Mathematical Methods

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Contents

1 Second Order Linear Differential Equations

1.1 Introduction

In this chapter we aim to find solutions to 2nd order linear differential equations.

Definition 1.1. A general **second order ODE** has the form

$$y'' = f(t, y, y')$$

where y is the unknown function and $f : \mathbb{R}^3 \mapsto \mathbb{R}$ is given.

The **order** of the differential equation is the highest derivative appearing in the ODE. The equation is **linear** if the function f is linear in both arguments y and y' i.e.,

$$f(t, y, y') = a(t)y + b(t)y' + c(t).$$

Definition 1.2. A second order **linear** differential equation in the unknown y is

$$a(t)y'' + b(t)y' + c(t) = h(t) \tag{1.1}$$

where $a, b, c, h : I \mapsto \mathbb{R}$ are given functions on the interval $I \subseteq \mathbb{R}$.

The equation is called **homogeneous** if the source term $h(t) = 0$ for all $t \in I$. The equation is called **constant coefficient** if $a(t), b(t)$ and $c(t)$ are constants otherwise the equation is called **variable coefficient**.

Example 1.1.

(a) A second order linear homogeneous constant coefficient equation is

$$y'' + 5y' + 6y = 0.$$

(b) A second order linear inhomogeneous constant coefficient equation is

$$y'' - 3y' + y = \cos(3t).$$

(c) A second order linear inhomogeneous variable coefficient equation is

$$y'' - 2ty' - \ln(t)y = e^{3t}.$$

(d) Newton's second law of motion for a point particle of mass m, moving in one space dimensions under a force f is given by

$$my'' = f(t)$$

(force = mass × acceleration).

1.2 Initial Value Problems

Theorem 1.3 (Existence and Uniqueness)**.** *If the functions $p(t), q(t), f(t) : I \mapsto \mathbb{R}$ are continuous on a closed interval $I \subseteq \mathbb{R}$ and $t_0 \in I$ and $b_0, b_1 \in \mathbb{R}$ are constants, then there* ***exists*** *a* ***unique*** *solution $y : I \mapsto \mathbb{R}$ to the* ***initial value problem*** *(IVP)*

$$\left.\begin{aligned} y'' + p(t)y' + q(t)y &= f(t) \\ y(t_0) = b_0, \quad y'(t_0) &= b_1 \end{aligned}\right\} \tag{1.2}$$

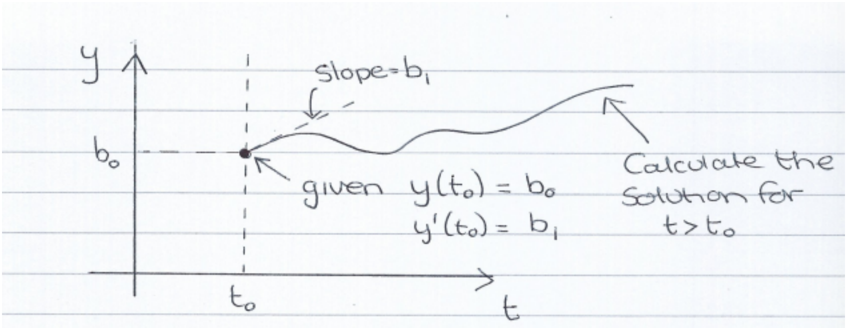

Figure 1: An initial value problem.

Proof. Omitted. It is actually an application of the contraction mapping theorem from metric spaces. $\square$

Example 1.2. Find the largest interval $I \in \mathbb{R}$ such that there exists a unique solution to the IVP:

$$(t-1)y'' - 3ty' + 4y = t(t-1)$$
$$y(-2) = 2, \quad y'(-2) = 1$$

Solution. Rewriting the ODE in the form of Theorem 1.3 we have

$$y'' - \frac{3t}{t-1}y' + \frac{4}{t-1}y = t.$$

The intervals where $\frac{3t}{t-1}, \frac{4}{t-1}$ and t are all continuous are

$$I_1 = (-\infty, 1) \quad \text{and} \quad I_2 = (1, \infty).$$

Since the initial condition lies in the interval I_1 the solution domain is

$$I_1 = (-\infty, 1).$$

We will look at an example of a homogeneous equation with constant coefficients

Example 1.3. Solve $y'' + 5y' + 6y = 0$ with $y(0) = 2$ and $y'(0) = 3$.

Solution. We try a solution of the form $y(t) = \exp(\lambda t)$ to derive the auxiliary equation

$$\lambda^2 + 5\lambda + 6 = 9$$

which factors to give $(\lambda + 2)(\lambda + 3) = 0$. Hence $\lambda = -2$ or $\lambda = -3$.
The general solution is then

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}, \qquad c_1, c_2 \in \mathbb{R}.$$

Applying the initial conditions we find that

$$\left.\begin{aligned} y(0) &= 2 = c_1 + c_2 \\ y'(0) &= 3 = 2c_1 - 3c_2 \end{aligned}\right\}$$

Hence $c_1 = 9$ and $c_2 = -7$ so the unique solution to the IVP is

$$y(t) = 9e^{-2t} - 7e^{-3t}.$$

This leaves us with a few questions

(i) Why did we add the two solutions together to get the general one?

(ii) Why is this the general solution?

(iii) What would the solution to the more general ODE (1.2) look like?

To answer these questions it is useful to introduce the definition of an operator, we will do this in the next section.

1.3 Linear Operators

A function $f : \mathbb{R} \mapsto \mathbb{R}$ takes a number as an input and produces another number as an output

$$x \xrightarrow{f} f(x)$$

i.e.,

$$\text{number} \xrightarrow{f} \text{another number.}$$

A simple example is $f(x) = x^2$ which maps

$$x \xrightarrow{f} x^2.$$

One can thing of differentiating in a similar way but with the inputs and outputs as functions rather than numbers, i.e.,

$$\text{function} \xrightarrow{\text{operator}} \text{another function.}$$

For example,

$$y(t) \xrightarrow{L} y'' + y$$

here $L = \frac{d^2}{dt^2} + 1$. L is called an **operator** or **functional**.

