

Нормы

(V, ρ)

$(\mathbb{R}^n, x \bar{y})$

Опр. норма $\|\cdot\|$

1) $\|\alpha \cdot x\| = |\alpha| \cdot \|x\|, \alpha \in \mathbb{R}$

2) $\|x+y\| \leq \|x\| + \|y\|$



3) $\|x\| = 0 \Rightarrow x = 0$

$\rho(x, y) = \|x - y\|$ - расстояние

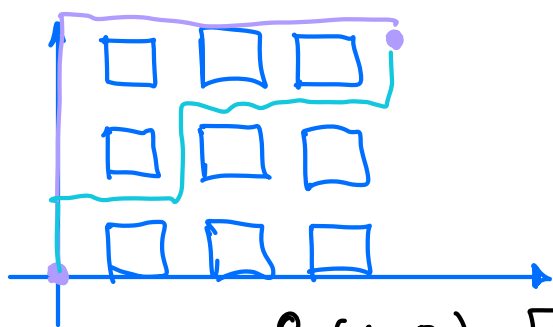
→ Евклидова норма (ℓ_2)

$$\|x\|_2 = \sqrt{\sum_{i=1}^n |x_i|^2}$$



→ Манхейттенская норма (ℓ_1)

$$\|x\|_1 = \sum_{i=1}^n |x_i|$$



→ Норма Минковского ℓ_p

$$\|x\|_p = \sqrt[p]{\sum_{i=1}^n |x_i|^p}$$

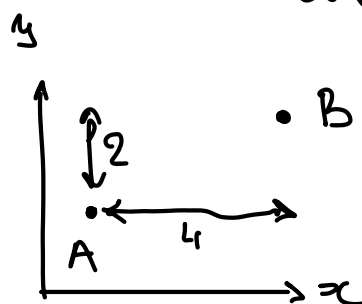
$\rho_2(A, B) = \sqrt{20}$

$\rho_1(A, B) = 6$

$\rho_\infty(A, B) = 4$

→ норма Чебышёва ℓ_∞

$$\|x\|_\infty = \max_i |x_i|$$



$$\|A\|_F = \left(\sum_{i,j} |a_{ij}|^2 \right)^{1/2}$$

QR - разложение

$$X = A B$$

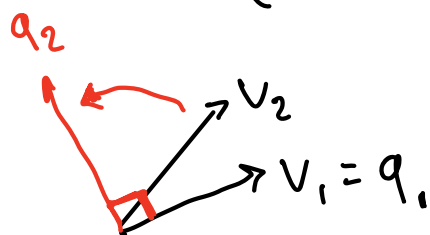
матричная
факторизация

$$\begin{array}{c} m \times n \\ \begin{array}{|c|c|c|} \hline A_1 & \dots & A_n \\ \hline \end{array} \\ A \end{array} = \begin{array}{c} m \times k \\ \begin{array}{|c|c|c|} \hline \frac{u_1}{\|u_1\|} & \dots & \frac{u_k}{\|u_k\|} \\ \hline \end{array} \\ Q \\ \text{ортогональная} \\ Q^T Q = I \end{array} \cdot \begin{array}{c} k \times n \\ \begin{array}{|c|c|c|} \hline * & & \\ \hline \end{array} \\ R \end{array}$$

* * *
появляется
если столбцы
A линеар. зав.

Упражнение

$$A = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$



$$q_i \perp q_j$$

$$\langle q_i, q_j \rangle = q_i^T q_j = 0$$

$$\cos(q_i, q_j) = \frac{\langle q_i, q_j \rangle}{\|q_i\| \cdot \|q_j\|}$$

$$\begin{cases} q_2 = v_2 - \lambda \cdot q_1 \\ q_2 \perp q_1 \end{cases}$$

$$\begin{aligned}
 \langle q_2, q_1 \rangle &= \langle v_2 - \lambda q_1, q_1 \rangle = \\
 &= \langle v_2, q_1 \rangle - \lambda \cdot \langle q_1, q_1 \rangle = 0 \\
 \Rightarrow \lambda &= \frac{\langle v_2, q_1 \rangle}{\langle q_1, q_1 \rangle}
 \end{aligned}$$

