Counting and Probability Week 6 Writing Problem

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Art of Problem Solving
Intermediate Counting and Probability

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Problem Statement

(a) Prove that given any set of 5 integers, there exist three of them whose sum is divisible by 3.

(b) Prove that given any set of 17 integers, there exist nine of them whose sum is divisible by 9.

Solution

(a)

Let the set of 5 integers be

$$S = \{a_1, a_2, a_3, a_4, a_5\}.$$

Consider their residues modulo 3. Each integer has residue 0, 1, or 2 (mod 3).

If any residue occurs at least 3 times, then choosing those three integers gives a sum congruent to $0 \pmod 3$, and we are done.

Otherwise, no residue appears more than twice. Since there are 5 integers and 3 residue classes, the only possible distribution of counts is $(2, 2, 1)$ in some order. In this case, all three residue classes are represented, so we can pick one integer of each residue. Their sum is

$$0 + 1 + 2 \equiv 0 \pmod 3.$$

Hence, in every possible case, we can find three integers whose sum is divisible by 3.

$$\boxed{\text{Therefore, among any five integers, there exist three whose sum is divisible by 3.}}$$

(b)

This is a special case of a well-known result in additive combinatorics called the **Erdős–Ginzburg–Ziv theorem**.

Theorem (Erdős–Ginzburg–Ziv). For any positive integer n, given any $2n-1$ integers, there exists a subset of n of them whose sum is divisible by n.

Applying this theorem with $n = 9$, we find that among any 17 integers, there exist 9 whose sum is divisible by 9.

$$\boxed{\text{Therefore, among any 17 integers, there exist nine whose sum is divisible by 9.}}$$