# Week two Writing Problem

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## Problem Statement

Let $n$ be a positive integer, and let $S = \{1, 2, 3, \ldots, n\}$. Three subsets $A$, $B$, $C$ of $S$ are chosen at random.

(a) Find the probability that $A \cup B \cup C = S$.

(b) Find the probability that $A \subseteq B \subseteq C$.

## Solution

**(a)** Each of the three subsets $A, B, C$ is chosen independently and uniformly at random from the power set of $S = \{1, 2, 3, \ldots, n\}$. For each element $x \in S$, the probability that $x$ is *not* contained in any of $A, B$, or $C$ is

$$\left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

Thus, the probability that $x$ is contained in at least one of the sets is

$$1 - \frac{1}{8} = \frac{7}{8}.$$

Since the inclusion of distinct elements is independent, we have

$$\Pr(A \cup B \cup C = S) = \left(\frac{7}{8}\right)^n.$$

**(b)** For a fixed element $x \in S$, the condition $A \subseteq B \subseteq C$ means that the membership indicator

$$(\mathbf{1}_{x\in A}, \mathbf{1}_{x\in B}, \mathbf{1}_{x\in C})$$

must be nondecreasing. The allowed membership patterns are:

$$(0,0,0), \quad (0,0,1), \quad (0,1,1), \quad (1,1,1).$$

There are 4 such patterns out of $2^3 = 8$ total possibilities, so the probability for a single element is

$$\frac{4}{8} = \frac{1}{2}.$$

By independence over all $n$ elements, we have

$$\Pr(A \subseteq B \subseteq C) = \left(\frac{1}{2}\right)^n.$$

**Final Answers:**

$$\boxed{\Pr(A \cup B \cup C = S) = \left(\frac{7}{8}\right)^n} \quad \text{and} \quad \boxed{\Pr(A \subseteq B \subseteq C) = \left(\frac{1}{2}\right)^n}.$$