

# 1 Hestenes Operator

## 1.1 Construction

We construct the Hestenes operator for domains  $\Omega \subset \mathbb{R}^n$  with  $C^m$  boundary mainly following paragraphs 6.2, 6.3 of [2]. First we consider a simple case where  $\Omega$  is a  $C^m$  half strip.

**Lemma 1.** Let  $l, n, m \in \mathbb{N}, m \geq l, 1 \leq p \leq \infty$  and  $W = \prod_{i=1}^{n-1} ]a_i, b_i[$  be an open cuboid of  $\mathbb{R}^{n-1}$ . Moreover define

$$S = W \times \mathbb{R}$$

$$\Omega = \{(\bar{x}, x_n) | \bar{x} \in W, x_n < \phi(\bar{x})\}$$

where  $\phi \in C^m(\overline{W}), m \geq l$ , and  $\|D^\alpha \phi\| \leq M < \infty$  for every  $1 \leq |\alpha| \leq l$ . Then there exists a bounded extension operator  $T$  from  $W^{l,p}(\Omega)$  to  $W^{l,p}(S)$ .

To prove Lemma 1 we prove first the case  $\phi \equiv 0$  in the following result, that is a generalization of Lemma 9.2 in [1].

**Lemma 2.** Let  $l, n \in \mathbb{N}, 1 \leq p \leq \infty$  and  $W = \prod_{i=1}^{n-1} ]a_i, b_i[$  be an open cuboid of  $\mathbb{R}^{n-1}$ . There exists a bounded extension operator

$$T : W^{l,p}(S^-) \rightarrow W^{l,p}(S)$$

where

$$S = W \times \mathbb{R}$$

$$S^- = W \times \mathbb{R}^-.$$

*Proof.* Let  $f \in W^{l,p}(S^-)$ . We define

$$Tf(\bar{x}, x_n) = \begin{cases} f(x), & \text{if } x_n < 0, \\ \sum_{k=1}^l \alpha_k f(\bar{x}, -\beta_k x_n), & \text{if } x_n > 0, \end{cases}$$

where  $\alpha_k, \beta_k$  are real numbers that satisfy  $\beta_k > 0$  and

$$\sum_{k=1}^l \alpha_k (-\beta_k)^s = 1 \tag{1}$$

for every  $s = 0, \dots, l-1$ . Notice that given  $\beta_1, \dots, \beta_l > 0$  pairwise distinct, we can always find  $\alpha_1, \dots, \alpha_l$  that satisfy the condition by solving a Vandermonde square system of linear equations. First we prove that  $Tf \in W^{l,p}(S)$ . We take any  $\phi \in C_c^\infty(S)$  and consider the integral

$$\int_S Tf(x)D^\alpha \phi(x)dx = \int_{S^+} Tf(x)D^\alpha \phi(x)dx + \int_{S^-} Tf(x)D^\alpha \phi(x)dx$$

where  $S^+ = \{(\bar{x}, x_n) \mid \bar{x} \in W, x_n > 0\}$  and  $\alpha \in \mathbb{N}_0^n, 1 \leq |\alpha| \leq l$ . Let's write  $\alpha = (\bar{\alpha}, \alpha_n)$ , with  $\bar{\alpha} \in \mathbb{N}_0^{n-1}$  and  $\alpha_n \in \mathbb{N}_0$ . By changing variables in the integrals we get

$$\begin{aligned} \int_S Tf(x)D^\alpha \phi(x)dx &= \int_{S^+} \sum_{k=1}^l \alpha_k f(\bar{x}, -\beta_k x_n) D^\alpha \phi(x)dx + \int_{S^-} f(x)D^\alpha \phi(x)dx \\ &= \int_{S^-} f(\bar{y}, y_n) D^\alpha \psi(\bar{y}, y_n) dy \end{aligned} \quad (*)$$

where  $\psi(\bar{x}, x_n) = \sum_{k=1}^l -\alpha_k (-\beta_k)^{\alpha_n-1} \phi(\bar{x}, -x_n/\beta_k) + \phi(\bar{x}, x_n)$ . Note that  $\psi$  belongs to  $C^\infty(S^-)$  but does not have compact support in  $S^-$ . To bypass this problem we use an auxiliary function  $\nu \in C^\infty(\mathbb{R})$  that satisfies

$$\begin{cases} \nu(x) = 0, & \text{if } x > -1/2, \\ \nu(x) = 1, & \text{if } x < -1, \end{cases}$$

and we define the functions  $\nu_k(t) = \nu(kt)$  for  $k \in \mathbb{N}$ . It's clear that  $\psi(x)\nu_k(x_n) \in C_c^\infty(S^-)$ , hence we can integrate by parts

$$\int_{S^-} f(x)D^\alpha(\psi(x)\nu_k(x_n))dx = (-1)^{|\alpha|} \int_{S^-} D_w^\alpha f(x)\psi(x)\nu_k(x_n)dx \quad (2)$$

By the Leibniz rule

$$\begin{aligned} D^\alpha(\psi(x)\nu_k(x_n)) &= \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} D^{\bar{\alpha}}(\psi(x)\nu_k(x_n)) \\ &= \nu(kx_n)D^\alpha \psi(x) + \sum_{i=1}^{\alpha_n} \binom{\alpha_n}{i} k^i \nu^{(i)}(kx_n) \frac{\partial^{\alpha_n-i}}{\partial x_n^{\alpha_n-i}} D^{\bar{\alpha}} \psi(x). \end{aligned}$$

By the Dominated Convergence Theorem

$$\int_{S^-} f(x)\nu(kx_n)D^\alpha \psi(x)dx \rightarrow \int_{S^-} f(x)D^\alpha \psi(x)dx \text{ as } k \rightarrow \infty,$$

because  $f \in L^1(S^- \cap \text{supp } \psi)$  since  $\text{supp } \psi$  is bounded. Next, we claim that for every  $i = 1, \dots, \alpha_n$

$$\int_{S^-} f(x) k^i \nu^{(i)}(kx_n) \frac{\partial^{\alpha_n-i}}{\partial x_n^{\alpha_n-i}} D^{\bar{\alpha}} \psi(x) dx \rightarrow 0 \quad (3)$$

as  $k \rightarrow \infty$ . To prove this first we notice that since  $\alpha_k, \beta_k$  satisfies (1) we have that

$$\frac{\partial^j}{\partial x_n^j} D^{\bar{\alpha}} \psi(\bar{x}, 0) = 0 ; j = 0, \dots, \alpha_n - 1,$$

hence by Taylor formula

$$\left| \frac{\partial^{\alpha_n-i}}{\partial x_n^{\alpha_n-i}} D^{\bar{\alpha}} \psi(\bar{x}, x_n) \right| \leq \frac{C |x_n|^i}{i!},$$

for all  $i = 1, \dots, \alpha_n$ , where  $C = \sup_{x \in S^-} |D^{\alpha} \psi(x)|$ . Therefore we get the following estimate

$$\begin{aligned} \int_{S^-} \left| f(x) k^i \nu^{(i)}(kx_n) \frac{\partial^{\alpha_n-i}}{\partial x_n^{\alpha_n-i}} D^{\bar{\alpha}} \psi(x) \right| dx &\leq \frac{\tilde{C} C}{i!} \int_{\{x \in S^- \cap \text{supp } f, -1/k < x_n < 0\}} |f(x)| k^i |x_n|^i dx \\ &\leq \frac{\tilde{C} C}{i!} \int_{\{x \in S^- \cap \text{supp } f, -1 < x_n < 0\}} |f(x)| dx \end{aligned}$$

where  $\tilde{C} = \sup_{\mathbb{R}} |\nu^{(i)}|$ . The second inequality comes from the fact that  $\nu^{(i)}(x) = 0$  for  $x < -1$  and  $i \geq 1$ . Hence we get (3) by Dominated Convergence Theorem. Passing to the limit in (2) we obtain

$$\int_{S^-} f(x) D^{\alpha} \psi(x) dx = (-1)^{|\alpha|} \int_{S^-} D_w^{\alpha} f(x) \psi(x) dx.$$

which, combined with (\*), implies

$$\int_S T f(x) D^{\alpha} \phi(x) dx = \int_{S^-} f(x) D^{\alpha} \psi(x) dx = (-1)^{|\alpha|} \int_{S^-} D_w^{\alpha} f(x) \psi(x) dx.$$

