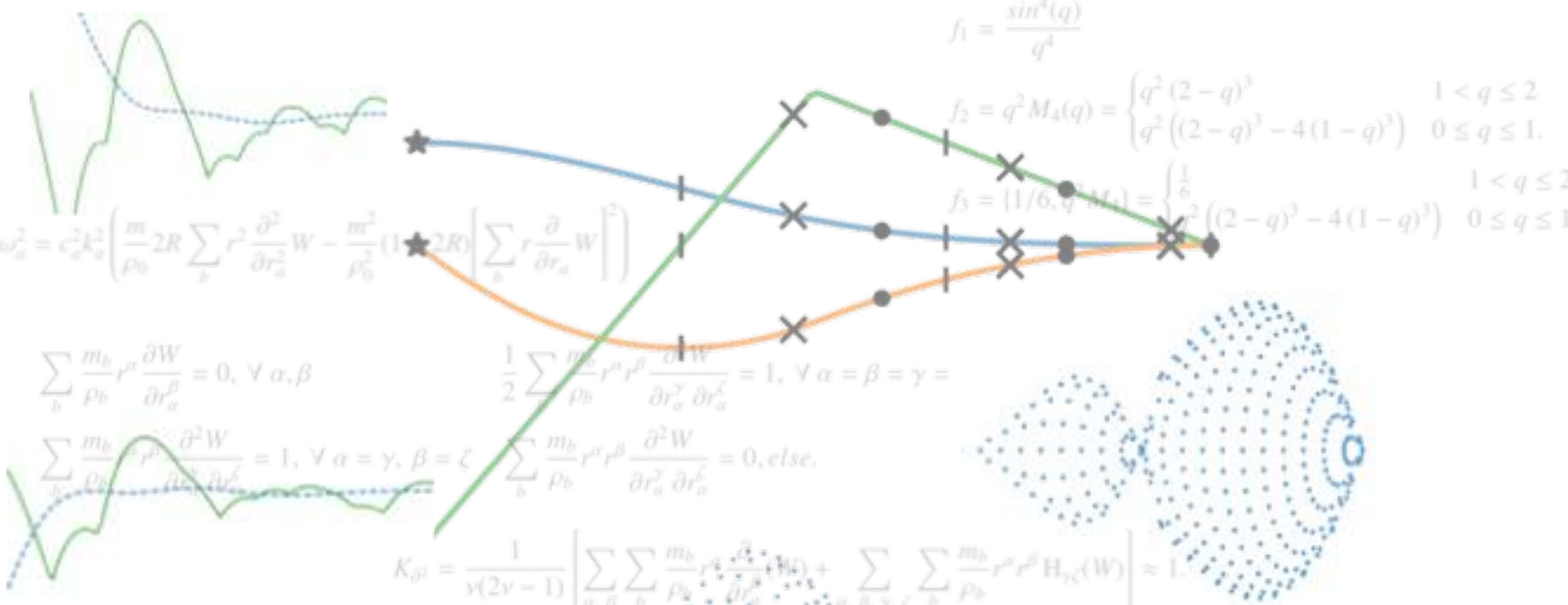
Kernels and Second Derivatives in SPH



Sergei Biriukov

Daniel Price

MONASH University

Diffusion

- Heat Conduction
- Radiative Transfer
- Gas-Dust Fraction
- Artificial Viscosity
- Magnetohydrodynamics

Isotropic Diffusion

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{1}{\rho}\nabla\cdot(\kappa\nabla T)$$

Analytical Equation

$$\frac{\mathrm{d}u_a}{\mathrm{d}t} = \sum_b \frac{m_b}{\rho_a\rho_b}\overline{\kappa}_{ab}(T_b - T_a)\nabla_a^2 W_{ab}$$

SPH Discretisation

Horribly Inaccurate (20% error)

