General Relativity
Quick Intro (refresher)

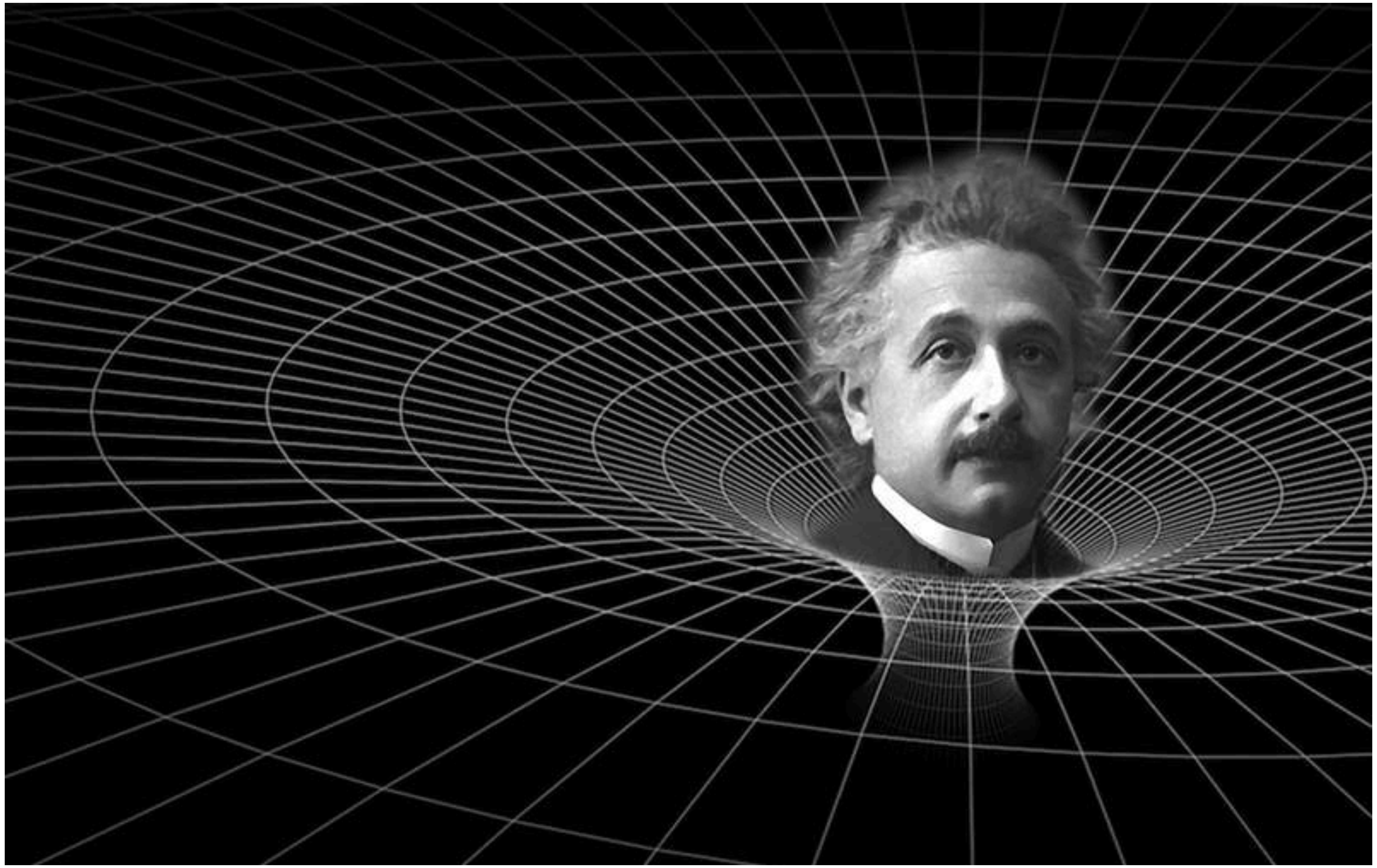


Image by Matt Payne
(https://www.sigmapisigma.org/sigmapisigma/radiations/fall/2016/journey-toward-general-relativity)

Newtonian Gravity

Poisson (field) equation:

$$\nabla^2 \Phi(\mathbf{r}) = 4\pi G \rho(\mathbf{r})$$

Gravitational potential field

Matter

General Relativity

Einstein's Field Equations:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Curvature

Matter/Energy

Curvature — nasty tensors

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$ Einstein tensor

Second derivative operator on the metric

$$R = g^{\mu\nu} R_{\mu\nu}$$ Ricci Scalar

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$$ Ricci Tensor

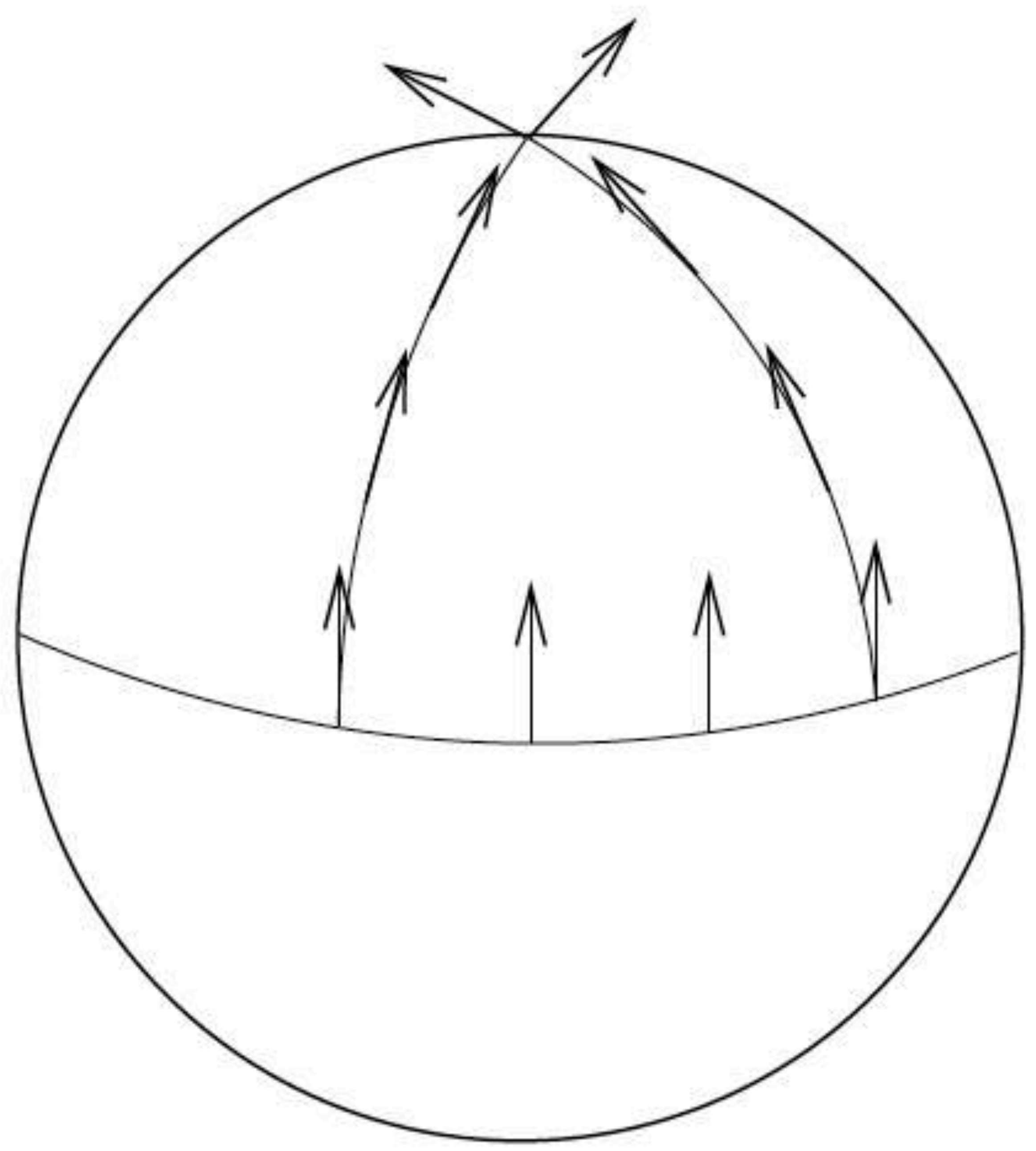
$$R^{\sigma}{}_{\mu\alpha\beta} = \partial_\alpha \Gamma^{\sigma}_{\mu\beta} - \partial_\beta \Gamma^{\sigma}_{\mu\alpha} + \Gamma^{\sigma}_{\alpha\lambda} \sigma^{\lambda}_{\mu\beta} - \Gamma^{\sigma}_{\beta\lambda} \Gamma^{\lambda}_{\mu\alpha}$$ Riemann Curvature Tensor

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu})$$ Connection Coefficient

(Christoffel Symbols)

Curved spacetime — Geodesics

Gravity is due to curvature



Geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

(Newtonian analogue)

$$\mathbf{a} = -\nabla\Phi \qquad (\mathbf{a} + \nabla\Phi = 0)$$

Carroll (1997)

Schwarzschild solution — spherically symmetric mass

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

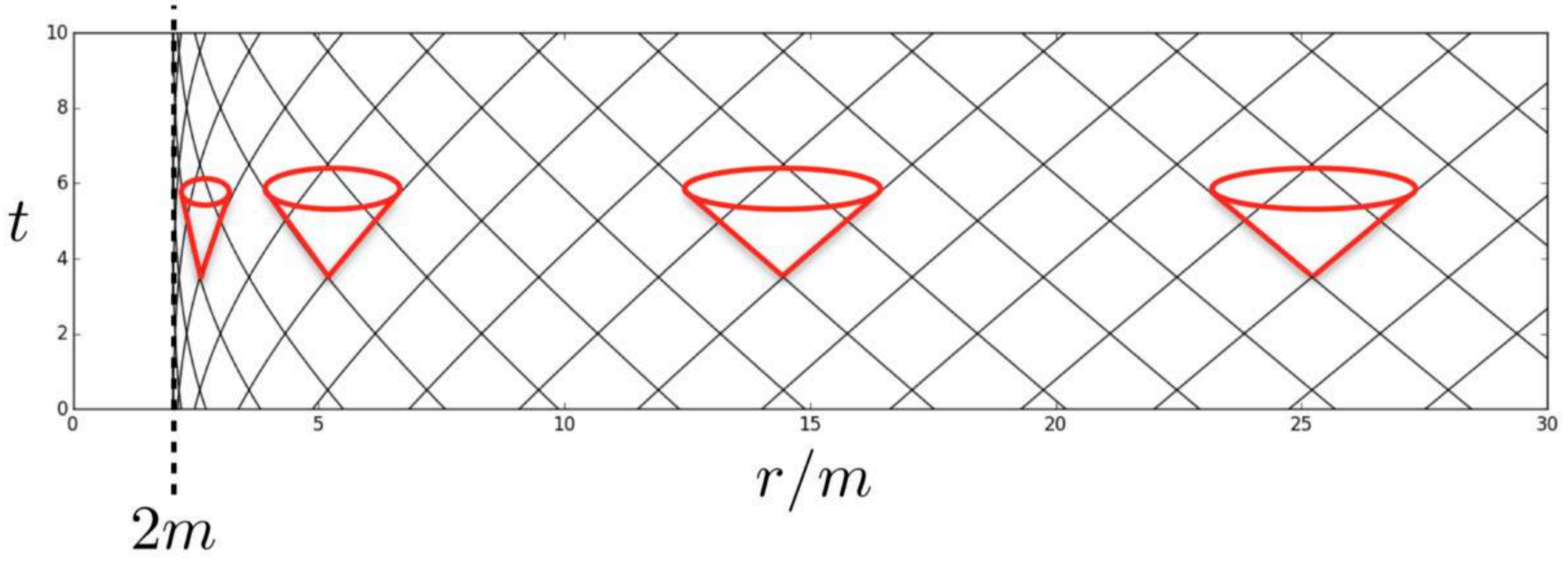


Figure 1: Radial, null geodesics in the Schwarzschild spacetime

Newtonian equivalent

$$\Phi(r) = -\frac{GM}{r}$$

P. Lasky (Monash ASP3051 Lecture notes)

Going through the horizon — Painleve-Gullstrand coordinates

Transformation from
Schwarzschild Coordinates

$$dt = dT + \sqrt{\frac{2M}{r}}\left(1 - \frac{2M}{r}\right)^{-1} dr$$

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dT^2 - 2\sqrt{\frac{2M}{r}} dT dr + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

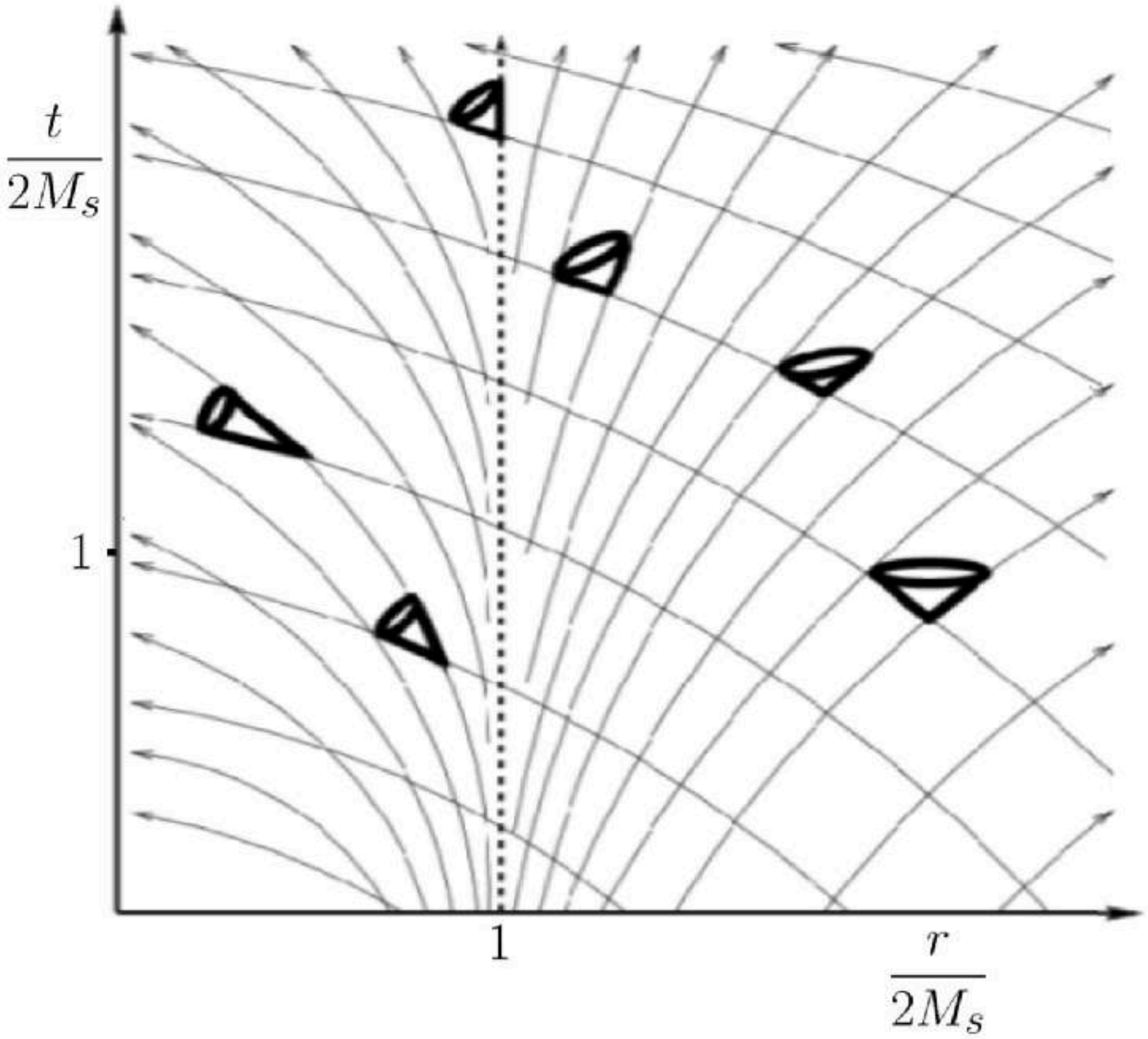
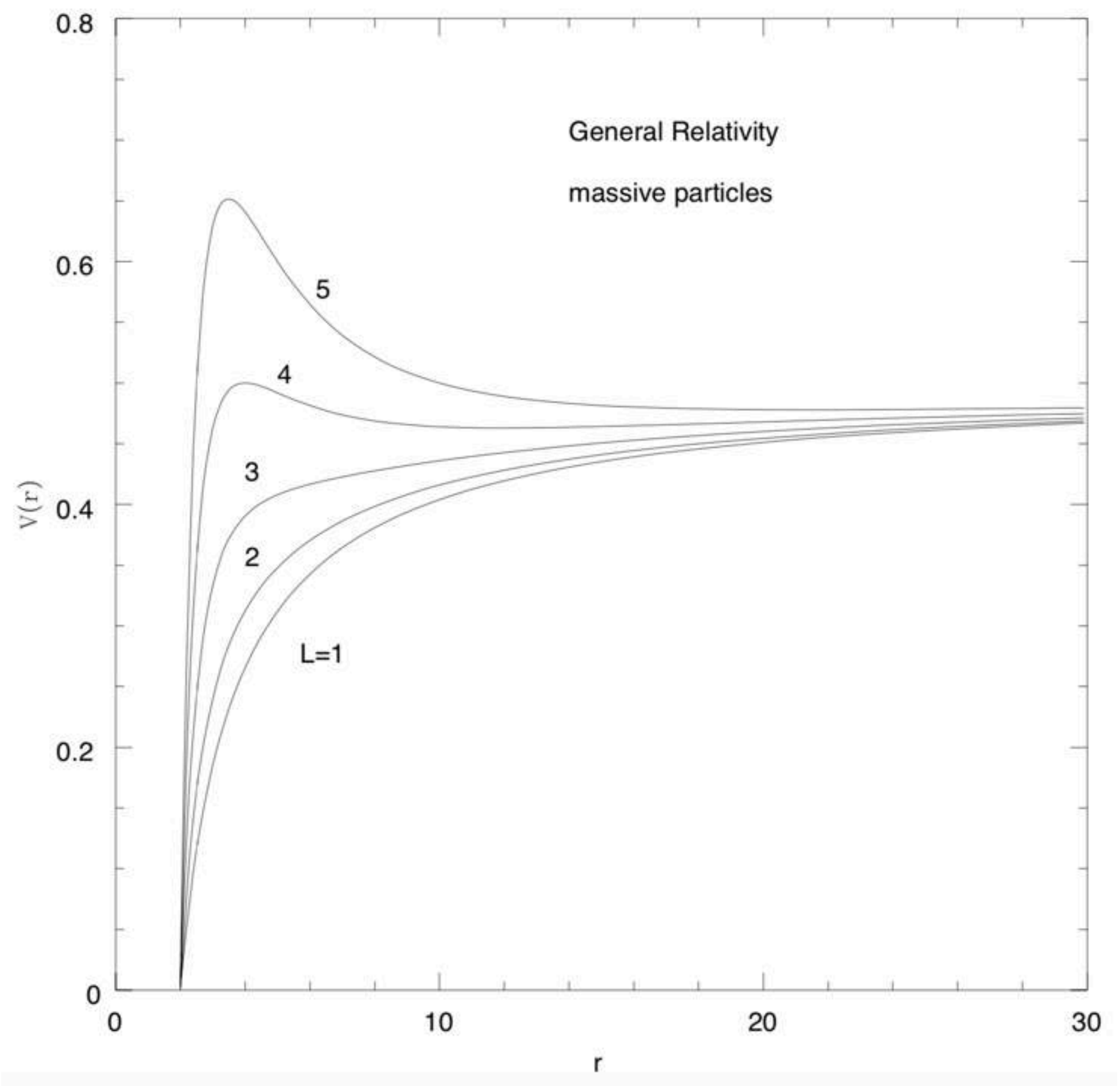


Figure 2: Radial, null geodesics in Painleve-Gullstrand coordinates. Ingoing geodesics pass through the event horizon without noticing its existence. Outgoing null geodesics that are launched at some $r < 2M$ can not escape out of the black hole.