Example 1.4. Let $L = \frac{d}{dx} + x^2$, then L sends the function $y(x)$ to the function $\frac{dy}{dx} + x^2 y$. We use square brackets $L[y]$ to denote the output of the operator so

$$L[y] = \frac{dy}{dx} + x^2 y.$$

The new function $L[y]$ can be evaluated at a number x, this is denoted by

$$L[y(x)] \quad \text{or} \quad L[y](x).$$

So for example

$$\begin{aligned} L\left[\sin(x)\right] &= \frac{d}{dx}\left(\sin x\right) + x^2 \sin x \\ &= \cos x + x^2 \sin x. \end{aligned}$$

Let us consider the operator

$$L[y] = y'' + p(t)y' + q(t)y$$

then equation (1.2) can be written as $L[y] = f$.

A particularly important type of operators are linear operators.

Definition 1.4. An operator L is called a **linear operator** if for every pair of function y_1 and y_2 and constants c_1, c_2 we have that

$$L\left[c_1 y_1 + c_2 y_2\right] = c_1 L[y_1] + c_2 L[y_2].$$

Example 1.5. Show that the operator $L = \frac{d^2}{dx^2} + p(t)\frac{d}{dt} + q(t)$ is linear.

Solution.

$$\begin{aligned} L[c_1y_1 + c_2y_2] &= \big(c_1y_1 + c_2y_2\big)'' + p(t)\big(c_1y_1 + c_2y_2\big)' + q(t)\big(c_1y_1 + c_2y_2\big) \\ &= \big(c_1y_1'' + p(t)c_1y_1' + q(t)c_1y_1\big) + \big(c_2y_2'' + p(t)c_2y_2' + q(t)c_2y_2\big) \\ &= c_1\big(y_1'' + p(t)y_1' + q(t)y_1\big) + c_2\big(y_2'' + p(t)y_2' + q(t)y_2\big) \\ &= c_1L[y_1] + c_2L[y_2] \end{aligned}$$

hence L is a linear operator.

Example 1.6. Show that the operator $L[y] = \left(\frac{dy}{dx}\right)^2$ is **not** linear.

Solution.

$$\begin{aligned} L[y_1 + y_2] = \left(\frac{d(y_1 + y_2)}{dx}\right)^2 &= \left(\frac{dy_1}{dx} + \frac{dy_2}{dx}\right)^2 \\ &= \left(\frac{dy_1}{dx}\right)^2 + 2\frac{dy_1}{dx}\frac{dy_2}{dx} + \left(\frac{dy_2}{dx}\right)^2 \\ &= L[y_1] + L[y_2] + 2\sqrt{L[y_1]L[y_2]} \\ &\neq L[y_1] + L[y_2] \end{aligned}$$

so L is not a linear operator.

The linearity of an operator L translates to into the superposition property (adding solutions) of the solutions to the homogeneous equation $L[y] = 0$.

Theorem 1.5 (The Principle of Superposition)**.** *If L is a linear operator and y_1, y_2 are solutions of the homogenenous equations*

$$L[y_1] = 0, \quad L[y_2] = 0,$$

then for every constant c_1, c_2 we have

$$L[c_1y_1 + c_2y_2] = 0.$$

Proof. We verify that $y = c_1y_1 + c_2y_2$ satisfies $L[y] = 0$ for every constant c_1, c_2. We have that

$$\begin{aligned} L[y] &= L[c_1y_1 + c_2y_2] \\ &= c_1L[y_1] + c_2L[y_2] \\ &= c_1 \cdot 0 + c_2 \cdot 0 \\ &= 0 \end{aligned}$$

since L is a linear operator. $\square$

1.4 Linear Dependence of Functions and Wronskians

We are going to digress away from solutions to ODEs in this section and think about linear dependance of functions.

Definition 1.6. Two continuous functions $y_1, y_2 : I \mapsto \mathbb{R}$ are called **linearly dependent** on the interval $I \subseteq \mathbb{R}$, if there exists constants $c_1, c_2 \in \mathbb{R}$ not both zero such that for all $t \in I$ we have

$$c_1y_1(t) + c_2y_2(t) = 0.$$

On the contrary, y_1, y_2 are **linearly independent** on the interval I if they are **not** linearly dependent, that is, the only constants c_1, c_2 that for all $t \in I$ satisfy $c_1y_1(t) + c_2y_2(t) = 0$ are the constants $c_1 = c_2 = 0$.

Example 1.7. Which of the following pairs of functions are linearly independent?

(a) $\sin t$ and $\cos(t - \pi/2)$,

(b) e^t and e^{2t}.

Solution. (a) Consider $c_1 \sin(t) + c_2 \cos(t - \pi/2) = 0, \quad c_1, c_2 \in \mathbb{R}$. Now by looking at the triangle in Figure 2 we can see that $\sin(t) = \cos(t - \pi/2)$.

Therefore by choosing $c_1 = -c_2$ we have

$$c_1 \sin(t) + c_2 \cos(t - \pi/2) = 0.$$

Thus $c_1, c_2 \neq 0$ and the two function are linearly dependent.

(b) Consider $c_1e^t + c_2e^{2t} = 0, \quad c_1, c_2 \in \mathbb{R}$. Differentiating gives $c_1e^t + 2c_2e^{2t} = 0$. Subtracting one equation from the other gives $c_2e^{2t} = 0$ so $c_2 = 0$. Thus $c_1 = 0$ and therefore the functions are linearly independent.

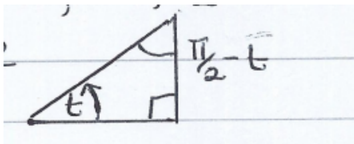

Figure 2: Angles in a right angled triangle

Example 1.8. If either function y_1 or y_2 is the zero function then y_1, y_2 are linearly independent. We can show this as follows, let $y_1(t) = 0$ for instance then

$$c_1y_1(t) + 0 \cdot y_2(t) = c_1 \cdot 0 = 0.$$

This is true for any value of c_2 therefore there exists c_1, c_2 not both zero such that $c_1y_1 + c_2y_2 = 0$.

Definition 1.7 (The Wronskian function)**.** The **Wronskian** of the differentiable functions y_1, y_2 is the function

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1(t)y_2'(t) - y_1'(t)y_2(t).$$

Example 1.9. Find the Wronskian of the functions

(a) $y_1(t) = \sin(t)$ and $y_2(t) = 2\sin(t)$,

(b) $y_2(t) = \sin(t)$ and $y_2(t) = t\sin(t)$.