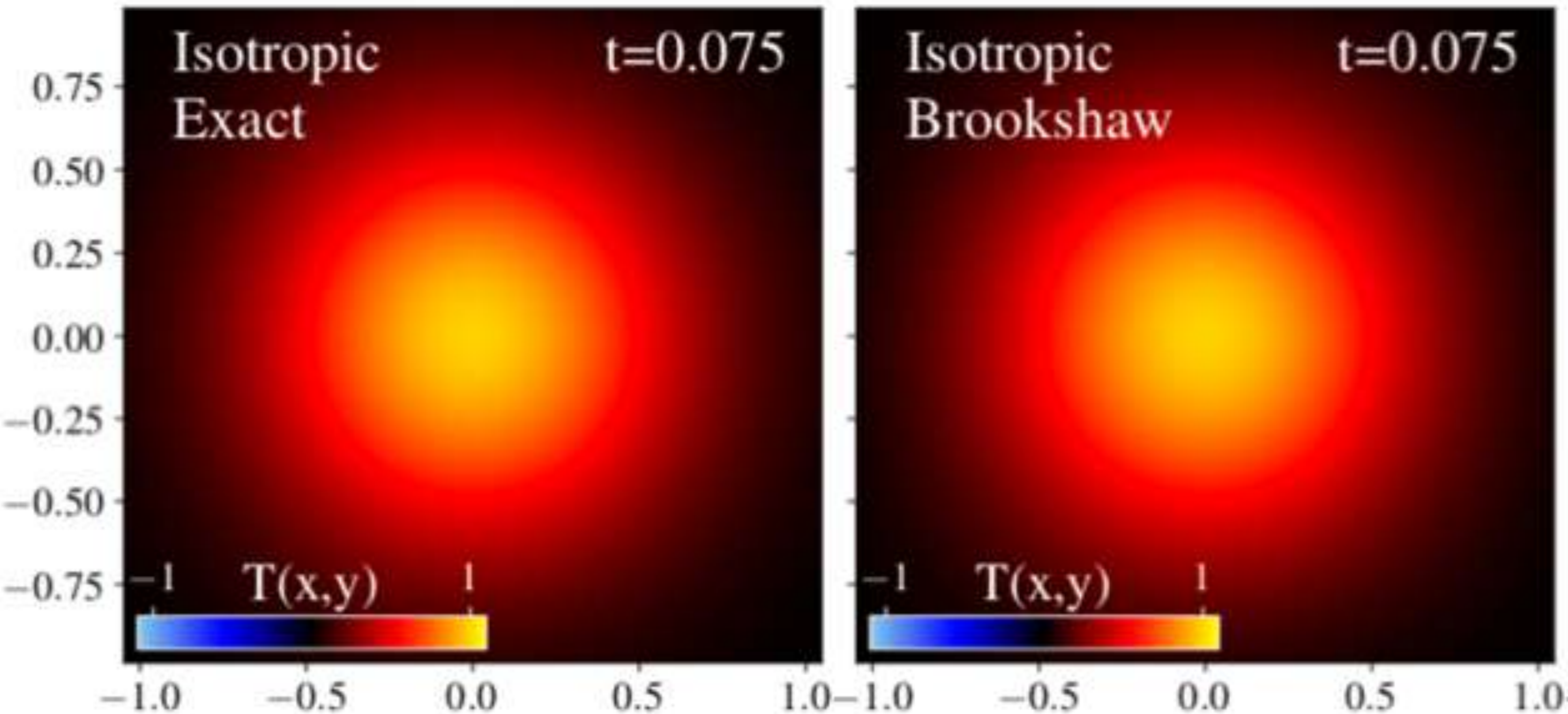
$$\frac{\mathrm{d}u_a}{\mathrm{d}t} = -2\sum_b \frac{m_b}{\rho_a\rho_b}\overline{\kappa}_{ab}(T_b - T_a)\frac{F_{ab}}{|r_{ab}|}$$

Magic

$$\nabla_a^2 W_{ab} \equiv \frac{-2F_{ab}}{|r_{ab}|} = \frac{-2f'(q_{ab})}{|r_{ab}|}$$

Brookshaw Method

Isotropic Diffusion



Brookshaw Method works just perfect

but why?!

Isotropic Diffusion

$$\nabla_a^2 W = -2\frac{(\mathbf{r}\cdot\nabla W)}{(\mathbf{r}\cdot\mathbf{r})} = \frac{-2}{C_\nu h^{\nu+2}}\frac{f'(q)}{q}$$