Finally going back to the original coordinates and using the definition of  $\psi$  we get

$$\begin{aligned} \int_S T f(x) D^{\alpha} \phi(x) dx &= (-1)^{|\alpha|} \int_{S^-} D_w^{\alpha} f(x) \left[ \sum_{k=1}^l -\alpha_k (-\beta_k)^{\alpha_n-1} \phi\left(\bar{x}, -\frac{x_n}{\beta_k}\right) + \phi(\bar{x}, x_n) \right] dx = \\ &= (-1)^{|\alpha|} \int_{S^+} \sum_{k=1}^l \alpha_k (-\beta_k)^{\alpha_n} D_w^{\alpha} f(\bar{y}, -\beta_k y_n) \phi(y) dy + (-1)^{|\alpha|} \int_{S^-} D_w^{\alpha} f(y) \phi(y) dy \end{aligned}$$

that implies that  $D_w^\alpha T f$  exists and

$$D_w^\alpha T f(x) = \begin{cases} D_w^\alpha f(x), & \text{if } x \in S^-, \\ \sum_{k=1}^l \alpha_k (-\beta_k)^{\alpha_n} D_w^\alpha f(\bar{x}, -\beta_k x_n) \phi(x), & \text{if } x \in S^+. \end{cases}$$

It remains to prove the boundedness of  $T$ . It's immediate to verify that

$$\|T f\|_{L^p(S^+)} \leq \sum_{i=1}^l |\alpha_k| \beta_k^{-1/p} \|f\|_{L^p(S^-)}$$

and that we have similar bounds for the norm of the weak derivatives of  $T f$ . Hence there exists a constant  $C$  depending only on  $\beta_k, \alpha_k, l$  such that  $\|T f\|_{W^{l,p}(S^+)} \leq C \|f\|_{W^{l,p}(S^-)}$ . Observing that  $\|T f\|_{W^{l,p}(S)}^p = \|T f\|_{W^{l,p}(S^+)}^p + \|f\|_{W^{l,p}(S^-)}^p$  the proof is concluded.  $\square$

**Lemma 3.** Let  $l \in \mathbb{N}$  and  $\Omega$  be a domain in  $\mathbb{R}^n$ . Suppose that  $f \in L_{loc}^1(\Omega)$  admits all the weak derivatives up to order  $l$  and that  $g : \Omega' \rightarrow \Omega$  is a diffeomorphism of class  $C^l$  with bounded derivatives  $|D^\alpha g_k| \leq M$  for all  $1 \leq |\alpha| \leq l$ . Then  $f \circ g$  admits weak derivative up to order  $l$ . Moreover for every  $1 \leq |\alpha| \leq l$  we have to following bounds

$$|D^\alpha (f \circ g)(x)| \leq C \sum_{1 \leq |\beta| \leq |\alpha|} |D^\beta f(g(x))|$$

where  $C$  depends only on  $M$  and  $l$ .

*Proof.* We prove the statement by induction on  $l$ . For  $l = 1$  we know that exists a sequence of functions  $\{f_k\}_k \in C^\infty(\Omega)$  such that

$$\begin{aligned} f_k &\rightarrow f && \text{in } L_{loc}^1(\Omega) \\ \frac{\partial f_k}{\partial x_i} &\rightarrow \frac{\partial f}{\partial x_i} && \text{in } L_{loc}^1(\Omega). \end{aligned}$$

Take  $\phi \in C_c^\infty(\Omega')$  and integrate by parts

$$\int_{\Omega'} f_k(g(x)) \frac{\partial \phi}{\partial x_i}(x) dx = - \int_{\Omega'} \left( \sum_{j=1}^n \frac{\partial f_k}{\partial x_j}(g(x)) \frac{\partial g_j}{\partial x_i}(x) \right) \phi(x) dx.$$

Since  $\phi(g^{-1}) \in C_c^l(\Omega)$  and the derivatives of  $g$  and  $g^{-1}$  are bounded, we can pass to the limit in the above equation

$$\int_{\Omega'} f(g(x)) \frac{\partial \phi}{\partial x_i}(x) dx = - \int_{\Omega'} \left( \sum_{j=1}^n \frac{\partial f}{\partial x_j}(g(x)) \frac{\partial g_j}{\partial x_i}(x) \right) \phi(x) dx.$$

Hence the case  $l = 1$  is proved. Now suppose that the statement is true for  $l$ . We prove the case  $l + 1$ , so we suppose that  $f$  admits weak derivatives up to order  $l + 1$  and that  $g$  is of class  $C^{l+1}$ . From the case  $l = 1$  we know that  $\frac{\partial(f \circ g)}{\partial x_i}$  exists and that

$$\frac{\partial(f \circ g)}{\partial x_i} = \sum_{j=1}^n \left( \frac{\partial f}{\partial x_j} \circ g \right) \frac{\partial g_j}{\partial x_i}$$

Since  $\frac{\partial f}{\partial x_j}$  admits weak derivatives up to order  $l$ , by induction hypothesis the functions  $\frac{\partial f}{\partial x_j} \circ g$  admit weak derivatives up to order  $l$ . Moreover  $\frac{\partial g_j}{\partial x_i}$  is of class  $C^l$ , thus by the Leibniz rule the functions  $(\frac{\partial f}{\partial x_j} \circ g) \frac{\partial g_j}{\partial x_i}$  admits weak derivatives of order  $l$ . In conclusion  $\frac{\partial(f \circ g)}{\partial x_i}$  admits derivatives up to order  $l$  and this conclude the proof of the case  $l + 1$ .

To prove the bounds we notice that the weak derivatives  $D^\alpha(f \circ g)$  can be computed using the chain rule for usual derivatives. Such formula can be found in [3, formula B]:

$$D_w^\alpha(f(g))(x) = \sum_{1 \leq |\beta| \leq |\alpha|} D_w^\beta(f(g(x))) Q_{\alpha, \beta}(g, x)$$

In this formula  $Q_{\alpha, \beta}(g, x)$  are homogeneous polynomials of degree  $|\beta| \leq l$  in the derivatives of order less than  $l$  of the components of  $g$ . Moreover the coefficients of these polynomials depend only on  $\alpha, l, n$ . Hence there exists a constant  $C$  depending only on  $l, n, M$  such that  $|Q_{\alpha, \beta}(g, x)| \leq C$  uniformly on  $x$ . This concludes the proof.  $\square$

*Proof of Lemma 1 .* Let  $f \in W^{l,p}(\Omega)$ . Consider the function  $g$  from  $S^-$  onto  $\Omega$  defined by

$$g(\bar{x}, x_n) = (\bar{x}, x_n + \phi(\bar{x}))$$

for all  $(\bar{x}, x_n) \in S^-$  and its inverse  $g^{-1}$

$$g^{-1}(\bar{x}, x_n) = (\bar{x}, x_n - \phi(\bar{x}))$$

where  $S^- = W \times \mathbb{R}^-$ . For all  $f \in W^{l,p}(\Omega)$  we set

$$Gf = f \circ g$$

Since  $g$  is a diffeomorphism between  $S^-$  and  $\Omega$  of class  $C^m$ , Lemma 3 guarantees that  $Gf$  admits weak derivatives up to order  $l$ . We claim that  $G$  defines a bounded operator from  $W^{l,p}(\Omega)$  to  $W^{l,p}(S^-)$ , with bounded inverse. To prove this, first we compute the Jacobian matrix of  $g^{-1}$

$$Jg^{-1}(x) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ \vdots & & & \ddots \\ -\frac{\partial \phi(\bar{x})}{\partial x_1} & -\frac{\partial \phi(\bar{x})}{\partial x_2} & \dots & \dots & 1 \end{bmatrix}$$

from which  $|\det(Jg^{-1}(x))| \equiv 1$ . Moreover, again by Lemma 3, we have

$$|D_w^\alpha(f(g))| \leq C(l, M) \sum_{1 \leq |\beta| \leq |\alpha|} |D_w^\beta f(g)|$$

where  $C(l, M)$  depends only on  $l$  and  $M$ , with  $M = \sup_{1 \leq |\alpha| \leq l} \|D^\alpha \phi\|$ . Next by the change of variable formula and Minkowski's inequality we get

$$\begin{aligned} \left( \int_{S^-} |D_w^\alpha(f(g))(x)|^p dx \right)^{\frac{1}{p}} &\leq \sum_{1 \leq |\beta| \leq |\alpha|} C(l, M) \left( \int_{S^-} |D_w^\beta f(g(x))|^p dx \right)^{\frac{1}{p}} \\ &= \sum_{1 \leq |\beta| \leq |\alpha|} C(l, M) \left( \int_{\Omega} |D_w^\beta f(y)|^p |\det Jg^{-1}|_{g(y)} dy \right)^{\frac{1}{p}} \\ &= \sum_{1 \leq |\beta| \leq |\alpha|} C(l, M) \|D_w^\beta f\|_{L^p(\Omega)} \end{aligned}$$

Thus, using the estimates for the intermediate derivatives, that

$$\|Gf\|_{W^{l,p}(S^-)} = \|f(g)\|_{W^{l,p}(S^-)} \leq C \|f\|_{W^{l,p}(\Omega)}$$

for a constant  $C$  independent of  $f$ . In a similar way we can also prove that

$$\|G^{-1}f\|_{W^{l,p}(\Omega)} = \|f(g^{-1})\|_{W^{l,p}(\Omega)} \leq D \|f\|_{W^{l,p}(S)}.$$

Now we can just define the operator  $T$  as

$$T = G^{-1} \circ \bar{T} \circ G$$

where  $\bar{T}$  is the extension operator from  $W^{l,p}(S^-)$  to  $W^{l,p}(S)$  defined in Lemma 2. Therefore  $T$  is bounded as composition of bounded operators. An explicit for for  $T$  is

$$Tf(x) = \begin{cases} f(x), & \text{if } x \in \Omega, \\ \sum_{i=1}^l \alpha_i f(\bar{x}, \phi(\bar{x}) - \beta_i(x_n - \phi(\bar{x}))), & \text{if } x \in S \setminus \bar{\Omega}. \end{cases}$$

□

We are now ready to define the Hestenes operator for a general domain  $\Omega$  with  $C^m$  boundary. First we write the precise definition for this kind of domains.