Solution.

(a)

$$W(y_1, y_2) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} \sin(t) & 2\sin(t) \\ \cos(t) & 2\cos(t) \end{vmatrix}$$
$$= \sin(t)2\cos(t) - \cos(t)2\sin(t) = 0$$

(Note that y_1, y_2 are linearly dependent.)

(b)

$$W(y_1, y_2) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} \sin(t) & t\sin(t) \\ \cos(t) & \sin(t) + t\cos(t) \end{vmatrix} = \sin^2(t)$$

(Note that y_1, y_2 are linearly independent.)

Theorem 1.8. *If the Wronskian $W(y_1(t_0), y_2(t_0)) \neq 0$ at a single point $t_0 \in I$ then the functions $y_1, y_2 : I \mapsto \mathbb{R}$ are linearly independent.*

Proof. Assume y_1, y_2 are linearly dependent then $\exists c \neq 0$ such that $y_1 = cy_2$, so

$$W(y_1, y_2) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = (cy_2(t))y_2'(t) - cy_2'(t)y_2(t) = 0 \quad \forall t \in I.$$

This is a contradiction as there exists a $t_0 \in I$ such that $W(y_1, y_2) \neq 0$ so y_1, y_2 are linearly independent. $\square$

In the steps of the previous proof we showed that if f and g are linearly dependent on I then $W(f, g) = 0$ for all $t \in I$. But this is a one way implication so we can have $W(f, g) = 0$ for all $t \in I$ and f, g linearly independent.

Example 1.10. Let $f(t) = t^2|t|$, $g(t) = t^3$. Show that f, g are linearly independent on $I = (-1, 1)$ but $W(f, g) = 0$, $\forall t \in I$.

Solution.

$$f'(t) = \begin{cases} 2t^2 + t^2, & t \geq 0 \\ -2t^2 - t^2, & t < 0 \end{cases} \quad \text{and} \quad g'(t) = 3t^2.$$

Thus for $c_1, c_2 \in \mathbb{R}$ if $c_1 f + c_2 g = 0$, i.e., $c_1 t^2|t| + c^2 t^3 = 0$ holds for all t then at $t = 1/2$ say

$$\frac{c_1}{8} = -\frac{c_2}{8} \implies c_1 = -c_2,$$

and at $t = -1/2$ say

$$\frac{c_1}{8} = \frac{c_2}{8} \implies c_1 = c_2$$

hence $c_1 = c_2 = 0$ and f, g are linearly independent.
Now calculating $W(f, g)$ gives

$$W(f, g) = \begin{cases} t^3 \cdot 3t^2 - 3t^2 \cdot t^3 = 0, & t \geq 0 \\ -t^3 \cdot 3t^2 + 3t^2 \cdot t^3 = 0, & t < 0 \end{cases}$$
$$= 0 \quad \forall t \in I.$$

Here is a summary of the implications here:

(i) If f, g are linearly dependent $\implies W(f, g) = 0$ for all t,

(ii) If $W(f, g) = 0 \; \forall t \;\not\Longrightarrow f, g$ are linearly dependent,

(iii) Contrapositive of (i). If $W(f,g) \neq 0$ for any $t_0 \in I$ then f,g are linearly independent.

1.5 Some Theorems About Solutions

Theorem 1.9 (Abel's Theorem)**.** *If y_1, y_2 are twice continuously differentiable solutions of*

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

where $p(t), q(t)$ are continuous on $I \subseteq \mathbb{R}$ then the Wronskian $W(y_1, y_2)$ satisfies

$$\frac{dW}{dt} + p(t)W = 0. \tag{1.3}$$

Moreover, for any $t_0 \in I$, the Wronskian $W(y_1, y_2)$ is given by

$$W(y_1, y_2) = W_0 \exp\left(-\int_{t_0}^{t} p(s)ds\right) \tag{Abel's Formula}$$

where $W_0 = W(y_1(t_0), y_2(t_0))$.

Proof. Firstly,

$$\frac{dW}{dt} = \Big(y_1y_2' - y_1'y_2\Big)' = y_1y_2'' - y_1''y_2.$$

Since y_1 and y_2 are solutions, $L[y] = 0$ gives

$$y_1'' = -p(t)y_1' - q(t)y_1$$

and similarly

$$y_2'' = -p(t)y_2' - q(t)y_2.$$

Substitute these in the expression for $\frac{dW}{dt}$ to get

$$\begin{aligned}\frac{dW}{dt} &= y_1\Big(-py_2' - qy_2\Big) - y_2\Big(-py_1' - qy_1\Big)\\ &= -p(y_1y_2' - y_1'y_2)\\ &= -pW.\end{aligned}$$

We can solve this equation by separating the variables.

$$\begin{aligned}\int_{t_0}^{t} \frac{1}{W}dW &= -\int_{t_0}^{t} p(s)ds\\ \ln(W(t)) - \ln(W(t_0)) &= -\int_{t_0}^{t} p(s)ds\\ W(y_1, y_2) &= W_0 \exp\left(-\int_{t_0}^{t} p(s)ds\right)\end{aligned}$$

where $W_0 = W(y_1(t_0), y_2(t_0))$. ☐

Theorem 1.10. *Let $y_1, y_2 : I \mapsto \mathbb{R}$ both be solutions of $L[y] = 0$ on I. If there exists a $t_0 \in I$ such that $W(y_1(t_0), y_2(t_0)) = 0$ then y_1 and y_2 are linearly dependent.*

Proof. Abel's Formula says that if the Wronskian $W(y_1(t_0), y_2(t_0)) = 0$ then $W(y_1, y_2) = 0$, $\forall t \in I$.

If either y_1 or y_2 is the the zero function then y_1, y_2 are linearly dependent. So we will assume that both y_1, y_2 are are **not** identically zero. Assume there exists a $t_1 \in I$ such that $y_1(t_1) \neq 0$. By continuity y_1 is non-zero in an open neighbourhood $I_1 \subset I$ of t_1 so

$$\frac{W(y_1, y_2)}{y_1^2} = \frac{y_1y_2' - y_1'y_2}{y_1^2} = 0.$$

Hence

$$\left(\frac{y_2}{y_1}\right)' = 0 \implies \frac{y_2}{y_1} = c$$

on I_1, where $c \in \mathbb{R}$ is an arbitrary constant. Therefore $y_2 = cy_1$, i.e. y_2 is proportional to y_1 on I_1.

