Week 18 Writing Problem

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Problem Statment

Let

$$f(x) = \frac{x - \sqrt{3}}{x\sqrt{3} + 1}.$$

What is $f^{2012}(x)$, where the function is being applied 2012 times? This notation indicates repeated composition of functions, not exponentiation of functions. For example,

$$f^2(x) = f(f(x))$$

and not $f(x) \cdot f(x)$. Similarly,

$$f^3(x) = f(f(f(x))).$$

Solution

The key observation is to note that $f(x)$ is a Möbius transformation, and repeated application of f can be analyzed by studying its properties.

Let us compute the first few iterations:

$$f(x) = \frac{x - \sqrt{3}}{x\sqrt{3} + 1}$$

$$f^2(x) = f(f(x)) = \frac{f(x) - \sqrt{3}}{f(x)\sqrt{3} + 1}$$

Substitute $f(x)$:

$$f^2(x) = \frac{\frac{x-\sqrt{3}}{x\sqrt{3}+1} - \sqrt{3}}{\frac{x-\sqrt{3}}{x\sqrt{3}+1}\sqrt{3} + 1}$$

$$
\begin{aligned}
&\frac{x-\sqrt{3}-\sqrt{3}(x\sqrt{3}+1)}{x\sqrt{3}+1} \\
&= \frac{x-\sqrt{3}-x\cdot 3-\sqrt{3}}{x\sqrt{3}+1} \\
&= \frac{x-3x-\sqrt{3}-\sqrt{3}}{x\sqrt{3}+1} \\
&= \frac{-2x-2\sqrt{3}}{x\sqrt{3}+1}
\end{aligned}
$$

Simplify denominator:

$$\frac{(x-\sqrt{3})\sqrt{3}}{x\sqrt{3}+1}+1=\frac{x\sqrt{3}-3}{x\sqrt{3}+1}+1=\frac{x\sqrt{3}-3+x\sqrt{3}+1}{x\sqrt{3}+1}=\frac{2x\sqrt{3}-2}{x\sqrt{3}+1}$$

So,

$$f^2(x)=\frac{-2(x+\sqrt{3})}{2(x\sqrt{3}-1)}=-\frac{x+\sqrt{3}}{x\sqrt{3}-1}$$

Now, compute $f^3(x)$:

$$f^3(x)=f(f^2(x))=\frac{f^2(x)-\sqrt{3}}{f^2(x)\sqrt{3}+1}$$

Let $y=f^2(x)=-\frac{x+\sqrt{3}}{x\sqrt{3}-1}$:

$$f^3(x)=\frac{-\frac{x+\sqrt{3}}{x\sqrt{3}-1}-\sqrt{3}}{-\frac{x+\sqrt{3}}{x\sqrt{3}-1}\sqrt{3}+1}$$

Numerator:

$$
\begin{aligned}
&-\frac{x+\sqrt{3}}{x\sqrt{3}-1}-\sqrt{3} \\
&= \frac{-(x+\sqrt{3})-\sqrt{3}(x\sqrt{3}-1)}{x\sqrt{3}-1} \\
&= \frac{-(x+\sqrt{3})-x\cdot 3+\sqrt{3}}{x\sqrt{3}-1} \\
&= \frac{-(x+\sqrt{3})-3x+\sqrt{3}}{x\sqrt{3}-1} \\
&= \frac{-x-\sqrt{3}-3x+\sqrt{3}}{x\sqrt{3}-1} \\
&= \frac{-4x}{x\sqrt{3}-1}
\end{aligned}
$$

Denominator:

$$\begin{aligned}
&-\frac{x+\sqrt{3}}{x\sqrt{3}-1}\sqrt{3}+1 \\
&= -\frac{(x+\sqrt{3})\sqrt{3}}{x\sqrt{3}-1}+1 \\
&= \frac{-(x\sqrt{3}+3)+(x\sqrt{3}-1)}{x\sqrt{3}-1} \\
&= \frac{-(x\sqrt{3}+3)+x\sqrt{3}-1}{x\sqrt{3}-1} \\
&= \frac{-3-1}{x\sqrt{3}-1} \\
&= \frac{-4}{x\sqrt{3}-1}
\end{aligned}$$

So,

$$f^3(x) = \frac{-4x}{-4} = x$$

Thus, $f^3(x) = x$, so f is of order 3 under composition. Therefore,

$$f^{2012}(x) = f^{2012 \bmod 3}(x)$$

Since $2012 \div 3 = 670$ remainder 2, $2012 \bmod 3 = 2$.

Therefore,

$$\boxed{f^{2012}(x) = f^2(x) = -\frac{x+\sqrt{3}}{x\sqrt{3}-1}}$$