**Definition 1.** Let  $0 < d \leq D < \infty, M > 0, \varkappa > 0$  We say that an open set  $\Omega$  in  $\mathbb{R}^n$  has a resolved boundary with parameters  $d, D, \varkappa$  if there exists a family of open cuboids  $V_i, i = 1, \dots, s$  (where  $s \in \mathbb{N}$  if  $\Omega$  is bounded and  $s = \infty$  otherwise) such that

1.  $(V_i)_d \cap \Omega \neq \emptyset$
2.  $\Omega \subset \bigcup_{j=1}^s (V_j)_d$
3. The multiplicity of the cover  $\{V_i\}_{i=1}^s$  is less than  $\varkappa$ .
4. There exist isometries  $\lambda_i$  of  $\mathbb{R}^n$  such that

$$\lambda_j(V_j) = \prod_{i=1}^n ]a_{ij}, b_{ij}[$$

and, if  $\partial V_j \cap \Omega \neq \emptyset$ ,

$$\lambda_j(V_j \cap \Omega) = \{(\bar{x}, x_n) \in \mathbb{R}^n | \bar{x} \in W_j, a_{nj} + d < x_n < \phi_j(\bar{x})\}$$

where  $W_j = \prod_{i=1}^{n-1} ]a_{ij}, b_{ij}[$  and  $\phi_j : W_j \rightarrow \mathbb{R}$ .

Moreover

- if  $\phi_j \in C^m(\overline{W}_i)$  with  $\|D^\alpha \phi_j\| \leq M < \infty$ , for every  $1 \leq |\alpha| \leq m$ , we say that  $\Omega$  has a resolved  $C^m$  boundary with parameters  $d, D, \varkappa, M$ .
- if  $\phi_j \in \text{Lip}(\overline{W}_i)$  with  $\text{Lip}(\phi) = M$ , we say that  $\Omega$  has a resolved Lipschitz boundary with parameters  $d, D, \varkappa, M$ .

Finally we will say that a domain  $\Omega$  has a resolved  $C^m$  (or Lipschitz) boundary if there exist parameters  $d, D, \varkappa, M$  for which  $\Omega$  has a  $C^m$  (or Lipschitz) boundary.

**Remark 1.** In the notation of Lemma 1, let  $a, b \in \mathbb{R}$  such that  $a < \phi(\bar{x}) < b$  for every  $\bar{x} \in W$ . We define  $S^{a,b} = W \times (a, b)$ ,  $\Omega_a = \Omega \cap (W \times (a, \infty))$  and  $\widehat{W}^{l,p}(\Omega_a) = \{f \in W^{l,p}(\Omega_a) \mid \text{supp } f \subset S\}$ . Then exists a bounded extension operator

$$T : \widehat{W}^{l,p}(\Omega_a) \rightarrow W^{l,p}(S^{a,b}).$$

To see this we can just extend  $f \in \widehat{W}^{l,p}(\Omega_a)$  naturally by 0 to  $f_0 \in W^{l,p}(\Omega)$  and then define

$$Tf = (\tilde{T}f_0)|_{S^{a,b}}$$

where  $\tilde{T}$  is the operator of the previous Lemmma .

**Theorem 1.** Let  $m, l \in \mathbb{N}, l \leq m$  and  $1 \leq p \leq \infty$ . If  $\Omega$  is a domain in  $\mathbb{R}^n$  has a  $C^m$  resolved boundary then there exists a bounded extension operator

$$T : W^{l,p}(\Omega) \rightarrow W^{l,p}(\mathbb{R}^n).$$

*Proof Sketch.* Let  $f \in W^{l,p}(\Omega)$ . Let  $\{V_i\}_{i=1}^s$  be the covering of cuboids for  $\Omega$  as in Definition 1. It's possible to construct functions  $\{\psi_i\}_{i=1}^s \subset C_c^\infty(\mathbb{R}^n)$  such that the functions  $\{\psi_i^2\}_{i=1}^s$  form a partition of the unity corresponding to the covering  $\{V_i\}_{i=1}^s$  and satisfying  $\|D^\alpha \psi_i\|_{L^\infty} \leq M_1$  with  $M_1$  depending only on  $n, l, d$ . If  $\partial\Omega \cap V_i \neq \emptyset$  by Remark 1 there exists a bounded operator

$$T_i : \widehat{W}^{l,p}(\lambda_i(\Omega \cap V_i)) \rightarrow W^{l,p}(\lambda_i(V_i))$$

where  $\widehat{W}^{l,p}(\lambda_i(V_i \cap \Omega)) = \{f \in W^{l,p}(V_i \cap \Omega) \mid \text{supp } f \subset \lambda_i(V_i)\}$ . If  $V_i \subset \Omega$  the operator  $T_i$  is defined to be just the identity. We set

$$Tf = \sum_{i=1}^s \psi_i T_i(\psi_i f(\lambda_i^{-1}))(\lambda_i).$$



assuming  $(\psi_i T_i(\psi_i f(\lambda_i^{-1})))(\lambda_i) = 0$  outside  $V_i$ . The functions  $\psi_i f \in W^{l,p}(V_i \cap \Omega)$  are such that  $\text{supp } \psi_i f \subset \bar{\Omega} \cap V_i$ , hence  $\psi_i f(\lambda_i) \in \widehat{W}^{l,p}(\lambda_i(V_i \cap \Omega))$  and so  $T$  is well defined. To see that  $T$  is an extension operator, take  $x \in \Omega$ : if  $x \in \text{supp } \psi_i$  then  $\psi_i(x) T_i(\psi_i f(\lambda_i^{-1}))(\lambda_i(x)) = \psi_i(x)^2 f(x)$ ; if  $x \notin \text{supp } \psi_i$  then  $0 = \psi_i(x) T_i(\psi_i f(\lambda_i^{-1}))(\lambda_i(x)) = \psi_i(x)^2 f(x)$ . So  $Tf(x) = \sum_{i=1}^s \psi_i^2(x) f(x) = f(x)$ .

We omit the proof of the boundedness of  $T$ , the details of which can be found in the proofs of Lemma 13-14 in [2].  $\square$

## 1.2 Hestenes operator on Morrey spaces

**Definition 2.** Let  $1 \leq p < \infty$ ,  $\phi$  a function from  $\mathbb{R}^+$  to  $\mathbb{R}^+$  and  $\Omega$  be a domain in  $\mathbb{R}^n$ . For a function  $f \in L_{loc}^p(\Omega)$  we define the Morrey space as

$$M_p^\phi(\Omega) = \{f \in L_{loc}^p(\Omega) \mid \|f\|_{M_p^\phi(\Omega)} < \infty\}$$

where

$$\|f\|_{M_p^\phi(\Omega)} := \sup_{B_r(x), x \in \Omega, r > 0} \left( \frac{1}{\phi(r)} \int_{B_r(x) \cap \Omega} |f(y)|^p dy \right)^{\frac{1}{p}}.$$

**Lemma 4.** Let  $k \geq 1$  and  $\Omega$  be set in  $\mathbb{R}^n$  with diameter  $D > 0$ . Then there exists an integer  $C_{n,k}$  depending only on  $k$  and  $n$  such that  $\Omega$  can be covered by a collection of open balls  $B_1, \dots, B_h$  centered in  $\Omega$  with radius  $D/k$  and  $h \leq C_{k,n}$ .