P. Lasky (Monash ASP3051 Lecture notes)

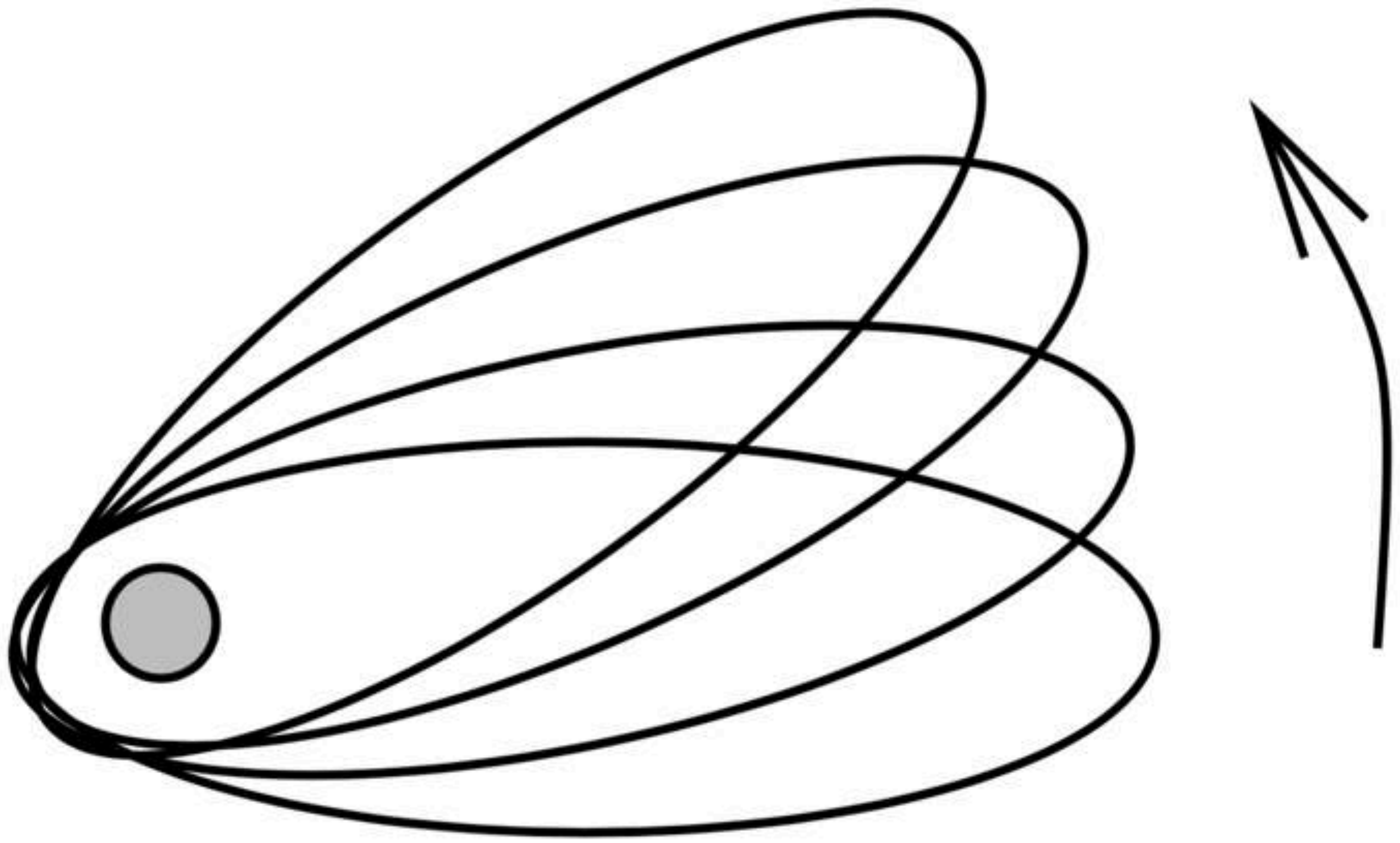
Orbital dynamics

- Last stable orbit



Carroll (1997)

- Apsidal advance



Carroll (1997)

Lense-Thirring precession
(rotating black hole)

Kerr solution — rotating black hole

$$
\begin{aligned}
ds^2 = &-\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4Mra\sin^2\theta}{\rho^2} dt\, d\phi + \frac{\rho^2}{\Delta} dr^2 \\
&+ \rho^2 d\theta^2 + \left(a^2 + r^2 + \frac{2Mr}{\rho^2} a^2 \sin^2\theta\right) \sin^2\theta\, d\phi^2
\end{aligned}
$$

$$
\begin{aligned}
\rho^2 &= r^2 + a^2 \cos^2\theta \\
\Delta &= r^2 - 2Mr + a^2
\end{aligned}
$$

When can we use Newtonian?

Newtonian limit Three requirements:

1. The particles are moving slowly, with respect to the speed of light.

$$\frac{dx^i}{d\tau} << \frac{dt}{d\tau}$$

2. The gravitational field is weak. It can be considered a perturbation of flat space.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| << 1$$

3. The field is static; unchanging with time.

When can we use Newtonian?

System	h_{00}
Earth	$\sim 10^{-9}$
Sun	$\sim 10^{-6}$
white dwarf	$\sim 10^{-4}$
neutron star	≈ 0.3
Black holes	> 0.3

Newtonian OK

GR effects important

Can we cheat?

Try and account for general relativistic effects e.g.

- Last stable orbit
- Apsidal advance
- Lense-Thirring precession

Modified relativistic potentials

$$\Phi^{\mathrm{K}} = -\frac{GM_{\mathrm{h}}}{R},$$

$$\Phi^{\mathrm{R}} = -\frac{GM_{\mathrm{h}}}{R} - \left(\frac{2R_{\mathrm{g}}}{R-2R_{\mathrm{g}}}\right)\left[\left(\frac{R-R_{\mathrm{g}}}{R-2R_{\mathrm{g}}}\right)v_{\mathrm{r}}^2 + \frac{v_{\mathrm{t}}^2}{2}\right]$$

Bonnerot et al. (2016)

Post Newtonian corrections

$$\frac{\mathrm{d}\boldsymbol{v}_i}{\mathrm{d}t} = \sum_j^{N_{\mathrm{nei}}} m_j\left(\frac{P_i}{\rho_i^2}+\frac{P_j}{\rho_j^2}+\Pi_{ij}\right)\nabla_j W(r_{ij},h_{ij})$$
$$-\sum_j^{N}\frac{GM(r_{ij})}{r_{ij}^2}\frac{\boldsymbol{r}_{ij}}{r_{ij}}$$
$$+\boldsymbol{a}_{i,0\mathrm{PN}}+\frac{1}{c^2}\boldsymbol{a}_{i,1\mathrm{PN}}+\frac{1}{c^3}\boldsymbol{a}_{i,1.5\mathrm{PN}}+\frac{1}{c^4}\boldsymbol{a}_{i,2\mathrm{PN}}$$

Corrections to acceleration

$$\boldsymbol{a}_{i,0\mathrm{PN}} = -\frac{GM_{\mathrm{BH}}}{r_{i\mathrm{BH}}^2}\boldsymbol{n}_{i\mathrm{BH}}$$

$$\boldsymbol{a}_{i,1\mathrm{PN}} = \left[\frac{5G^2 m_i M_{\mathrm{BH}}}{r_{i\mathrm{BH}}^3}+\frac{4G^2M_{\mathrm{BH}}^2}{r_{i\mathrm{BH}}^3}\right.$$
$$+\frac{GM_{\mathrm{BH}}}{r_{i\mathrm{BH}}^2}\left(\frac{3}{2}(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_{\mathrm{BH}})^2\right.$$
$$\left.\left.-v_i^2+4(\boldsymbol{v}_i\cdot\boldsymbol{v}_{\mathrm{BH}})-2v_{\mathrm{BH}}^2\right)\right]\boldsymbol{n}_{i\mathrm{BH}}$$
$$+\frac{GM_{\mathrm{BH}}}{r_{i\mathrm{BH}}^2}[4(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_i)-3(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_{\mathrm{BH}})]\boldsymbol{v}_{i\mathrm{BH}}$$

Hayasaki et al. (2016)

More corrections

$$
\begin{aligned}
\boldsymbol{a}_{i,2\mathrm{PN}} = &-\left[\frac{57G^3 m_i^2 M_{\mathrm{BH}}}{4r_{i\mathrm{BH}}^4} + \frac{69G^3 m_i M_{\mathrm{BH}}^2}{2r_{i\mathrm{BH}}^4} + \frac{9G^3 M_{\mathrm{BH}}^3}{r_{i\mathrm{BH}}^4}\right]\boldsymbol{n}_{i\mathrm{BH}} + \frac{GM_{\mathrm{BH}}}{r_{i\mathrm{BH}}^2}\left[-\frac{15}{8}(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_{\mathrm{BH}})^4\right. \\
&+ \frac{3}{2}(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_{\mathrm{BH}})^2\boldsymbol{v}_i^2 - 6(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_{\mathrm{BH}})^2(\boldsymbol{v}_i\cdot\boldsymbol{v}_{\mathrm{BH}}) \\
&- 2(\boldsymbol{v}_i\cdot\boldsymbol{v}_{\mathrm{BH}})^2 + \frac{9}{2}(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_{\mathrm{BH}})^2 v_{\mathrm{BH}}^2 \\
&+ 4(\boldsymbol{v}_i\cdot\boldsymbol{v}_{\mathrm{BH}})v_{\mathrm{BH}}^2 \\
&\left.- 2v_{\mathrm{BH}}^4\right]\boldsymbol{n}_{i\mathrm{BH}} + \frac{G^2 m_i M_{\mathrm{BH}}}{r_{i\mathrm{BH}}^3}\left[\frac{39}{2}(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_i)^2\right. \\
&- 39(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_i)(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_{\mathrm{BH}}) + \frac{17}{2}(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_{\mathrm{BH}})^2 \\
&\left.- \frac{15}{4}v_i^2 - \frac{5}{2}(\boldsymbol{v}_i\cdot\boldsymbol{v}_{\mathrm{BH}}) + \frac{5}{4}v_{\mathrm{BH}}^2\right]\boldsymbol{n}_{i\mathrm{BH}} \\
&+ \frac{GM_{\mathrm{BH}}^2}{r_{i\mathrm{BH}}^3}\left[\frac{4}{2}(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_i)^2\right. \\
&- 4(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_i)(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_{\mathrm{BH}}) + 6(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_{\mathrm{BH}})^2 \\
&\left.- 8(\boldsymbol{v}_i\cdot\boldsymbol{v}_{\mathrm{BH}}) + 4v_{\mathrm{BH}}^2\right]\boldsymbol{n}_{i\mathrm{BH}} + \frac{G^2 M_{\mathrm{BH}}^2}{r_{i\mathrm{BH}}^3} \\
&\left[-2(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_i) - 2(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_{\mathrm{BH}})\right]\boldsymbol{v}_{i\mathrm{BH}} \\
&+ \frac{G^2 m_i M_{\mathrm{BH}}}{r_{i\mathrm{BH}}^3}\left[-\frac{63}{4}(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_i)\right. \\
&\left.+ \frac{55}{4}(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_{\mathrm{BH}})\right]\boldsymbol{v}_{i\mathrm{BH}} \\
&+ \frac{GM_{\mathrm{BH}}}{r_{i\mathrm{BH}}^2}\left[-6(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_i)(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_{\mathrm{BH}})^2\right. \\
&+ \frac{9}{2}(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_{\mathrm{BH}})^3 + (\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_{\mathrm{BH}})v_i^2 \\
&- 4(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_i)(\boldsymbol{v}_i\cdot\boldsymbol{v}_{\mathrm{BH}}) \\
&+ 4(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_{\mathrm{BH}})(\boldsymbol{v}_i\cdot\boldsymbol{v}_{\mathrm{BH}}) \\
&\left.+ 4(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_i)v_{\mathrm{BH}}^2 - 5(\boldsymbol{n}_{i\mathrm{BH}}\cdot\boldsymbol{v}_{\mathrm{BH}})v_{\mathrm{BH}}^2\right]\boldsymbol{v}_{i\mathrm{BH}},
\end{aligned}
$$

Hayasaki et al. (2016)

More corrections

$$E_i = E_{i,0\mathrm{PN}} + \frac{1}{c^2}E_{i,1\mathrm{PN}} + \frac{1}{c^3}E_{i,1.5\mathrm{PN}} + \frac{1}{c^4}E_{i,2\mathrm{PN}},$$

$$E_{i,0\mathrm{PN}} = \frac{1}{2}(m_i v_i^2 + M_{\mathrm{BH}} v_{\mathrm{BH}}^2) - \frac{G m_i M_{\mathrm{BH}}}{r_{i\mathrm{BH}}},$$

$$\begin{aligned}
E_{i,1\mathrm{PN}} = &-\frac{G^2 m_i^2 M_{\mathrm{BH}}}{2 r_{i\mathrm{BH}}^2} + \frac{m_i v_i^4}{8} + \frac{G m_i M_{\mathrm{BH}}}{r_{i\mathrm{BH}}} \\
&\times \left[-\frac{1}{4}(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_i)(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_{\mathrm{BH}}) + \frac{3}{2}v_i^2 - \frac{7}{4}(\boldsymbol{v}_i \cdot \boldsymbol{v}_{\mathrm{BH}}) \right] \\
&- \frac{G^2 M_{\mathrm{BH}}^2 m_i}{2 r_{i\mathrm{BH}}^2} + \frac{M_{\mathrm{BH}} v_{\mathrm{BH}}^4}{8} + \frac{G M_{\mathrm{BH}} m_i}{r_{i\mathrm{BH}}} \\
&\times \left[-\frac{1}{4}(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_{\mathrm{BH}})(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_i) + \frac{3}{2}v_{\mathrm{BH}}^2 - \frac{7}{4}(\boldsymbol{v}_{\mathrm{BH}} \cdot \boldsymbol{v}_i) \right],
\end{aligned}$$

$$E_{i,1.5\mathrm{PN}} = \frac{G M_{\mathrm{BH}}}{r_{i\mathrm{BH}}^2}[\boldsymbol{S}_i \cdot (\boldsymbol{n}_{i\mathrm{BH}} \times \boldsymbol{v}_i)] - \frac{G m_i}{r_{i\mathrm{BH}}^2}[\boldsymbol{S}_{\mathrm{BH}} \cdot (\boldsymbol{n}_{i\mathrm{BH}} \times \boldsymbol{v}_{\mathrm{BH}})],$$

$$\begin{aligned}
E_{i,2\mathrm{PN}} = &-\frac{G^3 m_i^3 M_{\mathrm{BH}}}{2 r_{i\mathrm{BH}}^3} - \frac{19 G^3 m_i^2 M_{\mathrm{BH}}^2}{8 r_{i\mathrm{BH}}^3} + \frac{5}{16} m_i v_i^6 \\
&- \frac{G^3 M_{\mathrm{BH}}^3 m_i}{2 r_{i\mathrm{BH}}^3} - \frac{19 G^3 M_{\mathrm{BH}}^2 m_i^2}{8 r_{i\mathrm{BH}}^3} + \frac{5}{16} M_{\mathrm{BH}} v_{\mathrm{BH}}^6 \\
&+ \frac{G^2 m_i^2 M_{\mathrm{BH}}}{r_{i\mathrm{BH}}^2} \left[\frac{29}{4}(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_i)^2 - \frac{13}{4}(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_i)(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_{\mathrm{BH}}) \right. \\
&\left. + \frac{1}{2}(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_{\mathrm{BH}})^2 - \frac{3}{2}v_i^2 + \frac{7}{4}v_{\mathrm{BH}}^2 \right] + \frac{G^2 M_{\mathrm{BH}}^2 m_i}{r_{i\mathrm{BH}}^2} \\
&\times \left[\frac{29}{4}(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_{\mathrm{BH}})^2 - \frac{13}{4}(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_{\mathrm{BH}})(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_i) \right. \\
&\left. + \frac{1}{2}(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_i)^2 - \frac{3}{2}v_{\mathrm{BH}}^2 + \frac{7}{4}v_i^2 \right] + \frac{G m_i M_{\mathrm{BH}}}{r_{i\mathrm{BH}}} \\
&\times \left[\frac{3}{8}(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_i)^3(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_{\mathrm{BH}}) \right. \\
&+ \frac{3}{16}(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_i)^2(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_{\mathrm{BH}})^2 \\
&- \frac{9}{8}(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_i)(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_{\mathrm{BH}})v_i^2 - \frac{13}{8}(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_{\mathrm{BH}})^2 v_i^2 \\
&+ \frac{21}{8}v_i^4 + \frac{13}{8}(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_i)^2(\boldsymbol{v}_i \cdot \boldsymbol{v}_{\mathrm{BH}}) \\
&+ \frac{3}{4}(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_i)(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_{\mathrm{BH}})(\boldsymbol{v}_i \cdot \boldsymbol{v}_{\mathrm{BH}}) \\
&\left. - \frac{55}{8}v_i^2(\boldsymbol{v}_i \cdot \boldsymbol{v}_{\mathrm{BH}}) + \frac{17}{8}(\boldsymbol{v}_i \cdot \boldsymbol{v}_{\mathrm{BH}})^2 + \frac{31}{16}v_i^2 v_{\mathrm{BH}}^2 \right] \\
&+ \frac{G m_i M_{\mathrm{BH}}}{r_{i\mathrm{BH}}} \left[\frac{3}{8}(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_{\mathrm{BH}})^3(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_i) \right. \\
&+ \frac{3}{16}(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_{\mathrm{BH}})^2(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_i)^2 \\
&- \frac{9}{8}(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_{\mathrm{BH}})(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_i)v_{\mathrm{BH}}^2 \\
&- \frac{13}{8}(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_i)^2 v_{\mathrm{BH}}^2 + \frac{21}{8}v_{\mathrm{BH}}^4 \\
&+ \frac{13}{8}(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_{\mathrm{BH}})^2(\boldsymbol{v}_i \cdot \boldsymbol{v}_{\mathrm{BH}}) \\
&+ \frac{3}{4}(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_{\mathrm{BH}})(\boldsymbol{n}_{i\mathrm{BH}} \cdot \boldsymbol{v}_i)(\boldsymbol{v}_i \cdot \boldsymbol{v}_{\mathrm{BH}}) \\
&\left. - \frac{55}{8}v_{\mathrm{BH}}^2(\boldsymbol{v}_i \cdot \boldsymbol{v}_{\mathrm{BH}}) + \frac{17}{8}(\boldsymbol{v}_i \cdot \boldsymbol{v}_{\mathrm{BH}})^2 + \frac{31}{16}v_i^2 v_{\mathrm{BH}}^2 \right].
\end{aligned}$$

Hayasaki et al. (2016)

Let’s just do it properly!