$$q_1 = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} \quad q_2 = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} - 0.9 \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1.6 \\ 3.2 \\ 0 \end{pmatrix}$$

$$\lambda = \frac{4 \cdot 2 + 2 \cdot 5 + 0 \cdot 0}{4 \cdot 4 + 2 \cdot 2 + 0 \cdot 0} = \frac{18}{20} = 0.9$$

$$\langle q_1, q_2 \rangle = -4 \cdot 1.6 + 2 \cdot 3.2 = 0$$

$$\begin{cases} q_3 = v_3 - \lambda_1 q_1 - \lambda_2 q_2 \\ q_3 \perp q_2 \\ q_3 \perp q_1 \end{cases}$$

$$0 = \langle q_3, q_2 \rangle = \langle v_3, q_2 \rangle - \lambda_1 \langle q_1, q_2 \rangle - \lambda_2 \langle q_2, q_2 \rangle$$

$$\lambda_2 = \frac{\langle v_3, q_2 \rangle}{\langle q_2, q_2 \rangle}$$

$$0 = \langle q_3, q_1 \rangle \Rightarrow \lambda_1 = \frac{\langle v_3, q_1 \rangle}{\langle q_1, q_1 \rangle}$$

$$\langle q_2, q_2 \rangle = 1.6^2 + 3.2^2$$

$$\lambda_2 = 0$$

$$\langle v_3, q_2 \rangle = 0$$

$$\langle v_3, q_1 \rangle = 0$$

$$\lambda_1 = 0$$

$$\begin{pmatrix} 4 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix} = \begin{pmatrix} 4 & -1.6 & 0 \\ 2 & 3.2 & 0 \\ 0 & 0 & 7 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0.9 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$q_k = v_k - \sum_{i=1}^k \frac{\langle v_k, q_i \rangle}{\langle q_i, q_i \rangle} \cdot q_i$$

$$v_k = q_k + \sum_{i=1}^k \frac{\langle v_k, q_i \rangle}{\langle q_i, q_i \rangle} \cdot q_i$$

$$\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} \cdot \lambda + \begin{pmatrix} -1.6 \\ 3.2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 7 \end{pmatrix} = \begin{pmatrix} \frac{4}{\sqrt{20}} & \frac{-1.6}{\sqrt{20}} & 0 \\ \frac{2}{\sqrt{20}} & \frac{3.2}{\sqrt{20}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \sqrt{20} & 0.9 \cdot \sqrt{20} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

$$\|q_1\| = \left\| \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} \right\| = \sqrt{16+4} = \sqrt{20}$$

$$\begin{bmatrix} A_1 & \dots & A_n \end{bmatrix} = \begin{bmatrix} \frac{q_1}{\|q_1\|} & \dots & \frac{q_n}{\|q_n\|} \end{bmatrix} \cdot \begin{bmatrix} \|q_1\| & & \\ & \ddots & \\ 0 & & \|q_n\| \end{bmatrix} \cdot \begin{bmatrix} * & & \\ & \ddots & \\ 0 & & * \end{bmatrix}$$

$$\|x\| = \sqrt{\langle x, x \rangle}$$

$$* = \frac{\langle v_k, q_i \rangle}{\langle q_i, q_i \rangle} \cdot \|q_i\|$$

$$= \frac{\langle v_k, q_i \rangle}{\sqrt{\langle q_i, q_i \rangle}}$$

Спектральное разложение

A
[n x n]

Если у неё \exists n лнн. нез. соб. векторов
Тогда

$$A = \begin{bmatrix} | & & | \\ p_1 & \dots & p_n \\ | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} | & & | \\ p_1 & \dots & p_n \\ | & & | \end{bmatrix}^{-1}$$

