Basic Graph Transformations

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Art of Problem Solving

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1 Introduction

In this article, we will perform and describe some basic graph transformations. We are given a function defined by its graph and asked to plot a transformed version of the given graph. Furthermore, we will explore why these transformations cause the graph to shift as they do.

1.1 The Problem

The graph of $y = f(x)$ is shown in the figure below.

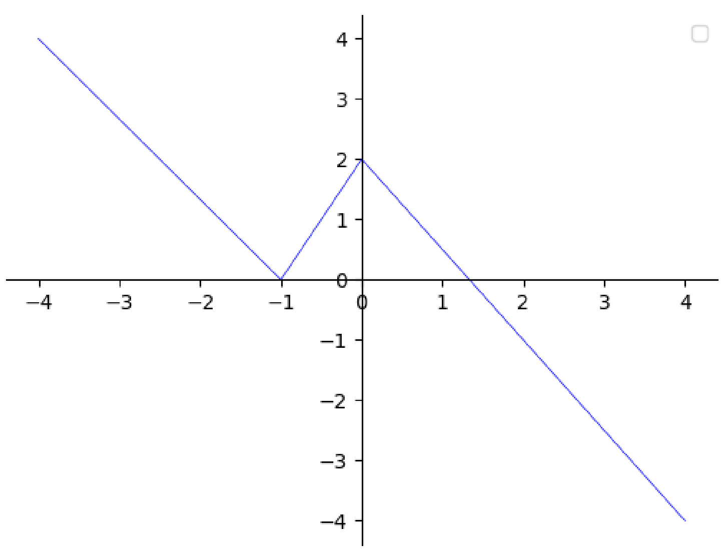

Figure 1: Graph of $y = f(x)$

For each point (a, b) that is on the graph of $y = f(x)$, the point $(2a + 1, 3b)$ is plotted, forming the graph of another function $y = g(x)$. As an example, the point $(0, 2)$ lies on the graph of $y = f(x)$, so the point $(2 \cdot 0 + 1, 3 \cdot 2) = (1, 6)$ lies on the graph of $y = g(x)$.

(a) Plot the graph of $y = g(x)$. Include the diagram as part of your solution.

(b) Express $g(x)$ in terms of $f(x)$.

(c) Describe the transformations that can be applied to the graph of $y = f(x)$ to obtain the graph of $y = g(x)$. For example, one transformation could be to stretch the graph vertically by a factor of 3.

2 Solution

2.1 Part (a)

Let's start by plotting the graph, $y = g(x)$. As these are all linear transformations, there is no need to generate or find additional points for a larger sample size. We can use the integer points given:

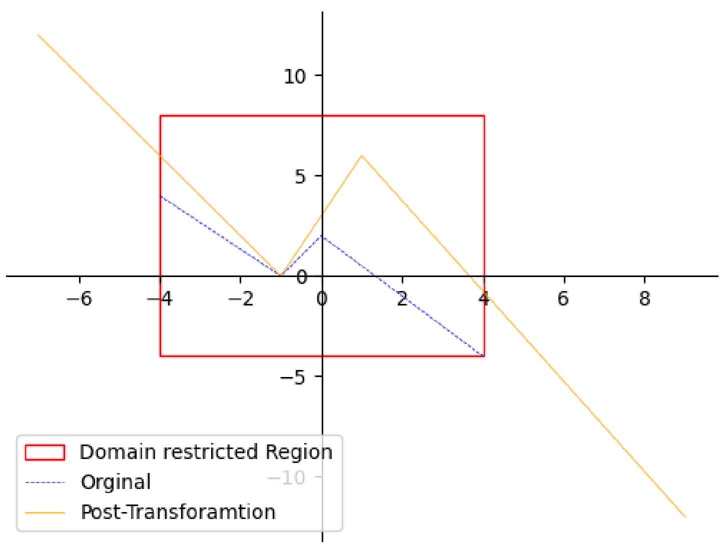

Figure 2: Graph of $y = g(x)$ vs $y = f(x)$

As the domain of $f(x)$ is $[-4, 4]$, the graph of $g(x)$ is subject to the same domain. This is shown by the red rectangle.

2.2 Part (b)

Let us define $g(x)$ in terms of $f(x)$:

$$\text{Any point on the graph of } g(x) \text{ satisfies the following:}$$

$$(a, b) = (2x + 1, 3f(x))$$

$$\text{Therefore,}$$

$$g(2x + 1) = 3f(x)$$

$$\text{Let } \phi = 2x + 1, \text{ so}$$

$$g(\phi) = 3f(x)$$

$$\text{Now, express } x \text{ in terms of } \phi:$$

$$\phi = 2x + 1$$

$$\phi - 1 = 2x$$

$$x = \frac{\phi - 1}{2}$$

$$\text{Substituting this back into the original equation:}$$

$$g(\phi) = 3f\left(\frac{\phi}{2} - \frac{1}{2}\right)$$

$$\text{Replacing } \phi \text{ with } x, \text{ we get:}$$

$$g(x) = 3f\left(\frac{x}{2} - \frac{1}{2}\right)$$

$$\text{Thus, we have arrived at our solution.}$$

2.3 Part (c)

Starting from the inside out, let's examine the effect of the transformation $y = f\left(\frac{x}{2} - \frac{1}{2}\right)$.