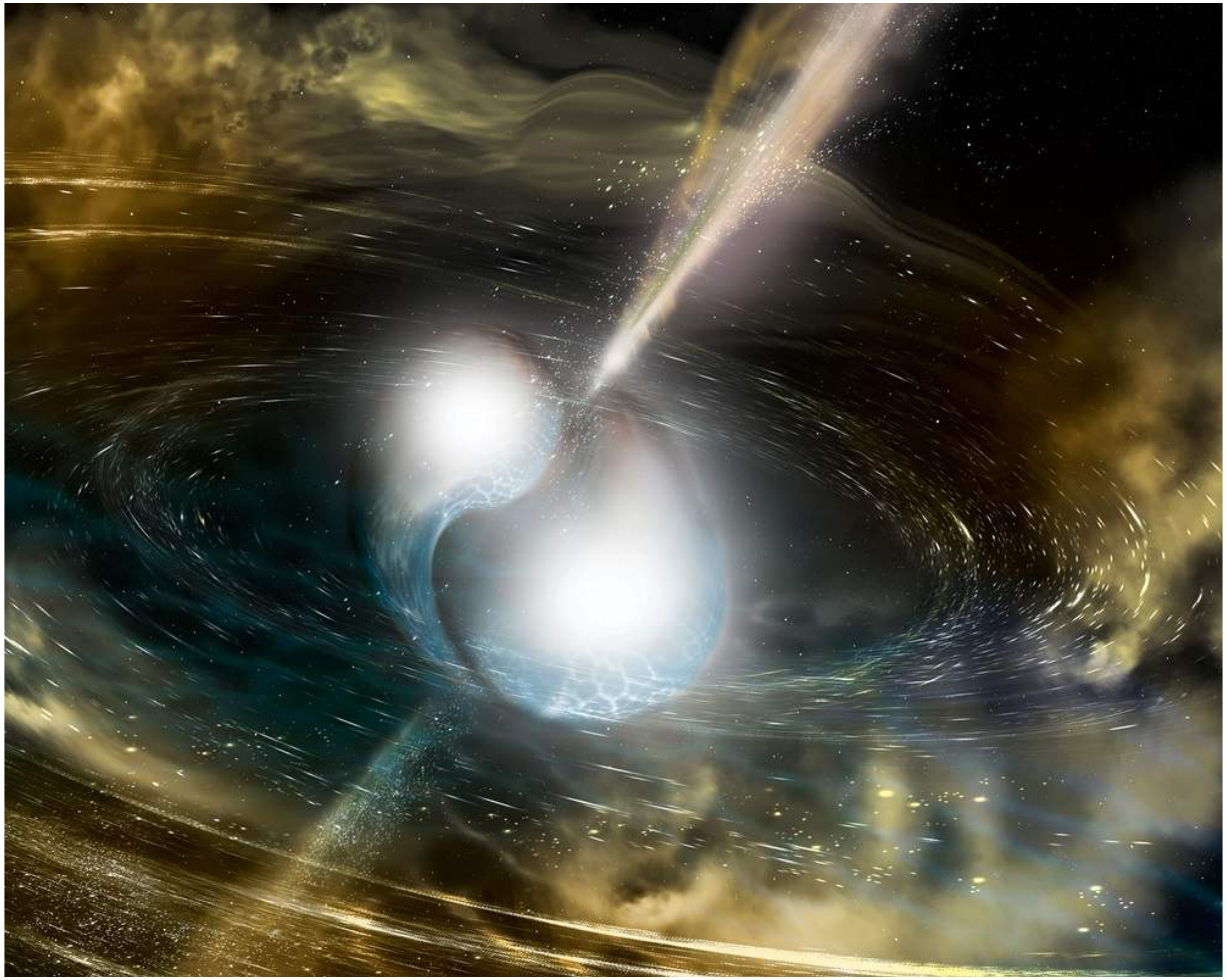
General Relativistic Smoothed Particle Hydrodynamics

(GR SPH)

David Liptai

Supervisors: Daniel Price and Paul Lasky

Motivations

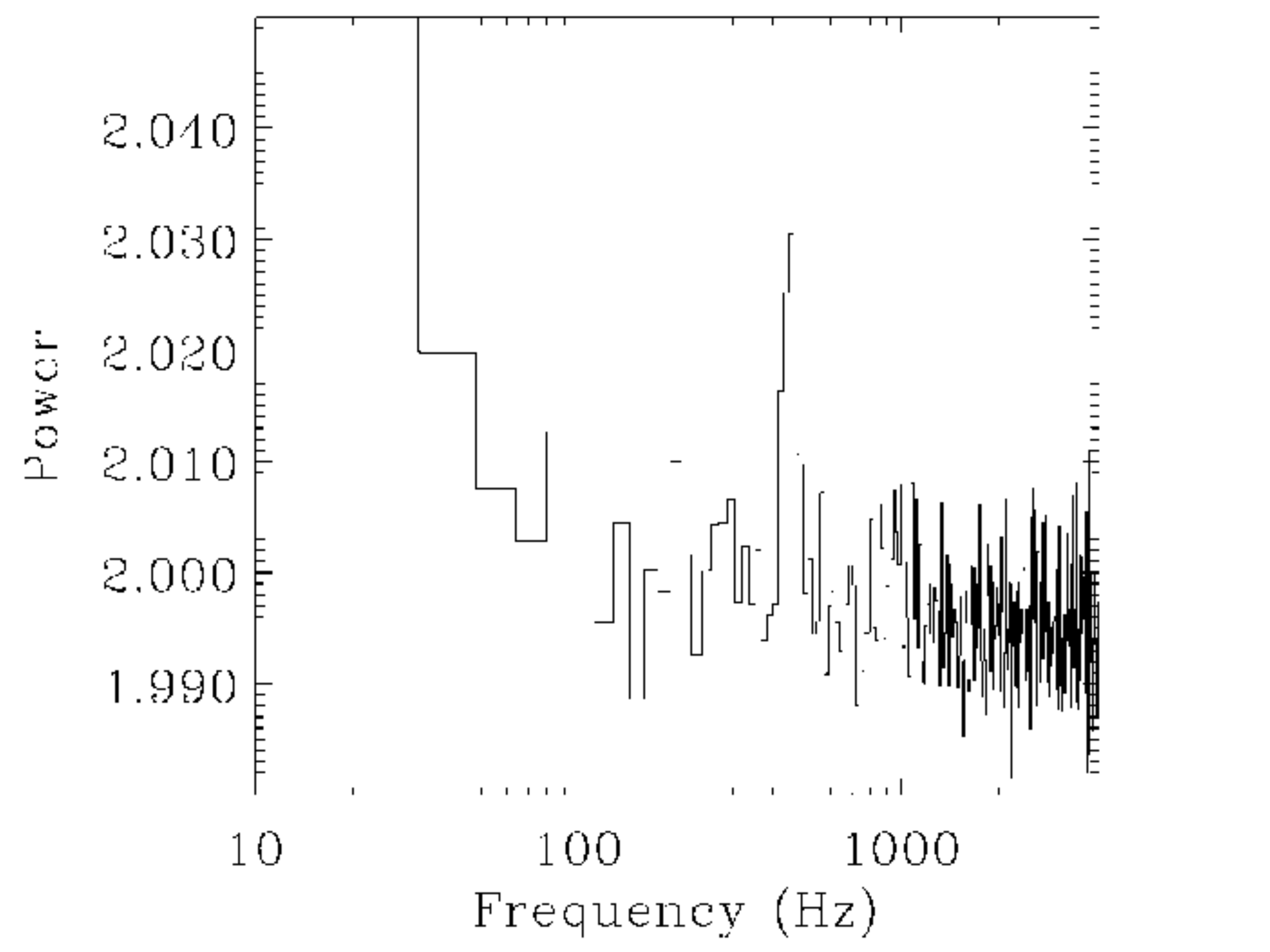

National Science Foundation/LIGO/Sonoma State University/A. Simonnet

Neutron star mergers

SPH is the **perfect tool!**

1) No preferred **geometry**
2) **Resolution** follows mass
3) **No** need for **background density** floor

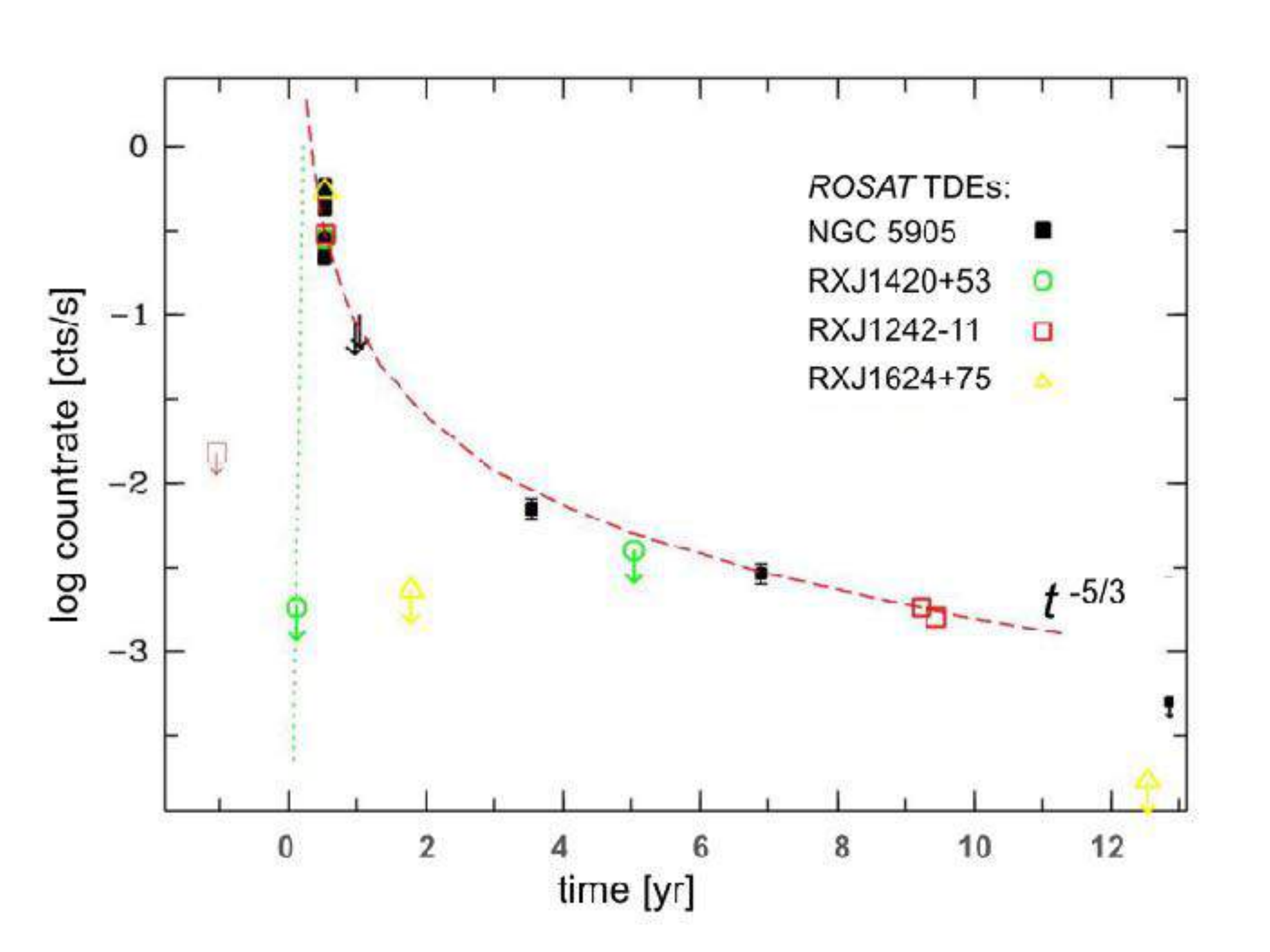
Except…. No GR!



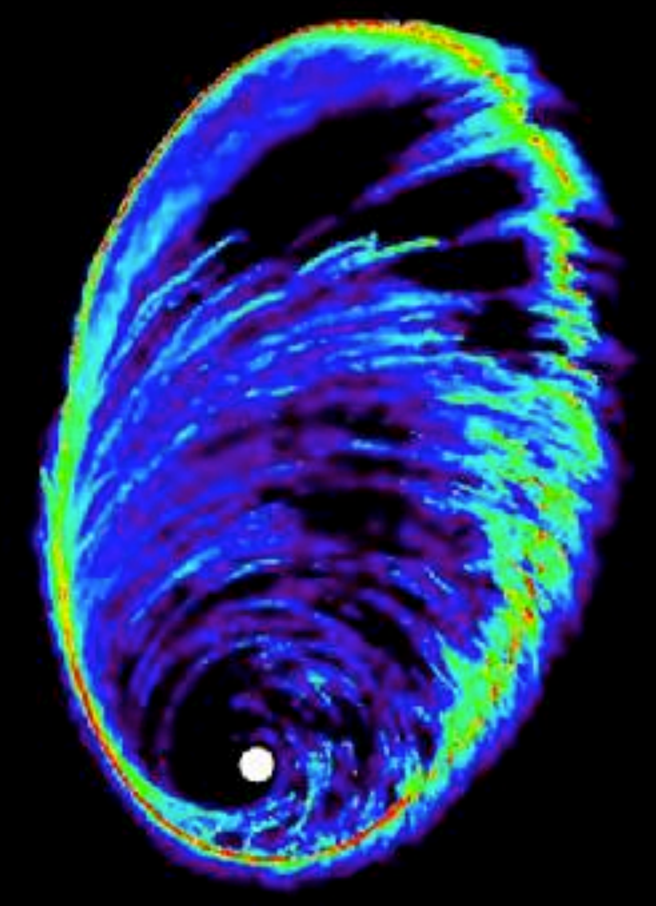
Tearing Discs and QPOs?

Nealon, Price and Nixon (2015)

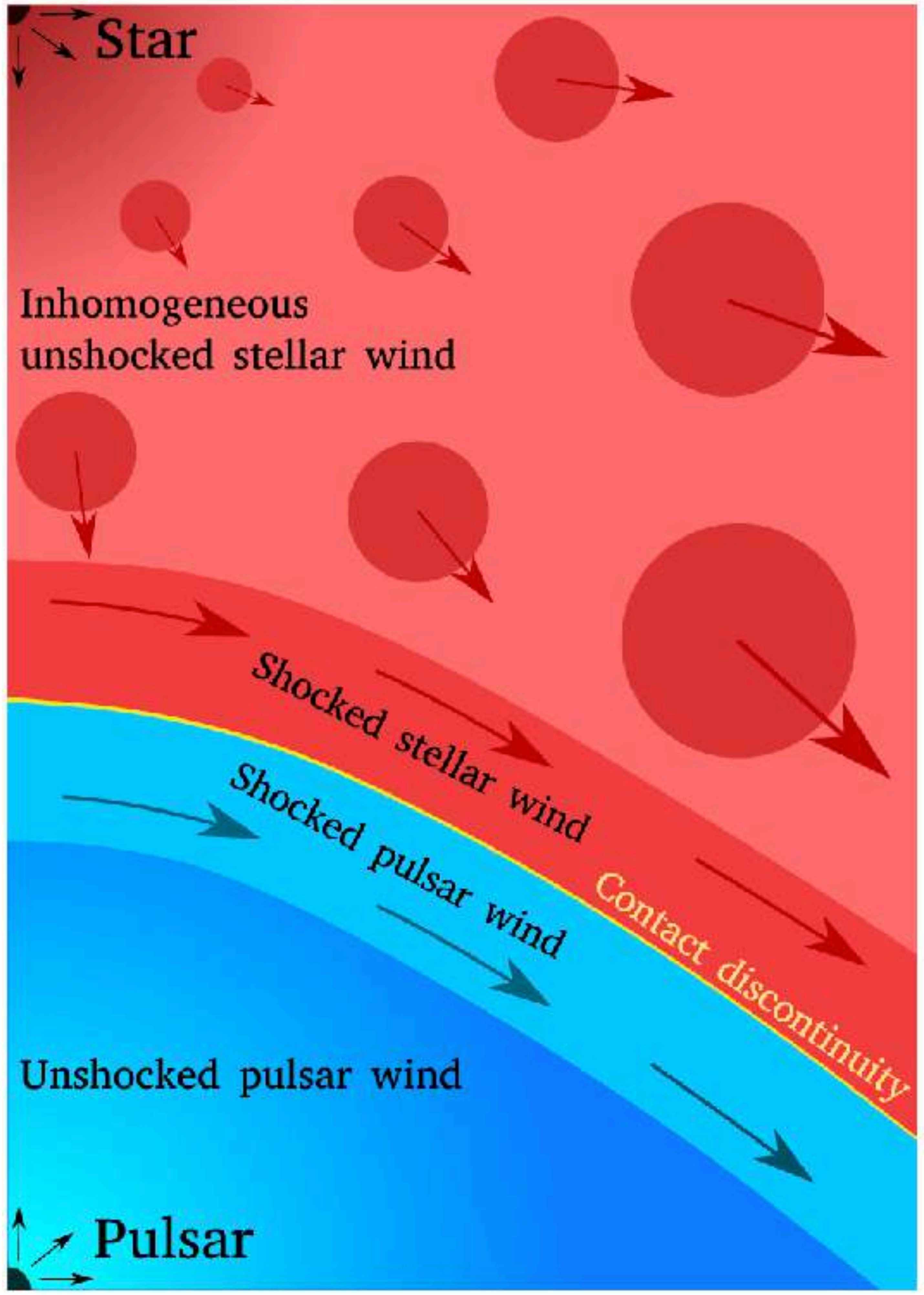
Approx. GR!



Tidal Disruption Events (TDEs)

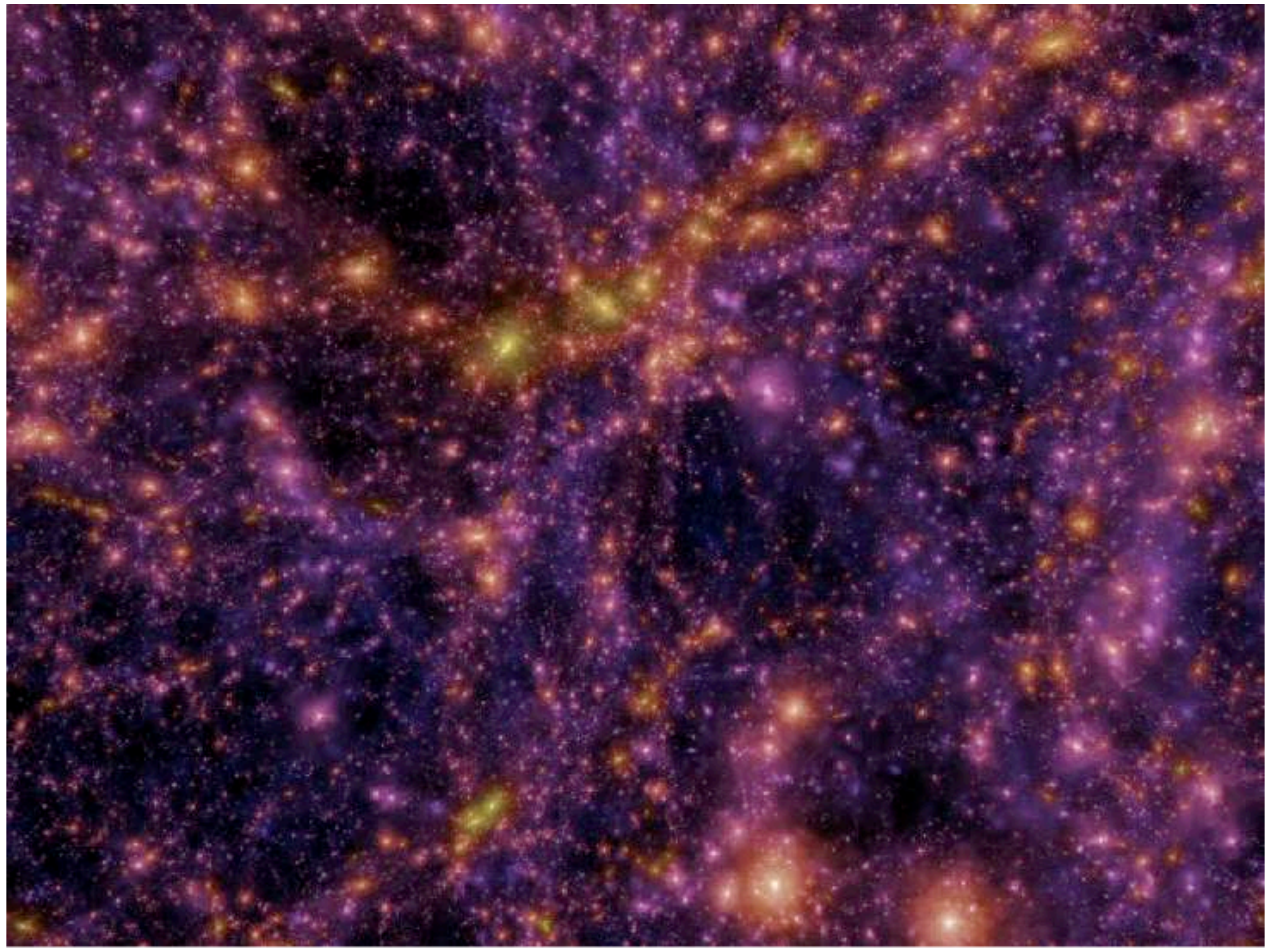


Bonnerot et al. (2016)



