Week 19 Writing Problem

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Problem Statement

Let

$$f(x) = \lfloor x \lfloor x \rfloor \rfloor \quad \text{for } x \geq 0.$$

(a) Find all $x \geq 0$ such that $f(x) = 1$.

(b) Find all $x \geq 0$ such that $f(x) = 3$.

(c) Find all $x \geq 0$ such that $f(x) = 5$.

(d) Find the number of possible values of $f(x)$ for $0 \leq x \leq 10$.

Solution

Let $n = \lfloor x \rfloor$, so $n \leq x < n + 1$ and n is a nonnegative integer. Then

$$f(x) = \lfloor x \lfloor x \rfloor \rfloor = \lfloor xn \rfloor.$$

(a) $f(x) = 1$

We want $\lfloor xn \rfloor = 1$.

If $n = 0$:	$xn = 0 \implies f(x) = 0.$
If $n = 1$:	$x \in [1, 2),\ xn = x \in [1, 2) \implies \lfloor xn \rfloor = 1.$
If $n \geq 2$:	$xn \geq 2n > 1.$

Thus, the solution is

$$\boxed{x \in [1, 2)}$$

(b) $f(x) = 3$

We want $\lfloor xn \rfloor = 3$.

$$\begin{aligned}
&\text{If } n = 0: && xn = 0.\\
&\text{If } n = 1: && x \in [1,2),\ xn \in [1,2) \implies \lfloor xn \rfloor = 1.\\
&\text{If } n = 2: && x \in [2,3),\ xn \in [4,6) \implies \lfloor xn \rfloor \geq 4.\\
&\text{If } n = 3: && x \in [3,4),\ xn \in [9,12) \implies \lfloor xn \rfloor \geq 9.
\end{aligned}$$

There is no $x \geq 0$ such that $f(x) = 3$.

$$\boxed{\text{No solution}}$$

(c) $f(x) = 5$

Try $n = 2$:

$$x \in [2,3),\ xn \in [4,6)$$

We want $xn \in [5,6)$, so $x \in [2.5, 3)$.

Thus,

$$\boxed{x \in [2.5, 3)}$$

(d) Number of possible values of $f(x)$ for $0 \leq x \leq 10$

For each integer n from 0 to 10:

$$\begin{aligned}
n = 0: &\quad x \in [0,1),\ f(x) = 0\\
n = 1: &\quad x \in [1,2),\ f(x) = 1\\
n = 2: &\quad x \in [2,3),\ f(x) = 4, 5\\
n = 3: &\quad x \in [3,4),\ f(x) = 9, 10, 11\\
n = 4: &\quad x \in [4,5),\ f(x) = 16, 17, 18, 19\\
n = 5: &\quad x \in [5,6),\ f(x) = 25, 26, 27, 28, 29\\
n = 6: &\quad x \in [6,7),\ f(x) = 36, 37, 38, 39, 40, 41\\
n = 7: &\quad x \in [7,8),\ f(x) = 49, 50, 51, 52, 53, 54, 55\\
n = 8: &\quad x \in [8,9),\ f(x) = 64, 65, 66, 67, 68, 69, 70, 71\\
n = 9: &\quad x \in [9,10],\ f(x) = 81, 82, 83, 84, 85, 86, 87, 88, 89\\
n = 10: &\quad x = 10,\ f(10) = 100
\end{aligned}$$

Listing all values:

0, 1, 4, 5, 9, 10, 11, 16, 17, 18, 19, 25, 26, 27, 28, 29, 36, 37, 38, 39, 40, 41, 49, 50, 51, 52, 53, 54, 55, 64, 65, 66, 67, 68, 69, 70, 71, 81, 82, 83, 84, 85, 86, 87, 88, 89, 100

Counting, we get 47 values.

$$\boxed{47}$$