Solving the Inequality

In this article, we aim to solve the inequality

$$\frac{x^2+2x+5}{3x^2-x-4} \geq 0,$$

using algebraic methods and casework analysis. By carefully handling the denominator and evaluating the behavior of the quadratic expressions, we will determine the solution set.

Step 1: Analyze the Inequality

The original inequality is:

$$\frac{x^2+2x+5}{3x^2-x-4} \geq 0.$$

To simplify, we multiply through by $3x^2-x-4$. However, this requires casework because the inequality flips when $3x^2-x-4<0$. Additionally, we must exclude points where the denominator is zero, as division by zero is undefined:

$$3x^2-x-4 \neq 0.$$

Thus, the inequality becomes:

$$x^2+2x+5 \geq 0, \quad \text{if } 3x^2-x-4>0,$$

and:

$$x^2+2x+5 \leq 0, \quad \text{if } 3x^2-x-4<0.$$

Step 2: Solve for $3x^2-x-4=0$

We find the roots of $3x^2-x-4$ by factoring:

$$\begin{gathered} 3x^2-x-4=0, \\ (3x-4)(x+1)=0, \\ x=-1, \quad x=\frac{4}{3}. \end{gathered}$$

The quadratic $3x^2-x-4$ changes sign at these points. We analyze the intervals:

$$3x^2-x-4 = \begin{cases} <0, & \text{if } x \in (-1, \frac{4}{3}), \\ >0, & \text{if } x \notin [-1, \frac{4}{3}]. \end{cases}$$

Step 3: Analyze x^2+2x+5

We now examine the behavior of the numerator x^2+2x+5:

$$x^2+2x+5=0.$$

Using the quadratic formula:

$$x=\frac{-2 \pm \sqrt{-16}}{2}.$$

The discriminant $\Delta=-16$ is negative, so the quadratic has no real roots. Since the coefficient of x^2 is positive, the parabola opens upwards, meaning:

$$x^2+2x+5>0 \quad \text{for all } x \in \mathbb{R}.$$

Step 4: Case Analysis

Case 1: $3x^2-x-4>0$

When $3x^2-x-4>0$, the inequality reduces to:

$$x^2+2x+5 \geq 0.$$

Since $x^2+2x+5>0$ for all x, this case holds true for all $x \notin [-1, \frac{4}{3}]$.

Case 2: $3x^2 - x - 4 < 0$

When $3x^2 - x - 4 < 0$, the inequality becomes:

$$x^2 + 2x + 5 \leq 0.$$

However, as previously established, $x^2 + 2x + 5 > 0$ for all x. Therefore, this case does not contribute any solutions.

Step 5: Exclude Undefined Points

We must exclude points where $3x^2 - x - 4 = 0$, i.e., $x = -1$ and $x = \frac{4}{3}$, as these make the denominator undefined.

Solution

Combining the results from both cases, the solution to the inequality is:

$$x \in (-\infty, -1) \cup \left(\frac{4}{3}, \infty\right).$$

Conclusion

By analyzing the numerator and denominator of the given inequality, we determined that $x^2+2x+5 > 0$ for all x, while the denominator $3x^2 - x - 4$ changes sign at $x = -1$ and $x = \frac{4}{3}$. Using casework, we excluded intervals where the denominator is negative or undefined. Thus, the final solution is:

$$x \in (-\infty, -1) \cup \left(\frac{4}{3}, \infty\right).$$