*Proof.* We start by claiming that if  $S$  is a set of points in  $\mathbb{R}^n$  satisfying

- i)  $S \subset \Omega$ ,
- ii)  $\|z_1 - z_2\| \geq D/k$  for every  $z_1, z_2 \in S$  with  $z_1 \neq z_2$ ,

then  $|S| \leq C_{n,k}$  where  $C_{k,n}$  is an integer depending only on  $k$  and  $n$ . To see this, first note that  $\Omega$  is contained in some closed cube  $Q$  of side  $2D$ . Then we choose  $m \in \mathbb{N}$  such that  $2^{m-1} > \sqrt{n}k$ . Next we cover  $Q$  with  $(2^m)^n$  small closed cubes of side  $2D/2^m$ . The diagonal of a small cube measures  $2D/2^m \cdot \sqrt{n} < D/k$ . Thus each of these cubes can contain at most one point of  $S$ , so  $|S| \leq (2^m)^n$ . Therefore it's enough to choose  $C_{n,k} = 2^{mn}$ . Set  $r := D/k$ , we'll prove that we can cover  $\Omega$  with a collection of balls  $B_1, \dots, B_h$  centered in  $\Omega$  of radius  $r$  and such that  $h \leq C_{n,k}$ . Choose  $x_1 \in \Omega$  and take

$B_1 = B_r(x_1)$ , the ball centered in  $x_1$  of radius  $r$ . If  $\Omega \subset B_1$  we are done, if not there exists  $x_2 \in \Omega \setminus B_1$  and we take  $B_2 = B_r(x_2)$ . Again, if  $\Omega \subset (B_1 \cup B_2)$  we stop, else we can pick  $x_3 \in \Omega \setminus (B_1 \cup B_2)$  and take  $B_3 = B_r(x_3)$ . We iterate this procedure : given  $B_1, \dots, B_i$  balls, if  $\Omega \subset (B_1 \cup \dots \cup B_i)$  we stop, else we can choose  $x_{i+1} \in \Omega \setminus (B_1 \cup \dots \cup B_i)$  and take  $B_{i+1} = B_r(x_{i+1})$ . We claim that this procedure stops with  $i \leq C_{n,k}$ . Suppose it doesn't, then we can find  $B_1, \dots, B_{C_{n,k}+1}$  balls centered respectively at  $x_1, \dots, x_{C_{n,k}+1}$ . Setting  $S = \{x_1, \dots, x_{C_{n,k}+1}\}$ , it's immediate to see that  $S$  satisfies i) and ii), but  $|S| = C_{n,k} + 1$ , that is a contradiction.  $\square$

**Lemma 5.** Let  $W \subset \mathbb{R}^{n-1}$  be open connected and define

$$\Omega = \{(\bar{x}, x_n) \mid \bar{x} \in W, x_n \leq \phi(\bar{x})\}$$

$$\Omega^+ = \{(\bar{x}, x_n) \mid \bar{x} \in W, x_n > \phi(\bar{x})\}$$

where  $\phi \in \text{Lip}(\overline{W})$ . Let  $\beta > 0$  and consider the function  $A_\beta$  from  $W \times \mathbb{R}$  to  $\Omega$  defined by

$$A_\beta(\bar{x}, x_n) = \begin{cases} (\bar{x}, \phi(\bar{x}) - \beta(x_n - \phi(\bar{x}))), & \text{if } (\bar{x}, x_n) \in \Omega^+, \\ (\bar{x}, x_n), & \text{if } (\bar{x}, x_n) \in \Omega. \end{cases}$$

Then for every  $x_0 \in W \times \mathbb{R}$  and  $r > 0$

$$A(B_r(x_0) \cap \Omega^+) \subset B_{cr}(A(x_0)) \cap \Omega$$

where  $c \geq 1$  is a constant depending only on  $\text{Lip } \phi$  and  $\beta$ .

*Proof.* Notice that it is sufficient to prove that for every  $x, y \in W \times \mathbb{R}$  we have

$$\|A(x) - A(y)\| \leq c\|x - y\|. \quad (4)$$

Set  $M = \text{Lip } \phi$ . We distinguish three cases: 1.  $x, y \in \Omega$  : in this case  $A(x) = x$  and  $A(y) = y$ , so  $\|x - y\| = \|A(x) - A(y)\|$  and there is nothing to prove.

2.  $x, y \in \Omega^+$  : we have

$$\begin{aligned} |A(x)_n - A(y)_n| &= |\phi(\bar{x}) - \beta(x_n - \phi(\bar{x})) - \phi(\bar{y}) + \beta(y_n - \phi(\bar{y}))| \\ &\leq (1 + \beta)|\phi(\bar{x}) - \phi(\bar{y})| + \beta|x_n - y_n| \\ &\leq M(1 + \beta)\|\bar{x} - \bar{y}\| + \beta|x_n - y_n| \end{aligned}$$

Hence

$$\begin{aligned}
\|A(x) - A(y)\|^2 &= \|\overline{A(x)} - \overline{A(y)}\|^2 + |A(x)_n - A(y)_n|^2 \\
&\leq \|\bar{x} - \bar{y}\|^2 + [M(1 + \beta)\|\bar{x} - \bar{y}\| + \beta|x_n - y_n|]^2 \\
&\leq (1 + 2M^2(1 + \beta)^2)\|\bar{x} - \bar{y}\|^2 + 2\beta^2|x_n - y_n|^2 \\
&\leq c_1^2(M, \beta)\|x - y\|^2
\end{aligned}$$

for some constant  $c_1(M, \beta)$ .

3.  $x \in \Omega^+, y \in \Omega$  : first notice that, since  $\phi(\bar{x}) < x_n$ , then  $x_n - y_n > \phi(\bar{x}) - y_n$ . Moreover  $\phi(\bar{y}) > y_n$ , hence  $M\|\bar{x} - \bar{y}\| \geq \phi(\bar{y}) - \phi(\bar{x}) > y_n - \phi(\bar{x})$ . This implies

$$|\phi(\bar{x}) - y_n| < |x_n - y_n| + M\|\bar{x} - \bar{y}\|.$$

Now

$$\begin{aligned}
|A(x)_n - A(y)_n| &= |\phi(\bar{x}) - \beta(x_n - \phi(\bar{x})) - y_n| \\
&= |(1 + \beta)(\phi(\bar{x}) - y_n) + \beta(y_n - x_n)| \\
&\leq M(1 + \beta)\|\bar{x} - \bar{y}\| + (1 + 2\beta)|x_n - y_n|
\end{aligned}$$

and

$$\begin{aligned}
\|A(x) - A(y)\|^2 &= \|\overline{A(x)} - \overline{A(y)}\|^2 + |A(x)_n - A(y)_n|^2 \\
&\leq \|\bar{x} - \bar{y}\|^2 + [M(1 + \beta)\|\bar{x} - \bar{y}\| + (1 + 2\beta)|x_n - y_n|]^2 \\
&\leq (1 + 2M^2(1 + \beta)^2)\|\bar{x} - \bar{y}\|^2 + 2(1 + 2\beta)^2|x_n - y_n|^2 \\
&\leq c_2^2(M, \beta)\|x - y\|^2.
\end{aligned}$$

for some constant  $c_2(M, \beta)$ . Then (4) by taking  $c = \max(\sqrt{c_1}, \sqrt{c_2}, 1)$ .  $\square$

**Lemma 6.** Let  $l, n, m \in \mathbb{N}, m \geq l, 1 \leq p \leq \infty, W = \prod_{i=1}^{n-1} ]a_i, b_i[$  be an open cuboid of  $\mathbb{R}^{n-1}$  and  $\phi$  a function from  $\mathbb{R}^+$  to  $\mathbb{R}^+$ . Moreover define

$$S = W \times \mathbb{R}$$

$$\Omega = \{(\bar{x}, x_n) | \bar{x} \in W, x_n < \phi(\bar{x})\}$$

where  $\phi \in C^m(\overline{W})$  and  $\|D^\alpha \phi\| \leq M < \infty$  for every  $1 \leq |\alpha| \leq l$ . Then for every  $f \in W^{l,p}(\Omega)$

$$\text{i)} \quad \|Tf\|_{M_p^\phi(S)} \leq C\|f\|_{M_p^\phi(\Omega)}$$

$$\text{ii)} \quad \|D_w^\alpha Tf\|_{M_p^\phi(S)} \leq C \sum_{1 \leq |\beta| \leq |\alpha|} \|D_w^\beta f\|_{M_p^\phi(\Omega)}, \quad 1 \leq |\alpha| \leq l$$

where  $T$  is the Hestenes operator defined in Lemma 1 and  $C$  is a constant independent of  $f$ .

