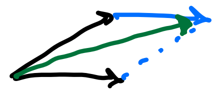
Нормы

(V, β)

$(\mathbb{R}^n, x^T y)$

Опр. норма $\|\cdot\|$

1) $\|\alpha \cdot x\| = |\alpha| \cdot \|x\|, \quad \alpha \in \mathbb{R}$
2) $\|x+y\| \leq \|x\| + \|y\|$

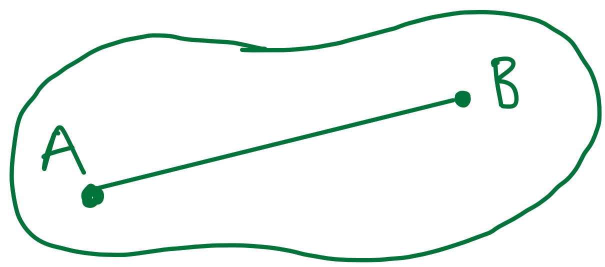


3) $\|x\| = 0 \Rightarrow x = 0$

$\rho(x,y) = \|x-y\|$ - расстояние

→ Евклидова норма (l_2)

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$



→ Манхеттенская норма (l_1)

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$\rho_2(A,B) = \sqrt{20}$

$\rho_1(A,B) = 6$

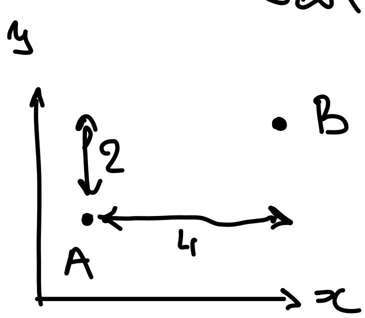
$\rho_\infty(A,B) = 4$

→ Норма Минковского l_p

$$\|x\|_p = \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}$$

→ норма Чебышёва l_∞

$$\|x\|_\infty = \max_i (x_i)$$



$$\|A\|_F = \left(\sum_{i,j} |a_{ij}|^2 \right)^{1/2}$$

QR – разложение

$$X = A \, B$$

матричная факторизация

$$\underset{m \times n}{A} = \begin{pmatrix} A_1 & \dots & A_n \end{pmatrix} = \underset{m \times k}{Q} \cdot \underset{k \times n}{R}$$

$$Q = \begin{pmatrix} \frac{u_1}{\|u_1\|} & \dots & \frac{u_k}{\|u_k\|} \end{pmatrix}, \qquad R = \begin{pmatrix} \ddots & * & \underbrace{* \;\; *} \\ 0 & \ddots & \end{pmatrix}$$

A   Q   R

Q ортогональная

$$Q^T Q = I$$

появляется если столбцы A лин. зав.

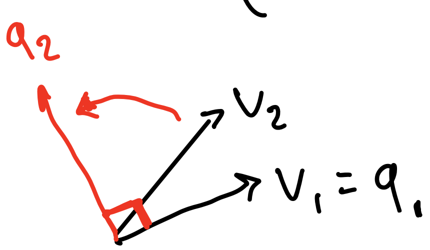
Упражнение

$$A = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

$$q_i \perp q_j$$

$$\langle q_i, q_j \rangle = q_i^T q_j = 0$$

$$\cos(q_i, q_j) = \frac{\langle q_i, q_j \rangle}{\|q_i\| \cdot \|q_j\|}$$



$$\begin{cases} q_2 = v_2 - \lambda \cdot q_1 \\ q_2 \perp q_1 \end{cases}$$

$$\langle q_2, q_1 \rangle = \langle V_2 - \lambda q_1, q_1 \rangle =$$

$$= \langle V_2, q_1 \rangle - \lambda \cdot \langle q_1, q_1 \rangle = 0$$

$$\Rightarrow \quad \lambda = \frac{\langle V_2, q_1 \rangle}{\langle q_1, q_1 \rangle}$$

$$q_1 = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} \qquad q_2 = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} - 0.9 \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1.6 \\ 3.2 \\ 0 \end{pmatrix}$$