$P \quad D \quad P^{-1}$

Если ещё $A^T = A$

тогда $A = P D P^T$ т.е. $P^T P = I$
 $P^T = P^{-1}$

Алгоритм поиска:

$$\det(A - \lambda I) = 0 \Rightarrow \begin{matrix} \lambda_1, \dots, \lambda_n \\ n_1, \dots, n_n \end{matrix}$$

$$(A - \lambda_i I)x = 0$$

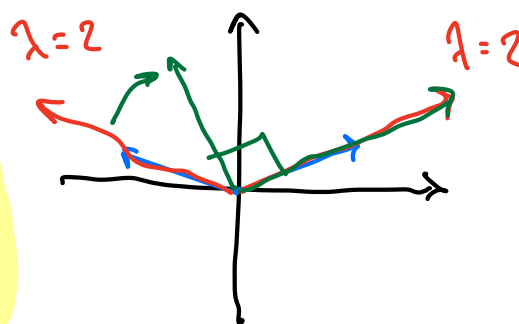
↓ решение

x_1, \dots, x_{n_i}

→
QR
+ нормировка

P

$$\chi_A(\lambda) = (1 - \lambda)^2 (2 - \lambda) = 0$$



Упрямление

a) $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $\begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 1 \end{matrix}$ $\begin{pmatrix} 1 - \lambda & 2 \\ 0 & 1 - \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

с не ктр.
разномерию
кет ☹

$$\begin{cases} 0 \cdot x + 2y = 0 \\ 0 \cdot x + 0 \cdot y = 0 \end{cases}$$

$$\begin{matrix} y = 0 \\ x = 1 \end{matrix}$$

b) $\begin{pmatrix} 5 & -1 & -1 \\ 0 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}$
||

$$\begin{matrix} \lambda_1 = 3 \\ \lambda_{2,3} = 5 \end{matrix} \rightarrow \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} x \\ -z \\ z \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \quad \checkmark$$

$$\begin{pmatrix} 1 & 0 & 1/\sqrt{3} \\ 0 & -1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \cdot \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 1/\sqrt{3} \\ 0 & -1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}^{-1}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \otimes$$

$$P^T P = I$$

SVD (сингулярное р.-е)

$$X = U \cdot \Sigma \cdot V^T$$

$n \times k$ $[n \times n]$ $[n \times k]$ $[k \times k]$

$$U^T U = I$$

$$V^T V = I$$

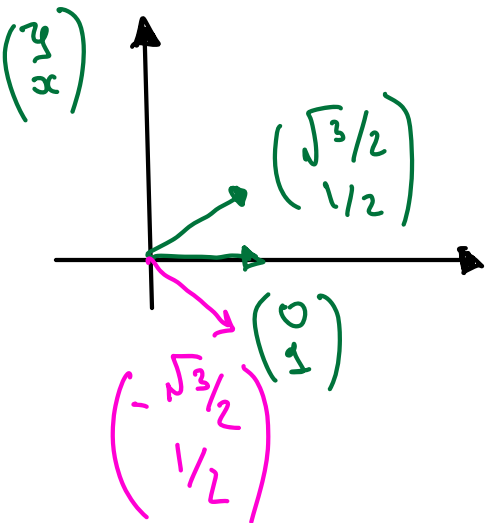
матрицы поворота

$$\Sigma = \text{diag}(\sigma_1, \dots, \sigma_k)$$

Упражнение

Пример матрицы поворота

$$\left\| \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\| = \left\| \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix} \right\|$$



$$P \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 \\ 1/2 \end{pmatrix}$$

$$P = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$$

Общий вид: $\begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}$

P^{-1} - отмена поворота

$$P^{-1} = \begin{pmatrix} \cos(-\alpha) & \sin(-\alpha) \\ -\sin(-\alpha) & \cos(-\alpha) \end{pmatrix} = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} = P^T$$