Brookshaw Method

+

$$f(q) = \exp\left(-q^2\right)$$

Use it with Gaussian kernel

$$= \frac{4}{h^2}\frac{\exp\left(-q^2\right)}{C_\nu\ h^\nu} = \frac{4W_f}{h^2}$$

Result is kernel itself

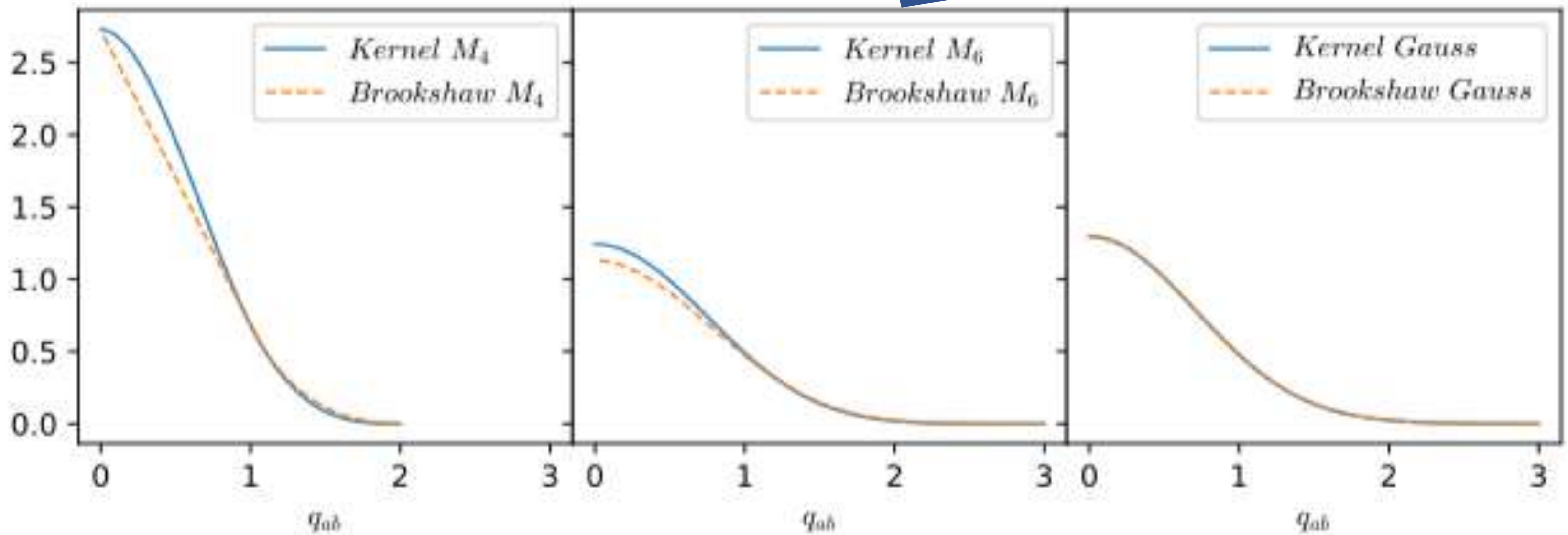
Isotropic diffusion can be estimated directly with kernel itself

Isotropic Diffusion

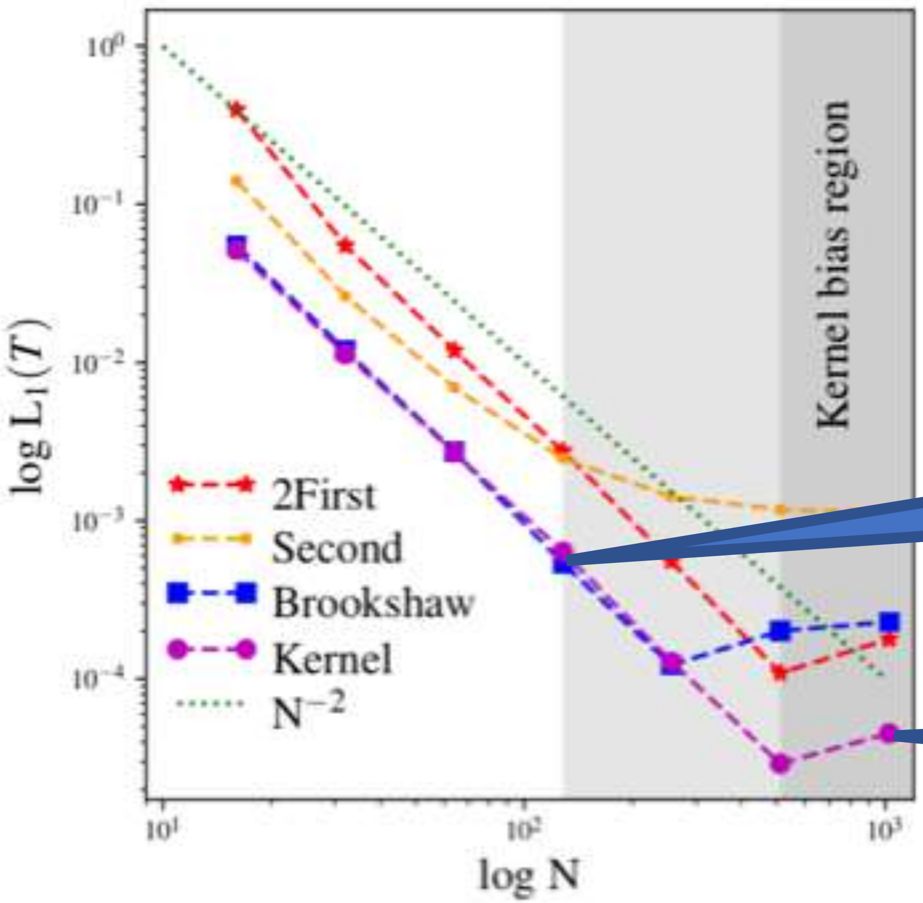
$$\nabla_a^2 W_{ab} \equiv \frac{4C_\nu^*}{h_a^2} W_{ab} = \frac{4C_\nu^*}{C_\nu \ h_a^{\nu+2}} f(q)$$

