Quadratic equations with aboslute value

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Problem

Find all solutions of the equation

$$|x^2 - 30x - 1| = 30.$$

Solution

To fins solutions to the equation $|x^2 - 30x - 1| = 30$, we need to consider two cases:

Case 1: $x^2 - 30x - 1 = 30$,
Case 2: $x^2 - 30x - 1 = -30$

Case 1: $x^2 - 30x - 1 = 30$

$$x^2 - 30x - 1 = 30,$$
$$x^2 - 30x - 31 = 0$$

We solve this quadratic equation using the quadratic formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$:

$$a = 1, \quad b = -30, \quad c = -31$$
$$x = \frac{30 \pm \sqrt{(-30)^2 - 4 \cdot 1 \cdot (-31)}}{2 \cdot 1},$$
$$x = \frac{30 \pm \sqrt{900 + 124}}{2},$$
$$x = \frac{30 \pm \sqrt{1024}}{2},$$
$$x = \frac{30 \pm 32}{2},$$
$$x = \frac{30 + 32}{2} \quad \text{or} \quad x = \frac{30 - 32}{2},$$
$$x = 31 \quad \text{or} \quad x = -1.$$

Case 2: $x^2 - 30x - 1 = -30$

$$x^2 - 30x - 1 = -30,$$
$$x^2 - 30x + 29 = 0$$

We solve this quadratic equation using the quadratic formula:

$$a = 1, \quad b = -30, \quad c = 29$$
$$x = \frac{30 \pm \sqrt{(-30)^2 - 4 \cdot 1 \cdot 29}}{2 \cdot 1},$$
$$x = \frac{30 \pm \sqrt{900 - 116}}{2},$$
$$x = \frac{30 \pm \sqrt{784}}{2},$$
$$x = \frac{30 \pm 28}{2},$$
$$x = \frac{30 + 28}{2} \quad \text{or} \quad x = \frac{30 - 28}{2},$$
$$x = 29 \quad \text{or} \quad x = 1.$$

Conclusion

The solutions to the equation $|x^2 - 30x - 1| = 30$ are:

$$x = 31, \quad x = -1, \quad x = 29, \quad x = 1$$