*Proof.* Define  $\Omega^+ = \{(\bar{x}, x_n) \mid \bar{x} \in W, x_n > \phi(\bar{x})\}$ . We recall the definition of  $T$

$$Tf(x) = \begin{cases} f(x) & x \in \Omega \\ \sum_{i=1}^l \alpha_k f(\bar{x}, \phi(\bar{x}) - \beta_k(x_n - \phi(\bar{x}))) & x \in \Omega^+ \end{cases}$$

and observe that we can rewrite it as

$$Tf(x) = \begin{cases} f(x), & \text{if } x \in \Omega, \\ \sum_{i=1}^l \alpha_k f(G_k(x)), & \text{if } x \in \Omega^+, \end{cases}$$

where  $G_k(\bar{x}, x_n) = (\bar{x}, \phi(\bar{x}) - \beta_k(x_n - \phi(\bar{x})))$ . Note that  $G_k : \Omega^+ \rightarrow \Omega$  defines a diffeomorphism from  $\Omega^+$  to  $\Omega$  of class  $C^m$  and satisfying  $|\det JG_k^{-1}| \equiv 1/\beta_k$ . First we prove ii). Let's fix  $x_0 \in S$  and a radius  $r > 0$ . We want to estimate the quantity

$$I = \left( \frac{1}{\phi(r)} \int_{B_r(x_0) \cap S} |D_w^\alpha Tf(x)|^p dx \right)^{\frac{1}{p}}$$

for  $1 \leq |\alpha| \leq l$ . To do this we estimate the integral as follows

$$I \leq \underbrace{\left( \frac{1}{\phi(r)} \int_{B_r(x_0) \cap \Omega^+} |D_w^\alpha Tf(x)|^p dx \right)^{\frac{1}{p}}}_{I_1} + \underbrace{\left( \frac{1}{\phi(r)} \int_{B_r(x_0) \cap \Omega} |D_w^\alpha Tf(x)|^p dx \right)^{\frac{1}{p}}}_{I_2}.$$

Since  $Tf(x) = f(x)$  when  $x \in \Omega$ , we have immediately

$$I_2 \leq \|D_w^\alpha f\|_{M_p^\phi(\Omega)}.$$

It remains to estimate  $I_1$ . We start by observing that from Lemma 3 there exists a constant  $C_k$  depending only on  $G_k$  and  $l$  such that

$$|D_w^\alpha(f \circ G_k)| \leq C_k \sum_{1 \leq |\beta| \leq |\alpha|} |D_w^\beta f(G_k)|.$$

By the previous inequality and Lemma 5 we are able to produce the following bound

$$\begin{aligned} \frac{\|D_w^\alpha(f \circ G_k)\|_{L^p(B_r(x_0) \cap \Omega^+)}}{\phi(r)^{\frac{1}{p}}} &\leq C_k \sum_{1 \leq |\beta| \leq |\alpha|} \left( \phi(r)^{-1} \int_{G_k(B_r(x_0) \cap \Omega^+)} |D_w^\beta f(y)|^p |\det JG_k^{-1}|_{G_k(y)} dy \right)^{\frac{1}{p}} \\ &\leq C_k \beta_k^{-\frac{1}{p}} \sum_{1 \leq |\beta| \leq |\alpha|} \left( \phi(r)^{-1} \int_{B_{c_k r}(A_{\beta_k}(x_0)) \cap \Omega} |D_w^\beta f(y)|^p dy \right)^{\frac{1}{p}} \end{aligned}$$

where  $A_{\alpha_k}$  is defined as in Lemma 5 and  $c_k$  depends only on  $\beta_k$  and  $M$ . By Lemma 4 the set  $B_{c_k r}(A_{\beta_k}(x_0)) \cap \Omega$  can be covered with a collection of open balls  $B_1, \dots, B_h$  centered in  $\Omega$  with radius  $r$  and  $h \leq m_k$ , where  $m_k$  depends only on  $c_k$ . Hence we get

$$\frac{\|D_w^\alpha(f \circ G_k)\|_{L^p(B_r(x_0) \cap \Omega^+)}}{\phi(r)^{\frac{1}{p}}} \leq C_k \beta_k^{-\frac{1}{p}} m_k \sum_{1 \leq |\beta| \leq |\alpha|} \|D_w^\beta f\|_{M_p^\phi(\Omega)}$$

Next we estimate  $I_1$ :

$$\begin{aligned} I_1 &= \phi(r)^{-\frac{1}{p}} \|D_w^\alpha T f\|_{L^p(B_r(x_0) \cap \Omega^+)} \leq \phi(r)^{-\frac{1}{p}} \sum_{k=1}^l \alpha_k \|D_w^\alpha f(G_k)\|_{L^p(B_r(x_0) \cap \Omega^+)} \\ &\leq \sum_{k=1}^l \alpha_k C_k \beta_k^{-\frac{1}{p}} m_k \left( \sum_{1 \leq |\beta| \leq |\alpha|} \|D_w^\beta f\|_{M_p^\phi(\Omega)} \right). \end{aligned}$$

Finally putting the estimates of  $I_1, I_2$  together

$$\begin{aligned} \|D_w^\alpha T f\|_{M_p^\phi(S)} &= \sup_{x_0 \in S, r > 0} \left( \frac{1}{\phi(r)} \int_{B_r(x_0) \cap S} |D_w^\alpha T f(x)|^p dx \right)^{\frac{1}{p}} \\ &\leq \|D_w^\alpha f\|_{M_p^\phi(\Omega)} + \sum_{k=1}^l \alpha_k C_k \beta_k^{-\frac{1}{p}} m_k \left( \sum_{1 \leq |\beta| \leq |\alpha|} \|D_w^\beta f\|_{M_p^\phi(\Omega)} \right) \\ &\leq \tilde{C} \sum_{1 \leq |\beta| \leq |\alpha|} \|D_w^\beta f\|_{M_p^\phi(\Omega)} \end{aligned}$$

where  $\tilde{C}$  depends only on  $\{b_k\}_k, \{\alpha_k\}_k, l, M, p$ . This proves ii). The proof of i) is exactly analogous to the proof of ii).  $\square$

**Theorem 2.** Let  $m, l \in \mathbb{N}, l \leq m, 1 \leq p \leq \infty$ ,  $\phi$  a function from  $\mathbb{R}^+$  to  $\mathbb{R}^+$  and  $\Omega$  a domain in  $\mathbb{R}^n$  with  $C^m$  resolved boundary. Then for every  $f \in W^{l,p}(\Omega)$

$$\text{i) } \|Tf\|_{M_p^\phi(\mathbb{R}^n)} \leq C\|f\|_{M_p^\phi(\Omega)}$$

$$\text{ii) } \|D_w^\alpha Tf\|_{M_p^\phi(\mathbb{R}^n)} \leq C \sum_{1 \leq |\beta| \leq |\alpha|} \|D_w^\beta f\|_{M_p^\phi(\Omega)}, \quad 1 \leq |\alpha| \leq l$$

where  $T$  is the Hestenes operator defined in Theorem 1 and  $C$  doesn't depend on  $f$ .

*Proof.* Let  $f \in W^{l,p}(\Omega)$  and  $\{V_i\}_{i=1}^s$  be the covering of cuboids for  $\Omega$  as in the definition of set with resolved boundary. We recall the definition of  $T$  :

$$Tf = \sum_{i=1}^s \psi_i T_i(\psi_i f(\lambda_i^{-1}))(\lambda_i)$$

where  $\{\psi_i^2\}_{i=1}^s$  form a partition of the unity corresponding to the covering  $\{V_i\}_{i=1}^s$  and satisfying  $\|D^\alpha \psi_i\|_{L^\infty} \leq M_1$ , with  $|\alpha| \leq l$  and  $M_1$  depending only on  $n, l, d$ . To make the notation simpler we will rewrite  $T$  as

$$Tf = \sum_{i=1}^s \psi_i \tilde{T}_i(\psi_i f)$$

where the operator  $\tilde{T}_i$  is defined as  $\tilde{T}_i f = T_i(f(\lambda_i^{-1}))(\lambda_i)$ . Before starting the proof we remark some facts that will be justified at the end of the proof:

a) Let  $C_i$  the constant such that

$$\begin{aligned} \|T_i g\|_{M_p^\phi(\lambda_i(V_i))} &\leq C_i \|g\|_{M_p^\phi(\lambda_i(\Omega \cap V_i))} \\ \|D_w^\alpha T_i g\|_{M_p^\phi(\lambda_i(V_i))} &\leq C_i \sum_{1 \leq |\beta| \leq |\alpha|} \|D_w^\beta g\|_{M_p^\phi(\lambda_i(\Omega \cap V_i))} \end{aligned}$$

for  $1 \leq |\alpha| \leq l$  and  $g \in \widehat{W}^{l,p}(\lambda_i(\Omega \cap V_i))$ . Then  $\sup_{i=1,\dots,s} C_i \leq M_2$ , where  $M_2$  depends only on  $\Omega, l, n$ .