Any non-Gaussian kernels have another normalisations

Still same shapes for different kernels



Isotropic Diffusion



Convergence for isotropic heat conduction with M_6 and various methods

Same accuracy for low resolution as Brookshaw method

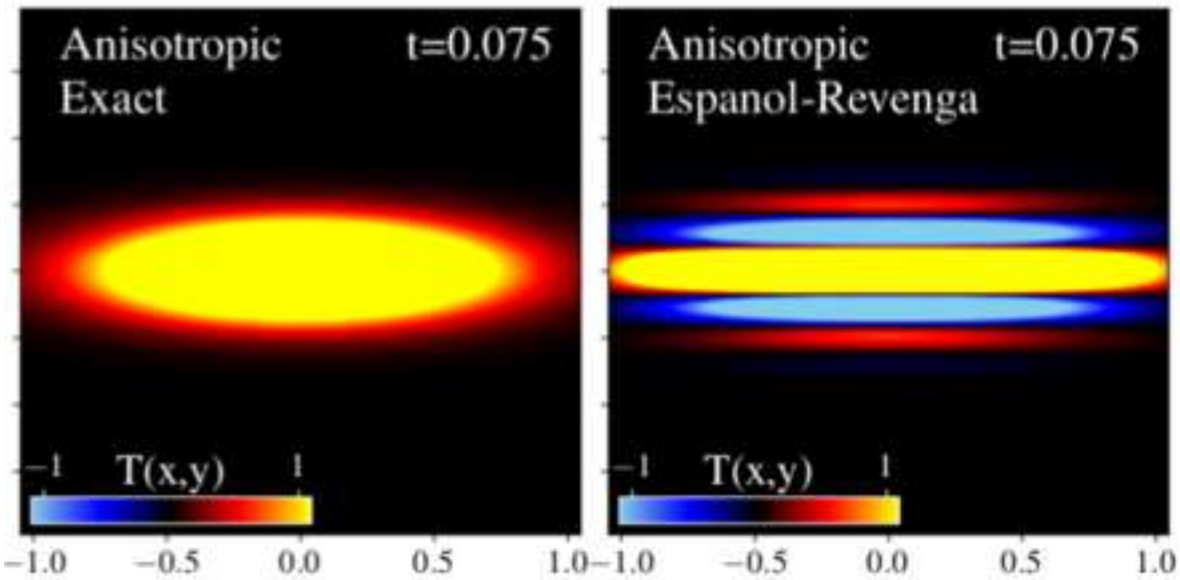
Twice better result for high resolution

Anisotropic Diffusion

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{1}{\rho}\nabla \cdot (\boldsymbol{\kappa}\nabla T)$$

Analytical Equation

$$\frac{\mathrm{d}u_a}{\mathrm{d}t} = -\sum_b \frac{m_b}{\rho_a \rho_b} T_{ba}\overline{\kappa}_{ij}^{ab}[(\nu+2)\hat{r}_{ab}^{i}\hat{r}_{ab}^{j} - \delta^{\alpha\beta}]\frac{F_{ab}}{|r_{ab}|}$$



