Expression of logarithmic functions in terms of other logarithms

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Art of Problem Solving,
Intermediate Algebra

June 3, 2025

Introduction

In this document, we will explore the expression of logarithmic functions in terms of other logarithms. We use an example to illustrate the process of rewriting logarithmic expressions.

Problem Statement

Let $P = \log_8 3$ and $Q = \log_3 5$. Express $\log_{10} 5$ in terms of P and Q. Your answer should no longer include any logarithms.

Solution 1

To express $\log_{10} 5$ in terms of P and Q, we start by using the change of base formula for logarithms:

$$\log_{10} 5 = \frac{\log_3 5}{\log_3 10}$$

Substituting Q for $\log_3 5$, we have:

$$\log_{10} 5 = \frac{Q}{\log_3 10}$$

Now, we use the change of base formula again to express $\log_3 10$ in terms of P:

$$\frac{Q}{\log_3 10} = \frac{Q}{\frac{\log_8 10}{\log_8 3}} = \frac{Q}{\frac{\log_8 10}{P}}$$

This simplifies to:

$$\begin{aligned}\log_{10} 5 &= \frac{QP}{\log_8 10}, \\ &= \frac{QP}{\frac{1}{3}\log_2 10}, \\ &= \frac{3QP}{\log_2 10}.\end{aligned}$$

Where $\log_2 10$ can be expressed as:

$$\begin{aligned}\log_2 10 &= \log_2(2 \cdot 5), \\ &= \log_2 2 + \log_2 5, \\ &= 1 + \log_2 5.\end{aligned}$$

Thus, we can express $\log_{10} 5$ as:

$$\log_{10} 5 = \frac{3QP}{1 + \log_2 5}$$

We can simplify further by expressing $\log_2 5$ in terms of Q using the change of base formula:

$$\log_2 5 = \frac{\log_3 5}{\log_3 2} = \frac{Q}{\log_3 2}$$

And once again using the change of base formula:

$$\log_3 2 = \frac{\log_8 2}{\log_8 3} = \frac{1/3}{P} = \frac{1}{3P}$$

Substituting this into our expression for $\log_2 5$ gives:

$$\log_2 5 = \frac{Q}{\frac{1}{3P}} = 3QP.$$

Thus, we can express $\log_{10} 5$ as:

$$\log_{10} 5 = \frac{3QP}{1 + 3QP}.$$

Or, equivalently:

$$\frac{3Q}{P + 3Q}$$

Solution 2

Another approach to express $\log_{10} 5$, P and Q in terms of the natural logarithm:

$$\log_{10} 5 = \frac{\ln 5}{\ln 10}, \quad P = \frac{\ln 3}{\ln 8}, \quad Q = \frac{\ln 5}{\ln 3}.$$

Using the product rule, we can express $\ln 10$ in terms of P and Q:

$$\ln 10 = \ln(2 \cdot 5) = \ln 2 + \ln 5.$$

Substituting $\ln 5 = Q \ln 3$ and $\ln 2 = \frac{\ln 8}{3} = \frac{P \ln 3}{3}$, we have:

$$\ln 10 = \frac{P \ln 3}{3} + Q \ln 3 = \left(\frac{P}{3} + Q\right) \ln 3.$$

Thus, we can express $\log_{10} 5$ as:

$$\log_{10} 5 = \frac{\ln 5}{\left(\frac{P}{3} + Q\right)} = \frac{Q}{\left(\frac{P}{3} + Q\right)} = \frac{3Q}{P + 3Q}.$$

Conclusion

In this article, we have successfully expressed $\log_{10} 5$ in terms of P and Q. We used the change of base formula and properties of logarithms to derive the final expression. The result is:

$$\log_{10} 5 = \frac{3Q}{P + 3Q}.$$