b) We have

$$\begin{aligned} \|\tilde{T}_i g\|_{M_p^\phi(V_i)} &\leq C_i \|g\|_{M_p^\phi(\Omega \cap V_i)} \\ \|D_w^\alpha \tilde{T}_i g\|_{M_p^\phi(V_i)} &\leq M_3 C_i \sum_{1 \leq |\beta| \leq |\alpha|} \|D_w^\beta g\|_{M_p^\phi(\Omega \cap V_i)} \end{aligned}$$

for  $1 \leq |\alpha| \leq l$  and  $g \in \widehat{W}^{l,p}(\Omega \cap V_i)$  and where  $M_3$  doesn't depend on  $i$ .

$$\begin{aligned} \left( \frac{1}{\phi(r)} |Tf(x)|^p dx \right)^{\frac{1}{p}} &\leq \left( \frac{1}{\phi(r)} \int_B \left| \sum_{i=1}^s \psi_i \tilde{T}_i(f(\psi_i))(x) \right|^p dx \right)^{\frac{1}{p}} \\ &\leq \sum_{i \in J} \left( \frac{1}{\phi(r)} \int_{B \cap V_i} |\tilde{T}_i(f(\psi_i))(x)|^p dx \right)^{\frac{1}{p}} \end{aligned}$$

Let's now prove a),b),c),d),e).

a)  $\Omega$  has a resolved  $C^m$  boundary with parameters  $\varkappa, d, D, M$ . Hence, if  $\phi_i$  are the  $C^m$  functions of Definition 1, we have  $\|D^\alpha \phi_i\| \leq M$  for every  $i$  and for every  $1 \leq |\alpha| \leq l$ . Therefore by the proof of Lemma 6 we deduce that  $C_i$  depends only on  $l, n, M$  and on the choice of the constants  $\alpha_k, \beta_k$ , which can be chosen to be the same for every  $T_i$ . b) We notice that since  $\lambda_i$  are isometries, they are smooth and their derivatives are uniformly bounded with a bound depending only on  $n$ . Then the result follows from a straightforward computation using a change of variable and the Leibniz rule for derivatives. c) We have that

$$\sum_k |f|^p \mathbb{1}_{X_k} \leq \delta |f|^p.$$

Then it's enough to integrate on  $X$  and raise to the power  $1/p$ . d) A proof can be found in [2, Lemma 13]. e) For every  $N \in \mathbb{N}$  and every  $t \in T$  we have

$$\sum_{n=1}^N a_n(t) \leq \sum_{n=1}^N \sup_{t \in T} a_n(t),$$

which letting  $N \rightarrow \infty$  gives

$$\sum_{n=1}^{\infty} a_n(t) \leq \sum_{n=1}^{\infty} \sup_{t \in T} a_n(t).$$

Applying the sup on the left-hand side we obtain the result. □

## 2 Hestenes operator

### 2.1 Construction

In this section we will define the Stein extension operator for Lipschitz domains in  $\mathbb{R}^n$ . The details of the construction and the proofs of all the following results can be found in [4, Section 2-3, Ch. VI]. We start by introducing the notion of regularized distance with the following theorem.

**Theorem 3.** Let  $F$  be a closed set in  $\mathbb{R}^n$  and denote  $d(x, F)$  the distance of  $x$  from  $F$ . Then there exists a function  $\Delta(x) = \Delta(x, F)$  defined in  $F^c$  such that

$$\text{a) } c_1 d(x, F) \leq \Delta(x) \leq c_2 d(x, F),$$

$$\text{b) } \Delta(x) \text{ is } C^\infty \text{ in } F^c \text{ and}$$

$$|D^\alpha \Delta(x)| \leq B_\alpha d(x, F)^{1-|\alpha|},$$

where  $B_\alpha, c_1, c_2$  are constants independent of  $F$  and  $d(x, F)$  is the distance of  $x$  from  $F$ .

Next we give the definition of a Lipschitz subgraph

**Definition 3.** An subset  $\Omega$  of  $\mathbb{R}^n$  it's said to be a Lipschitz subgraph if exists a Lipschitz function  $\psi$  defined from  $\mathbb{R}^{n-1}$  to  $\mathbb{R}$  such that

$$\Omega = \{(\bar{x}, y) \in \mathbb{R}^n \mid \psi(\bar{x}) < y\}.$$

Moreover the constant  $\text{Lip } \psi$  is said to be the Lipschitz bound of  $\Omega$ .

It's convenient to define first the Stein extension operator in the case of a Lipschitz subgraph, to do this we need the following two lemmas.

**Lemma 7.** Let  $\Omega$  be a Lipschitz subgraph of  $\mathbb{R}^n$  and set  $F = \bar{\Omega}$ . Suppose  $\Delta(\bar{x}, y)$  is the regularized distance from  $F$  as given in Theorem 3. Then there exists a constant  $c$ , which depends only on the Lipschitz bound of  $\Omega$ , so that if  $(\bar{x}, y) \in F^c$ , then  $c\Delta(\bar{x}, y) \geq \psi(\bar{x}) - y$ .

**Lemma 8.** There exists a continuous function  $\tau$  defined in  $[1, \infty)$  satisfying

$$\text{i) } \tau(\lambda) = O(\lambda^N), \text{ as } \lambda \rightarrow \infty \text{ for every } N,$$



ii)  $\int_1^\infty \tau(\lambda) d\lambda = 1$ ,  $\int_1^\infty \lambda^k \tau(\lambda) d\lambda = 0$ , for every  $k = 1, 2, \dots$

**Definition 4.** Let  $\Omega$  be a Lipschitz subgraph of  $\mathbb{R}^n$ . Moreover let  $\tau$  be the function in Lemma 8 and  $c$  the constant of Lemma 7. For every function  $f$  that is  $C^\infty$  in  $\Omega$ , continuous and bounded in  $\overline{\Omega}$  together with all its partial derivatives, we define

$$Tf(\overline{x}, y) = \begin{cases} f(\overline{x}, y), & \text{if } y \geq \psi(\overline{x}) \\ \int_1^\infty f(\overline{x}, y + \lambda \delta^*(\overline{x}, y)) \tau(\lambda) d\lambda, & \text{if } y < \psi(\overline{x}), \end{cases}$$

where  $\delta^*(\overline{x}, y) = c\Delta(\overline{x}, y)$ .

**Definition 5.** Let  $l \in \mathbb{N}$ ,  $1 \leq p \leq \infty$  and  $\Omega$  be a Lipschitz subgraph of  $\mathbb{R}^n$  with Lipschitz bound  $M$ . Denote with  $\Gamma$  the cone with vertex at the origin given by  $\Gamma = \{(\overline{x}, y) \in \mathbb{R}^n \mid M|\overline{x}| < |y|, y < 0\}$ . Suppose now that  $\eta \in C_c^\infty(\mathbb{R}^n)$  is a non-negative function with total integral 1 and which support is contained in  $\Gamma$ . Then for every  $f \in W^{l,p}(\Omega)$  we define

$$Sf = \lim_{\varepsilon \rightarrow 0} Tf_\varepsilon$$

where  $f_\varepsilon(x) = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} f(x - y) \eta(y/\varepsilon) dy$  and the convergence of  $Tf_\varepsilon$  is the convergence in the norm  $\|\cdot\|_{W^{l,p}(\mathbb{R}^n)}$ .

**Theorem 4.** Let  $l \in \mathbb{N}$ ,  $1 \leq p \leq \infty$  and  $\Omega$  be a Lipschitz subgraph of  $\mathbb{R}^n$ . Then the function  $Sf$  is well defined for every  $f \in W^{l,p}(\Omega)$  and defines a bounded extension operator from  $W^{l,p}(\Omega)$  to  $W^{l,p}(\mathbb{R}^n)$ .

## 2.2 Stein operator in Sobolev-Morrey spaces

**Definition 6.** Let  $1 \leq p < \infty$ ,  $\phi$  a function from  $\mathbb{R}^+$  to  $\mathbb{R}^+$  and  $\Omega$  be a domain in  $\mathbb{R}^n$ . For a function  $f \in L_{loc}^p(\Omega)$  we define the cubic-Morrey norm  $\|\cdot\|_{M_{p,Q}^\phi(\Omega)}$  as

$$\|f\|_{M_{p,Q}^\phi(\Omega)} := \sup_{Q_r(x), x \in \Omega, r > 0} \left( \frac{1}{\phi(r)} \int_{Q_r(x) \cap \Omega} |f(y)|^p dy \right)^{\frac{1}{p}}$$

where  $Q_r(x)$  is the open cube centered in  $x$  of side  $2r$ .