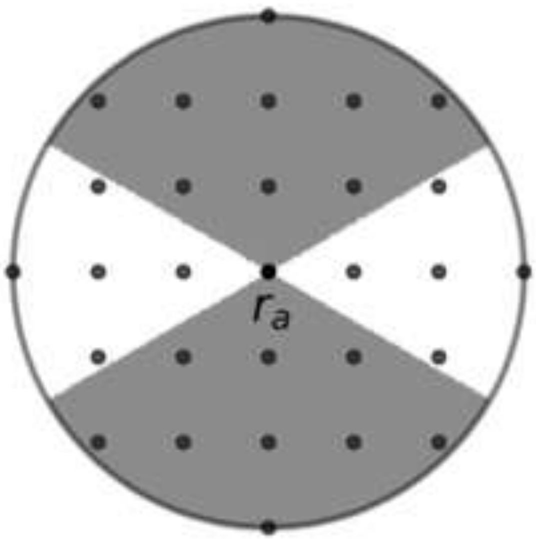
Espanol-Revenga Method: Brookshaw Method generalisation for anisotropic case

Unstable

Anisotropic Diffusion

Stability condition in 2D

$$-\kappa_{xx}(4\hat{r}^x_{ab}\hat{r}^x_{ab} - 1) - \kappa_{yy}(4\hat{r}^y_{ab}\hat{r}^y_{ab} - 1) \geq 0$$



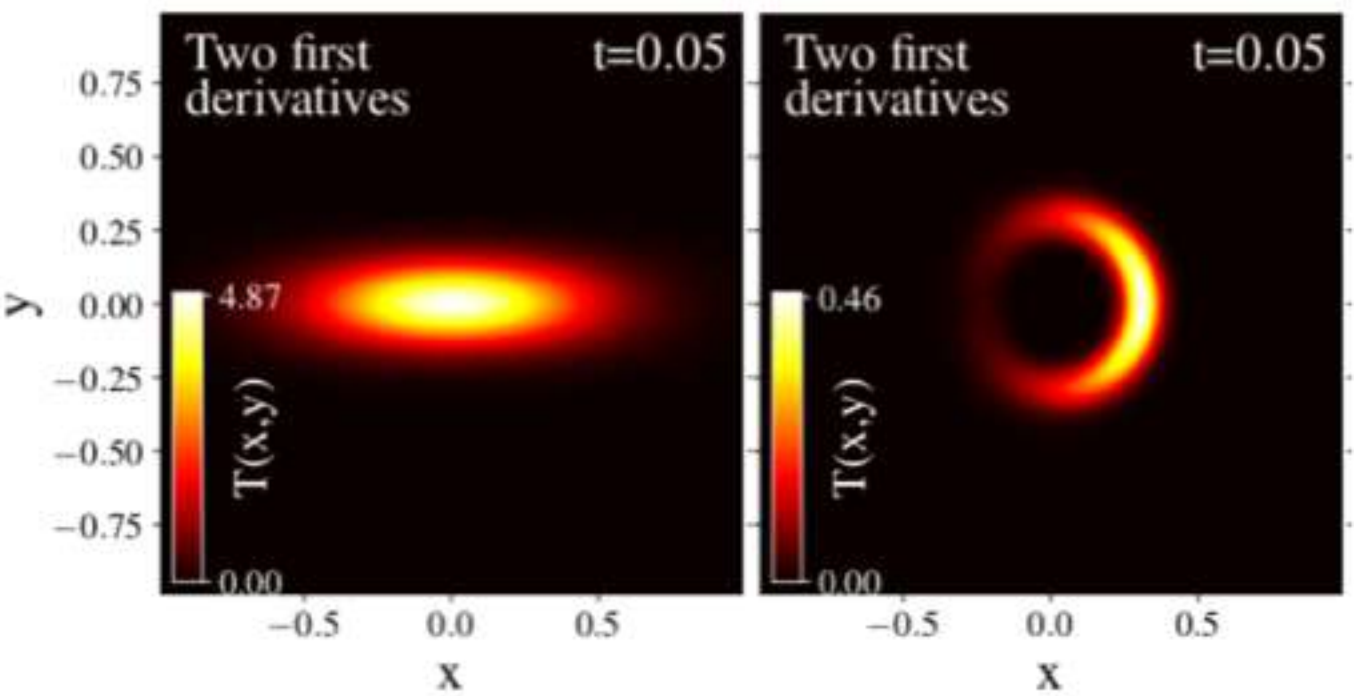
M_6 kernel with particles placed on uniform lattice in 2D.

