University of the Basque Country

PhD Thesis

LOWER BOUNDS ON QUANTUM METROLOGICAL PRECISIONS

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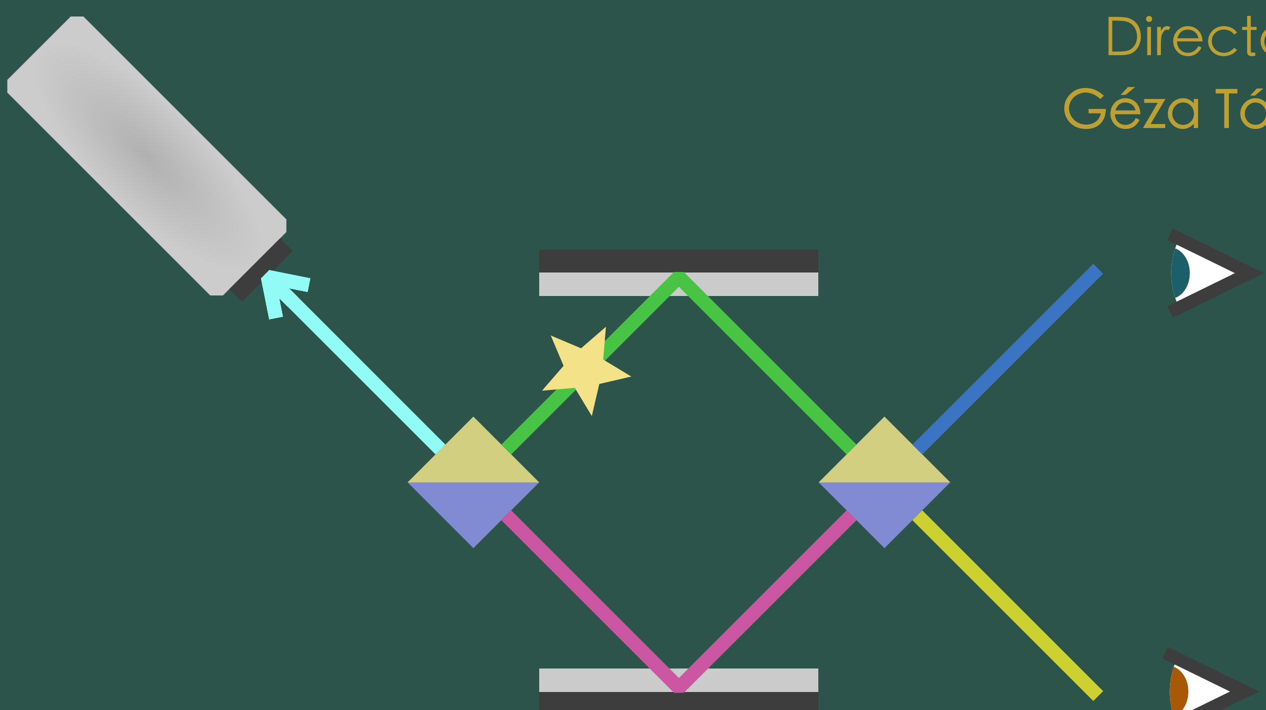

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Prologue

This work is part of the doctoral project of M. Sc. Iagoba Apellaniz in order to obtain the necesary qualification to promote himself to become a PhD. This work also tries to collect almost al the research discoveries done by the author on those previous years in a clear and concise way to make it understable for a general reader with a basic background in mathematics and physics.

The aim of this thesis is to present to the reader some important results of quantum metrology as well as guide possible interested ones into the fascinating field that is quantum metrology and its applications.

This is the prologe

Publications

Iagoba Apellaniz *et al*, *New J. Phys.* **17** 083027 (2015)

Detecting metrologically useful entanglement in the vicinity of Dicke states

Géza Tóth and Iagoba Apellaniz, *J. Phys. A: Math. Theor.* **47** 424006 (2014)

Quantum metrology from a quantum information science prespective

Preprints

Out of the scope of this thesis

Giuseppe Vitagliano *et al* 2014 *Phys. Rev. A* **89** 032307

Spin squeezing and entanglement for an arbitrary spin

Contents

Tables, figures and abbreviations

[Insert in a table]

SLD - Symmetric logarithmic derivative.

