Counting and Probability Week 8 Writing Problem

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Art of Problem Solving,
Itermediate Counting and Probability

December 5, 2025

Problem Statement

In class, we proved that the number of ways of tiling a $1 \times n$ rectangle with 1×1 and 1×2 tiles is F_{n+1}.
(a) Use a tiling argument to give a combinatorial proof that

$$F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1.$$

(b) Use a tiling argument to give a combinatorial proof that

$$\binom{n}{1}F_1 + \binom{n}{2}F_2 + \binom{n}{3}F_3 + \cdots + \binom{n}{n}F_n = F_{2n}.$$

Solution

(a)

First We refactor the expression in part (a) using Binet's formula for the n^{th} Fibonacci number.

$$\begin{aligned}
F_1 + F_2 + F_3 + \cdots + F_n &= F_{n+2} - 1, \\
\frac{\phi - \psi}{\sqrt{5}} + \frac{\phi^2 - \psi^2}{\sqrt{5}} + \cdots + \frac{\phi^n - \psi^n}{\sqrt{5}} &= \frac{\phi^{n+2} - \psi^{n+2}}{\sqrt{5}} - 1, \\
(\phi - \psi) + (\phi^2 - \psi^2) + (\phi^3 - \psi^2) + \cdots + (\phi^n - \psi^n) &= \phi^{n+2} - \psi^{n+2} - \sqrt{5}.
\end{aligned}$$

We proceed henceforth with induction.

Base Case

$$
\begin{aligned}
\phi - \psi &= \phi^3 - \psi^3 - \sqrt{5}, \\
\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} &= \left(\frac{1+\sqrt{5}}{2}\right)^3 - \left(\frac{1-\sqrt{5}}{2}\right)^3 - \sqrt{5}, \\
1 &= \left(\frac{\left(1+\sqrt{5}\right)^3}{8}\right) - \left(\frac{\left(1-\sqrt{5}\right)^3}{8}\right) - \sqrt{5},
\end{aligned}
$$