Grey domain is the region where the inequality doesn't work for $\kappa_{xx} = 1, \kappa_{yy} = 0$

Anisotropic Diffusion

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \frac{1}{\rho}\nabla^i(\kappa_{ij}F^j)$$
$$F^j = \nabla^j T$$

The only stable way to the moment is the two first derivatives

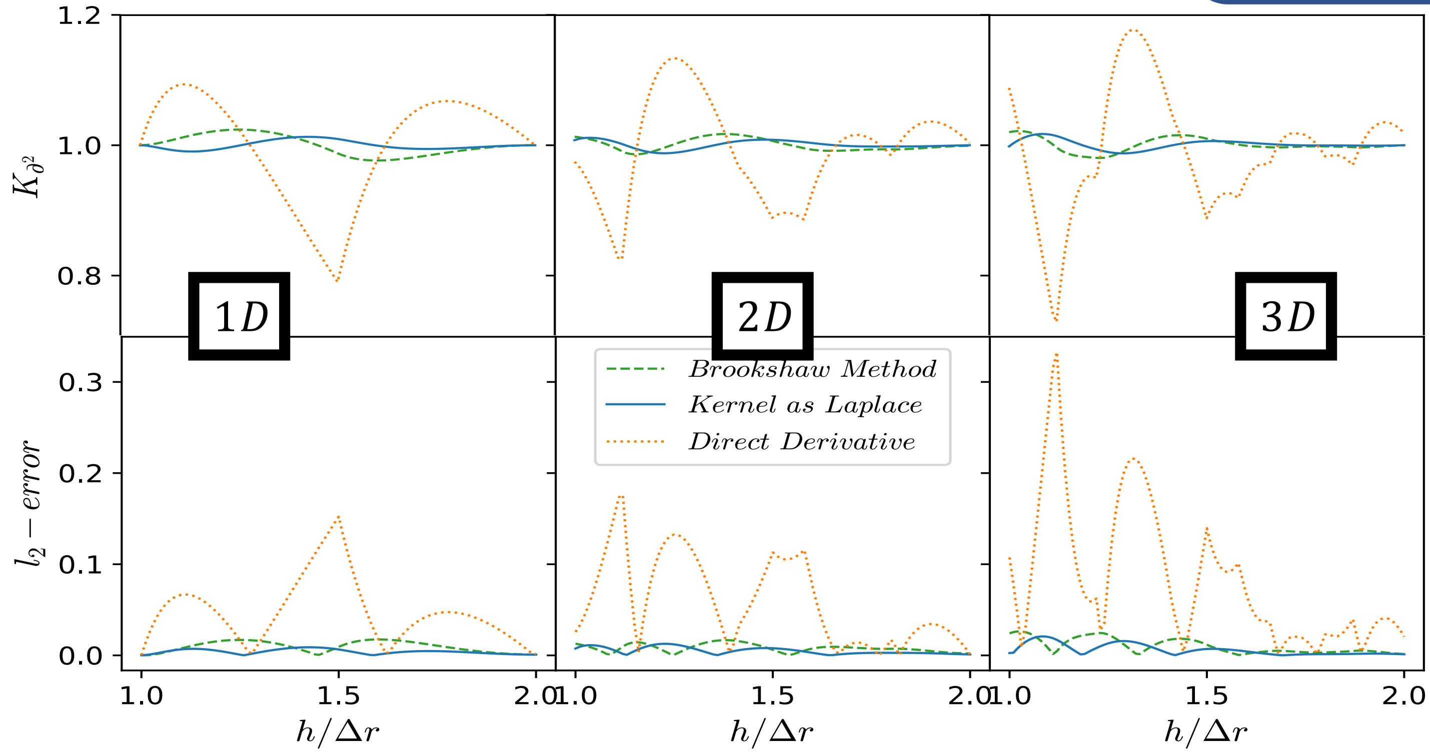


Kernels

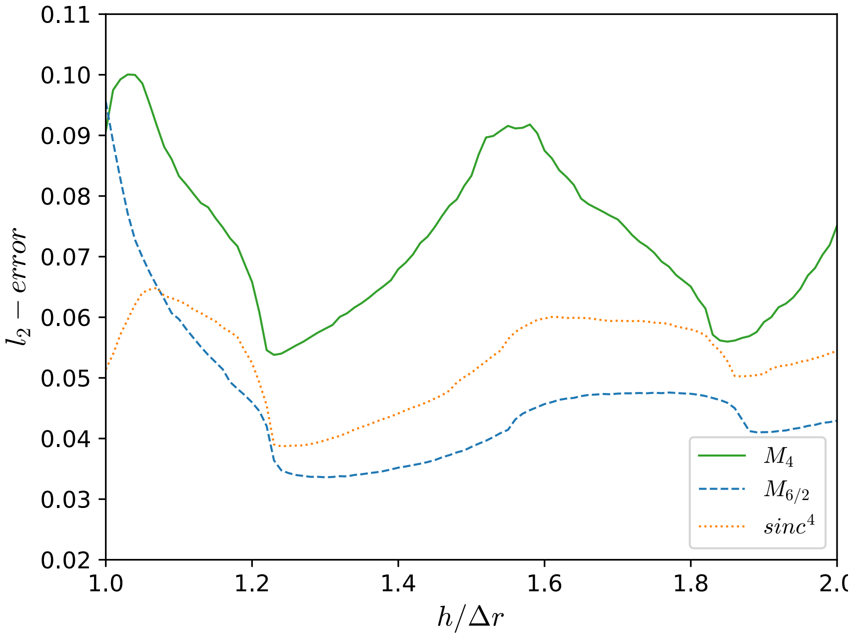
$$K_{\partial^2} \equiv \frac{1}{\nu}\left[\sum_{\alpha}\sum_{b}\frac{m_b}{\rho_b} r_{ba}^{\alpha}\nabla_a^2 W_{ab} + \frac{1}{2}\sum_{\alpha}\sum_{\beta}\sum_{b}\frac{m_b}{\rho_b} r_{ba}^{\alpha} r_{ba}^{\beta}\nabla_a^2 W_{ab}\right]$$