QFI - Quantum Fisher information

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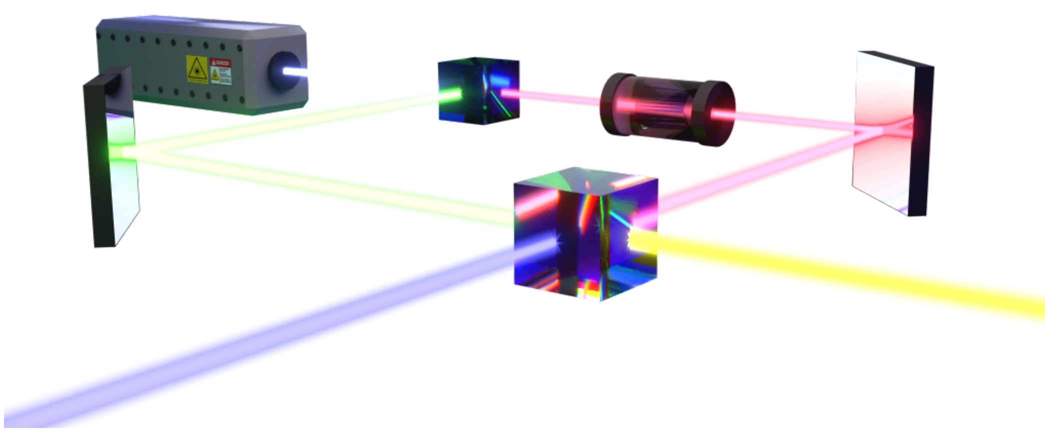

May 10, 2016

To my parents, my family

and to all the people

I have had around me those years.

Acknowledgments

I want to thank the people that support me in this endeavor especially my office and discussion mates and my director Géza Tóth for without whom my work would not be even started. I also want to thank more people especially those from the theoretical physics department of the University of the Basque Country and the unique very especial group for me from the Theoretical Quantum Optics group at the University of Siegen. I would really like to mention the names of all of them but I think it would be quite heavy for the average reader of this thesis. Thank you guys!

On the other hand I also felt very comfortable at my university, the University of the Basque Country, but I want to thank especially the people that make me grow in all ways as person.

1 Introduction

IN the recent years...

The figure of merit for the precision is the inverse of the variance normalized with the number of particles, $(\Delta\Theta)^{-2}/N$. It has the following properties:

(i) The bigger it is the bigger is the precision

(ii) It is normalized so for the best separable state it is 1. For greater values than 1 it would be a non-classical sign.

SQL

$$(\Delta\Theta)^{-2} \leq N \tag{1.1}$$

HL

$$(\Delta\Theta)^{-2} \leq N^2 \tag{1.2}$$

This thesis consists of 4 well differentiated parts, apart from the current introduction, on which different topics are developed. In the first part, we will introduce the reader onto the research field of quantum metrology.

Brief comments on the notation: c_Θ and s_Θ stand for $\cos\Theta$ and $\sin\Theta$ respectively, probably some other trigonometry function is shortened.

2 Development of Quantum Metrology

METROLOGY, as the science of measuring, has played an essential role for the development of the technology as we know it today. It studies several aspects of the estimation process, such as which strategy to follow in order to improve the precision of an estimation. One of the most important figure of merit in this context is the achievable precision for a given system, independently of the strategy. We will show how to characterize it in the subsequents sections. And we will show as well different strategies to achieve the desired results. The metrology science also covers from the design aspects of a precise measuring device, until the most basic concepts of nature which lead in ultimate instance to the better understanding of the whole process.

In this sense, with the discovery of the Quantum Physics and the development of Quantum Mechanics, new doors for advances in metrology were open on the earlies decades of the 19th century. Later on, the Quantum Theory lead to the so-called field of Quantum Information which merges the notions of the theory of information and computer science, among others subfields, with the quantum mechanics. The role of the so-named entanglement, an exclusive feature of Quantum Mechanics, is essential in this context. Its complete understanding has integrated efforts of many researches world wide. Said this, the entanglement also is in the center of theoretical concepts included in Quantum

Metrology.

On the other hand and with the aim of interpreting raw data, there are the statistics, without which many descriptions of the actual and past physical findings would lack of the rigorous interpretation needed for the complexity of data samples.

2.1 Background on statistics and theory of estimation

The main mathematical tools used by the metrology science belong to statistics. Moreover we are also interested on estimation theory. The statistics main characteristic is that makes the raw data under consideration comprehensible. The data can be anything, from a set of different heights of a basketball team, to the outcomes of a coin toss or the ages of a hundred students or even the outcomes of a thousand times repeated measurement of the electric field at some spatial point. The aim of this section is to give the reader sufficient material to follow this thesis and make it comprehensible from the beginning.