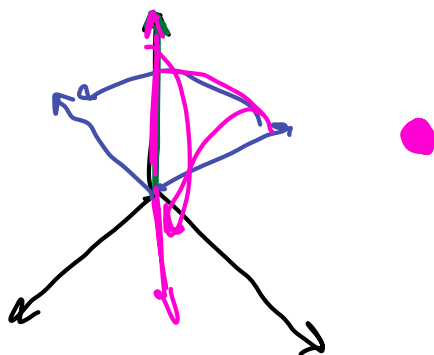
$$\underbrace{P^{-1}P}_I A = A$$

Hand-drawn diagram of a 2D lattice with 8 sites. The top-left site is labeled $+1$, the top-right site is labeled -1 , and the bottom-right site is labeled $+1$. The center of the lattice is labeled $\cos 2 - \sin 2$ and $\sin 2 \cos 2$. The sites are arranged in a square grid with a central cross.

$$\begin{pmatrix} 1 & 0 \\ 0 & \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Травга м?



$$A = U \Sigma V^T$$

$$A^T A = V D_1 V^T$$

$$[4 \times 4]$$

$$\begin{bmatrix} n \times h \end{bmatrix}$$

$$\begin{aligned} \underbrace{AA^T}_{[2 \times 2]} &= U \Sigma V^T (U \Sigma V^T)^T = U \Sigma \underbrace{V^T V}_{I} \Sigma^T U^T = \\ &= U \mathcal{D} U^T \quad \mathcal{D} = \Sigma \Sigma^T \end{aligned}$$

$$U = \text{соб. вект. } AA^T$$

$$D = \text{соб. значения}$$

$$\Sigma = ? \quad \sigma_i = \sqrt{\lambda_i}$$

$$V = ?$$

$$\Sigma = \begin{pmatrix} \sigma_{\max} & & & & \\ & \sigma_{(2)} & & & \\ & & \dots & & \\ & & & \sigma_{\min} & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A = U \Sigma V^T \quad | \times V$$

$$AV = U \Sigma \quad \text{система ур.}$$

$$u_i = \frac{AV_i}{\sigma_i}$$

Упражнение

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

U

D

U^T

$$AA^T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$\chi_{AA^T}(\lambda) = (2-\lambda)^2 - 1 = 0$$

$$\det(AA^T - \lambda I)$$

$$\lambda_1 = 3 \quad \lambda_2 = 1$$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$u_2 - u_1 = 0$$

$$u_1 + u_2 = 0$$

$$\begin{pmatrix} u_1 \\ u_1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ -u_1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$A = U \Sigma V^T$$

$$AV = U \Sigma$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \cdot \sqrt{3}$$

$$A v_i = u_i \sigma_i$$

$$\begin{cases} v_1 + v_3 = \sqrt{3/2} \\ v_1 + v_2 = \sqrt{3/2} \end{cases}$$

$$v_1 = 0$$

$$\begin{pmatrix} 0 \\ \sqrt{3/2} \\ \sqrt{3/2} \end{pmatrix} = \tilde{v}_1$$

$$\|\tilde{v}_1\| = 6$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \cdot 1$$

$$\tilde{v}_1 \perp \tilde{v}_2$$

$$\begin{cases} v_1 + v_3 = \sqrt{1/2} \\ v_1 + v_2 = -\sqrt{1/2} \end{cases}$$

$$\begin{pmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \tilde{v}_2$$

$$\langle \tilde{v}_1, \tilde{v}_2 \rangle = 0$$

$$\|\tilde{v}_2\| = 1$$

\tilde{v}_3 - ? Дополните до ортон. базиса

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1/\sqrt{6} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{2} & 0 \end{pmatrix}^T$$

$[2 \times 2]$ $[2 \times 3]$ 3×3

Алгоритм:

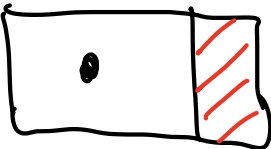
$A \rightarrow A A^T \rightarrow$ спектр. разл.-е