**Lemma 9.** Let  $1 \leq p \leq \infty$ ,  $\phi$  a function from  $\mathbb{R}^+$  to  $\mathbb{R}^+$  and  $\Omega$  be a domain in  $\mathbb{R}^n$ . Then the cubic-Morrey norm  $\|\cdot\|_{M_{p,Q}^\phi(\Omega)}$  is equivalent to the classical Morrey norm  $\|\cdot\|_{M_p^\phi(\Omega)}$ . In particular

$$\|\cdot\|_{M_p^\phi(\Omega)} \leq \|\cdot\|_{M_{p,Q}^\phi(\Omega)} \leq 2^{n^2} \|\cdot\|_{M_p^\phi(\Omega)}.$$

**Lemma 10.** Let  $1 \leq p < \infty, n \geq 2$ ,  $\phi$  a function from  $\mathbb{R}^+$  to  $\mathbb{R}^+$  and  $\psi$  a Lipschitz function from  $\mathbb{R}^{n-1}$  to  $\mathbb{R}$ . Define

$$\Omega = \{(\bar{x}, y) \in \mathbb{R}^n \mid \bar{x} \in \mathbb{R}^{n-1}, y > \psi(\bar{x})\}.$$

Then  $T$  defines a bounded extension operator from  $M_p^\phi(\mathbb{R}_+^n)$  to  $M_p^\phi(\mathbb{R}^n)$ .

*Proof.* We will prove that for an arbitrary open cube  $Q$  of side  $r$  contained in  $\mathbb{R}^n$  we have

$$\left( \frac{1}{\phi(r/2)} \int_Q |Tf(x)|^p dx \right)^{\frac{1}{p}} \leq C \|f\|_{M_{p,Q}^\phi(\Omega)} \quad (5)$$

for a constant  $C$  independent of  $f$ , then the main statement follows from Lemma 1. Let's define  $\Omega^- = \{(\bar{x}, y) \in \mathbb{R}^n \mid \bar{x} \in \mathbb{R}^{n-1}, y < \psi(\bar{x})\}$ . There are three cases: 1.  $Q \subset \Omega$  2.  $Q \subset \Omega^-$  3.  $Q \cap \{y = \psi(\bar{x})\} \neq \emptyset$ .

1. Since  $Tf = f$  in  $\Omega$

$$\left( \frac{1}{\phi(r/2)} \int_Q |Tf(x)|^p dx \right)^{\frac{1}{p}} = \left( \frac{1}{\phi(r/2)} \int_Q |f(x)|^p dx \right)^{\frac{1}{p}} \leq \|f\|_{M_{p,Q}^\phi(\Omega)}$$

and we are done.

2. Let's write  $Q$  as  $Q = \{(\bar{x}, y) \in \mathbb{R}^n \mid \bar{x} \in F, y \in (a-r, a)\}$  where  $F$  is an open cube of  $\mathbb{R}^{n-1}$  of side  $r$  and  $a < \phi(\bar{x})$  for every  $\bar{x} \in F$ . Fix now  $(\bar{x}, y) \in Q$ , from the definition of  $Tf$  we have

$$|Tf(\bar{x}, y)| \leq \int_1^\infty |f(\bar{x}, y + \lambda \delta^*(\bar{x}, y))| |\tau(\lambda)| d\lambda \leq A \int_1^\infty |f(\bar{x}, y + \lambda \delta^*(\bar{x}, y))| \frac{1}{\lambda^3} d\lambda$$

Let's apply the change of variable  $s = y + \lambda \delta^*(\bar{x}, y)$

$$|Tf(\bar{x}, y)| \leq \int_{y+\delta^*}^\infty |f(\bar{x}, s)| \frac{(\delta^*)^2}{(s-y)^3} ds \leq c^2 \int_{2\psi(\bar{x})-y}^\infty |f(\bar{x}, s)| \frac{(\psi(x)-y)^2}{(s-y)^3} ds$$

because  $c(\psi(x) - y) \geq \delta^* \geq 2(\psi(x) - y)$ . Let's now decompose the last integral as follows

$$|Tf(\bar{x}, y)| \leq \sum_{k=0}^{\infty} c^2 \int_{2\psi(\bar{x})-y+kr}^{2\psi(\bar{x})-y+(k+1)r} |f(\bar{x}, s)| \frac{(\psi(\bar{x}) - y)^2}{(s - y)^3} ds.$$

Now by applying Minkowski's inequality for an infinite sum we get

$$\left( \int_{a-r}^a |Tf(\bar{x}, y)|^p dy \right)^{\frac{1}{p}} \leq c^2 \sum_{k=0}^{\infty} \left( \int_{a-r}^a \left( \int_{2\psi(\bar{x})-y+kr}^{2\psi(\bar{x})-y+(k+1)r} \frac{|f(\bar{x}, s)|(\psi(x) - y)^2}{(s - y)^3} ds \right)^p dy \right)^{\frac{1}{p}} (*)$$

Next we plan to estimate each summand. In the right-hand side of (\*) we apply the change of variable  $y = \psi(\bar{x}) - z$

$$\left( \int_{\psi(x)-a}^{\psi(x)-a+r} \left( \int_{\psi(x)+z+kr}^{\psi(x)+z+(k+1)r} |f(\bar{x}, s)| \frac{z^2}{(s - \psi(x) + z)^3} ds \right)^p dz \right)^{\frac{1}{p}}$$

and the change of variable  $u = s - \psi(x)$

$$\left( \int_{\psi(x)-a}^{\psi(x)-a+r} \left( \int_{z+kr}^{z+(k+1)r} |f(\bar{x}, u + \psi(x))| \frac{z^2}{(u + z)^3} du \right)^p dz \right)^{\frac{1}{p}}.$$

Then we apply the change of variable  $t = u/z$

$$\left( \int_{\psi(\bar{x})-a}^{\psi(\bar{x})-a+r} \left( \int_{1+kr/z}^{1+(k+1)r/z} |f(\bar{x}, tz + \psi(x))| \frac{1}{(t+1)^3} dt \right)^p dz \right)^{\frac{1}{p}}.$$

that can be rewritten as

$$\left( \int_{\psi(\bar{x})-a}^{\psi(\bar{x})-a+r} \left( \int_{1+kr/(\psi(\bar{x})-a+r)}^{1+(k+1)r/(\psi(\bar{x})-a)} |f(\bar{x}, tz + \psi(x))| \mathbb{1}_{(1+kr/z, 1+(k+1)r/z)}(t) \frac{1}{(t+1)^3} dt \right)^p dz \right)^{\frac{1}{p}}.$$

By Minkowski's integral inequality and setting  $\alpha = r/(\psi(\bar{x}) - a)$

$$\begin{aligned} & \left( \int_{a\psi(\bar{x})-a}^{\psi(\bar{x})-a+r} \left( \int_{1+k\alpha/(\alpha+1)}^{1+(k+1)\alpha} |f(\bar{x}, tz + \psi(x))| \mathbb{1}_{(1+kr/z, 1+(k+1)r/z)}(t) \frac{1}{(t+1)^3} dt \right)^p dz \right)^{\frac{1}{p}} \\ & \leq \int_{1+k\alpha/(\alpha+1)}^{1+(k+1)\alpha} \left( \int_{\psi(\bar{x})-a}^{\psi(\bar{x})-a+r} |f(\bar{x}, tz + \psi(x))|^p \mathbb{1}_{(1+kr/z, 1+(k+1)r/z)}(t) \frac{1}{(t+1)^{3p}} dz \right)^{\frac{1}{p}} dt. \end{aligned}$$