2.1.1 Data sample, average, variance and central moments

As we mentioned above, everything in statistics starts with a data set. A dataset is defined as a set of values, we will restrict ourselves to quantitative values, representing some physical quantity. So in our framework the set can be written as $X = \{x_i\}_{i=1}^{M}$, where all different M values are collected.

With this data at hand, we are able to compute the *arithmetic average* of the set, namely $\mathrm{E}[X] := \frac{1}{M}\sum_{i=1}^{M} x_i$. There are other types of averages such as the geometric mean, the root mean square or the harmonic mean, see reef. [XXX]. For us will be enough with the arithmetic average.

The second quantity one can compute is how spread the data is. Namely the *variance*, which can be defined easily as $\mathrm{V}[X] := \mathrm{E}[(X - \mathrm{E}[X])^2]$, where X can be seen as a vector and when subtracting the scalar $\mathrm{E}[X]$ and X^2 stands for the elements wise squared of X, namely $X^2 := \{x_i^2\}_{i=1}^{M}$. Similarly can be done for higher orders. The variance is also a well known quantity on many fields of science. It can also be written alternatively as $\mathrm{V}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2$. The definition of the *standard deviation* follows directly from the variance, $\sigma_X = \sqrt{\mathrm{V}[X]}$. Many quantities on statistics require operations like the one above.

At this point, let us illustrate thees quantities with an example. It will also help us for introducing another concept. Let us have the outcomes of, let us say, measurements of the heights of 18 trees. Let us have those heights as integer values and in meters, see Tab. [XXX]. Then, the average height of the set is simply $\mathrm{E}[X] = \frac{1}{18}\sum_{i=1}^{18} x_i = 6\mathrm{m}$. Now let us see what happens with the variance,

Tree #	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Heights (m)	6	7	5	7	2	11	7	4	7	2	7	7	5	5	5	8	3	10

Table 1: A set of values for the heights of 18 trees. All measurements were rounded to integers for simplicity.

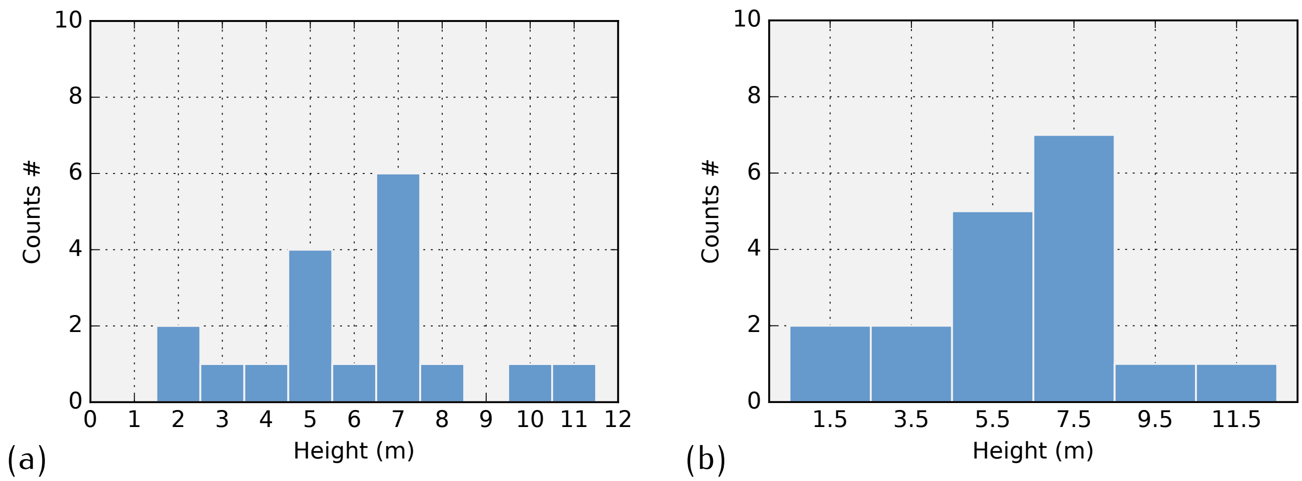

Figure 1: (a) p_i for different values of i. All bars have width as 1, so it is drawn how many times each data value appears on the data sample. The width of the bars is called *bin*, and it can change so the bars would represent a wider range of values. (b) Yo can see the same data represented in this case by a histogram with the bin size equal to 2. To produce this histogram we have summed 2 adjacent p_i values starting from $p_1 + p_2$, $p_3 + p_4$ and so on. Those new bars we have chosen to represent a value in between, for instance $(3 + 4)/2 = 3.5$ and so on.