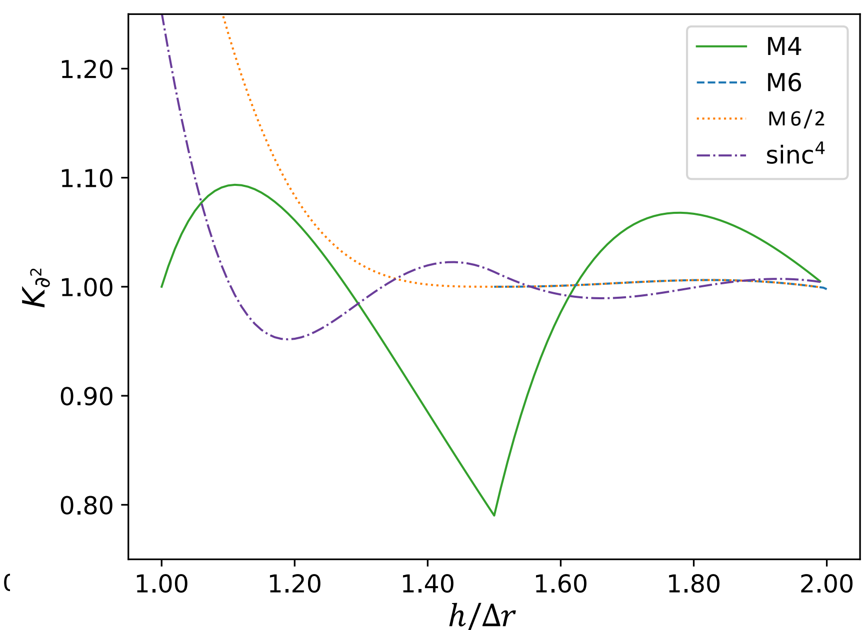
2^nd Derivative Normalisation Condition

The closer to “1”, the better Laplacian. Same as l_2 error.



Kernels





l_2-error in sod shock problem for various kernels with respect to h_{fac}

2nd Derivative Normalisation Condition for various kernels with respect to h_{fac}

Summary

1. The kernel itself instead of the Brookshaw method gives up to twice more accurate results
2. Two first derivatives is the only stable way for anisotropic diffusion
3. 2nd derivative matters even if it is not explicitly presented in problem
4. M_6 with $h_{fac} = 0.9$ is twice better then M_4 with $h_{fac} = 1.35$ with the same cost.