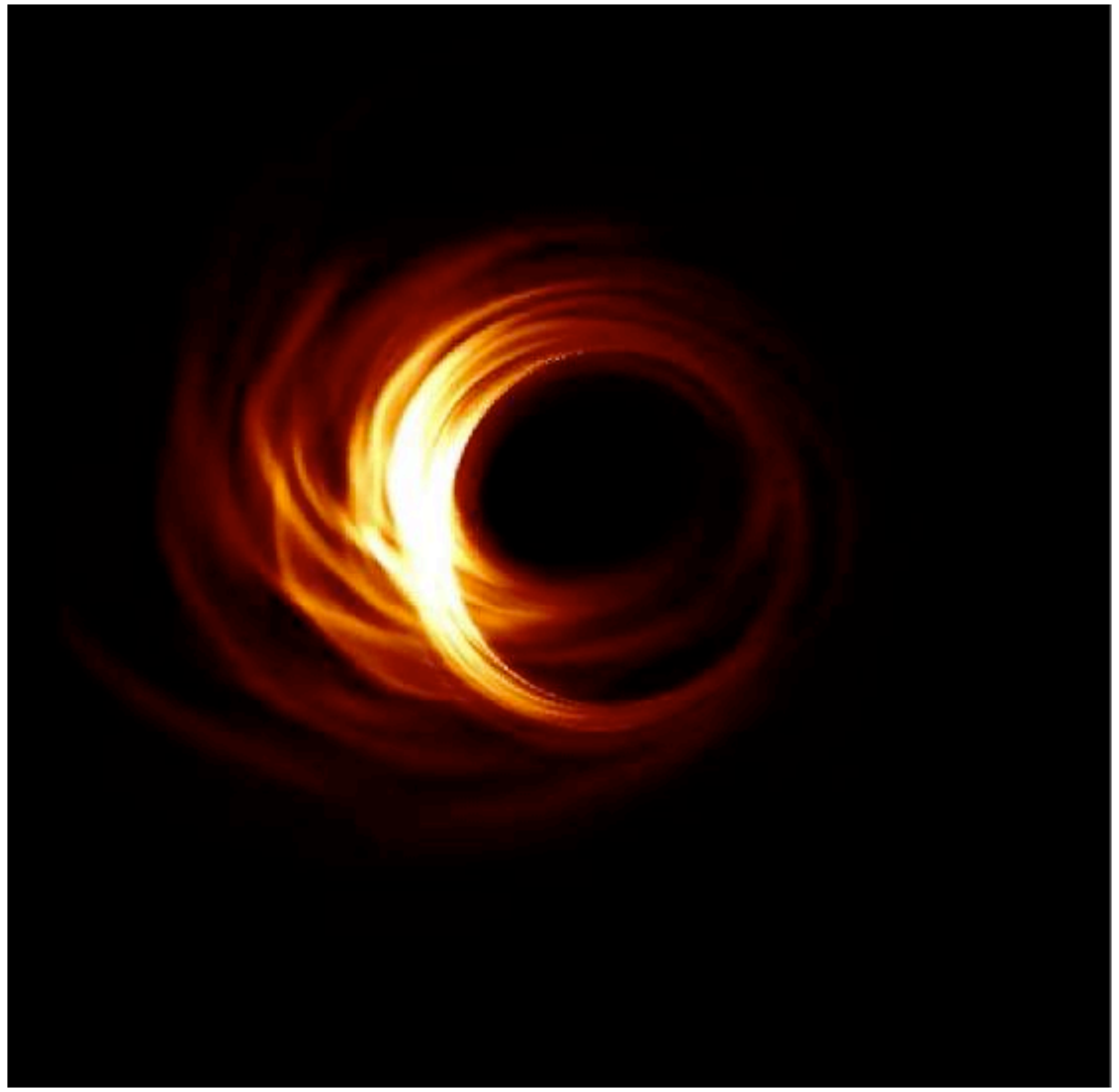
Paredes-Fortuny et al. (2015)

Relativistic Pulsar Winds



Springel et al. (2005)

Cosmological
Simulations with full GR



Credit: Hotaka Shiokawa

Event Horizon Telescope

Equations of relativistic hydrodynamics

Continuity:

$$\frac{\mathrm{d}\rho^*}{\mathrm{d}t} = \underbrace{-\rho^* \frac{\partial v^i}{\partial x^i}}_{}$$

Momentum:

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = \underbrace{-\frac{1}{\rho^*}\frac{\partial(\sqrt{-g}P)}{\partial x^i}}_{} + \underbrace{\frac{\sqrt{-g}}{2\rho^*}\left(T^{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial x^i}\right)}_{\text{“GR”}}$$

Energy:

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \underbrace{-\frac{1}{\rho^*}\frac{\partial(\sqrt{-g}Pv^i)}{\partial x^i}}_{\text{“Hydro”}} + \xcancel{\frac{-\sqrt{-g}}{2\rho^*}\left(T^{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial t}\right)}$$

Equations of relativistic hydrodynamics

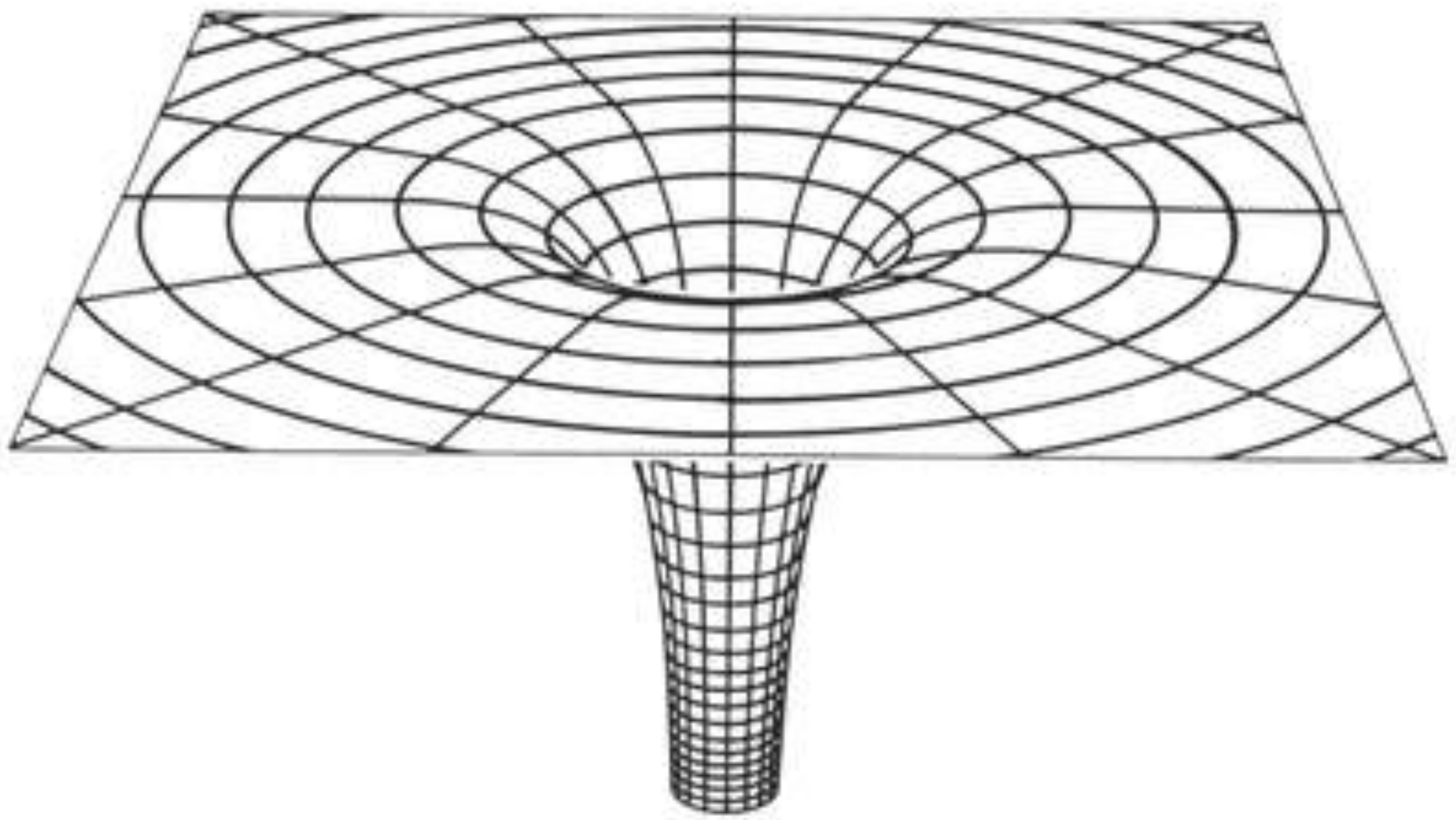
Continuity:
$$\cancel{\frac{\mathrm{d}\rho_a}{\mathrm{d}t} = \underbrace{\frac{1}{\Omega_a}\sum_b m_b (v_a^i - v_b^i)\frac{\partial W_{ab}(h_a)}{\partial x^i}}},\qquad \rho_a^* = \sum_b m_b W_{ab}(h_a)$$

Momentum:
$$\frac{\mathrm{d}p_i^a}{\mathrm{d}t} = \underbrace{-\sum_b m_b\left[\frac{\sqrt{-g_a}P_a}{\Omega_a\rho_a^{*2}}\frac{\partial W_{ab}(h_a)}{\partial x^i} + \frac{\sqrt{-g_b}P_b}{\Omega_b\rho_b^{*2}}\frac{\partial W_{ab}(h_b)}{\partial x^i}\right]} + \underbrace{f_i^a}_{\text{“GR”}},$$

Energy:
$$\frac{\mathrm{d}e_a}{\mathrm{d}t} = -\sum_b \underbrace{m_b\left[\frac{\sqrt{-g_a}P_a v_b^i}{\Omega_a\rho_a^{*2}}\frac{\partial W_{ab}(h_a)}{\partial x^i} + \frac{\sqrt{-g_b}P_b v_a^i}{\Omega_b\rho_b^{*2}}\frac{\partial W_{ab}(h_b)}{\partial x^i}\right]}_{\text{“Hydro”}} + \cancel{\Lambda_a},$$

Checklist: Metrics and Coordinates

- Minkowski, Schwarzschild and **Kerr**
- Need in **Cartesian**-like coordinates
- A way to compute **derivatives**
- Choice of **frame?** (which observer?)



$$ds^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2M}{r}} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)$$

$$= -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \ \ldots\ \mathrm{d}x^2 + \ \ldots\ \mathrm{d}x\,\mathrm{d}y + \ \ldots\ \mathrm{d}x\,\mathrm{d}z + \ \ldots \quad \textbf{?}$$

Checklist: Time Integration

- Preserve the Hamiltonian properties of the system
- Operator splitting approach
- Time reversible (conserves energy)
- Cost effective for 2nd order

Modified Leapfrog algorithm

$$p_i^{n+\frac{1}{2}} = p_i^n + \frac{\Delta t}{2} f_i^{\,\mathrm{sph}}(p_i^n, x^{i,n}),$$

$$\left\{ \begin{aligned} p_i^{m+\frac{1}{2}} &= p_i^m + \frac{\Delta t_{\mathrm{ext}}}{2} f_i^{\,\mathrm{ext}}(p_i^{m+\frac{1}{2}}, x^{i,m}), \\ x^{i,m+1} &= x^{i,m} + \frac{\Delta t_{\mathrm{ext}}}{2} \left[\frac{\mathrm{d}x^i}{\mathrm{d}t}(p_i^{m+\frac{1}{2}}, x^{i,m}) \right. \\ &\qquad \left. + \frac{\mathrm{d}x^i}{\mathrm{d}t}(p_i^{m+\frac{1}{2}}, x^{i,m+1}) \right], \\ p_i^{m+1} &= p_i^{m+\frac{1}{2}} + \frac{\Delta t_{\mathrm{ext}}}{2} f_i^{\,\mathrm{ext}}(p_i^{m+\frac{1}{2}}, x^{i,m+1}), \end{aligned} \right\}$$

$$p_i^{n+1} = p_i^{n+\frac{1}{2}} + \frac{\Delta t}{2} f_i^{\,\mathrm{sph}}(p_i^{n+1}, x^{i,n+1})$$

Checklist: Recovery of Primitive Variables

- Every time-step
- Rigorous and cheap
- Cannot solve explicitly
- Newton-Raphson scheme (Tejeda 2012)

$$\rho^* = \sqrt{-g}\rho U^0,$$

$$p_i = U^0 w g_{i\mu} v^\mu,$$

$$e = U^0 \left[w g_{i\mu} v^\mu v^i - (1+u) g_{\mu\nu} v^\mu v^\nu \right],$$

$$\Downarrow$$

$$\rho =$$

$$v_i = \qquad ?$$

$$u =$$

Testing

Metric terms, time integration, conservative to primitive

Three parts:

