Week 20 Writing Problem

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Art Of Problem Solving,
Intermediate Algebra

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Problem Statement

Let x be a positive real number. Show that

$$\frac{1}{x} \geq 3 - 2\sqrt{x}.$$

Describe when we have equality.

Solution

We are asked to prove that for all $x > 0$,

$$\frac{1}{x} \geq 3 - 2\sqrt{x}.$$

We define the function

$$f(x) = \frac{1}{x} + 2\sqrt{x} - 3,$$

and we want to show that $f(x) \geq 0$ for all $x > 0$, and determine when equality occurs.

Step 1: Analyze the derivative

First, compute the derivative of $f(x)$:

$$f'(x) = \frac{d}{dx}\left(\frac{1}{x} + 2\sqrt{x} - 3\right) = -\frac{1}{x^2} + \frac{1}{\sqrt{x}}.$$

Set the derivative equal to 0 to find critical points:

$$-\frac{1}{x^2} + \frac{1}{\sqrt{x}} = 0 \Rightarrow \frac{1}{\sqrt{x}} = \frac{1}{x^2} \Rightarrow x^2 = \sqrt{x} \Rightarrow x^4 = x^{1/2} \Rightarrow x^{7/2} = 1 \Rightarrow x = 1.$$

So the only critical point in the domain $x > 0$ is at $x = 1$.

Step 2: Consider where the derivative is undefined

The derivative $f'(x)$ is undefined at $x = 0$, but since $f(x)$ is only defined for $x > 0$, we are not concerned about $x = 0$ itself. However, we examine the behavior as $x \to 0^+$.

Step 3: Boundary behavior as $x \to 0^+$ and $x \to \infty$

As $x \to 0^+$,

$$\frac{1}{x} \to \infty, \quad \sqrt{x} \to 0, \quad \text{so } f(x) \to \infty.$$

As $x \to \infty$,

$$\frac{1}{x} \to 0, \quad \sqrt{x} \to \infty, \quad \text{so } f(x) \to \infty.$$

Step 4: Evaluate the critical point

We evaluate $f(1)$:

$$f(1) = \frac{1}{1} + 2\sqrt{1} - 3 = 1 + 2 - 3 = 0.$$

Since $f(x) \to \infty$ as $x \to 0^+$ and as $x \to \infty$, and the only critical point is a minimum at $x = 1$, we conclude that

$$f(x) \geq 0 \quad \text{for all } x > 0,$$

with equality if and only if $x = 1$.

Conclusion

$$\frac{1}{x} \geq 3 - 2\sqrt{x}$$

for all $x > 0$, with equality if and only if $x = 1$.