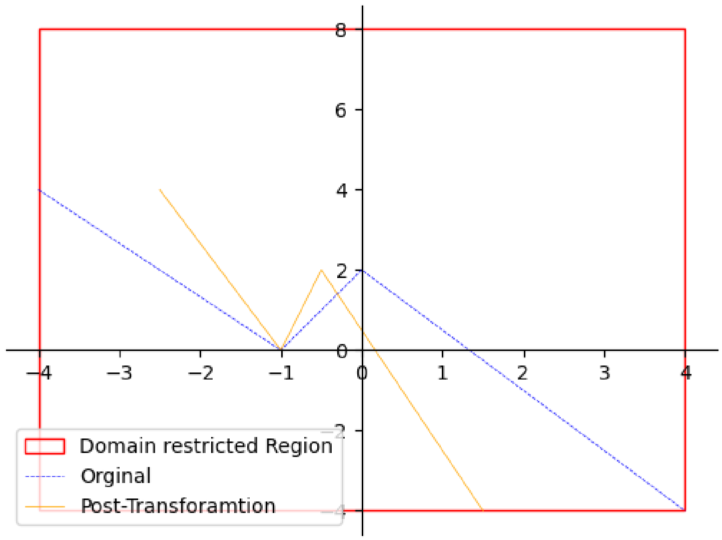

Figure 3: Graph of $y = f\left(\frac{x}{2} - \frac{1}{2}\right)$ vs $y = f(x)$

It appears that the graph is horizontally compressed by a factor of 2, followed by a translation by the vector $(-\frac{1}{2}, 0)$. Now, if we plot only $y = f\left(\frac{x}{2}\right)$:

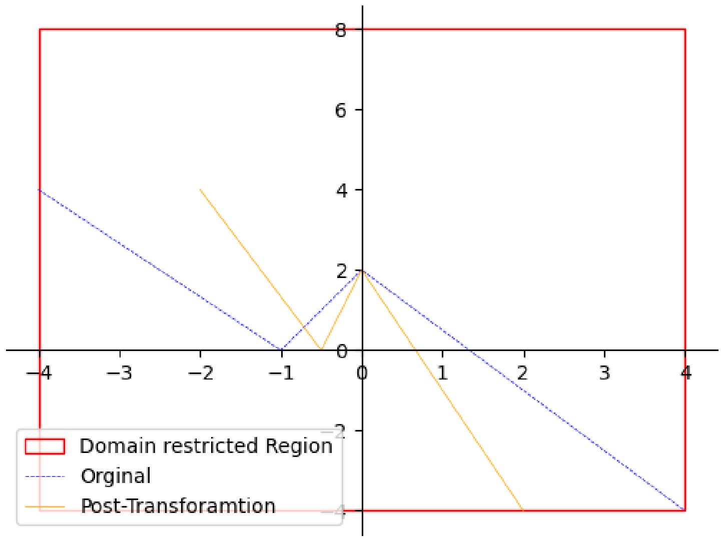

Figure 4: Graph of $y = f\left(\frac{x}{2}\right)$ vs $y = f(x)$

We see that the graph has only been compressed horizontally. Thus, scaling the input is inversely proportional to the horizontal stretch, and adding a constant translates the graph by a vector of $(c, 0)$, where c is the constant.

Multiplying by three is fairly straghtforward. it stretches the graph vertically by a factor of 3. More precisely, it triples the function's derivative.

Let $\phi = \frac{x}{2} - \frac{1}{2}$, and differentiate:

$$\frac{d}{dx}\left[3f(\phi)\right] = 3\frac{d}{dx}\left[f(\phi)\right] = 3f'(\phi)$$

Therefore, the final transformation consists of a horizontal compression by a factor of 2, a leftward translation by $\frac{1}{2}$, and a vertical stretch by a factor of 3, leaving us with figure 5 below.

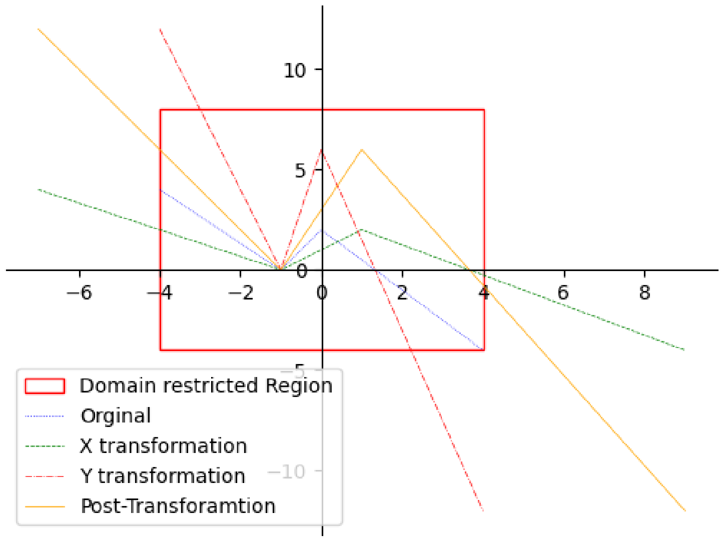

Figure 5: Graph of $y = f\left(\frac{x}{2}\right)$ vs $y = f(x)$