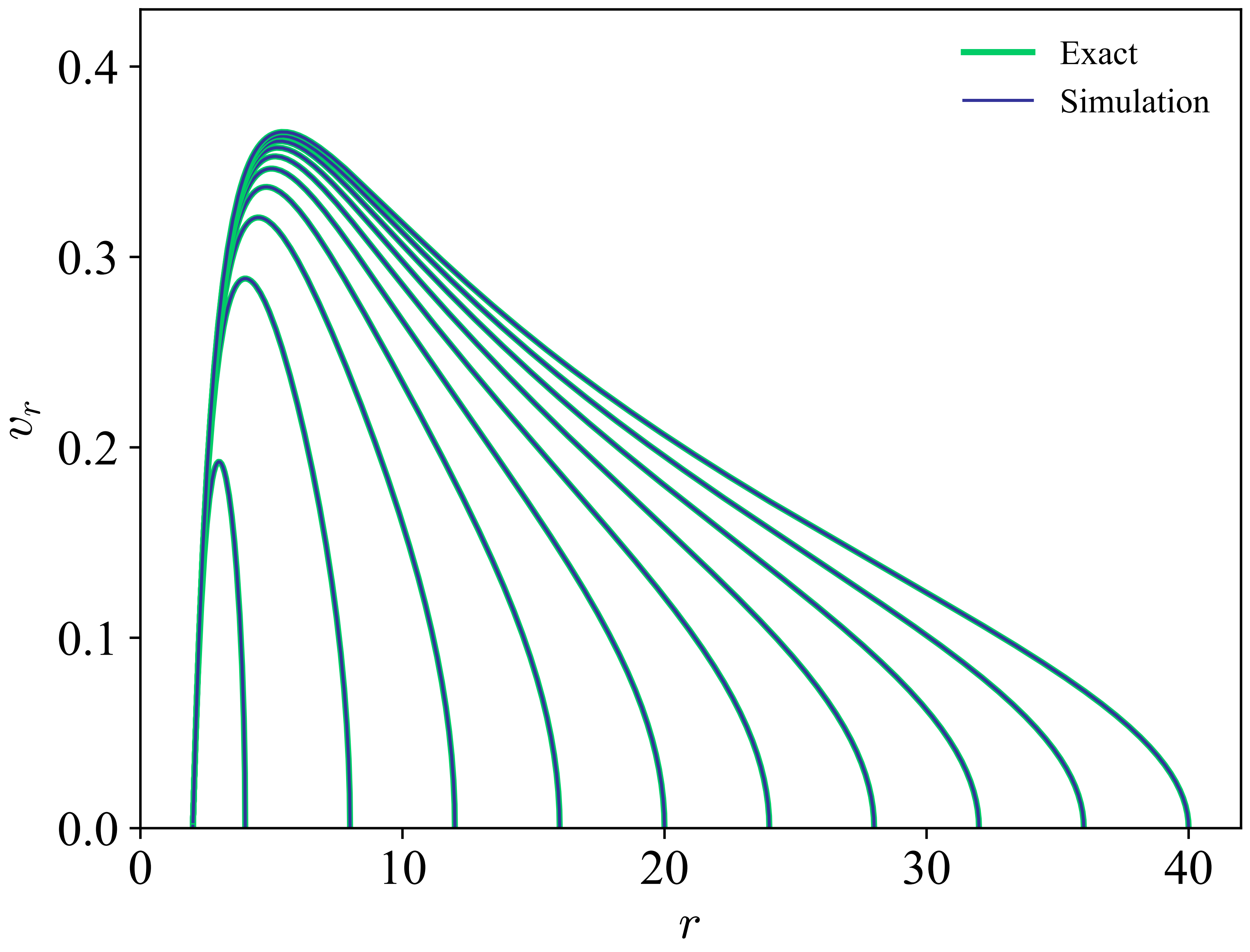
- 1) Orbital dynamics

- 2) Shocks and special relativity

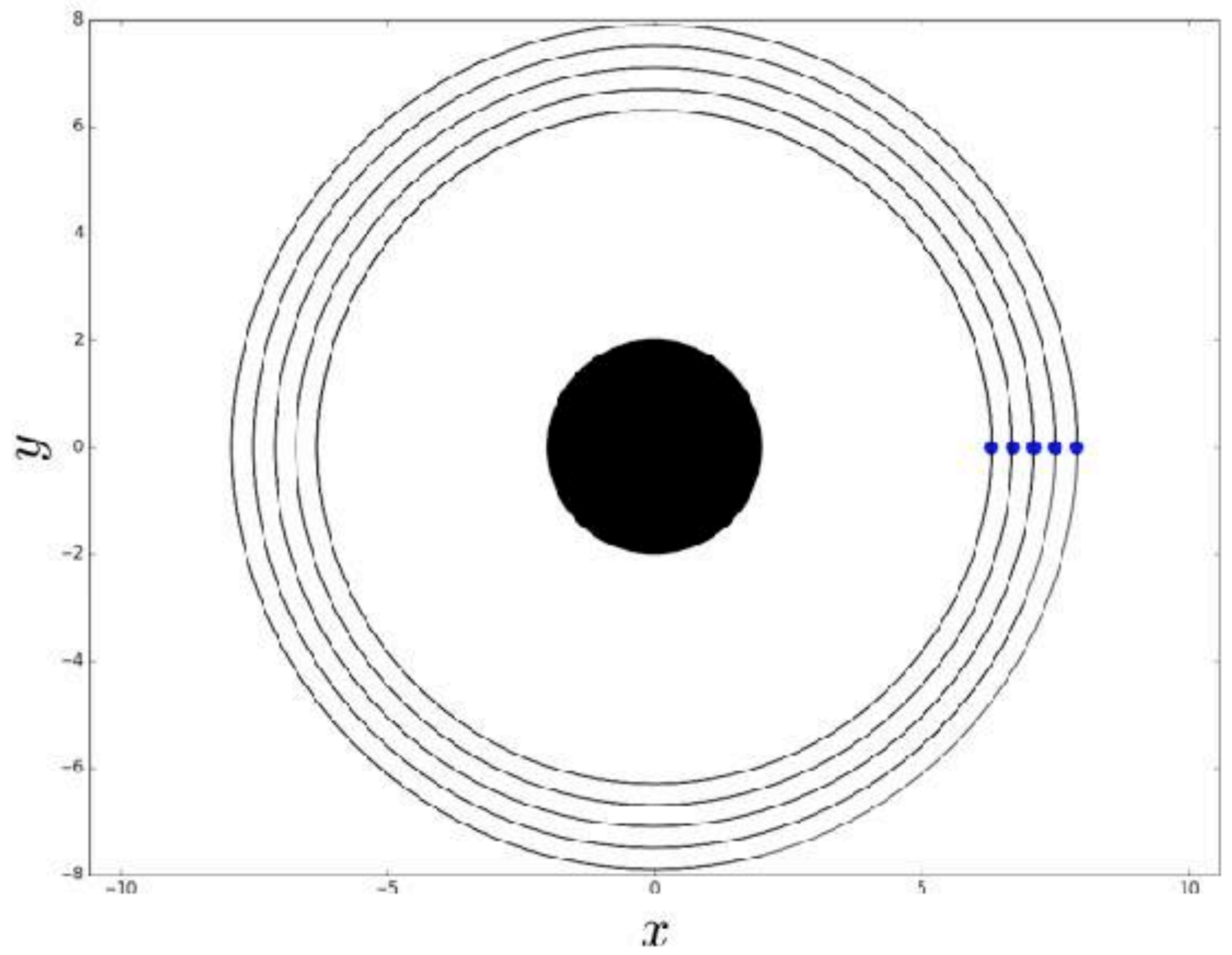
- 3) 3D GR hydrodynamics

Tests: Schwarzschild metric

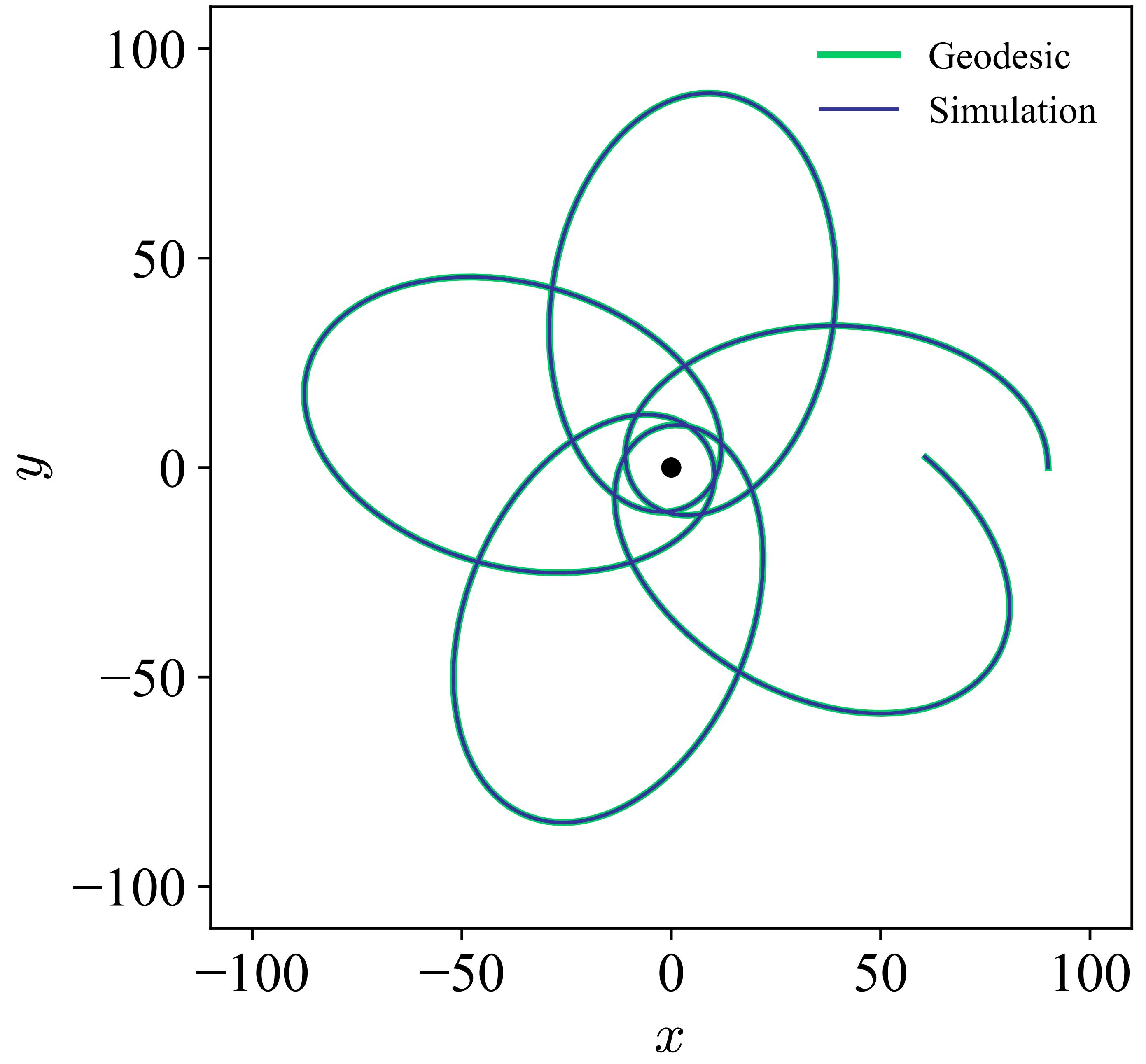
Radial Infall

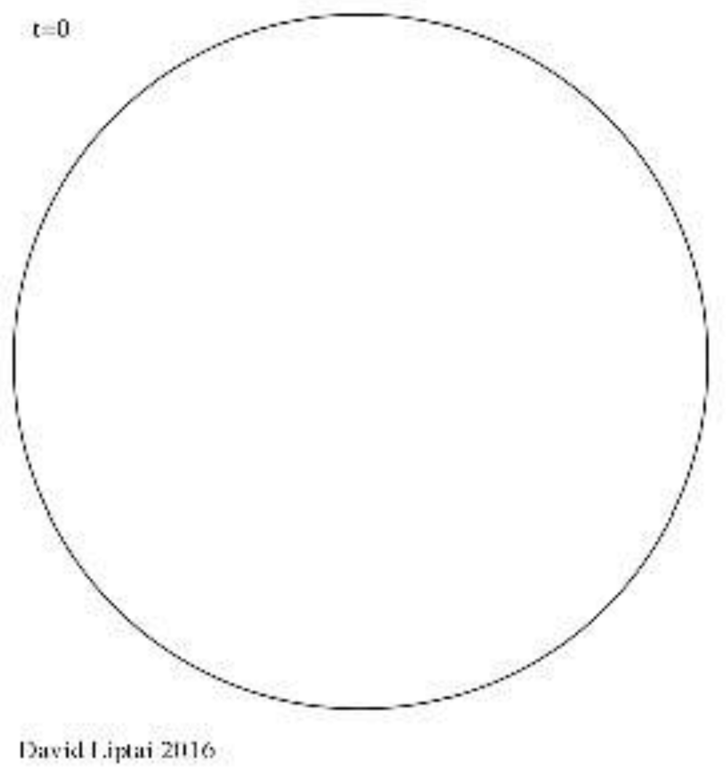


Circular orbits



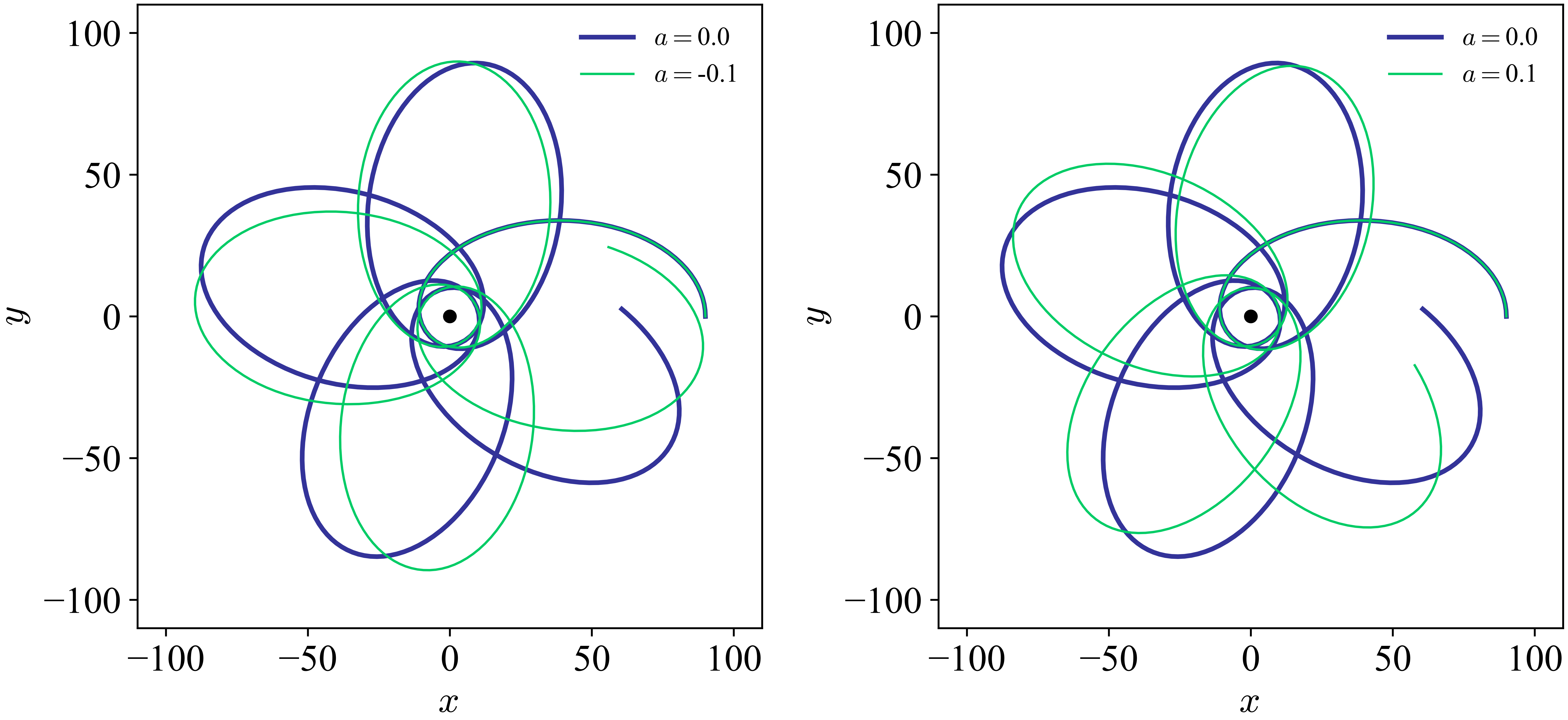
Precession





Tests: Kerr metric

Apsidal precession



Retrograde **Prograde**

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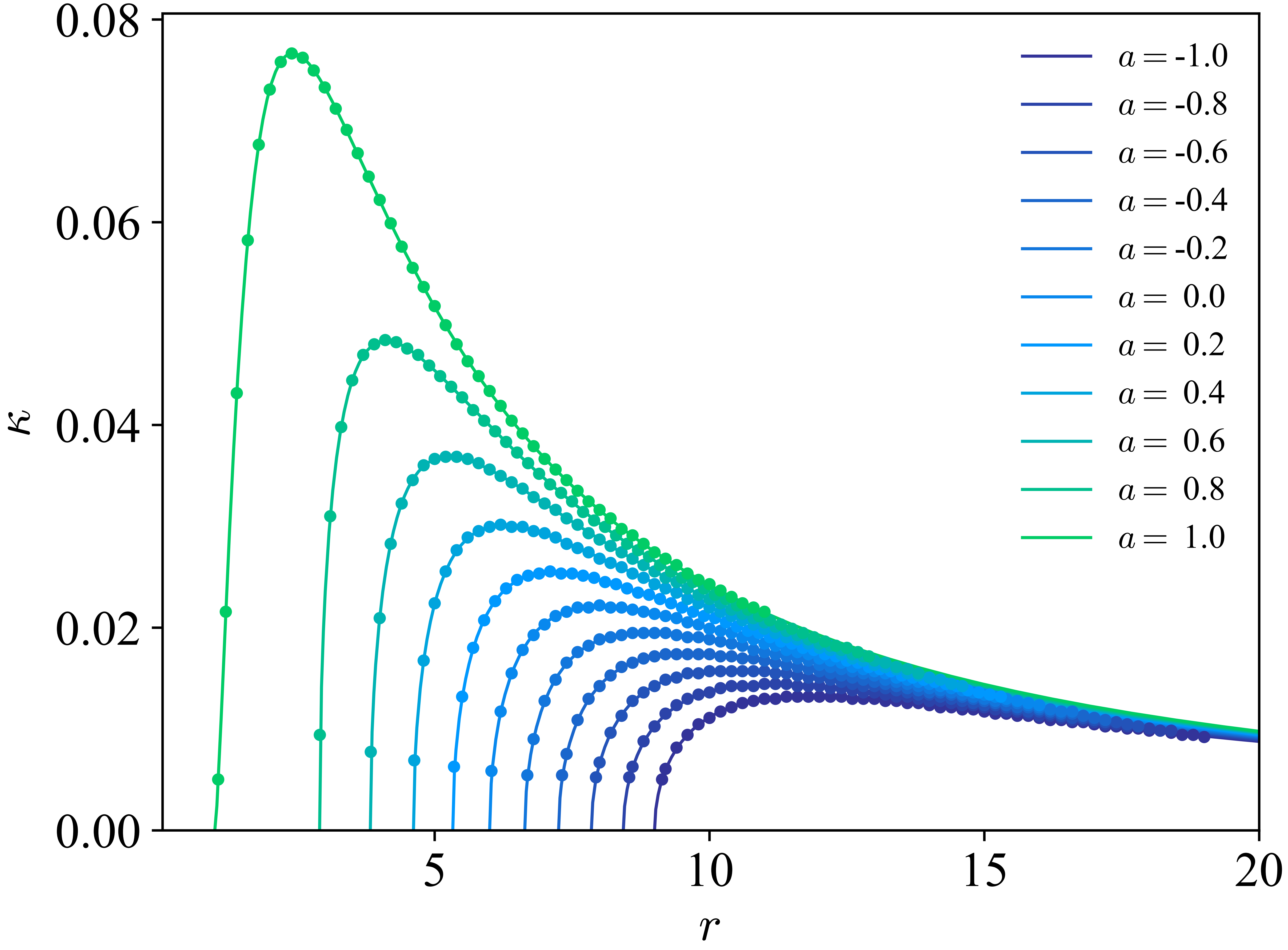
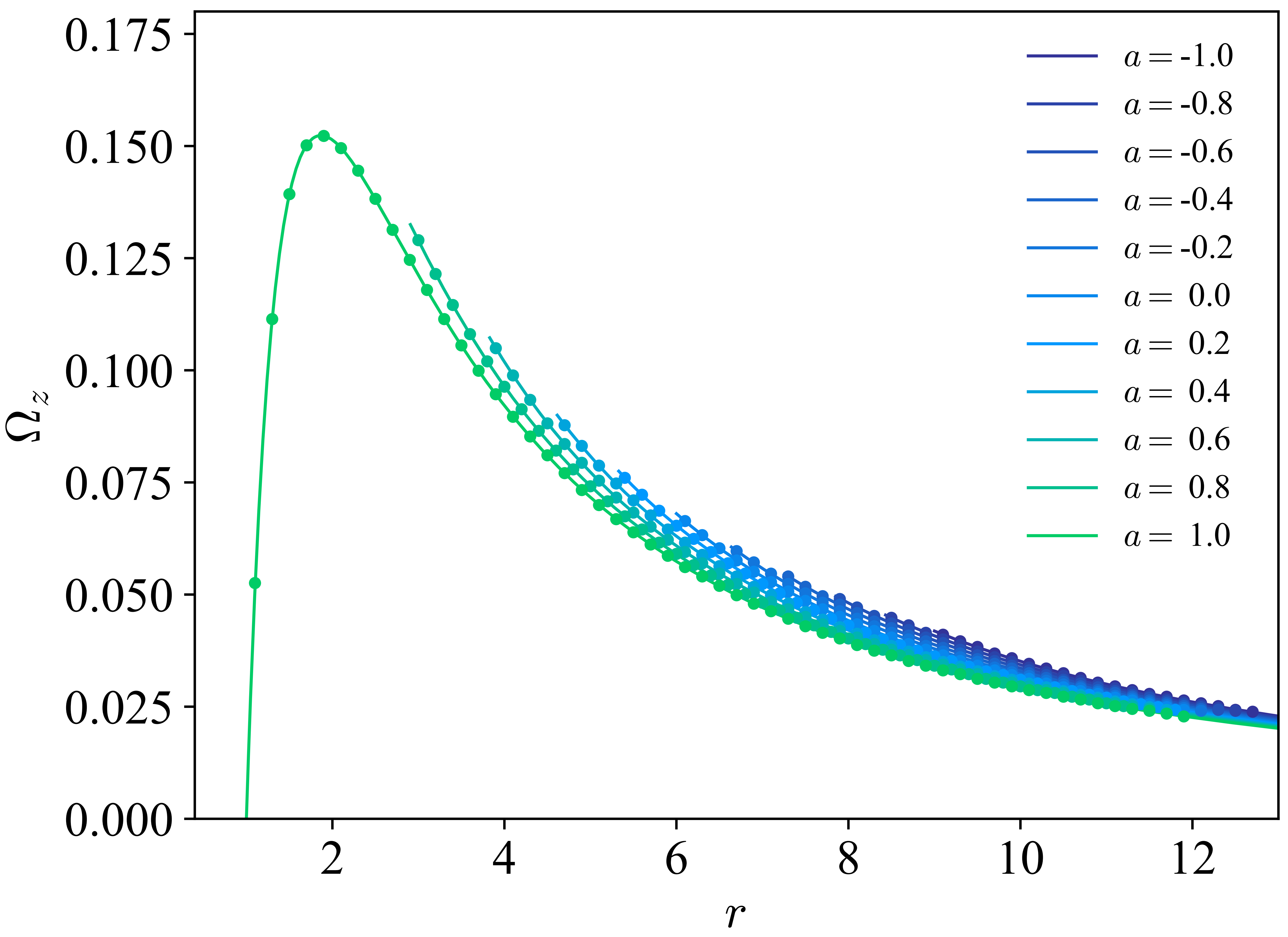
Tests: Kerr metric