The zero function also satisfies $L[y] = 0, y(t_1) = 0, y'(t_1) = 0$. By the existence and uniqueness theorem (Theorem 1.3) we must have that $y(t)$ is the zero function on I. So y_1 is proportional to y_2 on I and y_1, y_2 are linearly dependent as required. ☐

Corollary 1.11 (Contrapositive of above theorem). *Let $y_1, y_2 : I \mapsto \mathbb{R}$ both be solutions of $L[y] = 0$ on I. If y_1, y_2 are linearly independent on I then their Wronskian $W(y_1, y_2) \neq 0$ for all $t \in I$.*

We comment that in an earlier example, Example 1.10, with 2 linear independent functions which had a zero Wronskian. This corollary does not contradict that example. This just tells us that the 2 functions f, g in the example are not both solutions to some ODE, $y'' + p(t)y' + q(t)y = 0$, with p, q continuous on I.

1.6 General Solutions

Definition 1.12.

(a) The functions y_1 and y_2 are called **fundamental solutions** of the equation $L[y] = 0$ if $L[y_1] = 1$ and $L[y_2] = 0$ **and** y_1, y_2 are linearly independent.

(b) The **general solution** of the homogeneous second order linear equation $L[y] = 0$ is two parameter family of functions y_{gen} given by:

$$y_{\text{gen}} = c_1 y_1(t) + c_2 y_2(t)$$

where the arbitrary constants c_1 and c_2 are the parameters of the family and y_1, y_2 are fundamental solutions of $L[y] = 0$.

Theorem 1.13 (General Solutions). *If y_1 and y_2 are linearly independent solutions of the equation $L[y] = 0$ on the interval $I \subseteq \mathbb{R}$ where*

$$L[y] = y'' + p(t)y' + q(t)y$$

*and $p(t), q(t)$ are continuous functions on I, then there exists **unique** constants c_1, c_2 such that **every** solution y of the differential equation $L[y] = 0$ can be written as a linear combination*

$$y(t) = c_1 y_1(t) + c_2 y_2(t).$$

Proof. We need to show that given any fundamental solution pair y_1, y_2 **any** other solution y to the homogeneous equation $L[y] = 0$ must be a **unique** linear combination of the fundamental solutions $y(t) = c_1 y_1(t) + c_2 y_2(t)$ for appropriately chosen constants.

(i) The super position property gives us that any function $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is a solution of $L[y] = 0$ for every pair of constants c_1 and c_2.

(ii) Given a solution y, if there exists c_1, c_2 such that $y(t) = c_1 y_1(t) + c_2 y_2(t)$ holds then these constants are unique. We can show this as follows:

If there are other constants $\tilde{c_1}, \tilde{c_2}$ so that $y(t) = \tilde{c_1} y_1(t) + \tilde{c_2} y_2(t)$ then if we subtract this equation from the eqn with c_1 and c_2 we get

$$0 = (c_1 - \tilde{c_1})y_1 + (c_2 - \tilde{c_2})y_2 \implies c_1 - \tilde{c_1} = 0 \quad \text{and} \quad c_2 - \tilde{c_2} = 0$$

since y_1, y_2 are linearly independent therefore $c_1 = \tilde{c_1}, c_2 = \tilde{c_2}$ and hence the linear combination is **unique**.

(iii) It remains to show that given a solution y we can express it in the form

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

We shall be using the existence and uniqueness theorem to do this.

Given $c_1 y_1(t) + c_2 y_2(t)$ is a solution, the unique solution also satisfies the initial conditions. If we have $y(t_0) = b_0, y'(t_0) = b_1$ then

$$\left.\begin{array}{l} b_0 = c_1 y_1(t_0) + c_2 y_2(t_0) \\ b_1 = c_1 y_1'(t_0) + c_2 y_2'(t_0) \end{array}\right\}$$

this is a 2×2 system of eqns in unknowns c_1 and c_2

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$$

Let $A = \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix}$ the this system has solutions c_1, c_2 as long as $W(y_1, y_2) = \det A = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \neq 0$. We require $W \neq 0 \forall t$ since t_0 was chosen arbitrarily.

So $c_1y_1(t) + c_2y_2(t)$ satisfies the initial conditions if $W \neq 0$. Since our solution y satisfies the initial conditions too, the existence and uniqueness theorem tells us that $y = c_1y_1(t) + c_2y_2(t) \quad \forall t \in I$ by uniqueness. Since this is true for arbitrary initial conditions we conclude that any solution y can be written as a linear combination of y_1, y_2.

To complete the we need to show that given 2 linearly independent solutions y_1, y_2 of $L[y]$ then $W(y_1(t), y_2(t)) \neq 0 \quad \forall t \in I$, But we have already proved this earlier in the corollary in the last section.

$\square$

To summarise. Let y_1, y_2 be solutions of $L[y] = y'' + p(t)y' + q(t)y$ on $I \subseteq \mathbb{R}$, then the following are equivalent.

(i) y_1, y_2 are a fundamental set of solutions,

(ii) y_1, y_2 are linearly independent on I,

(iii) $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in I$,

(iv) $W(y_1, y_2)(t) \neq 0$ for all $t \in I$.

1.7 Reduction of Order

Reduction of Order provides a way to obtain a second solution to the homogeneous equation $L[y] = 0$ if we know one solution.

Let y_1 be a solution of $L[y] = y'' + p(t)y' + q(t)y = 0$. Using Abel's theorem to find the Wronskian W then if $W \neq 0$ for some t we can find y_2 such that $L[y_2] = 0$ and y_1, y_2 are linearly independent. Hence we can construct the general solution as follows:

$W(y_1, y_2) = y_1y_2' - y_1'y_2$, so provided $y_1 \neq 0$ we can divide by y_1

$$\frac{W}{y_1} = y_2' - \frac{y_1'}{y_1}y_2 \qquad \text{ie} \quad y_2' - \frac{y_1'}{y_1}y_2 = \frac{W}{y_1}.$$

This is a first order ODE in y_2. We can solve this using the integrating factor

$$\mu(t) = \exp(-\int \frac{y_1'}{y_1}dt) = \exp(-\log(y_1)) = \frac{1}{y_1},$$

therefore the equation can be rewritten as

$$\left(\frac{y_2}{y_1}\right)' = \frac{W}{y_1}$$

giving

$$y_2 = y_1(t) \int \frac{W}{(y_1(t))^2} dt = y_1(t) \int^t \frac{1}{y_1^2(x)} \exp\left(-\int^x p(s)ds\right) dt.$$

Example 1.11. $y_1(t) = \exp(t)$ is a solution to the ODE

$$y'' - 2y' + y = 0.$$

Find the general solution.

