x},z)|^p dz\right)^{\frac{1}{p}} \\ &= \frac{s_k(a,r)}{2}\left(\int_{a+kr}^{a+(k+2)r}|f(\overline{x},z)|^p dz\right)^{\frac{1}{p}}.\end{aligned}$$

Plugging in this estimate in the infinite sum we get

$$\left(\int_{-a-r}^{-a} |Tf(\overline{x},y)|^p dy\right)^{\frac{1}{p}} \leq \frac{c^2}{2}\sum_{k=0}^{\infty} s_k(a,r)\left(\int_{a+kr}^{a+(k+2)r} |f(\overline{x},z)|^p dz\right)^{\frac{1}{p}}.$$

Integrating on F and applying again Minkowski inequality

$$\begin{aligned}\left(\int_F\int_{-a-r}^{-a} |Tf(\overline{x},y)|^p dy\right)^{\frac{1}{p}} &\leq \frac{c^2}{2}\sum_{k=0}^{\infty} s_k(a,r)\left(\int_F\int_{a+kr}^{a+(k+2)r} |f(\overline{x},z)|^p dz\right)^{\frac{1}{p}} \\ &\leq \frac{c^2}{2}\sum_{k=0}^{\infty} s_k(a,r)\left[\left(\int_{Q_k} |f(\overline{x},z)|^p dz\right)^{\frac{1}{p}} + \left(\int_{Q_{k+1}} |f(\overline{x},z)|^p dz\right)^{\frac{1}{p}}\right]\end{aligned}$$

where Q_i is the open cube $F \times (a+ir, a+(i+1)r)$. Dividing both sides by $\phi(r/2)^{\frac{1}{p}}$ we obtain

$$\left(\frac{1}{\phi(r/2)}\int_Q |Tf(\overline{x},y)|^p dy\right)^{\frac{1}{p}} \leq c^2\sum_{k=0}^{\infty} s_k(a,r)\|f\|_{M_{p,Q}(\mathbb{R}^n_+)}$$

We want now to estimate the series $\sum_{k=0}^{\infty} s_k(a,r)$, to do this we define $x = r/a$ that allows us to rewrite it as

$$\sum_{k=0}^{\infty} s_k(a,r) = \sum_{k=1}^{\infty} \frac{x(x+2)}{(kx+1)^2}.$$

To bound this series we distinguish two cases, when $x \leq 1$ and when $x > 1$. In the first case we can bound the series using a Riemann Sum

$$\sum_{k=1}^{\infty} \frac{x(x+2)}{(kx+1)^2} \leq 3\sum_{k=1}^{\infty} \frac{x}{(kx+1)^2} \leq 3\int_0^{\infty} \frac{1}{(t+1)^2} dt = 3.$$

In the second case

$$\sum_{k=1}^{\infty} \frac{x(x+2)}{(kx+1)^2} \leq \sum_{k=1}^{\infty} \frac{x(x+2)}{k^2x^2} = \sum_{k=1}^{\infty} \frac{1+\frac{2}{x}}{k^2} \leq 3\frac{\pi^2}{6} < 5.$$

Hence we get

$$\left(\frac{1}{\phi(r/2)}\int_Q |Tf(\overline{x},y)|^p dy\right)^{\frac{1}{p}} \leq 5c^2\|f\|_{M_{p,Q}(\mathbb{R}^n_+)}$$

that shows (1).

3. It's sufficient to notice that, up to a set of measure 0, we can cover Q with two open cubes Q_+, Q_- of side r with $Q_+ \subset \mathbb{R}^n_+$ and $Q_- \subset \mathbb{R}^n_-$. Hence

$$\left(\frac{1}{\phi(r/2)}\int_Q |Tf(x)|^p dx\right)^{\frac{1}{p}} \leq \left(\frac{1}{\phi(r/2)}\int_{Q_+} |f(x)|^p dx\right)^{\frac{1}{p}} + \left(\frac{1}{\phi(r/2)}\int_{Q_-} |Tf(x)|^p dx\right)^{\frac{1}{p}}$$

and we can conclude by part 1. and 2. $\square$