Epicyclic frequency

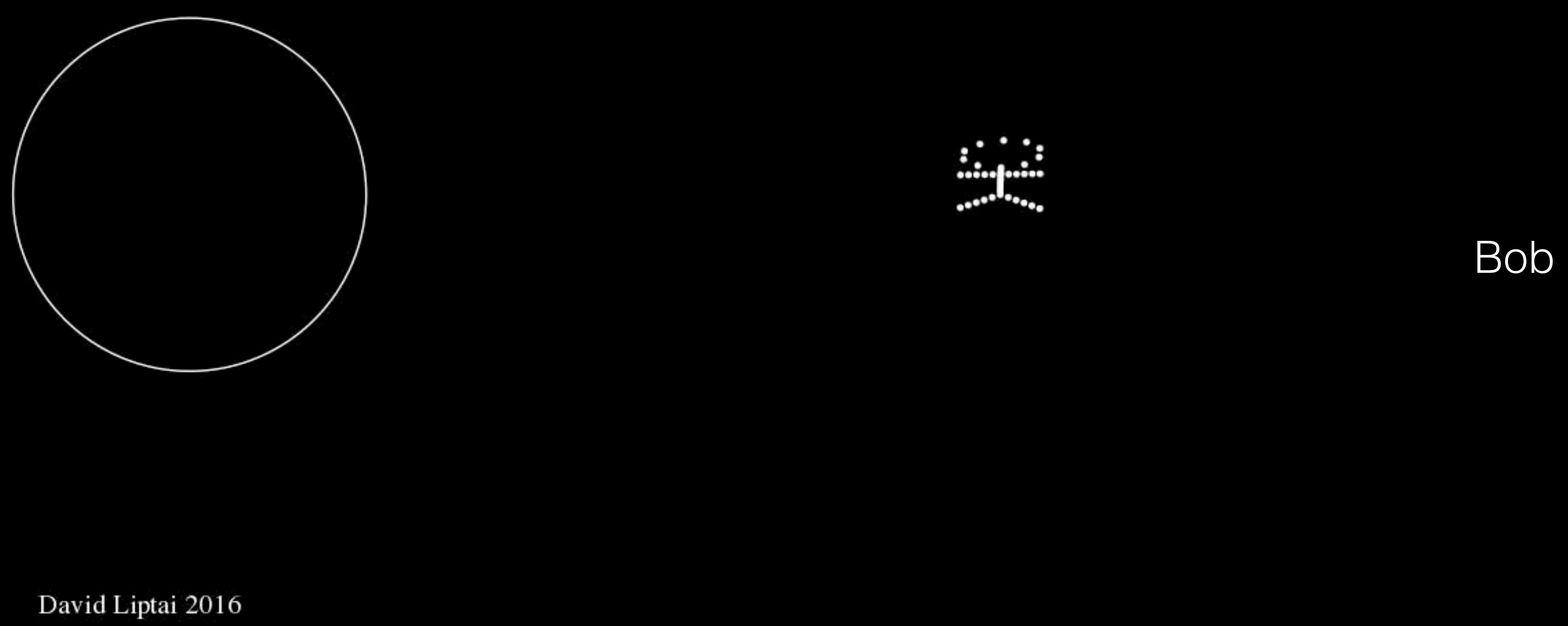


Vertical-oscillation frequency

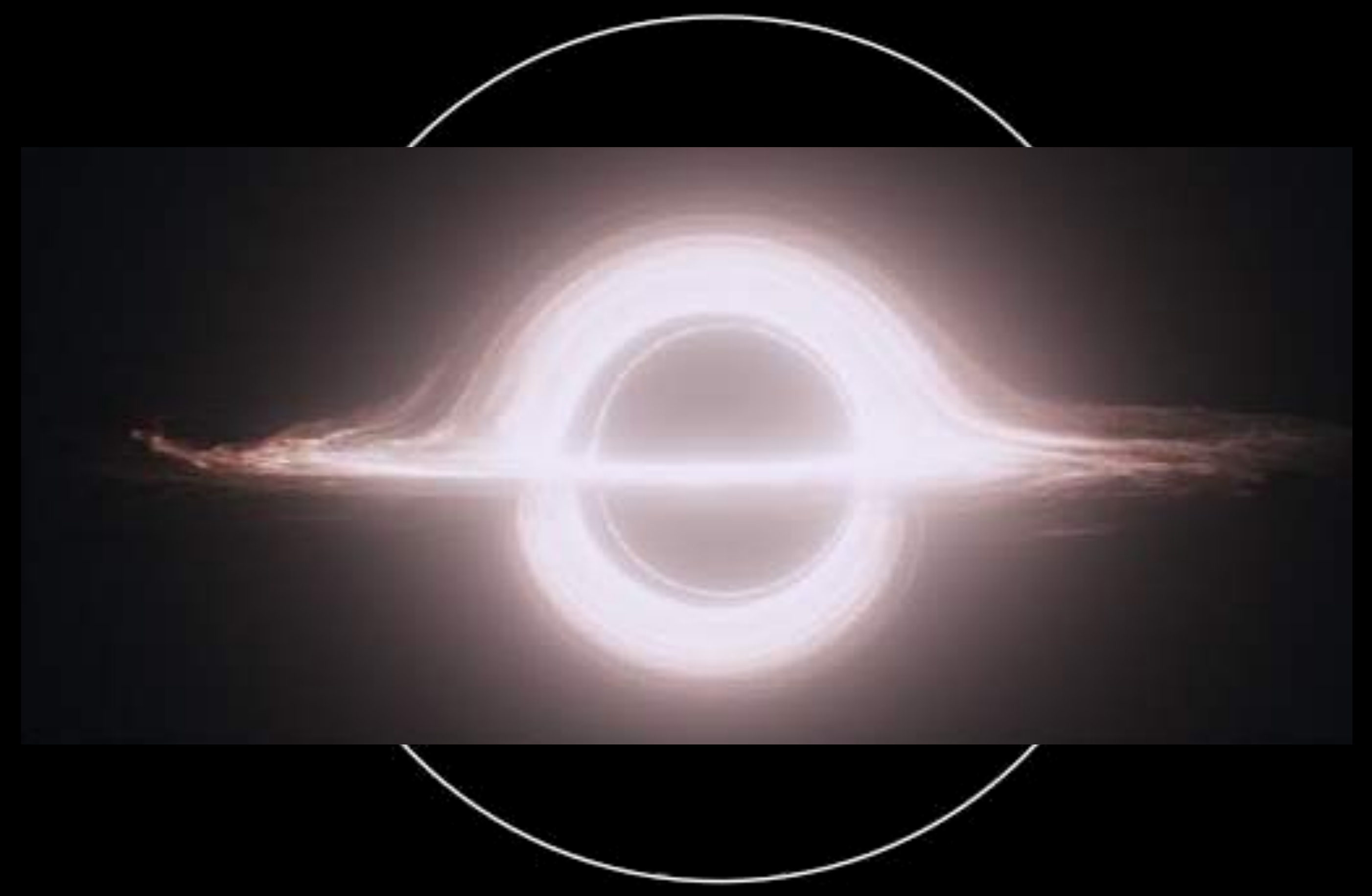


Liptai and Price 2018 (In prep.)

