Function invertability

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AoPS Intermediate Algebra

February 22, 2025

1 Introduction

In this article, we will discuss the invertibility of functions. A function $f : A \to B$ is said to have an inverse function if there exists a function $g : B \to A$ such that $f(g(x)) = x$ for all $x \in B$ and $g(f(x)) = x$ for all $x \in A$.

2 Show that the Function $f(x) = x - \frac{1}{x}$ is not invertible

Let $f : (-\infty, 0) \cup (0, \infty) \to \mathbb{R}$ be defined by

$$f(x) = x - \frac{1}{x}.$$

Show that f has no inverse function.

Solution:

To show that f has no inverse function, we need to show that f is not one-to-one. This means that there exist distinct x_1 and x_2 such that $f(x_1) = f(x_2)$.

Consider $f(x_1) = x_1 - \frac{1}{x_1}$ and $f(x_2) = x_2 - \frac{1}{x_2}$. Suppose $f(x_1) = f(x_2)$, then:

$$x_1 - \frac{1}{x_1} = x_2 - \frac{1}{x_2}.$$

Rearranging terms, we get:

$$x_1 - x_2 = \frac{1}{x_1} - \frac{1}{x_2}.$$

Multiplying both sides by x_1x_2, we obtain:

$$x_1x_2(x_1 - x_2) = x_2 - x_1.$$

This simplifies to:

$$x_1x_2(x_1 - x_2) = -(x_1 - x_2).$$

If $x_1 \neq x_2$, we can divide both sides by $(x_1 - x_2)$:

$$x_1 x_2 = -1.$$

Thus, for any $x_1 \in (-\infty, 0) \cup (0, \infty)$, there exists $x_2 = -\frac{1}{x_1}$ such that $f(x_1) = f(x_2)$. Since $x_1 \neq x_2$, f is not one-to-one and therefore does not have an inverse function.

3 Show that the Function $g(x) = x - \frac{1}{x} : (0, \infty)$ is invertible

Let $g : (0, \infty) \to \mathbb{R}$ be defined by

$$g(x) = x - \frac{1}{x}.$$

Show that g has an inverse function.

Solution:

To show that g has an inverse function, we need to show that g is one-to-one and onto.

First, we show that g is one-to-one. Suppose $g(x_1) = g(x_2)$ for $x_1, x_2 \in (0, \infty)$. Then:

$$x_1 - \frac{1}{x_1} = x_2 - \frac{1}{x_2}.$$

Rearranging terms, we get:

$$x_1 - x_2 = \frac{1}{x_1} - \frac{1}{x_2}.$$

Multiplying both sides by $x_1 x_2$, we obtain:

$$x_1 x_2 (x_1 - x_2) = x_2 - x_1.$$

This simplifies to:

$$x_1 x_2 (x_1 - x_2) = -(x_1 - x_2).$$

If $x_1 \neq x_2$, we can divide both sides by $(x_1 - x_2)$:

$$x_1 x_2 = -1,$$

which is a contradiction since $x_1, x_2 > 0$. Therefore, $x_1 = x_2$, and g is one-to-one.

Next, we show that g is onto. For any $y \in \mathbb{R}$, we need to find $x \in (0, \infty)$ such that $g(x) = y$. Consider the equation:

$$y = x - \frac{1}{x}.$$

Rearranging terms, we get:

$$x^2 - yx - 1 = 0.$$

This is a quadratic equation in x, and its discriminant is:

$$\Delta = y^2 + 4.$$

Since $\Delta > 0$ for all $y \in \mathbb{R}$, there are two real solutions for x:

$$x = \frac{y \pm \sqrt{y^2 + 4}}{2}.$$

Since we are considering $x \in (0, \infty)$, we take the positive solution:

$$x = \frac{y + \sqrt{y^2 + 4}}{2}.$$

Thus, for any $y \in \mathbb{R}$, there exists $x \in (0, \infty)$ such that $g(x) = y$. Therefore, g is onto.

Since g is both one-to-one and onto, it has an inverse function.