Solution. $W = y_1 y_2' - y_2 y_1'$. If y_2 is a second solution to the ODE then

$$\frac{dW}{dt} + P(t)W = 0.$$

In this example $P = -2$ therefore $\frac{dW}{dt} = 2W$. We can solve this

$$\int \frac{dW}{w} = 2\int dt$$

giving $W = Ae^{2t} \neq 0$ if $A \neq 0$.
Now

$$\begin{aligned} W(y_1, y_2) &= y_1 y_2' - y_2 y_1' \\ &= e^t y_2' - e^t y_2 = Ae^{2t} \end{aligned}$$

Therefore

$$y_2' - y_2 = Ae_t.$$

This is a first order linear eqn which we can solve. The integrating factor is $\mu(t) = e^{-\int 1 dt} = e^{-t}$ so

$$\left(e^{-t} y_2\right)' = A \implies y_2 = (At + B)e^t$$

Since Be^t is a constant multiple of y_1 we define $y_2 = te^t$ (i.e. set $B = 0, A = 1$) so our general solution is

$$y(t) = c_1 e^t + c_2 te^t.$$

Example 1.12. Verify $y_1(t) = t)$ is a solution of the equation

$$(1-t^2)y'' - 2ty' + 2y = 0 \quad t > 1$$

and find using the Wronskian, the second linearly independent solution.

Solution. We have $y_1(t) = t, y_1'(t) = 1, y_1''(t) = 0$ so

$$(1-t^2) \cdot 0 - 2t \cdot 1 + 2t = 0$$

as required. So $y_1(t)$ is a solution to the ODE.
Let $y_2(t)$ be a second solution to the ODE and let $W = y_1 y_2' - y_2 y_1'$. Then

$$\begin{aligned}
(1-t^2)W' &= (y_1 y_2'' - y_2 y_1'')(1-t^2) \\
&= y_1(1-t^2)y_2'' - y_2(1-t^2)y_1'' \\
&= y_1(2ty_2' - 2y_2) - y^2(2ty_1' - 2y_1) \\
&= 2t(y_1 y_2' - y_2 y_1') \\
&= 2tW
\end{aligned}$$

Hence $W' = \frac{2t}{1-t^2}W$ i.e.

$$\int \frac{W'}{W}dt = \int \frac{2t}{1-t^2}dt \quad \Longrightarrow \quad W = \exp\left(\int \frac{2t}{1-t^2}dt\right) = \frac{A}{1-t^2} \qquad A \in \mathbb{R}\backslash\{0\}$$

Since $W = y_1 y_2' - y_2 y_1'$ and $y_1 = t$ we have

$$ty_2' - y_2 = \frac{A}{1-t^2} \quad \text{i.e.} \quad y_2' - \frac{1}{t}y_2 = \frac{A}{t(1-t^2)}.$$

This is a first order equation which we can solve. It has integrating factor $\mu = e^{\int -\frac{1}{t}dt} = e^{-\ln(t)} = \frac{1}{t}$ so

$$\begin{aligned}
\left(\frac{y_2}{t}\right)' &= \frac{A}{t^2(1-t^2)} \\
y_2 &= t\int \frac{A}{t^2(1-t^2)}dt.
\end{aligned}$$

Using partial fractions we get

$$\begin{aligned}
y_2 &= At\left[\int \left(\frac{1}{t^2} + \frac{1}{2(1+t)} + \frac{1}{2(1-t)}\right)dt\right] \\
&= At\left[-\frac{1}{t} + \frac{1}{2}\ln|1+t| - \frac{1}{2}\ln|t-1| + b\right] \\
&= A\left[-1 + \frac{t}{2}\ln\left(\frac{1+t}{t-1}\right) + Bt\right].
\end{aligned}$$

where A, B are constants of integration. Now ABt is a multiple of $y_1(t) = t$ so define

$$y_2(t) = -1 + \frac{t}{2}\ln\left(\frac{t+1}{t-1}\right)$$

where we are setting $A = 1, B = 0$.

2 Power Series Solutions

In Section 1.4 we could find the general solution to the homogeneous ODE

$$L[y] = a(x)y'' + b(x)y' + c(x)y = 0 \tag{2.1}$$

when we already know one solution to the ODE.

Now we address the issue of how to find the first solution if we can't find one by inspection alone. If $a(x)$ and $b(x)$ and $c(x)$ are polynomials in x then we can use the power series solutions of (2.1) to find a general solution of the form

$$y(t) = \sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$$

Where determining the a_n coefficients is at the heart of the solution technique.

2.1 Review of Power and Taylor Series

Taylor series are an example of a power series and they are a generalisation of Maclaurin series.

if we have a function $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ then

$$\begin{aligned}
f(x) &= a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + a_3(x-x_0)^3 + \cdots; & f(x_0) &= a_0 \\
f'(x) &= a_1 + \cdot 2a_2(x-x_0) + 3a_3(x-x_0)^2 + \cdots; & f'(x_0) &= a_2 \\
f''(x) &= 2a_2 + 3\cdot 2a_3(x-x_0) + 4\cdot 3a_4(x-x_0)^2 + \cdots; & f''(x_0) &= 2a_3 \\
&\vdots
\end{aligned}$$

So the Taylor series expansion of $f(x)$ about some point x_0 is

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots$$

In order to talk about power series solutions we need the series to converge.