$$\lambda = \frac{4 \cdot 2 + 2 \cdot 5 + 0 \cdot 0}{4 \cdot 4 + 2 \cdot 2 + 0 \cdot 0} = \frac{18}{20} = 0.9$$

$$\langle q_1, q_2 \rangle = -4 \cdot 1.6 + 2 \cdot 3.2 = 0$$

$$4 \cdot 1.6$$

$$\begin{cases} q_3 = V_3 - \lambda_1 q_1 - \lambda_2 q_2 \\ q_3 \perp q_2 \\ q_3 \perp q_1 \end{cases}$$

$$0 = \langle q_3, q_2 \rangle = \langle V_3, q_2 \rangle - \lambda_1 \underbrace{\langle q_1, q_2 \rangle}_{= 0} - \lambda_2 \langle q_2, q_2 \rangle$$

$$\lambda_2 = \frac{\langle V_3, q_2 \rangle}{\langle q_2, q_2 \rangle}$$

$$0 = \langle q_3, q_1 \rangle \Rightarrow \quad \lambda_1 = \frac{\langle V_3, q_1 \rangle}{\langle q_1, q_1 \rangle}$$

$$\langle q_2, q_2 \rangle = 1.6^2 + 3.2^2$$

$$\lambda_2 = 0$$

$$\langle v_3, q_2 \rangle = 0$$

$$\langle v_3, q_1 \rangle = 0 \qquad \lambda_1 = 0$$

$$\begin{pmatrix} 4 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix} = \begin{pmatrix} 4 & -1.6 & 0 \\ 2 & 3.2 & 0 \\ 0 & 0 & 7 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0.9 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$q_k = v_k - \sum_{i=1}^{k} \frac{\langle v_k, q_i \rangle}{\langle q_i, q_i \rangle} \cdot q_i$$

$$v_k = q_k + \sum_{i=1}^{k} \frac{\langle v_k, q_i \rangle}{\langle q_i, q_i \rangle} \cdot q_i$$

$$\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} \cdot \lambda + \begin{pmatrix} -1.6 \\ 3.2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix} = \begin{pmatrix} 4/\sqrt{20} & -1.6/\sqrt{\dots} & 0 \\ 2/\sqrt{20} & 3.2/\sqrt{\dots} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{20} & 0.9 \cdot \sqrt{20} & 0 \\ 0 & \sqrt{\dots} & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

$$\|q_1\| = \left\| \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} \right\| = \sqrt{16+4} = \sqrt{20}$$

$$
\begin{bmatrix} A_1 & \cdots & A_n \end{bmatrix} = \begin{bmatrix} \dfrac{q_1}{\|q_1\|} & \cdots & \dfrac{q_n}{\|q_n\|} \end{bmatrix} \cdot \begin{bmatrix} \|q_1\| & & * \\ & \ddots & \\ 0 & & \|q_n\| \end{bmatrix}
$$

$$
\|x\| = \sqrt{\langle x, x \rangle} \qquad * = \frac{\langle v_k, q_i \rangle}{\langle q_i, q_i \rangle} \cdot \|q_i\| = \frac{\langle v_k, q_i \rangle}{\sqrt{\langle q_i, q_i \rangle}}
$$

Спектральное разложение

A
$[n \times n]$

Если у неё $\exists$ n лин. нез. соб. векторов

Тогда

$$
\boxed{A} = \underbrace{\begin{bmatrix} | & & | \\ p_1 & \cdots & p_n \\ | & & | \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} | & & | \\ p_1 & \cdots & p_n \\ | & & | \end{bmatrix}^{-1}}_{P^{-1}}
$$

Если ещё $A^T = A$

тогда $A = P D P^T$ т.е. $P^T P = I$

$P^T = P^{-1}$

Алгоритм поиска:

