Proof of Admissibility of the Heuristic

To prove the admissibility of the heuristic used in the A* algorithm, we must show that the heuristic function $h(n)$ (the estimated cost to reach the goal from a node n) never **overestimates** the true cost $h^*(n)$, i.e.,

$$h(n) \leq h^*(n) \quad \forall n$$

where:

- $h(n)$: heuristic value computed by the algorithm.
- $h^*(n)$: true cost of the optimal path from node n to the goal.

The heuristic in the given implementation can be summarised as follows:

1. **Straight-Line Distance**: If the straight line from the current node n to the goal node does not intersect any no-fly zone, $h(n)$ is simply the Euclidean distance between n and the goal g.
2. **No-Fly Zone Adjustments**: If the straight line intersects one or more no-fly zones, $h(n)$ is calculated as:

$$h(n) = \min_{p \in \text{vertices of no-fly zone}} (\text{distanceTo}(n, p) + \text{distanceTo}(p, g))$$

Case 1: Straight-Line Distance

If the straight line from n to g does not intersect any no-fly zone, the heuristic is defined as:

$$h(n) = \text{distanceTo}(n, g)$$

The straight-line distance is always the **shortest possible path** in the absence of obstacles. Since $h^*(n)$ (the actual cost of the optimal path) cannot be smaller than this straight-line distance, we have:

$$h(n) = \text{distanceTo}(n, g) \leq h^*(n)$$

Thus, $h(n)$ is admissible in this case.

Case 2: Intersection with No-Fly Zones

If the straight line from n to g intersects a no-fly zone, the heuristic is defined as:

$$h(n) = \min_{p \in \text{vertices of no-fly zone}} (\text{distanceTo}(n, p) + \text{distanceTo}(p, g))$$

Admissibility Property: The path from n to g must avoid the no-fly zone. The true cost $h^*(n)$ must include the additional cost of avoiding the no-fly zone by travelling to a point outside it (e.g., p). Since $h(n)$ calculates the cost of travelling through such a point p, it cannot overestimate the true cost. Thus:

$$h(n) \leq h^*(n)$$

Triangle Inequality: The path $n \to p \to g$ satisfies the triangle inequality:

$$\text{distanceTo}(n, g) \leq \text{distanceTo}(n, p) + \text{distanceTo}(p, g)$$

Hence, $h(n)$, which uses the path $n \to p \to g$, is a conservative estimate of the true cost $h^*(n)$.

General Case

For any node n, $h(n)$ is:

- The straight-line distance to the goal g if the line does not intersect any no-fly zone.
- The shortest possible detour around a no-fly zone if an intersection exists.

In both cases, $h(n)$ does not overestimate the true cost $h^*(n)$, satisfying the condition for admissibility.

Conclusion: Admissibility of the Heuristic

Since $h(n) \leq h^*(n)$ for all n, the heuristic is admissible.