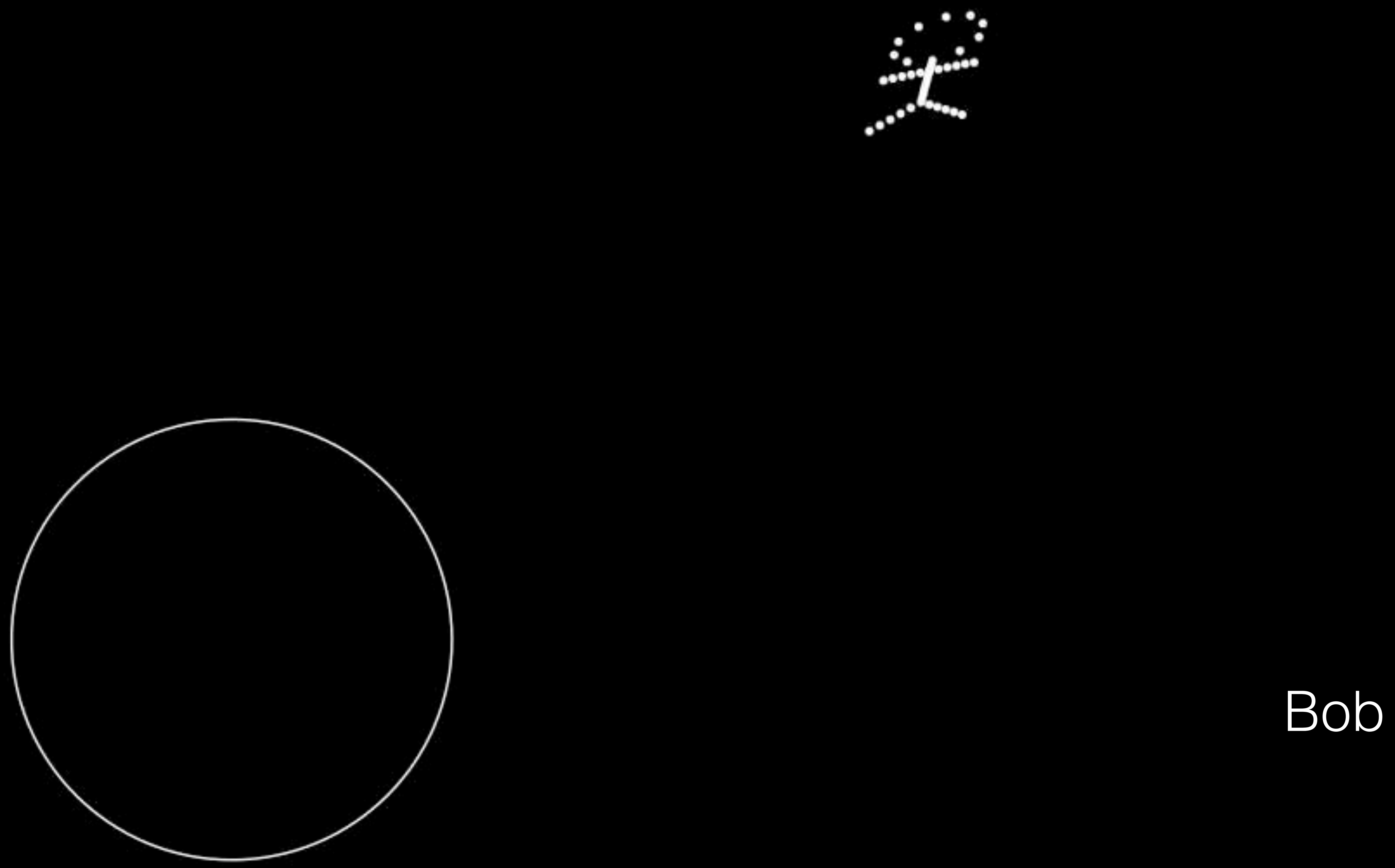
Spaghettification



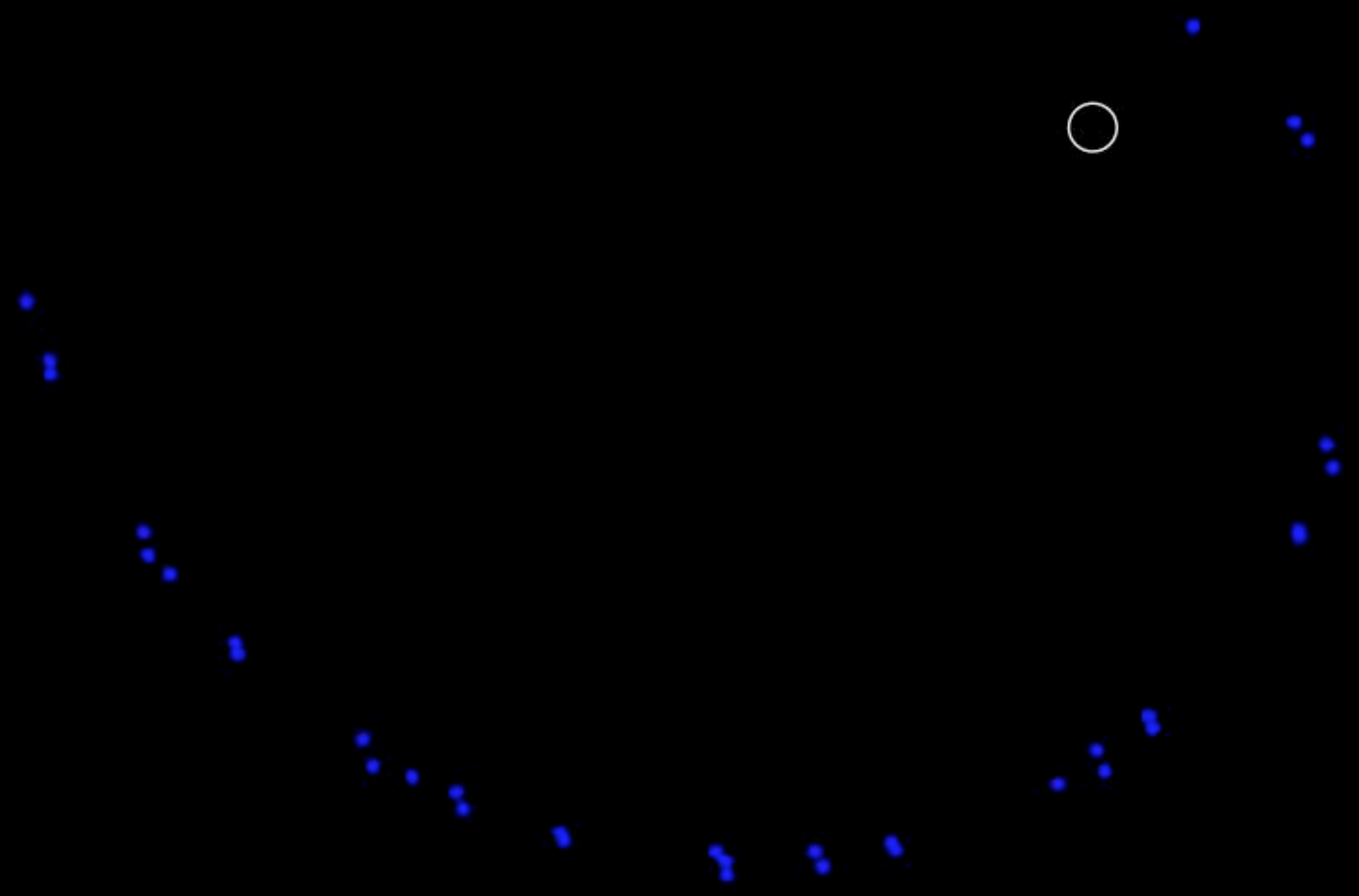
Matthew McConaughey



David Liptai 2016

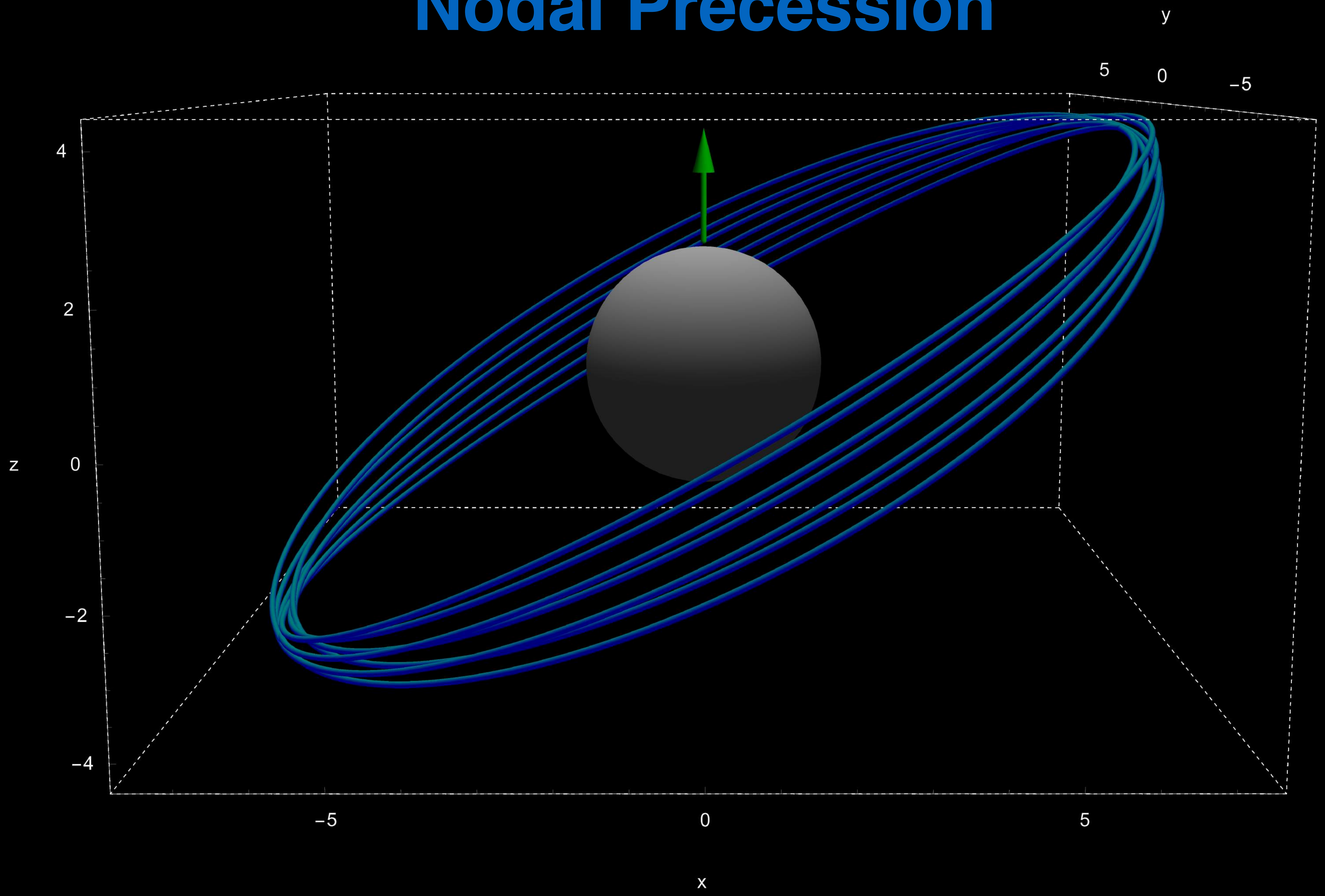


David Liptai 2016

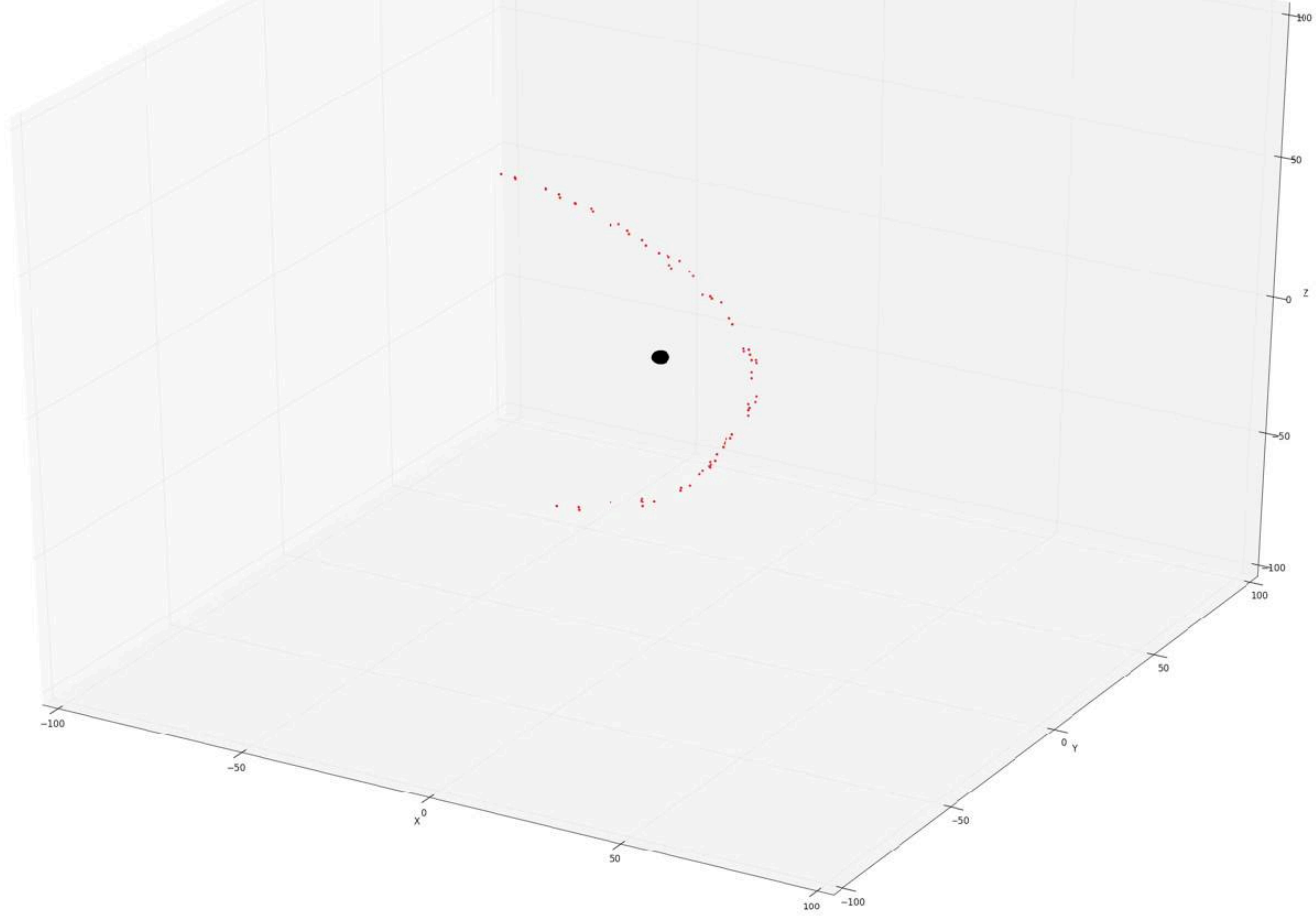


David Liptai 2016

Nodal Precession



A mock tidal disruption event (TDE)
in the Kerr metric



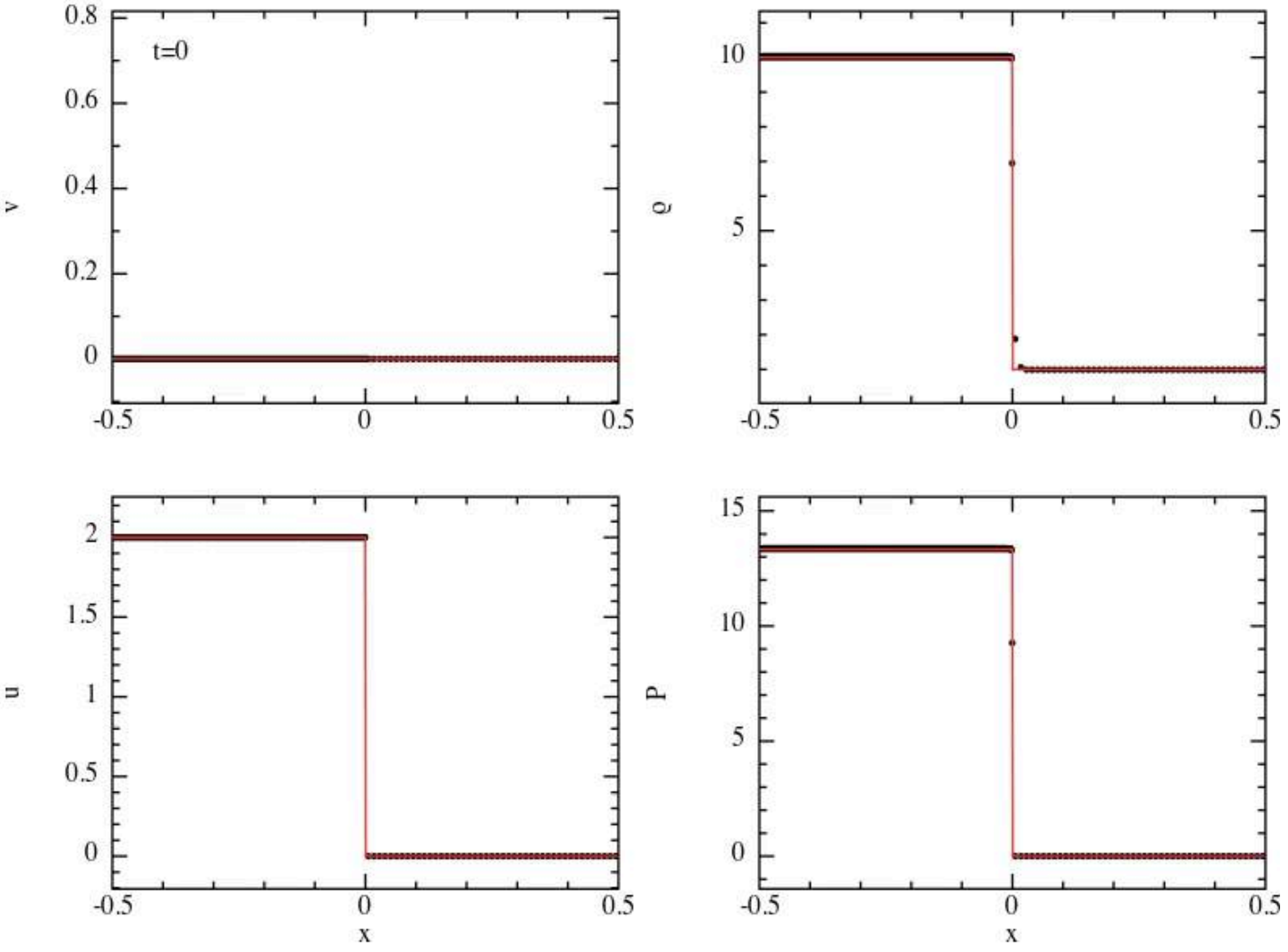
Tests: shock capturing

- 1D shock tubes
- Minkowski metric (special rel)

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = -\sum_b m_b \left[\frac{\sqrt{-g_a}P_a}{\Omega_a \rho_a^{*2}} \frac{\partial W_{ab}(h_a)}{\partial x^i} + \frac{\sqrt{-g_b}P_b}{\Omega_b \rho_b^{*2}} \frac{\partial W_{ab}(h_b)}{\partial x^i} \right] + \left(\frac{\mathrm{d}p_i}{\mathrm{d}t}\right)_{\mathrm{diss}}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = -\sum_b m_b \left[\frac{\sqrt{-g_a}P_a}{\Omega_a \rho_a^{*2}} v_b^i \frac{\partial W_{ab}(h_a)}{\partial x^i} + \frac{\sqrt{-g_b}P_b}{\Omega_b \rho_b^{*2}} v_a^i \frac{\partial W_{ab}(h_b)}{\partial x^i} \right] + \left(\frac{\mathrm{d}e}{\mathrm{d}t}\right)_{\mathrm{diss}}$$

What should we use?



Attempts at artificial dissipation in SR

Chow & Monaghan (1997)
Overly dissipative

Siegler & Riffert (2000)
No artificial conductivity

1D special relativistic shock tubes

Artificial viscosity only

Artificial viscosity AND conductivity

Liptai and Price 2018 (In prep.)

Evolving Entropy

- **Split dissipation** —artificial viscosity from conductivity
- **Positive definite** contribution to entropy
- Evolve entropy:
  - robust — **no negative pressures**

$$K = P\rho^{\gamma_{\rm ad}}$$

$$\begin{aligned}\frac{\mathrm{d}K}{\mathrm{d}t} &= \frac{U^0 K}{u}\left[\frac{\mathrm{d}e}{\mathrm{d}t} + \frac{\sqrt{-g}}{2\rho^*}T^{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial t}\right.\\ &\qquad \left. - v^i\left(\frac{\mathrm{d}p_i}{\mathrm{d}t} - \frac{\sqrt{-g}}{2\rho^*}T^{\mu\nu}\frac{\partial g_{\mu\nu}}{\partial x^i}\right) - \frac{P\sqrt{-g}}{{\rho^*}^2}\frac{\mathrm{d}\rho^*}{\mathrm{d}t}\right]\\ &= 0\end{aligned}$$

$$\frac{\mathrm{d}K_a}{\mathrm{d}t} = \frac{U^0_a K_a}{u_a}\,[\text{ dissipation terms }]$$

1D special relativistic shock tubes

Ultra-relativistic 1D

Sine wave perturbation 1D

Liptai and Price 2018 (In prep.)

3D Hydrodynamics

Publications of the Astronomical Society of Australia (PASA)
doi: 10.1017/pas.2017.xxx.

PHANTOM: A smoothed particle hydrodynamics and magnetohydrodynamics code for astrophysics

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Abstract
We present PHANTOM, a fast, parallel, modular and low-memory smoothed particle hydrodynamics and magnetohydrodynamics code developed over the last decade for astrophysical applications in three dimensions. The code has been developed with a focus on stellar, galactic, planetary and high energy astrophysics and has already been used widely for studies of accretion discs and turbulence, from the birth of planets to how black holes accrete. Here we describe and test the core algorithms as well as modules for magnetohydrodynamics, self-gravity, sink particles, H_2 chemistry, dust-gas mixtures, physical viscosity, external forces including numerous galactic potentials as well as implementations of Lense-Thirring precession, Poynting-Robertson drag and stochastic turbulent driving. PHANTOM is hereby made publicly available.

Keywords: hydrodynamics — methods: numerical — magnetohydrodynamics (MHD) — accretion, accretion discs — ISM: general

1 Introduction

Numerical simulations are the 'third pillar' of astrophysics, standing alongside observations and analytic theory. Since it is difficult to perform laboratory experiments in the relevant physical regimes and over the correct range of length and time-scales involved in most astrophysical problems, we turn instead to 'numerical experiments' in the computer for understanding and insight. As algorithms and simulation codes become ever more sophisticated, the public availability of simulation codes has become crucial to ensure that these experiments can be both verified and reproduced.

PHANTOM is a smoothed particle hydrodynamics (SPH) code, written in Fortran 90, developed over the last decade. It has been used widely for studies of accretion (Lodato & Price, 2010; Nixon et al., 2012a; Rosotti et al., 2012; Nixon, 2012; Nixon et al., 2012b; Facchini et al., 2013; Nixon et al., 2013; Martin et al., 2014a,b; Nixon & Lubow, 2015; Coughlin & Nixon, 2015; Forgan et al., 2017) and turbulence (Kitsionas et al., 2009; Price & Federrath, 2010; Price et al., 2011; Price, 2012b; Tricco et al., 2016b) as well as for studies of the Galaxy

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10 months later…

3D Special relativistic shocktubes

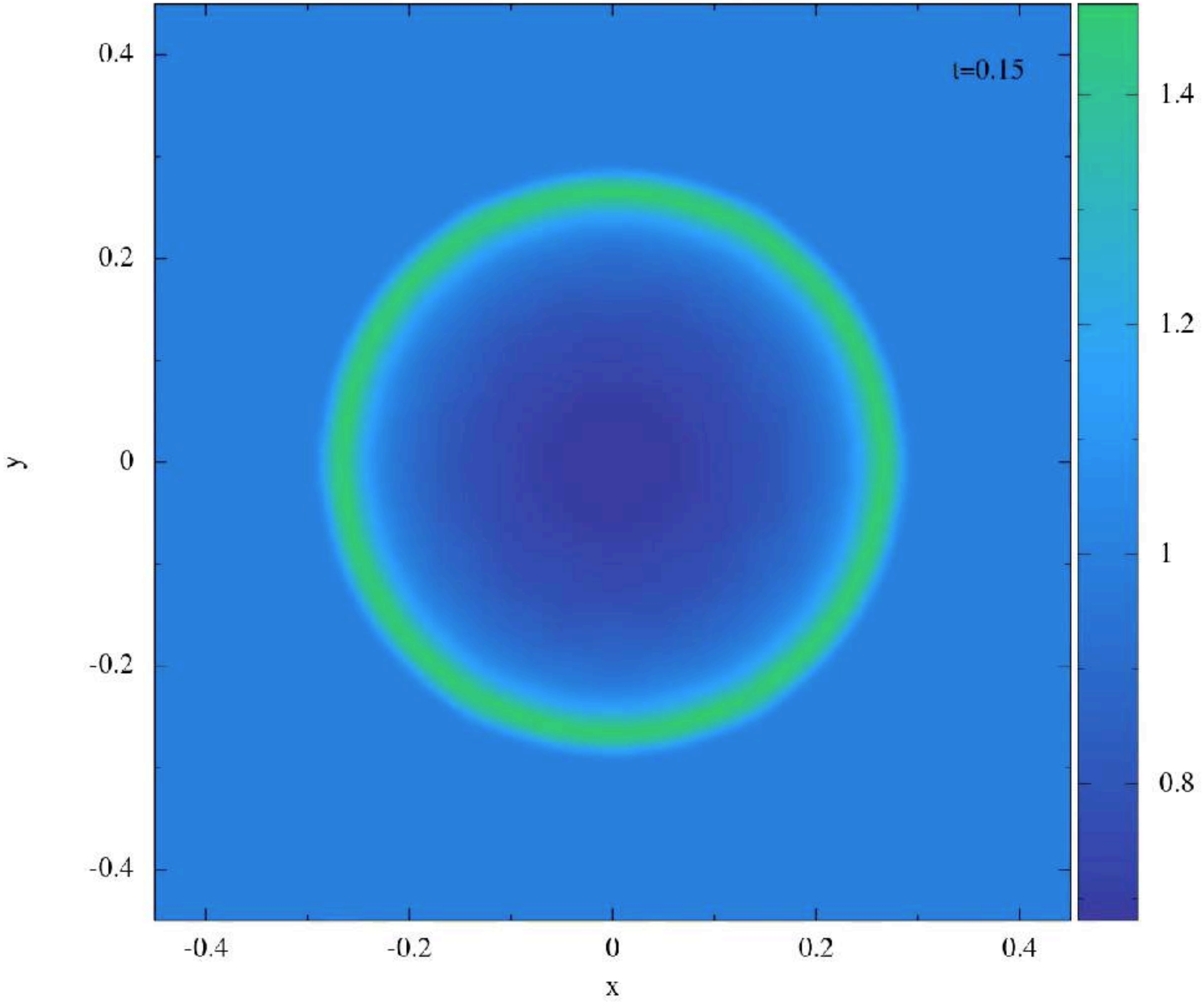
Mildly-relativistic 3D

Ultra-relativistic 3D

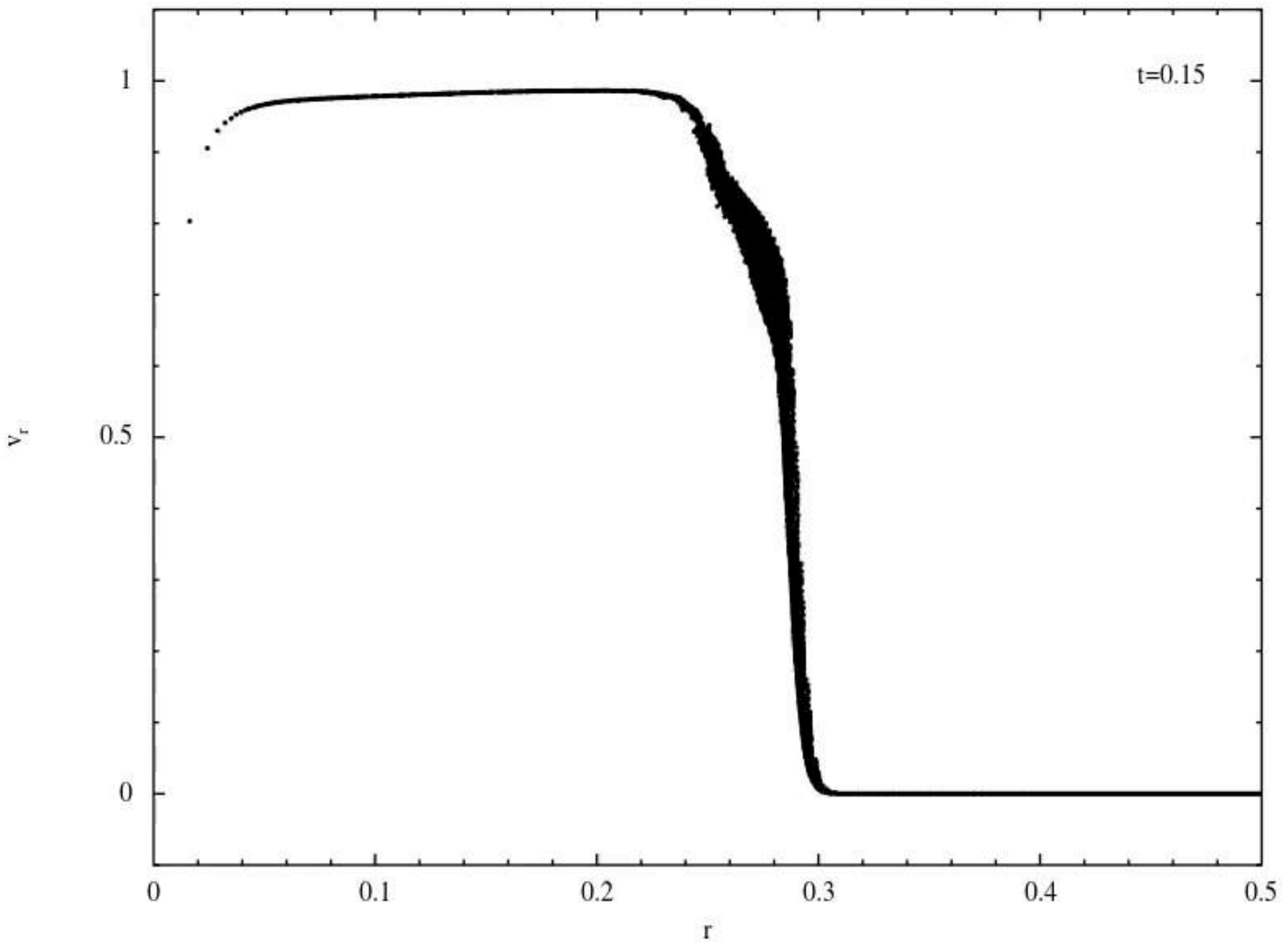
Liptai and Price 2018 (In prep.)

3D spherical blast wave

Column Density



Velocity



(Maximum Lorentz factor ~6.4)