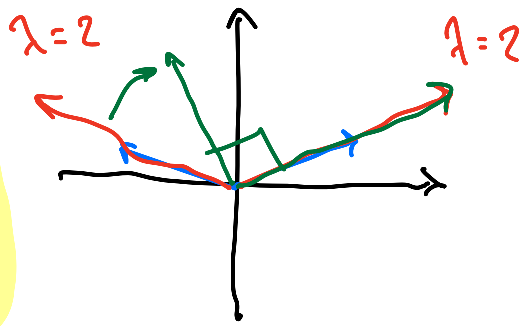
$$\det(A - \lambda I) = 0 \Rightarrow \begin{matrix}\lambda_1, \dots, \lambda_r \\ n_1, \dots, n_r\end{matrix}$$

$$(A - \lambda_i I)x = 0$$

$$\Big\downarrow \text{решение}$$

$$x_1, \dots, x_{n_i} \xrightarrow[\text{QR + нормировка}]{} P$$

$$\chi_A(\lambda) = (1-\lambda)^2(2-\lambda) = 0$$



Упражнение

а) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $\quad \lambda_1 = 1$, $\lambda_2 = 1$ $\quad \begin{pmatrix} 1-\lambda & 2 \\ 0 & 1-\lambda \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

спектр. разложения нет :(

$$\begin{cases} 0\cdot x + 2y = 0 \\ 0\cdot x + 0\cdot y = 0 \end{cases} \qquad \begin{matrix} y = 0 \\ x = 1 \end{matrix}$$

б) $\begin{pmatrix} 5 & -1 & -1 \\ 0 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}$

$$\lambda_1 = 3 \rightarrow \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$$\lambda_{2,3} = 5 \rightarrow \begin{pmatrix} x \\ -z \\ z \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \checkmark$$

$$\begin{pmatrix} 1 & 0 & 1/\sqrt{3} \\ 0 & -1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \cdot \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1/\sqrt{3} \\ 0 & -1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}^{-1}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \quad X$$

$$P^T P = I$$

SVD (Сингулярное р.-е)

$$\underset{n \times k}{X} = \underset{[n \times n]}{U} \cdot \underset{[n \times k]}{\Sigma} \cdot \underset{[k \times k]}{V^T}$$

$$U^T U = I$$

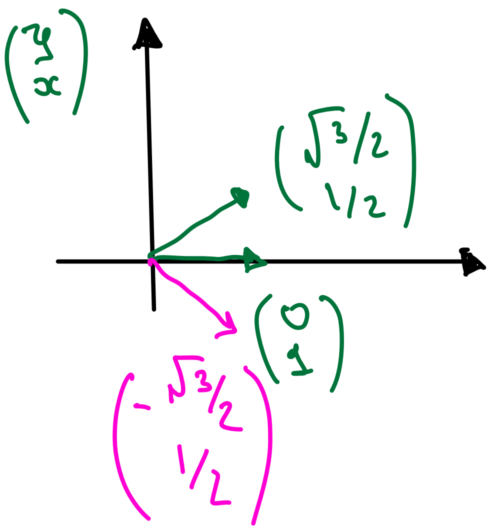
$$V^T V = I$$

матрицы поворотов

$$\Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_k)$$

Упражнение

Пример матрицы поворота



$$P \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}$$

$$\left\| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \right\|$$

$$P = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$$

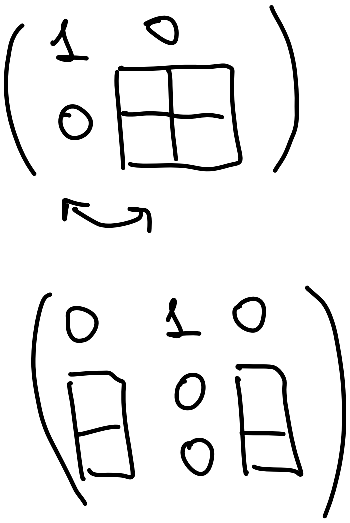
Общий вид: $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}$