$\Sigma \quad U$



Тогда V

+ QR и коррепорвка

$V =$ 

каноническая
координатная

(дополнение до \perp)
базиса

Разложение SVD:

$$\begin{matrix} n \\ \boxed{A} \\ k \end{matrix} = \begin{matrix} n & k \\ \boxed{\begin{matrix} | & | & | \\ u_1 & u_2 & \dots & u_n \\ | & | & | \end{matrix}} \\ n \end{matrix} \begin{matrix} k \\ \boxed{\begin{matrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n & 0 \end{matrix}} \\ k \end{matrix} \begin{matrix} k \\ \boxed{\begin{matrix} - & v_1^T & - \\ - & v_2^T & - \\ & \dots & \\ - & v_k^T & - \end{matrix}} \\ k \end{matrix}$$

Усечённое SVD

$$\begin{array}{c} n \\ \boxed{A} \\ k \end{array} = \begin{array}{c} \begin{array}{c} | \quad | \quad | \\ u_1 \quad u_2 \dots u_n \\ | \quad | \quad | \end{array} \\ n \end{array} \begin{array}{c} \begin{array}{c} \sigma_1 \quad \dots \quad 0 \\ \vdots \quad \ddots \quad \vdots \\ 0 \quad \dots \quad \sigma_n \quad 0 \end{array} \\ \begin{array}{c} k \\ n \end{array} \end{array} \begin{array}{c} \begin{array}{c} \text{--- } v_1^T \text{ ---} \\ \text{--- } v_2^T \text{ ---} \\ \vdots \\ \text{--- } v_k^T \text{ ---} \end{array} \\ \begin{array}{c} k \\ n \end{array} \end{array}$$

Diagram illustrating the truncated SVD decomposition of matrix A (size $n \times k$). The decomposition is shown as $A = U \Sigma V^T$. The matrix U (size $n \times n$) contains columns u_1, u_2, \dots, u_n . The matrix Σ (size $n \times k$) contains singular values $\sigma_1, \dots, \sigma_n$ on the diagonal, with the bottom-right $(k-n) \times (k-n)$ submatrix shaded blue. The matrix V^T (size $k \times k$) contains rows $v_1^T, v_2^T, \dots, v_k^T$, with the bottom-right $(k-n) \times (k-n)$ submatrix shaded blue.

Компактное SVD

$$\begin{array}{c} n \\ \boxed{A} \\ k \end{array} = \begin{array}{c} \begin{array}{c} | \quad | \quad | \\ u_1 \quad u_2 \dots u_k \\ | \quad | \quad | \end{array} \\ n \end{array} \begin{array}{c} \begin{array}{c} \sigma_1 \quad \dots \quad 0 \\ \vdots \quad \ddots \quad \vdots \\ 0 \quad \dots \quad \sigma_k \quad 0 \end{array} \\ \begin{array}{c} k \\ n \end{array} \end{array} \begin{array}{c} \begin{array}{c} \text{--- } v_1^T \text{ ---} \\ \text{--- } v_2^T \text{ ---} \\ \vdots \\ \text{--- } v_k^T \text{ ---} \end{array} \\ \begin{array}{c} k \\ n \end{array} \end{array}$$

Diagram illustrating the compact SVD decomposition of matrix U (size $n \times k$). The decomposition is shown as $A = U \Sigma V^T$. The matrix U (size $n \times k$) contains columns u_1, u_2, \dots, u_k , with the bottom-right $(k-n) \times (k-n)$ submatrix shaded blue. The matrix Σ (size $k \times k$) contains singular values $\sigma_1, \dots, \sigma_k$ on the diagonal, with the bottom-right $(k-n) \times (k-n)$ submatrix shaded blue. The matrix V^T (size $k \times k$) contains rows $v_1^T, v_2^T, \dots, v_k^T$, with the bottom-right $(k-n) \times (k-n)$ submatrix shaded blue.