Trinomial roots and arithmetic progressions

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Art of Problem Solving,
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1 Problem

1.1 (a)

Prove that if the roots of

$$x^3 + ax^2 + bx + c = 0$$

form an arithmetic sequence, then $2a^3 + 27c = 9ab$.

1.2 (b)

Prove that if $2a^3 + 27c = 9ab$, then the roots of

$$x^3 + ax^2 + bx + c = 0$$

form an arithmetic sequence.

2 Solution

2.1 (a)

Let the roots of the polynomial be r_1, r_2, r_3. Since they are in arithmetic progression, we can express them as:

$$r_1 = r$$

$$r_2 = r + d$$

$$r_3 = r + 2d$$

where r is the first term and d is the common difference. By Vieta's formulas, we have:

$$a = -(r_1 + r_2 + r_3) = -3r - 3d$$

$$b = r_1r_2 + r_2r_3 + r_3r_1 = 3r^2 + 6rd + 2d^2$$
$$c = -r_1r_2r_3 = -r(r+d)(r+2d)$$

Substituting these expressions into the equation $2a^3 + 27c = 9ab$, we get:

$$\begin{align} -2(3r+3d)^3 - 27r(r+d)(r+2d) &= -9(3r^2+6dr+2d^2)(3r+3d), \tag{1}\\ -2\cdot 27(r+d)^3 - 27r(r+d)(r+2d) &= -27(3r^2+6dr+2d^2)(r+d), \tag{2}\\ 2(r+d)^3 + r(r+d)(r+2d) &= (3r^2+6dr+2d^2)(r+d), \tag{3}\\ 2(r+d)^2 + r(r+2d) &= 3r^2+6dr+2d^2, \tag{4}\\ 2(r^2+2dr+d^2) + r^2 + 2dr &= 3r^2+6dr+2d^2, \tag{5}\\ 2r^2+4dr+2d^2+r^2+2dr &= 3r^2+6dr+2d^2, \tag{6}\\ 3r^2+6dr+2d^2 &= 3r^2+6dr+2d^2. \tag{7} \end{align}$$

This is an identity, which means that the equation holds for all values of r and d. Thus, we have shown that if the roots of the polynomial are in arithmetic progression, then $2a^3 + 27c = 9ab$.

2.2 (b)

Prove that if $2a^3 + 27c = 9ab$, then the roots of

$$x^3 + ax^2 + bx + c = 0$$

form an arithmetic sequence. Assume that $2a^3 + 27c = 9ab$. We can express c in terms of a and b:

$$c = \frac{9ab - 2a^3}{27}$$

Using Vieta's formulas, we have:

$$r_1 + r_2 + r_3 = -a$$
$$r_1r_2 + r_2r_3 + r_3r_1 = b$$
$$r_1r_2r_3 = -c$$

Substituting the expression for c into the third equation, we get:

$$r_1r_2r_3 = -\frac{9ab - 2a^3}{27}$$

Now, let us consider the general case for the roots r_1, r_2, r_3 of the cubic. By Vieta's formulas, we have:

$$r_1 + r_2 + r_3 = -a, \quad r_1r_2 + r_2r_3 + r_3r_1 = b, \quad r_1r_2r_3 = -c.$$

Given $2a^3 + 27c = 9ab$, substitute $c = \frac{9ab-2a^3}{27}$ into the third Vieta equation:

$$r_1r_2r_3 = -\frac{9ab - 2a^3}{27}.$$

Now, consider the elementary symmetric sums $S_1 = r_1 + r_2 + r_3$, $S_2 = r_1 r_2 + r_2 r_3 + r_3 r_1$, $S_3 = r_1 r_2 r_3$. The only way for the relation $2a^3 + 27c = 9ab$ to always hold is if the roots are in arithmetic progression (as shown in part (a)). Therefore, the roots must be of the form $r, r + d, r + 2d$ for some r and d. Thus, the roots form an arithmetic sequence.

3 Conclusion

We have shown that if the roots of the polynomial $x^3 + ax^2 + bx + c = 0$ form an arithmetic sequence, then $2a^3 + 27c = 9ab$. Conversely, if $2a^3 + 27c = 9ab$, then the roots of the polynomial also form an arithmetic sequence. Thus, we have proven both parts of the problem.