Function Analysis

Michael Xie
Art of Problem Solving

December 2024

Exploring the Differences Between Two Functions

In this article, we explore the differences between the following two functions:

$$f(x) = \sqrt{\frac{x+5}{x-7}} \quad \text{and} \quad h(x) = \frac{\sqrt{x+5}}{\sqrt{x-7}}.$$

At first glance, these functions may seem very similar. However, there are key differences in their domains and ranges, which we will analyze in detail.

Domain and Range Analysis

Domain of $f(x)$

The function $f(x) = \sqrt{\frac{x+5}{x-7}}$ is defined only when both of the following conditions hold:

- The denominator $x - 7 \neq 0$ (to avoid division by zero).
- The expression $\frac{x+5}{x-7} \geq 0$ (to ensure the square root is defined and real).

Step 1: Avoiding Division by Zero

The denominator $x - 7 \neq 0$ implies:

$$x \neq 7.$$

Step 2: Ensuring Non-Negativity

To ensure $\frac{x+5}{x-7} \geq 0$, we analyze the signs of the numerator $(x + 5)$ and denominator $(x - 7)$ over different intervals of x:

- When $x + 5 \geq 0$ and $x - 7 > 0$, the fraction is non-negative. This occurs when $x \geq 7$.
- When $x + 5 \leq 0$ and $x - 7 < 0$, the fraction is also non-negative. This occurs when $x \leq -5$.

Combining these conditions, the domain of $f(x)$ is:

$$x \in (-\infty, -5] \cup [7, \infty).$$

Step 3: Removing $x = 7$ (Division by Zero)

Finally, removing $x = 7$ from the domain:

$$\text{Domain of } f(x): \quad x \in (-\infty, -5] \cup (7, \infty) \subset \mathbb{R}.$$

Domain of $h(x)$

The function $h(x) = \frac{\sqrt{x+5}}{\sqrt{x-7}}$ imposes stricter constraints due to the square roots in both the numerator and denominator. It is defined only when:

- The numerator $\sqrt{x+5}$ is defined, which requires $x + 5 \geq 0 \implies x \geq -5$.
- The denominator $\sqrt{x-7}$ is defined and nonzero, which requires $x - 7 > 0 \implies x > 7$.

Combining these conditions, the domain of $h(x)$ is:

$$\text{Domain of } h(x): \quad x \in (7, \infty) \subset \mathbb{R}.$$

Comparison of Domains

The key difference between the domains of $f(x)$ and $h(x)$ lies in the inclusion of the interval $(-\infty, -5]$ for $f(x)$, which is not part of the domain of $h(x)$. This is because the two square roots in $h(x)$ impose stricter constraints, requiring both $x + 5 \geq 0$ and $x - 7 > 0$ simultaneously, compared to $f(x)$'s one, encompassing square root.