P^{-1} – отмена поворота

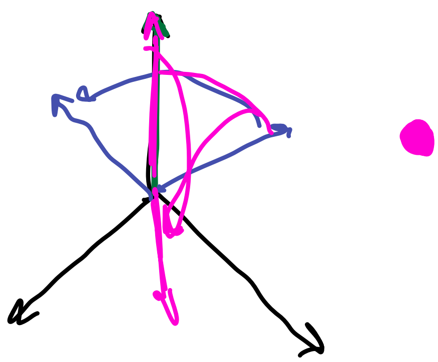
$$P^{-1} = \begin{pmatrix}\cos(-\alpha) & \sin(-\alpha)\\ -\sin(-\alpha) & \cos(-\alpha)\end{pmatrix} = \begin{pmatrix}\cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha\end{pmatrix} = P^T$$

$$\underbrace{P^{-1}P}_{I}A = A$$

$$\begin{pmatrix}
\pm1 & & & & & 0 & 0\\
 & \ddots & & & & & \\
 & & \pm1 & & & & \\
0 & & & \cos\alpha & -\sin\alpha & & 0 \\
 & & & \sin\alpha & \cos\alpha & & \\
 & & & & & \pm1 & \\
0 & & 0 & & & \ddots & \\
 & & & & & & \pm1
\end{pmatrix}$$



Правда ли?



$$\underset{[2\times4]}{A} = U\Sigma V^T$$

$[n\times n]$

$$\underset{[4\times4]}{A^TA} = VDV^T$$

$[n\times k][k\times n]$

$$\underset{[2\times2]}{AA^T} = U\Sigma V^T(U\Sigma V^T)^T = U\Sigma \overbrace{V^TV}^{I}\Sigma^T U^T =$$

$$= UDU^T \qquad D = \Sigma\Sigma^T$$

$U =$ соб. вект. AA^T

$D =$ соб. значения

Σ_1 - ?

$\sigma_i = \sqrt{\lambda_i}$

$$\Sigma_1 = \begin{pmatrix} \sigma_{max} & & & \\ & \sigma_{(2)} & & \\ & & \ddots & \\ & & & \sigma_{min} \\ 0 & 0 & 0 \; 0 & 0 \end{pmatrix}$$

V - ?

$A = U \Sigma V^T \quad | \times V$

$AV = U\Sigma_1$ система ур.

$$u_i = \frac{A v_i}{\sigma_i}$$

Упражнение

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad U \qquad D \qquad U^T$$

$$AA^T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$\chi_{AA^T}(\lambda) = (2-\lambda)^2 - 1 = 0$$

$\det(AA^T - \lambda I)$ $\quad \lambda_1 = 3 \quad \lambda_2 = 1$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

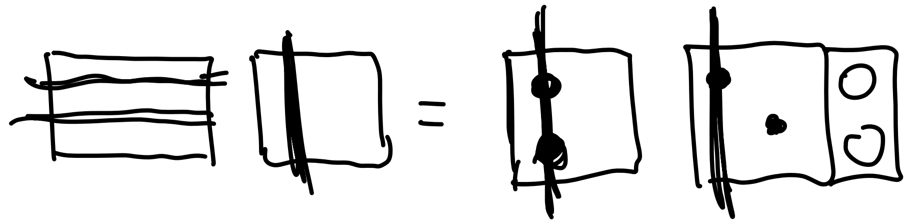
$u_2 - u_1 = 0 \qquad u_1 + u_2 = 0$

$\begin{pmatrix} u_1 \\ u_1 \end{pmatrix}$ $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ $\qquad$ $\begin{pmatrix} u_1 \\ -u_1 \end{pmatrix}$ $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

$$\Sigma_1 = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$A = U\Sigma_1 V^T$

$AV = U\Sigma_1$



$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \cdot \sqrt{3} \qquad A v_i = u_i \sigma_i$$

$$\begin{cases} v_1 + v_3 = \sqrt{3/2} \\ v_1 + v_2 = \sqrt{3/2} \end{cases} \qquad v_1 = 0 \qquad \begin{pmatrix} 0 \\ \sqrt{3/2} \\ \sqrt{3/2} \end{pmatrix} = \tilde{v}_1$$

$\|\tilde{v}_1\| = 6$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \cdot 1 \qquad \tilde{v}_1 \perp \tilde{v}_2$$

$$\begin{cases} v_1 + v_3 = \sqrt{1/2} \\ v_1 + v_2 = -\sqrt{1/2} \end{cases} \qquad \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \tilde{v}_2 \qquad \langle \tilde{v}_1, \tilde{v}_2 \rangle = 0$$

