Polynomial Composition

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In this article, we aim to prove that **there exist infinitely many positive integers n such that a nonconstant polynomial $P(n)$ is composite.** We will do this by constructing a polynomial $P(x)$ and showing that it can be evaluated at infinitely many integers to yield composite numbers.

1 Defining the Polynomial

Let $P(x) = \sum_{k=0}^{m} a_k x^k$, where $a_k \geq 0$. This is a nonconstant polynomial with nonnegative integer coefficients. We will show that there are infinitely many positive integers n such that $P(n)$ is composite.

2 Observing Limiting Behavior

Because $P(x)$ is a nonconstant polynomial with positive integer coefficients, the behavior of $P(n)$ as $n \to \infty$ tends to infinity. This means that for sufficiently large n, $P(n)$ will be a large positive integer. Thus, $P(n)$ cannot always be prime, as there are infinitely many integers n but only finitely many primes less than any given bound.

3 Constructing Composite Values

In this section, we will prove that there are infinitely many positive integers n such that $P(n)$ is composite.

Using our definition for $P(N)$, we split into two cases.

3.1 Case 1: $P(1)$ is Composite

If $P(1)$ is composite, then $n = 1$ is one such n. Since $P(x)$ is nonconstant, $P(n)$ will take on infinitely many values as $n \to \infty$, and we can find infinitely many n such that $P(n)$ is composite.

3.2 Case 2: $P(1)$ is Prime

If $P(1)$ is prime, consider the sequence $P(kP(1))$ for $k = 1, 2, 3, \ldots$. Since $P(x)$ is a polynomial with integer coefficients, $P(kP(1))$ is divisible by $P(1)$ for all k. Specifically:

$$P(kP(1)) \equiv 0 \pmod{P(1)}.$$

Thus, $P(kP(1))$ is composite for all $k \geq 2$, because it is divisible by $P(1)$ and greater than $P(1)$.

Therefore, we have constructed infinitely many positive integers n such that $P(n)$ is composite.