$\mathrm{V}[X] = \frac{1}{18}\sum_{i=1}^{18} x_i^2 - 6^2 = 5.5\bar{5}\mathrm{m}^2$. Using now the standard deviation one can say about the original distribution that most values are around 6m with a deviation of 2.357m. Of course, some values are outside this range, but nevertheless the description is quite accurate, note that 12 values from 18 are inside the range 6m $\pm$ 2.357m.

Can we simplify the way we represent the data? One may notice that the value 7m is repeated 6 times, as the value 5m is 4 times and so on. For that we will introduce the distribution function p_i, which in this case is for discrete values of the outcomes but which can be generalized for continuous variables. This function takes the value of how many times the outcome i has appeared on the data sample. Let us plot the distribution function on what is usually referred as *histogram*.

Now some question arises immediately: How this is fully connected with the previous picture? How can one compute the average and other interesting quantities? The answer is simple but it has to be considered carefully. First of all, notice that p_i is defined for all the natural numbers including zero, see that in the example of the trees p_9 equals zero, so it can be extended for other heights too setting them to zero. This will depend on the physical property that our data sample represents but in this case it is the height of some trees, so the values cannot be negative. Second, notice that the sum of all the repetitions, all the values of p_i, is exactly 18 the number of data samples in the set. So we have that $M = \sum_{i=0}^{\infty} p_i$. Now we can formulate the ensemble average as $\mathrm{E}[X] = \sum_{i=0}^{\infty} p_i i / \sum_{j=0}^{\infty} p_j$. The variance and with this the standard deviation immediately follow this approach. It is convenient to notice that the total measurement outcomes has not contribute anything but to normalize the quantities.

We can now without losing generality redefine the distribution function to be the number of repetitions corresponding to the variable divided by the total outcomes, in this case M. It would have the same properties of a probability distribution function (PDF). For instance, now we have that the sum of all p_i–s equal to one, $\sum_{i=0}^{\infty} p_i = 1$, and the average is directly obtained by computing $\mathrm{E}[X] = \sum_{i=0}^{\infty} p_i i$. This is the approach we will follow to represent data samples. The variance and other quantities also are simpler in this way.

2.1.2 Probabilities and frecuentist vs. bayesian approach

The trivial values to compute from the data

We can estimate the average height of the trees of that forest.

histogram

estimator

MLH

CLT

2.2 Quantum Mechanics

The ubiquitous probabilistic nature of quantum mechanics is reflected on each corner of the field. This *ambiguity* of a system been quantum is exploited by many recent technologies.

The state

product states

Entanglement

Evolution

Unitary evolution

Markov

Limblad

Measurements

2.3 Quantum Metrology

Histograms of quantum states

Merging the probabilistic features of quantum mechanics with the estimation theory is not tibial.

The figure of merit for this purpose is the so-called Quantum Fisher Information.

3 Quantum metrology with states on the vicinity of Dicke states

In this section we will show some results regarding the metrological usefulness of unpolarized states [1]. All together, one can also show that the unpolarized states are able to perform better than the polarized ones. The more important figure of merit of such unpolarized but still useful states is the so-called unpolarized Dicke state, which consists of an equal number of particles pointing up and pointing down and it lives on the symmetric subspace.

One of the most particular features that this state has is that since it is a eigenstate of the collective operator J_z with eigenvalue zero, it mus have a very large uncertainty on the operators J_x and J_y. See Fig.

Since one cannot use the expectation value of the collective J_l operators to see how the state evolve under the magnetic field, one have to go at least a new level and consider the evolution of the variances.

3.1 Evolution of the expectation values

Here we show how the expectation values of collective operators needed in this section evolve under the unitary dynamics because of the influence of the homogeneous magnetic field.

$$\mathrm{tr}\, J_x(\Theta) = J_x \mathrm{c}_\Theta - J_y \mathrm{s}_\Theta \tag{3.1}$$

4 Bounding the QFI with few initial expectation values

In the previous section we have shown how with few expectation values one would be able too bound from below the QFI. In this section we will show how the approach of bounding the precision with some expectation values can be further generalized.

Another approach must be taken to bound from below the QFI.

5 Accuracy bound for gradient field estimation with atomic ensembles

Hello it's me

6 Conclusions

Hello it's me again

References

[1] Iagoba Apellaniz, Bernd Lücke, Jan Peise, Carsten Klempt, and Géza Tóth. Detecting metrologically useful entanglement in the vicinity of dicke states. *New Journal of Physics*, 17(8):083027, 2015.