We have absolute convergence if

$$\lim_{N\to\infty} \sum_{n=0}^{N} |a_n(x-x_0)^n| \quad \text{converges.}$$

We can check for convergence using the ratio test:

$$\lim_{N\to\infty} \left| \frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n} \right| = |x-x_0| \lim_{N\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

If

$$\begin{cases} L < 1 & \text{the series converges for that } x \\ L = 1 & \text{convergence/divergence cannot be determined} \\ L > 1 & \text{the series diverges for that } x \end{cases}$$

The radius of convergence is all x for which $L < 1$. So if $L < 1$ for some $|x - x_0| < p$ this gives the radius of convergence p.

Example 2.1. Does $S = \sum_{n=1}^{\infty} \frac{(-1)^2 x^{2n}}{(2n)!}$, near $x = 0$ converge absolutely?

Solution.

$$L = \lim_{N\to\infty} \left| \frac{(-1)^{n+1}\ x^{2n+2}\ (2n)!}{(-1)^n\ x^{2n}\ (2n+2)!} \right| = |x|^2 \lim_{N\to\infty} \left| \frac{1}{(2n+1)(2n+2)} \right| = 0 < 1.$$

So S converges absolutely. In fact S is the series expansion for $\cos(x)$.

2.2 Power Series Solution Techniques

Consider the differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0$$

where $a(x), b(x), c(x)$ are polynomials. We look for power series solutions near the point x_0. The power series solution is valid if the series converges.

2.2.1 Ordinary Points

Definition 2.1. A point $x_0 \in \mathbb{R}$ is called an *ordinary point* of equation (2.1) if $a(x_0) \neq 0$ and otherwise x_0 is called a *singular point*.

Example 2.2. Solve $y'' - xy' - y = 0$ about $x_0 = 0$.

Solution. We first note that every point is an ordinary point since $a(x) = 1$. Next we seek a power series solution $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$, where $x_0 = 0$. Hence

$$y' = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1}, y'' = \sum_{n=0}^{\infty} n(n-1) a_n (x - x_0)^{n-2}.$$

Substituting into the ODE gives:

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0.$$

Combining the sums

$$\sum_{n=0}^{\infty} \left[n(n-1) a_n x^{n-2} - n a_n x^n - a_n x^n \right] = 0$$

$x^0, x^1, x^2, \ldots, x^n$ are linearly independent so the sum is zero provided the coefficients of each power of x are zero.
The coefficient of x^N is:

$$(N+2)(N+1) a_{N+2} - N a_N - a_N = 0$$

Rearranging:

$$a_{N+2} = a_N \left[\frac{1}{N+2} \right] \qquad \text{(Recurrence relation)}$$

Given a_0 we can use the recurrence relation to determine the coefficients of all the even powers of x. Likewise we can determine all the odd powers given a_1. So

$$a_2 = \frac{a_0}{2}, \quad a_4 = \frac{a_2}{4} = \frac{a_0}{2 \cdot 4}, \quad a_6 = \frac{a_4}{6} = \frac{a_0}{2 \cdot 4 \cdot 6}, \quad \cdots$$

and

$$a_3 = \frac{a_1}{3}, \quad a_5 = \frac{a_3}{5} = \frac{a_1}{3 \cdot 5}, \quad a_7 = \frac{a_5}{7} = \frac{a_1}{3 \cdot 5 \cdot 7}, \quad \cdots$$

We can group the even and odd terms to get

$$y = a_0 \left(1 + \frac{x^2}{2} + \frac{x^4}{2 \cdot 4} + \frac{x^6}{2 \cdot 4 \cdot 6} + \cdots \right) + a_1 \left(1 + \frac{x^3}{3} + \frac{x^5}{3 \cdot 5} + \frac{x^7}{3 \cdot 5 \cdot 7} + \cdots \right)$$

or simplifying:

$$y = a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} + a_1 \sum_{n=0}^{\infty} \frac{2^n n! x^{2n+1}}{(2n+1)!}$$

We expect two linearly independent solutions to the ODE with two arbitrary coefficients c_1, c_2 to give the general solution $y = c_1 y_1 + c_2 y_2$. This is what we have above with $a_0 = c_1$ and $a_1 = c_2$ and the two solutions y_1, y_2 are given by the two power series.
Suppose we are given initial conditions $y(0) = A$ and $y'(0) = B$, we can readily see that $a_0 = A$ and $a_1 = B$. So the unique solution to the IVP is then

$$y = A \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} + B \sum_{n=0}^{\infty} \frac{2^n n! x^{2n+1}}{(2n+1)!}$$

We use the ratio test to check the convergence of each power series. For the first series:

$$\lim_{n \to \infty} \left| \frac{\frac{x^{2(n+1)}}{2^{n+1}(n+1)!}}{\frac{x^{2n}}{2^n n!}} \right| = \lim_{n \to \infty} \left| \frac{1}{2(n+1)} \right| x^2 = 0.$$

So $L = 0$ and the series converges for all x. For the second series

$$\lim_{n \to \infty} \left| \frac{\frac{x^{2(n+1)+1} \; 2^{n+1}(n+1)!}{(2(n+1)+1)!}}{\frac{x^{2n+1} \; 2^n n!}{(2n+1)!}} \right| = \lim_{n \to \infty} \left| \frac{1}{2n+3} \right| x^2 = 0.$$

So $L = 0$ and the series converges for all x.

Example 2.3. Find the recurrence relation corresponding to the power series solution of $y''+xy = 0$ around the point $x_0 = 2$.

Solution. Since $a(x) = 1$ every point is an ordinary point. Looking for a power series solution of the form $y = \sum_{n=0}^{\infty} a_n(x-2)^n$ gives

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n(x-2)^{n-2}.$$

Substituting these expressions into the ODE yields:

$$\sum_{n=0}^{\infty} \left[n(n-1)a_n(x-2)^{n-2} + xa_n(x-2)^n\right] = 0,$$

rewriting $x = x-2+2$ we have

$$\sum_{n=0}^{\infty} \left[n(n-1)a_n(x-2)^{n-2} + a_n(x-2)^{n+2} + 2a_n(x-2)^n\right] = 0.$$

The coefficient of $(x-2)^N$ is:

$$(N+2)(N+1)a_{N+2} + a_{N-1} + 2a_N = 0,$$

so the recurrence relation for the coefficients is

$$\begin{cases} a_{N+2} = \frac{-2a_N - a_{N-1}}{(N+2)(N+1)} & \text{for } N \geq 1 \\ a_2 = a_0 & \text{for } N = 0. \end{cases}$$

