Week 14 Writing Problem

Michael Xie,
Art of Problem Solving,
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Problem

The sequence (a_n) is defined by $a_1 = \frac{1}{2}$ and

$$a_n = a_{n-1}^2 + a_{n-1}$$

for all $n \geq 2$.

Prove that

$$\frac{1}{a_1+1} + \frac{1}{a_2+1} + \cdots + \frac{1}{a_k+1} < 2$$

for all $k \geq 1$.

Solution

Observe first that

$$a_{n+1} \;=\; a_n^2 + a_n \quad\Longrightarrow\quad a_{n+1} - a_n = a_n^2,$$

so

$$\frac{1}{a_n} - \frac{1}{a_{n+1}} \;=\; \frac{a_{n+1}-a_n}{a_n\,a_{n+1}} \;=\; \frac{a_n^2}{a_n\,a_{n+1}} \;=\; \frac{1}{a_{n+1}} \;=\; \frac{1}{a_n+1}.$$

Hence for each $n \geq 1$,

$$\frac{1}{a_n+1} \;=\; \frac{1}{a_n} - \frac{1}{a_{n+1}},$$

and the sum telescopes:

$$\sum_{n=1}^{k} \frac{1}{a_n+1} \;=\; \left(\tfrac{1}{a_1} - \tfrac{1}{a_2}\right) + \left(\tfrac{1}{a_2} - \tfrac{1}{a_3}\right) + \cdots + \left(\tfrac{1}{a_k} - \tfrac{1}{a_{k+1}}\right) \;=\; \frac{1}{a_1} - \frac{1}{a_{k+1}}.$$

Since $a_1 = \frac{1}{2}$, this becomes

$$\sum_{n=1}^{k} \frac{1}{a_n+1} \;=\; 2 - \frac{1}{a_{k+1}} \;<\; 2,$$

as claimed.

Conclusion

The sequence (a_n) is defined recursively, and we have shown that the sum of the series converges to a value less than 2. This demonstrates the power of telescoping series in simplifying complex recursive definitions.