$\|\tilde{v}_2\| = 1$

$\tilde{v}_3$ - ? Дополним до ортог. базиса

$$A=\begin{pmatrix}1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}=\underset{[2\times 2]}{\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}}\underset{[2\times 3]}{\begin{pmatrix}\sqrt{3} & 0 & 0\\ 0 & 1 & 0\end{pmatrix}}\underset{3\times 3}{\begin{pmatrix}0 & 0 & 1\\ 1/\sqrt{6} & -1/\sqrt{2} & 0\\ 1/\sqrt{6} & 1/\sqrt{2} & 0\end{pmatrix}^T}$$

Алгоритм:

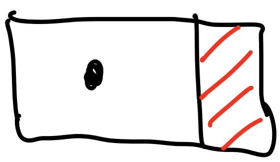
$A \to AA^T \to$ спектр. разл-е

$\Sigma \quad U$

$\downarrow$

Гаусс $\quad V$

+ QR и нормировка

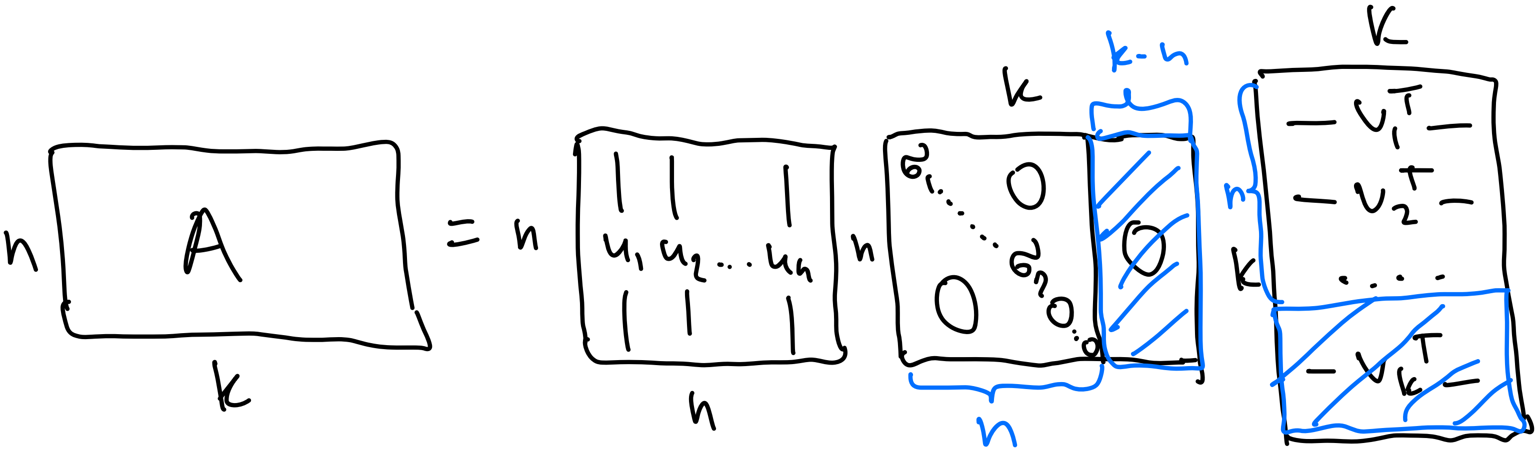
$V =$ 

находится неоднозначно
(дополнение до $\perp$ базиса)

Разновидности SVD:

$$\underset{n\times k}{A} = \underset{n\times n}{\begin{pmatrix} | & | & & | \\ u_1 & u_2 & \dots & u_n \\ | & | & & | \end{pmatrix}} \underset{n\times k}{\left(\begin{array}{ccc|c} \sigma_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & \sigma_n & \end{array}\right)} \underset{k\times k}{\begin{pmatrix} - v_1^T - \\ - v_2^T - \\ \dots \\ - v_k^T - \end{pmatrix}}$$

Усечённое SVD



Компактное SVD