Probability that at least one triangle appears among five random segments

Problem

Twelve points are given in the plane, with no three on a line. Five distinct segments joining pairs of these points are chosen at random (all $\binom{12}{2} = 66$ segments equally likely). Find the probability that the chosen segments contain the three edges of at least one triangle whose vertices are among the twelve points.

Solution

Let the total number of ways to choose five segments be $N_{\text{total}} = \binom{66}{5}$.

We count the number of 5-edge sets that contain at least one triangle; denote this by $N_{\geq 1}$. We use inclusion–exclusion:

- First, count 5-edge sets that contain a fixed triangle. There are $\binom{12}{3}$ choices for the triangle (three vertices), and for each such triangle we must choose the remaining 2 edges from the other $66 - 3 = 63$ edges. So the first-term contribution is $\binom{12}{3}\binom{63}{2}$.

- Next we subtract those 5-edge sets that were counted twice because they contain two (distinct) triangles. Observe that two distinct triangles cannot both be subgraphs of a 5-edge set unless they share exactly one edge. (If they were disjoint or shared only a vertex their union would require at least 6 edges.) Two triangles sharing an edge live on exactly 4 vertices and together use exactly 5 distinct edges. Thus every such pair of triangles corresponds to a unique 5-edge set: the union of the two triangles.

  How many such unions ("bow-ties") are there? Choose any set of 4 vertices out of 12: $\binom{12}{4}$ choices. On those 4 vertices there are $\binom{4}{2} = 6$ choices for which edge is the common edge of the two triangles. Hence the number of 5-edge sets that contain two triangles (i.e. that were double-counted) is $6\binom{12}{4}$.

Therefore $N_{\geq 1} = \binom{12}{3}\binom{63}{2} - 6\binom{12}{4}$.

The desired probability is $P = \frac{N_{\geq 1}}{\binom{66}{5}} = \frac{\binom{12}{3}\binom{63}{2} - 6\binom{12}{4}}{\binom{66}{5}}$.

We can simplify this to a reduced rational and a decimal approximation: $P = \frac{2155}{45136} \approx 0.0477446 \approx 4.774\%$.