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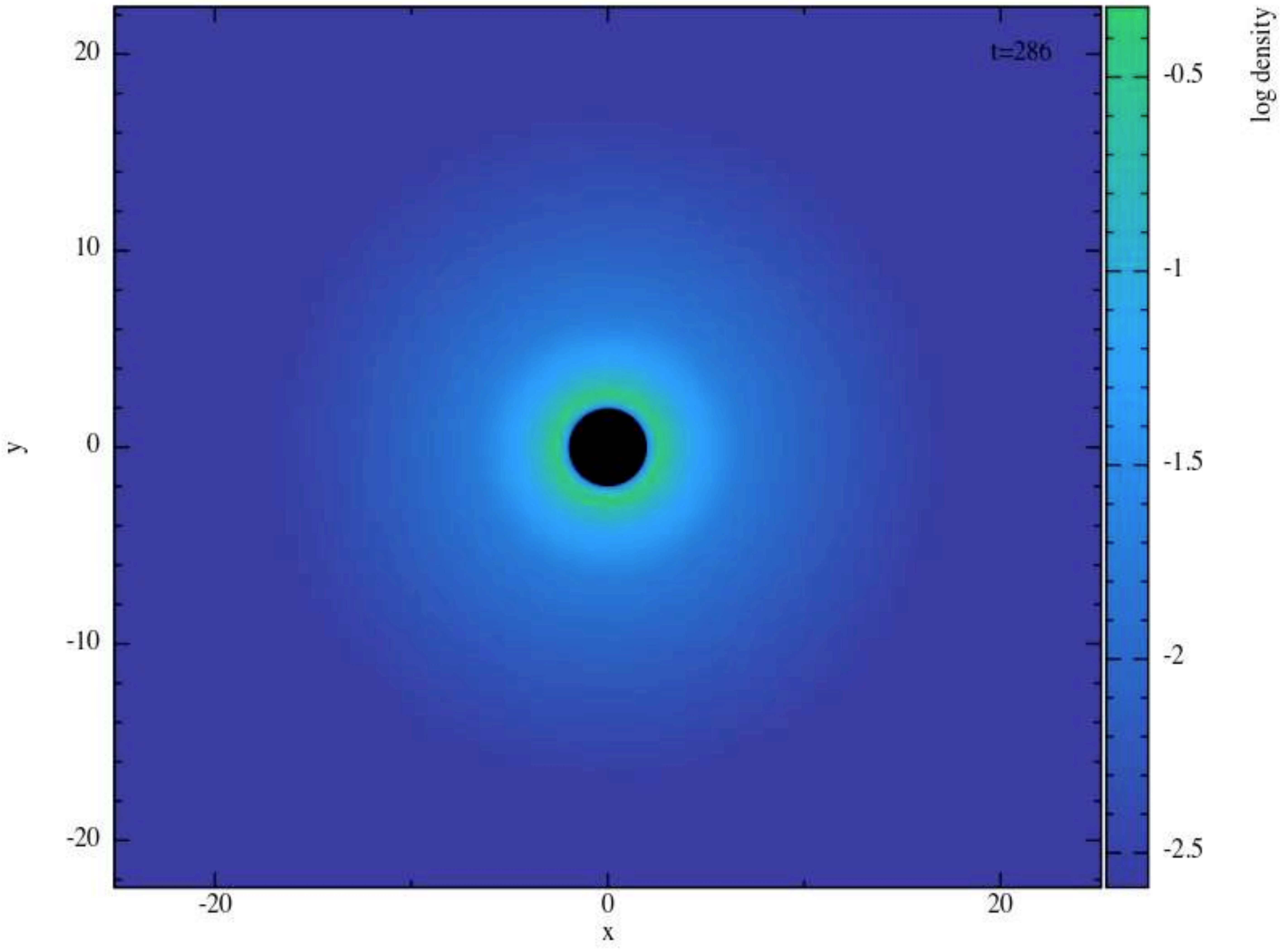
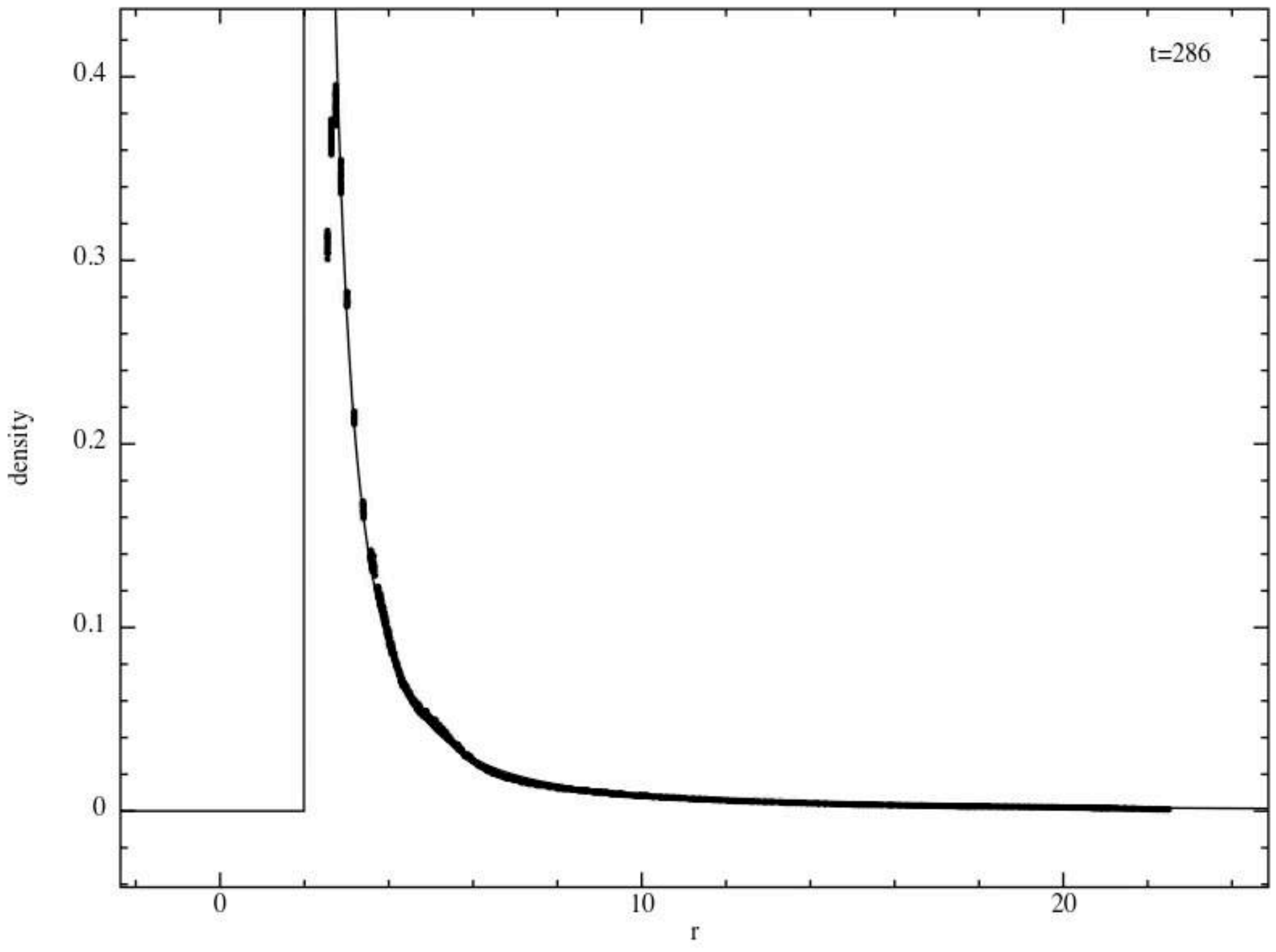
Generalised Bondi flow (Schwarzschild)





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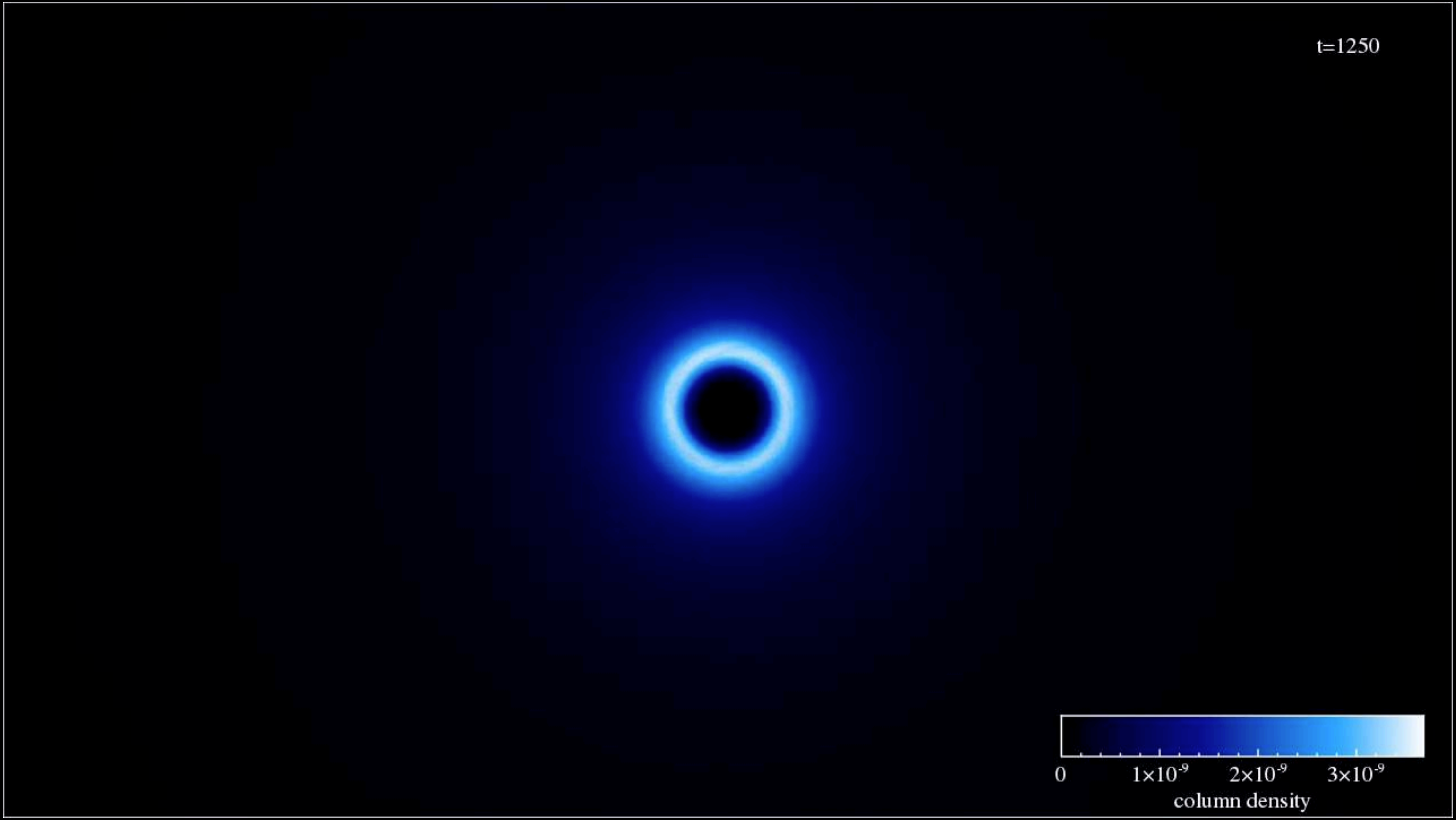
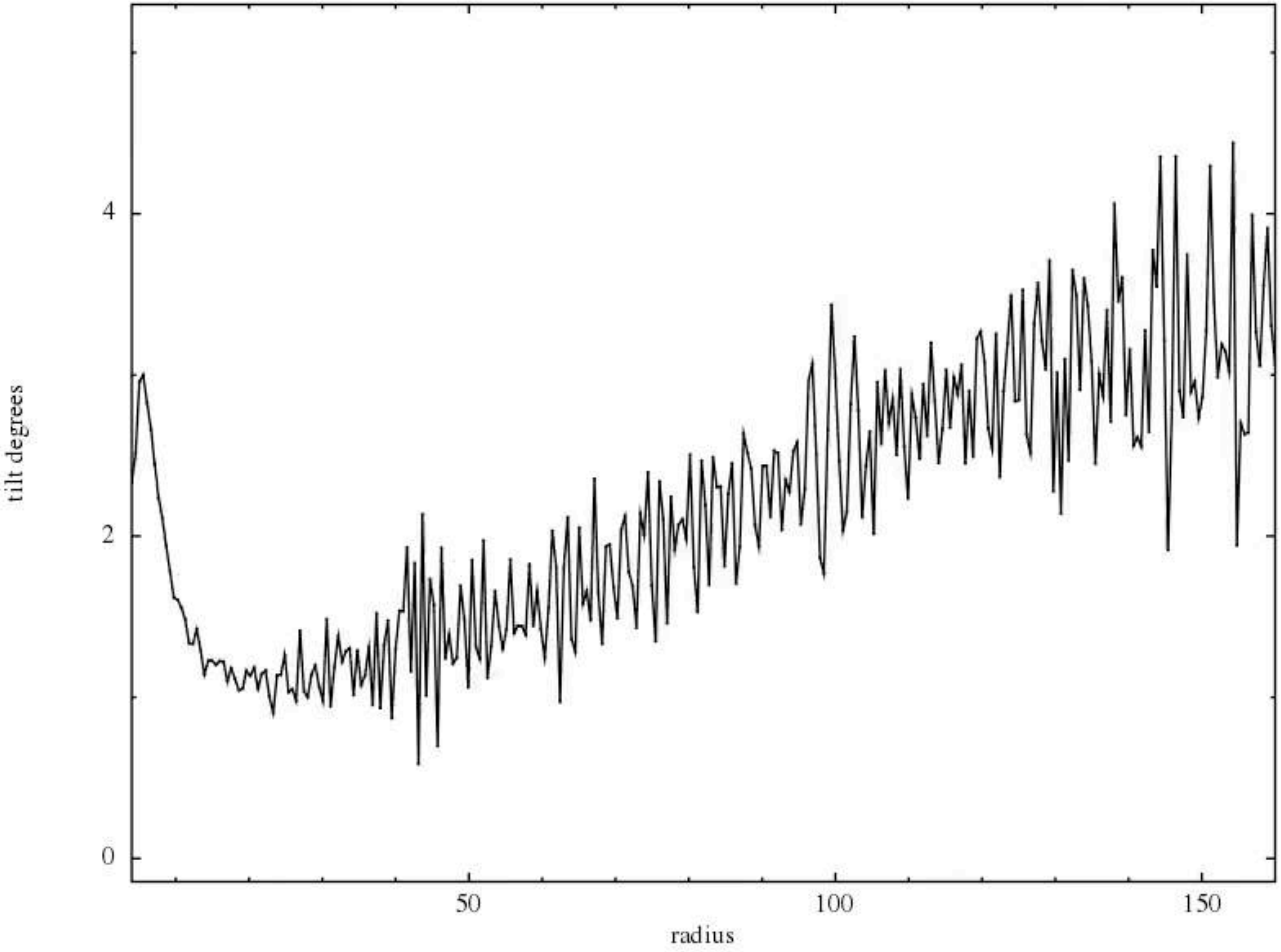
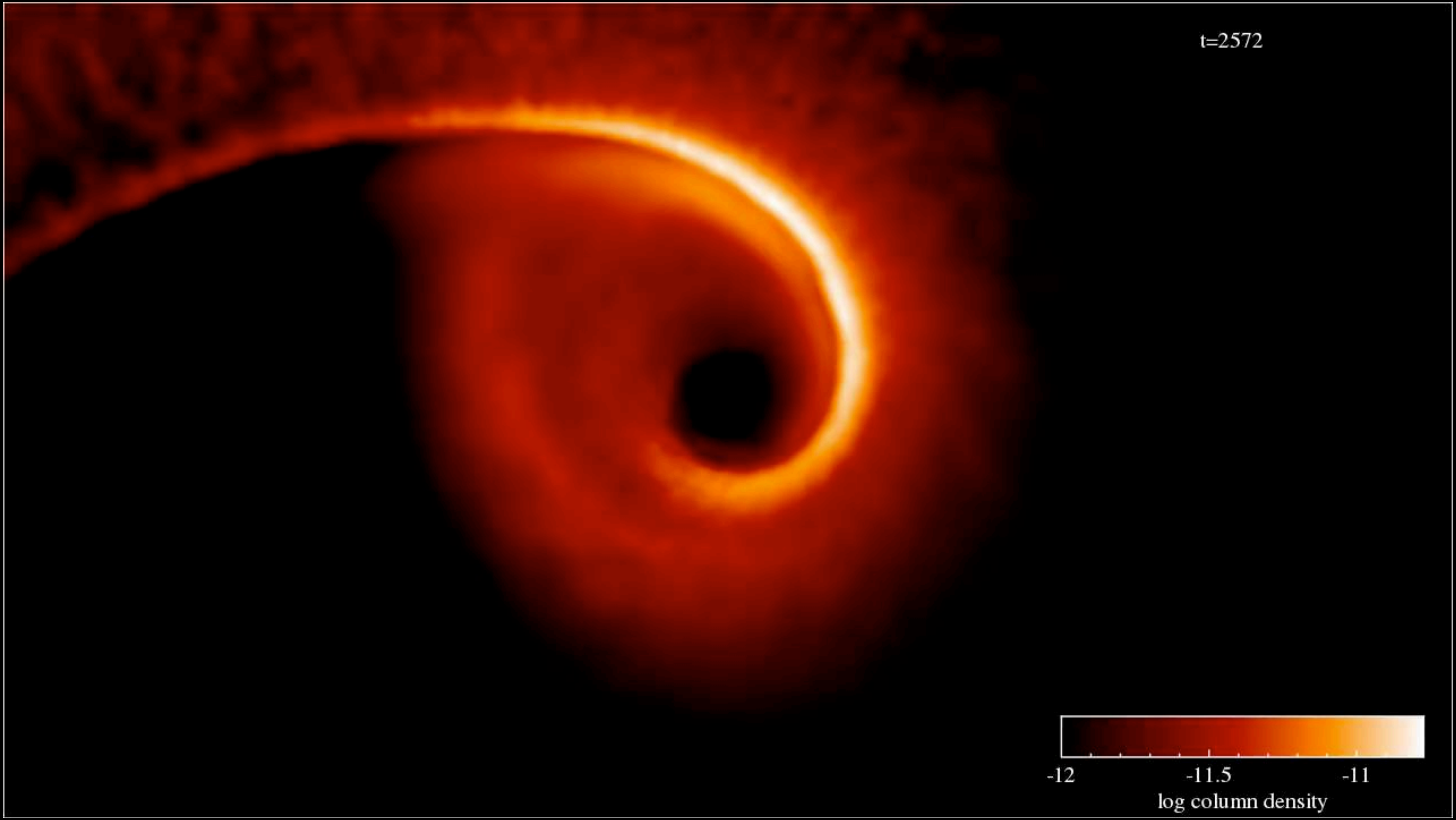
Preliminary black hole accretion disc



Preliminary black hole accretion disc — tilt



Preliminary TDE in GR



Conclusions

PHANTOMSPH

- **Orbital tests** (Schwarzschild AND Kerr) are in excellent agreement with theory

- We can handle **relativistic shocks** very well

- We have **split artificial dissipation** into viscosity and conductivity

- Merged with **PHANTOM** to do full **3D-GRSPH** simulations

Bonus slide: The Kerr metric… But which frame?

Boyer-Lindquist

- Far away observer
- Has event horizon singularity

$$
\begin{aligned}
ds^2 &= -\left[1 - \frac{2mr}{r^2 + a^2\cos^2\theta}\right] dt^2 - \frac{4mra\sin^2\theta}{r^2 + a^2\cos^2\theta}\, dt\, d\phi \\
&\quad + \left[\frac{r^2 + a^2\cos^2\theta}{r^2 - 2mr + a^2}\right] dr^2 + (r^2 + a^2\cos^2\theta)\, d\theta^2 \\
&\quad + \left[r^2 + a^2 + \frac{2mra^2\sin^2\theta}{r^2 + a^2\cos^2\theta}\right] \sin^2\theta\, d\phi^2.
\end{aligned}
$$

Kerr-Schild "Cartesian"

- In-falling observer
- No singularity at event horizon

$$
\begin{aligned}
ds^2 &= -dt^2 + dx^2 + dy^2 + dz^2 \\
&\quad + \frac{2mr^3}{r^4 + a^2z^2}\left[dt + \frac{r(x\, dx + y\, dy)}{a^2 + r^2} + \frac{a(y\, dx - x\, dy)}{a^2 + r^2} + \frac{z}{r}dz\right]^2
\end{aligned}
$$

Doran

- Can also "go through" the horizon
- Lapse = 1 everywhere

$$
\begin{aligned}
ds^2 &= -dt^2 + (r^2 + a^2\cos^2\theta)\, d\theta^2 + (r^2 + a^2)\sin^2\theta\, d\phi^2 \qquad (90) \\
&\quad + \left[\frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}\right] \left\{dr + \frac{\sqrt{2mr(r^2 + a^2)}}{r^2 + a^2\cos^2\theta}(dt - a\sin^2\theta\, d\phi)\right\}^2.
\end{aligned}
$$

Controlling artificial conductivity

$$\left(\frac{\mathrm{d}\mathbf{p}_a}{\mathrm{d}t}\right)_{\rm diss} \sim \sum_b \frac{m_b}{\bar{\rho}_{ab}} v_{\rm sig}\ \hat{\mathbf{r}}_{ab} \cdot (\mathbf{p}_a - \mathbf{p}_b)\ \overline{\nabla W}_{ab}$$

$$\left(\frac{\mathrm{d}e_a}{\mathrm{d}t}\right)_{\rm diss} \sim \sum_b \frac{m_b}{\bar{\rho}_{ab}} v_{\rm sig}\ (e_a - e_b)\ \hat{\mathbf{r}}_{ab} \cdot \overline{\nabla W}_{ab}$$

Non-relativistic

$$e = \frac{1}{2} v^2 + u$$

$$e_a - e_b = \underbrace{\frac{1}{2}\alpha_{\rm visc}\left(v_a^2 - v_b^2\right)}_{\text{Viscosity}} + \underbrace{\alpha_{\rm cond}\left(u_a - u_b\right)}_{\text{Conductivity}}$$

Relativistic

$$e = \frac{v^2}{\sqrt{1-v^2}}(1 + u + P/\rho) + \sqrt{1-v^2}(1+u)$$

$$e_a - e_b = \quad \ldots \quad ??? \quad \ldots$$

$$\boxed{\underbrace{\alpha_{\rm visc}\left[\varpi\left(\gamma_a v_a^2 - \gamma_b v_b^2\right) + \left(\frac{1}{\gamma_a} - \frac{1}{\gamma_b}\right)\right]}_{\text{Viscosity}} + \underbrace{\alpha_{\rm cond}\left[\frac{u_a}{\gamma_a} - \frac{u_b}{\gamma_b}\right]}_{\text{Conductivity}}}$$