2.2.2 Regular Singular Points

Returning to equation (2.1):

$$a(x)y'' + b(x)y' + c(x)y = 0$$

on dividing by $a(x)$ we get

$$y'' + p(x)y' + q(x)y = 0, \quad p(x) = \frac{b(x)}{a(x)},\ q(x) = \frac{c(x)}{a(x)}. \tag{2.2}$$

If we can write the solution as a Taylor series

$$y(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$$

then we have seen $a_0 = y(x_0),\ a_1 = y'(x_0),\ a_2 = \frac{y''(x_0)}{2},\ \ldots$

We are given $y(x_0)$ and $y'(x_0)$ in an IVP so from (2.1) we have

$$y''(x_0) = -p(x_0)y'(x_0) - q(x_0)y(x_0)$$

We can calculate higher derivatives of y and hence find a_n by repeatedly differentiating the ODE (2.1). This works provided $p(x), q(x)$ are infinitely differentiable at x_0. So we need $p(x), q(x)$ to be analytic at x_0, that is they need to have a convergent Taylor series expansion in some interval about x_0:

$$p(x) = p_0 + p_1(x-x_0) + p_2(x-x_0)^2 + \cdots = \sum_{k=0}^{\infty} p_k(x-x_0)^k,$$

$$q(x) = q_0 + q_1(x-x_0) + q_2(x-x_0)^2 + \cdots = \sum_{k=0}^{\infty} q_k(x-x_0)^k.$$

If x_0 is an ordinary point then this is not a problem. To solve (2.1) in the neighbourhood of a singular point x_0 we have to restrict ourselves to the case where the singularity in $\frac{b(x)}{a(x)}$ and $\frac{c(x)}{a(x)}$ is not too severe. This means the singularity of $\frac{b}{a}$ is no worse than $(x-x_0)^{-1}$ and the singularity of $\frac{c}{a}$ is no worse than $(x-x_0)^{-2}$. Such a point is a regular singular point.

Definition 2.2. Given a singular point x_0, it is a *regular singular point* of equation (2.1) if

$$\lim_{x\to x_0}(x-x_0)\frac{b(x)}{a(x)} \quad \text{and} \quad \lim_{x\to x_0}(x-x_0)^2\frac{c(x)}{a(x)}$$

are both finite. If these conditions do not hold, then the singular point is *irregular*.

So we want $(x-x_0)p(x)$ and $(x-x_0)q(x)$ to be analytic about x_0, that is they have a convergent power series of the form

$$(x-x_0)p(x)=\sum_{k=0}^{\infty}p_k(x-x_0)^k, \qquad (x-x_0)^2q(x)=\sum_{k=0}^{\infty}q_k(x-x_0)^k$$

on some $|x-x_0|<\rho,\ \rho>0$.

Multiplying (2.2) by $(x-x_0)^2$ gives:

$$(x-x_0)^2y''+(x-x_0)^2p(x)y'+(x-x_0)^2q(x)y=0$$

i.e.

$$(x-x_0)^2y''+(x-x_0)\left[\sum_{k=0}^{\infty}p_k(x-x_0)^k\right]y'+\left[\sum_{k=0}^{\infty}q_k(x-x_0)^k\right]y=0$$

If all of p_k,q_k are zero except p_0,q_0 then we have **Euler's equation**.

$$\boxed{(x-x_0)^2y''+(x-x_0)p_0y'+q_0y=0} \tag{2.3}$$

2.2.3 Euler's equation

It is easy to see Euler's equation has a singular point at x_0, moreover

$$\lim_{x\to x_0}(x-x_0)\frac{p_0}{(x-x_0)}=p_0 \quad \text{and} \quad \lim_{x\to x_0}(x-x_0)^2\frac{q_0}{(x-x_0)^2}=q_0$$

are both finite so x_0 is a regular singular point.

Solving Euler's equation is done by using $y=(x-x_0)^r$ so $y'=r(x-x_0)^{r-1}, y''=r(r-1)(x-x_0)^{r-2}$, substituting these into (2.3) gives

$$(x-x_0)^2\left(r(r-1)(x-x_0)^{r-2}\right)+p_0(x-x_0)r(x-x_0)^{r-1}+q(x-x_0)^r=0$$

i.e. $(x-x_0)^r\Big[(r-1)r+p_0r+q_0\Big]=0$. This has solutions provided

$$r^2+r(p_0-1)+q_0=0$$

i.e.

$$r_\pm=\frac{1}{2}\left[-(p_0-1)\pm\sqrt{(p_0-1)^2-4q_0}\right]$$

- if $(p_0-1)^2-4q_0>0$ then $r_\pm$ are **distinct real roots** and

$$y(x)=c_1(x-x_0)^{r_+}+c_2(x-x_0)^{r_-}, \quad c_1,c_2\in\mathbb{R}.$$

- if $(p_0-1)^2-4q_0<0$ then $r_\pm$ are **complex**. Let $r_\pm=\lambda\pm i\mu$ then

$$\begin{aligned}(x-x_0)^{\lambda\pm i\mu}&=(x-x_0)^\lambda(x-x_0)^{\pm i\mu}\\&=(x-x_0)^\lambda e^{\ln(x-x_0)^{\pm i\mu}}\\&=(x-x_0)^\lambda e^{\pm i\mu\ln(x-x_0)}.\end{aligned}$$

Thus

$$y(x)=c_1(x-x_0)^\lambda\cos\left(\mu\ln|x-x_0|\right)+c_2(x-x_0)^\lambda\sin\left(\mu\ln|x-x_0|\right).$$

- if $(p_0-1)^2 - 4q_0 = 0$ then $r_+ = r_-$ and we have a **double root**. Then $y_1 = (x-x_0)^{r_+}$ and we use the Wronskian approach from Chapter 1 to find the second linearly independent solution.

Example 2.4. Solve $x^2y'' - 2y = 0$.

Solution. The ODE is an Euler equation with $x_0 = 0$. We look for solutions of the form $y = x^r$; thus

$$r(r-1) - 2 = 0 \implies (r-2)(r+1) = 0.$$

Hence $r_+ = 2, r_- = -1$. The general solution is therefore

$$y(x) = c_1x^2 + c_2x^{-1}, \quad c_1, c_2 \in \mathbb{R}.$$

Notice the solution blows up at $x = 0$ as expected as $x = 0$ is a singular point.

