Application of the Cauchy-Schwarz Inequality and the Arithmetic-Geometric Mean Inequality

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Intermediate Algebra,
Art of Problem Solving

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Problem Statement

(a) Show that if a, b, c, d, e, f are nonnegative real numbers, then

$$(a^2+b^2)^2(c^4+d^4)(e^4+f^4) \geq (ace+bdf)^4.$$

(b) Show that if a, b, c, d, e, f are nonnegative real numbers, then

$$(a^2+b^2)(c^2+d^2)(e^2+f^2) \geq (ace+bdf)^2.$$

Solution

Both parts of the problem can be solved using the Cauchy-Schwarz inequality and the Arithmetic-Geometric Mean (AM-GM) inequality.

Part (a)

We begin by applying the Cauchy-Schwarz inequality:

$$(c^4+d^4)(e^4+f^4) \geq (c^2e^2+d^2f^2)^2.$$

Multiplying both sides by $(a^2+b^2)^2$ gives:

$$(a^2+b^2)^2(c^4+d^4)(e^4+f^4) \geq (a^2+b^2)^2(c^2e^2+d^2f^2)^2.$$

Now we expand the right-hand side:

$$(a^2+b^2)^2(c^2e^2+d^2f^2)^2 = \left(a^2c^2e^2+a^2d^2f^2+b^2c^2e^2+b^2d^2f^2\right)^2.$$

Let us group the terms:

$$\left(a^2c^2e^2+b^2d^2f^2+a^2d^2f^2+b^2c^2e^2\right)^2.$$

Apply AM-GM to the middle two terms:

$$a^2d^2f^2 + b^2c^2e^2 \geq 2\sqrt{a^2d^2f^2 \cdot b^2c^2e^2} = 2abcdef.$$

Hence,

$$\left(a^2c^2e^2 + b^2d^2f^2 + 2abcdef\right)^2 \geq (ace + bdf)^4.$$

Thus, we conclude:

$$(a^2 + b^2)^2(c^4 + d^4)(e^4 + f^4) \geq (ace + bdf)^4.$$

Part (b)

We expand the left-hand side:

$$\begin{aligned}&(a^2 + b^2)(c^2 + d^2)(e^2 + f^2) = \\ &\quad a^2c^2e^2 + a^2c^2f^2 + a^2d^2e^2 + a^2d^2f^2 + b^2c^2e^2 + b^2c^2f^2 + b^2d^2e^2 + b^2d^2f^2.\end{aligned}$$

Group $a^2c^2e^2$ and $b^2d^2f^2$ separately. Apply AM-GM to the remaining six terms:

$$a^2c^2f^2+a^2d^2e^2+a^2d^2f^2+b^2c^2e^2+b^2c^2f^2+b^2d^2e^2 \geq 6\sqrt[6]{(a^2b^2c^4d^4e^4f^4)} = 6abcdef.$$

So the total expression satisfies:

$$(a^2 + b^2)(c^2 + d^2)(e^2 + f^2) \geq a^2c^2e^2 + b^2d^2f^2 + 6abcdef.$$

Now, observe that:

$$a^2c^2e^2 + b^2d^2f^2 + 6abcdef \geq (ace + bdf)^2,$$

since

$$(ace + bdf)^2 = a^2c^2e^2 + b^2d^2f^2 + 2abcdef \leq a^2c^2e^2 + b^2d^2f^2 + 6abcdef.$$

Therefore, we conclude:

$$(a^2 + b^2)(c^2 + d^2)(e^2 + f^2) \geq (ace + bdf)^2.$$