Writing Problem for Week 17

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Art of Problem Solving

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Problem Statement

Find all real solutions for x in

$$2(2^x - 1)x^2 + (2^{x^2} - 2)x = 2^{x+1} - 2.$$

Solution

Let us define

$$f(x) = 2(2^x - 1)(x^2 - 1) + (2^{x^2-1} - 1)x.$$

We claim that the original equation is equivalent to $f(x) = 0$.

Step 1: Sign analysis.

Note that for any real y, $2^y - 1$ and y have the same sign (since $2^y - 1 = 0$ if $y = 0$, and $2^y - 1 > 0$ if $y > 0$, $2^y - 1 < 0$ iff $y < 0$). Thus,

$$\operatorname{sgn}(2^x - 1) = \operatorname{sgn}(x),$$

$$\operatorname{sgn}(2^{x^2-1} - 1) = \operatorname{sgn}(x^2 - 1).$$

Therefore,

$$\operatorname{sgn}\left(2(2^x - 1)(x^2 - 1)\right) = \operatorname{sgn}(x)\operatorname{sgn}(x^2 - 1),$$

$$\operatorname{sgn}\left((2^{x^2-1} - 1)x\right) = \operatorname{sgn}(x^2 - 1)\operatorname{sgn}(x).$$

So both terms have the same sign, and thus

$$\operatorname{sgn}(f(x)) = \operatorname{sgn}(x)\operatorname{sgn}(x^2 - 1).$$

But $\operatorname{sgn}(x^2 - 1) = \operatorname{sgn}(x - 1)\operatorname{sgn}(x + 1)$, so

$$\operatorname{sgn}(f(x)) = \operatorname{sgn}(x)\operatorname{sgn}(x - 1)\operatorname{sgn}(x + 1).$$

Step 2: Zeros of $f(x)$.

The sign function is zero if and only if its argument is zero. Therefore, $f(x) = 0$ if and only if $x = 0$, $x = 1$, or $x = -1$.

Step 3: Conclusion.

Thus, the only real solutions to the original equation are

$$\boxed{x = -1,\ 0,\ 1}$$