We notice that for every  $t, z \in \mathbb{R}$  with  $\psi(\bar{x}) - a \leq z \leq \psi(\bar{x}) - a + r$

$$\mathbb{1}_{(1+kr/z, 1+(k+1)r/z)}(t) \leq \mathbb{1}_{(\psi(\bar{x})-a+kr, \psi(\bar{x})-a+(k+2)r)}(tz)$$

hence using the change of variable  $w = tz$

$$\begin{aligned} & \int_{1+k\alpha/(\alpha+1)}^{1+(k+1)\alpha} \left( \int_{\psi(\bar{x})-a+kr}^{\psi(\bar{x})-a+(k+2)r} |f(\bar{x}, tz + \psi(x))|^p \mathbb{1}_{(1+kr/z, 1+(k+1)r/z)}(t) \frac{1}{(t+1)^{3p}} dz \right)^{\frac{1}{p}} dt \\ & \leq \int_{1+k\alpha/(\alpha+1)}^{1+(k+1)\alpha} \left( \int_{\psi(\bar{x})-a+kr}^{\psi(\bar{x})-a+(k+2)r} |f(\bar{x}, w + \psi(\bar{x}))|^p \frac{1}{t(t+1)^{3p}} dw \right)^{\frac{1}{p}} dt \\ & = \int_{1+k\alpha/(\alpha+1)}^{1+(k+1)\alpha} \frac{1}{t^{\frac{1}{p}}(t+1)^3} dt \left( \int_{\psi(\bar{x})-a+kr}^{\psi(\bar{x})-a+(k+2)r} |f(\bar{x}, w + \psi(\bar{x}))|^p dw \right)^{\frac{1}{p}} \\ & \leq \int_{1+k\alpha/(\alpha+1)}^{1+(k+1)\alpha} \frac{1}{(t+1)^3} dt \left( \int_{\psi(\bar{x})-a+kr}^{\psi(\bar{x})-a+(k+2)r} |f(\bar{x}, w + \psi(\bar{x}))|^p dw \right)^{\frac{1}{p}} \\ & = \frac{1}{2} \left[ \frac{1}{(1+(k+1)\alpha)^2} - \frac{1}{(1+k\alpha/(\alpha+1))^2} \right] \left( \int_{\psi(\bar{x})-a+kr}^{\psi(\bar{x})-a+(k+2)r} |f(\bar{x}, w + \psi(\bar{x}))|^p dw \right)^{\frac{1}{p}} \\ & = \frac{s_k(\alpha)}{2} \left( \int_{\psi(\bar{x})-a+kr}^{\psi(\bar{x})-a+(k+2)r} |f(\bar{x}, w + \psi(\bar{x}))|^p dw \right)^{\frac{1}{p}}. \end{aligned}$$

Plugging in this estimate in (\*) we get

$$\begin{aligned} \left( \int_{a-r}^a |Tf(\bar{x}, y)|^p dy \right)^{\frac{1}{p}} & \leq \frac{c^2}{2} \sum_{k=0}^{\infty} s_k(\alpha) \left( \int_{\psi(\bar{x})-a+kr}^{\psi(\bar{x})-a+(k+2)r} |f(\bar{x}, w + \psi(\bar{x}))|^p dw \right)^{\frac{1}{p}} \\ & = \frac{c^2}{2} \sum_{k=0}^{\infty} s_k(\alpha) \left( \int_{2\psi(\bar{x})-a+kr}^{2\psi(\bar{x})-a+(k+2)r} |f(\bar{x}, y)|^p dy \right)^{\frac{1}{p}}. \end{aligned}$$

Taking the  $L^p$  norm on  $F$  on both sides and applying again Minkowski in-

equality we obtain

$$\begin{aligned} \left( \int_F \int_{a-r}^a |Tf(\bar{x}, y)|^p dy d\bar{x} \right)^{\frac{1}{p}} &\leq \frac{c^2}{2} \sum_{k=0}^{\infty} s_k(\alpha) \left( \int_F \int_{2\psi(\bar{x})-a+kr}^{2\psi(\bar{x})-a+(k+2)r} |f(\bar{x}, y)|^p dy d\bar{x} \right)^{\frac{1}{p}} \\ &= \frac{c^2}{2} \sum_{k=0}^{\infty} s_k(\alpha) \|f\|_{L^p(S_k)}. \end{aligned} \quad (**)$$

where  $S_k = \{(\bar{x}, y) \in \mathbb{R}^n \mid \bar{x} \in F, 2\psi(\bar{x}) - a + kr < y < 2\psi(\bar{x}) - a + (k+2)r\}$ . Clearly the set  $S_k$  has diameter less than  $cr$ , where  $c$  is a constant depending only on  $n$  and on the Lipschitz constant of  $\psi$ . Hence by Lemma 4 there exists a collection of open cubes  $Q_1, \dots, Q_m$  centered in  $S_k$  of side  $r$  that covers  $S_k$ , with  $m \in \mathbb{N}$  depending only on  $\text{Lip } \psi$  and  $n$ . Moreover for every  $(\bar{x}, y) \in S_k$  we have  $y > 2\psi(\bar{x}) - a > \psi(\bar{x})$ , so  $S_k \subset \Omega$ . This implies that

$$S_k \subset \bigcup_{i=1}^m (Q_i \cap \Omega)$$

and that every cube  $Q_i$  is centered in  $\Omega$ . Therefore by  $(**)$

$$\|Tf\|_{L^p(Q)} \leq \frac{c^2}{2} \sum_{k=0}^{\infty} s_k(\alpha) (\|f\|_{L^p(Q_1 \cap \Omega)} + \dots + \|f\|_{L^p(Q_m \cap \Omega)}),$$

then dividing in both sides by  $\psi(r/2)^{\frac{1}{p}}$  we obtain

$$\left( \frac{1}{\phi(r/2)} \int_Q |Tf(x)|^p dx \right)^{\frac{1}{p}} \leq \frac{c^2 m}{2} \sum_{k=0}^{\infty} s_k(a, r) \|f\|_{M_{p,Q}(\Omega)}$$

We want now to estimate the series  $\sum_{k=0}^{\infty} s_k(\alpha)$ . First we notice that can be rewritten as as

$$\sum_{k=0}^{\infty} s_k(\alpha) = \sum_{k=1}^{\infty} \frac{\alpha(\alpha+2)}{(k\alpha+1)^2}.$$

To bound this series we distinguish two cases, when  $\alpha \leq 1$  and when  $\alpha > 1$ . In the first case we can bound the series using a Riemann Sum

$$\sum_{k=1}^{\infty} \frac{\alpha(\alpha+2)}{(k\alpha+1)^2} \leq 3 \sum_{k=1}^{\infty} \frac{\alpha}{(k\alpha+1)^2} = 3 \leq 3 \sum_{k=1}^{\infty} \int_{\mathbb{R}} \mathbb{1}_{(\alpha(k-1), \alpha)}(t) \frac{1}{(\alpha k+1)^2} dt \leq 3 \int_{\mathbb{R}} \frac{1}{(t+1)^2} dt = 3.$$

In the second case

$$\sum_{k=1}^{\infty} \frac{\alpha(\alpha+2)}{(k\alpha+1)^2} \leq \sum_{k=1}^{\infty} \frac{\alpha(\alpha+2)}{k^2\alpha^2} = \sum_{k=1}^{\infty} \frac{1+\frac{2}{\alpha}}{k^2} \leq 3\frac{\pi^2}{6} < 5.$$

Hence we get

$$\left( \frac{1}{\phi(r/2)} \int_Q |Tf(x)|^p dx \right)^{\frac{1}{p}} \leq 5c^2 \|f\|_{M_{p,Q}^{\phi}(\Omega)}$$

that shows (1).

3. We write  $Q$  as  $F \times (a-r, a)$  and we define  $Q^+ = Q \cap \Omega$  and  $Q^- = Q \cap \Omega^-$ . Then

$$\|Tf\|_{L^p(Q)} \leq \|f\|_{L^p(Q^+)} + \|Tf\|_{L^p(Q^-)}.$$

Moreover  $Q^+$  can be written as  $\{(\bar{x}, y) \mid \bar{x} \in S, a-r < y < \min(\psi(\bar{x}), a)\}$  for some set  $S \subset F$ . Hence

$$\int_{Q^-} |Tf(x)|^p dx = \int_S \int_{a-r}^{\min(\psi(\bar{x}), a)} |Tf(\bar{x}, y)|^p dy d\bar{x}.$$

We can then proceed as in 2. to obtain

$$\begin{aligned} \left( \int_S \int_{a-r}^a |Tf(\bar{x}, y)|^p dy d\bar{x} \right)^{\frac{1}{p}} &\leq \frac{c^2}{2} \sum_{k=0}^{\infty} s_k(\alpha) \left( \int_S \int_{2\psi(\bar{x}) - \min(a, \psi(\bar{x})) + kr}^{2\psi(\bar{x}) - a + (k+2)r} |f(\bar{x}, y)|^p dy d\bar{x} \right)^{\frac{1}{p}} \\ &= \frac{c^2}{2} \sum_{k=0}^{\infty} s_k(\alpha) \|f\|_{L^p(S'_k)}. \end{aligned}$$

One can observe that the sets  $S'_k$  have the same property as the sets  $S_k$  in 2. Therefore

$$\frac{1}{\psi(r/2)^{\frac{1}{p}}} \|Tf\|_{L^p(Q^-)} \leq c \|f\|_{M_p^{\phi}(\Omega)}$$

for some constant  $c$  depending only on  $n$  and  $\text{Lip}\psi$ . Finally it's immediate to verify that  $\|f\|_{L^p(Q^+)} \leq \phi(r/2)^{\frac{1}{p}} \|f\|_{M_p^{\phi}(\Omega)}$ . This concludes the proof of case 3.  $\square$

## References

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