Solving Exponential Equations

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1 Introduction

In this article, we aim to solve the exponential equation

$$(3^x - 27)^3 + (27^x - 3)^3 = (3^x + 27^x - 30)^3.$$

Our approach will use substitution and factoring.

2 Solution

2.1 Substitution

We begin by letting

$$t = 3^x.$$

Since $27^x = (3^3)^x = 3^{3x} = (3^x)^3 = t^3$, the equation becomes

$$(t - 27)^3 + (t^3 - 3)^3 = (t + t^3 - 30)^3.$$

Now, define

$$a = t - 27 \quad \text{and} \quad b = t^3 - 3.$$

Then our equation can be written as

$$a^3 + b^3 = (a + b)^3.$$

This key observation will allow us to factor the equation.

2.2 Factoring

Recall the expansion of a cube:

$$(a + b)^3 = a^3 + b^3 + 3a^2b + 3ab^2.$$

Substituting into our equation, we have

$$a^3 + b^3 = a^3 + b^3 + 3a^2b + 3ab^2.$$

Subtracting $a^3 + b^3$ from both sides yields

$$3a^2b + 3ab^2 = 0.$$

Factor out $3ab$:

$$3ab(a + b) = 0.$$

Thus, at least one of the following must hold:

$$a = 0, \quad b = 0, \quad \text{or} \quad a + b = 0.$$

Rewriting these conditions in terms of t:

1. $a = 0$:
$$t - 27 = 0 \implies t = 27.$$

2. $b = 0$:
$$t^3 - 3 = 0 \implies t^3 = 3 \implies t = \sqrt[3]{3}.$$

3. $a + b = 0$:
$$(t - 27) + (t^3 - 3) = 0 \implies t^3 + t - 30 = 0.$$

We now focus on solving the third equation,

$$t^3 + t - 30 = 0.$$

Notice that $t = 3$ is a solution since

$$3^3 + 3 - 30 = 27 + 3 - 30 = 0.q$$

To factor $t^3 + t - 30$, perform polynomial division or factor by grouping. We can write:

$$\begin{aligned}
t^3 + t - 30 &= t^3 - 9t + 10t - 30 \\
&= t(t^2 - 9) + 10(t - 3) \\
&= t(t - 3)(t + 3) + 10(t - 3) \\
&= (t - 3)\big[t(t + 3) + 10\big] \\
&= (t - 3)(t^2 + 3t + 10).
\end{aligned}$$

Since the quadratic $t^2 + 3t + 10$ has discriminant

$$\Delta = 3^2 - 4 \cdot 1 \cdot 10 = 9 - 40 = -31,$$

its roots are non-real. Therefore, the only real solution from this factorization is

$$t - 3 = 0 \implies t = 3.$$

2.3 Back-Substitution

Recall that $t = 3^x$. The three real cases are:

1. $t = 27$:
$$3^x = 27 \implies 3^x = 3^3 \implies x = 3.$$
2. $t = \sqrt[3]{3}$:
$$3^x = \sqrt[3]{3} = 3^{1/3} \implies x = \frac{1}{3}.$$
3. $t = 3$:
$$3^x = 3 \implies 3^x = 3^1 \implies x = 1.$$

3 Conclusion

The real solutions to the exponential equation

$$(3^x - 27)^3 + (27^x - 3)^3 = (3^x + 27^x - 30)^3$$

are

$$x = \frac{1}{3}, \quad x = 1, \quad \text{and} \quad x = 3.$$