Example of a Functional Equation

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Problem Statement

The function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$f(x)f(y) = f(x + y) + xy$$

for all real numbers x and y. Find all possible functions f.

Solution

Let $P(x, y)$ denote the given assertion:

$$f(x)f(y) = f(x + y) + xy$$

Step 1: Find $f(0)$.
Let $y = 0$ in $P(x, 0)$:

$$f(x)f(0) = f(x) + 0 \implies f(x)f(0) = f(x)$$

If $f(0) = 0$, then $f(x) = 0$ for all x, but plugging this into the original equation gives $0 = 0 + xy$, which is not true for all x, y. Thus, $f(0) \neq 0$, and we can divide both sides by $f(0)$ to get $f(0) = 1$.

Step 2: Find $f(-x)$.
Let $y = -x$ in $P(x, -x)$:

$$f(x)f(-x) = f(0) + x(-x) = 1 - x^2$$

Step 3: Try linear solutions.
Suppose $f(x) = ax + b$. Plug into the original equation:

$$(ax + b)(ay + b) = a(x + y) + b + xy$$

$$a^2xy + abx + aby + b^2 = a(x + y) + b + xy$$

Comparing coefficients:

- xy: $a^2 = 1 \implies a = 1$ or $a = -1$

- x: $ab = a \implies b = 1$ (if $a \neq 0$)
- Constant: $b^2 = b \implies b = 0$ or $b = 1$

So the only possible linear solutions are $f(x) = x + 1$ and $f(x) = -x + 1$.

Check $f(x) = x + 1$:

$$(x+1)(y+1) = (x+y+1) + xy \implies xy + x + y + 1 = x + y + 1 + xy$$

True.

Check $f(x) = -x + 1$:

$$(-x+1)(-y+1) = (-(x+y)+1) + xy$$

$$(xy - x - y + 1) = (-x - y + 1) + xy$$

True.

Step 4: Check for other solutions.

Suppose f is not linear. Try $f(x) = c$ (constant):

$$c^2 = c + xy$$

This cannot hold for all x, y.

Try quadratic: $f(x) = px^2 + qx + r$. Plug into the original equation:

$$(px^2 + qx + r)(py^2 + qy + r) = p(x+y)^2 + q(x+y) + r + xy$$

The left side contains a $p^2x^2y^2$ term (degree 4), but the right side's highest degree is xy (degree 2). For the equation to hold for all x, y, the coefficients of x^2y^2 and all higher-degree terms must be zero, so $p = 0$. Thus, f cannot be quadratic or of higher degree.

This argument generalizes: if f is a polynomial of degree $n > 1$, the left side will have degree $2n$ terms, but the right side will have degree at most n. Thus, all higher-degree coefficients must be zero, forcing f to be linear.

Conclusion:

$$\boxed{f(x) = x + 1 \quad \text{or} \quad f(x) = -x + 1}$$