Example 2.5. Solve $x^2y'' + 5xy' + 4y = 0$.

Solution. The ODE is an Euler equation with $x_0 = 0$. Looking for solutions of the form $y = x^r$ yields

$$r(r-1) + 5r + 4 = (r+2)^2 = 0$$

so $r_\pm = -2$ (repeated root).
We have found one solution $y1 = x^{-2}$, we can use the Wronskian to find the second linearly independent solution y_2:

$$\begin{aligned} W' = y_1y_2'' - y_2y_1'' &= y_1\left(\frac{-5}{x}y_2' - \frac{4}{x^2}y_2\right) - y_2\left(\frac{-5}{x}y_1' - \frac{4}{x^2}y_1\right) \\ &= \frac{5}{x}(y_1y_2' - y_2y_1') = \frac{-5}{x}W. \end{aligned}$$

Solving this ODE in W:

$$W = A\exp\left(-\int \frac{5}{x}dx\right) = Ax^{-5}, \quad A \in \mathbb{R}.$$

Using $W = y_1y_2' - y_1'y_2 = x^{-2}y_2' + 2x^{-3}y_2 = Ax^{-5}$, hence

$$y_2' + 2x^{-1}y_2 = Ax^{-1}.$$

Using the integrating factor $\mu(x) = \exp\left(\int 2x^{-1}dx\right) = x^2$ we find

$$y_2(x) = \frac{1}{x^2}\int \frac{A}{x}dx = \frac{A}{x^2}\ln x + Bx^{-2}$$

where A, B are constants of integration. Since Bx^{-2} is a constant multiple of y_1, we can take $B = 0$. The general solution is therefore

$$y(x) = c_1y_1 + c_2y_2 = c_1x^{-2} + c_2x^{-2}\ln x.$$

Notice the solution is singular at $x = 0$ as expected.

2.2.4 Method of Frobenius

Returning back to equation (2.2) we had

$$(x-x_0)^2y'' + (x-x_0)\left[(x-x_0)p(x)\right]y' + \left[(x-x_0)^2q(x)\right]y = 0$$

If p, q are analytic around x_0 then there exists convergent power series such that

$$(x-x_0)^2y'' + (x-x_0)\left[\sum_{k=0}^{\infty} p_k(x-x_0)^k\right]y' + \left[\sum_{k=0}^{\infty} q_k(x-x_0)^k\right]y = 0$$

i.e.

$$\underbrace{(x-x_0)^2y'' + (x-x_0)p_0y' + q_0y}_{\text{LHS of Euler's Equation}} + \underbrace{(x-x_0)\Big\{p_1(x-x_0)y' + q_1y + p_2(x-x_0)^2y' + q_2(x-x_0)y + \cdots\Big\}}_{\text{higher order terms}} = 0$$

The higher order terms cannot introduce more singular terms but rather corrections that are higher powers of $(x-x_0)$ so we look for solutions of the form

$$y(x) = (x-x_0)^r \sum_{k=0}^{\infty} a_n(x-x_0)^n = \sum_{k=0}^{\infty} a_n(x-x_0)^{n+r}$$

which is known as a Frobenius series.

Method of Frobenius:

To determine a solution of (2.2) we need to know

1) the values of r,

2) a recurrence relation for the a_n's,

3) determine the radius of convergence of the resulting power series.

Example 2.6. Solve $2x^2y'' - xy' + (1+x)y = 0$.

Solution. $a(x) = 2x^2$ hence $x = 0$ is a singular point.

$$\lim_{x\to 0} x\frac{b(x)}{a(x)} = \lim_{x\to 0} \frac{x(-x)}{2x^2} = \frac{-1}{2} \quad \text{and} \quad \lim_{x\to 0} x^2\frac{c(x)}{a(x)} = \lim_{x\to 0} x^2\frac{(1+x)}{2x^2} = \frac{1}{2}.$$

Thus $x = 0$ is a regular singular point. We look for a Frobenius series solution $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, where $a_0 \neq 0$.
We have

$$y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-2}.$$

Substituting these into the ODE gives:

$$\sum_{n=0} \infty \left[2x^2 a_n(n+r)(n+r-1)x^{n+r-2} - xa_n(n+r)x^{n+r-1} + (1+x)a_n x^{n+r}\right] = 0.$$

To find an equaiont defining r we consider the coefficient of the lowest power of x (x^r) which gives:

$$a_0 2r(r-1) - a_0 r + a_0 = 0, \qquad \text{where } a_0 \neq 0$$

.

$$\implies (2r-1)(r-1) = 0 \qquad \boxed{\text{Indicial Equation}}$$

So $r = \frac{1}{2}$ or $r = 1$. The indicial equation is responsible for capturing the correct behaviour of the singularity. The coefficient of a general power of x (x^{N+r}) is:

$$a_N 2(N+r)(N+r-1) - a_N(N+r) + a_N + a_{N-1} = 0$$

giving

$$a_N = \frac{-a_{N-1}}{(N+r-1)(2(N+r)-1)} \qquad \boxed{\text{Recurrence Relation.}}$$

This allows us to determine each of the coefficients in the power series expansion for a given value of r.
Case 1$(r = 1)$. Recurrence relation: $a_n = \frac{-a_{n-1}}{n(2n+1)}$.
The corresponding power series solution which is valid for $x > 0$ is:

$$y_1(x) = x\left[\sum_{n=0}^{\infty} \frac{(-1)^n\ x^n\ 2^n\ n!}{(2n+1)!\ n!}\right]$$

Note: we arbitrarily took $a_0 = 1$. Recall that the general solution is $y = c_1y_1 + c_2y_2$ where c_1, c_2 account for the arbitrary value of a_0.
Case 2$(r = \frac{1}{2})$. Recurrence relation: $a_n = \frac{-a_{n-1}}{(2n-1)n}$.
The corresponding power series solution which is valid for $x > 0$ is:

$$y_2 = x^{\frac{1}{2}}\left[\sum_{n=0}^{\infty} \frac{(-1)^n\ 2^n\ x^n}{(2n)!}\right]$$

To determine the exact range of validity of the series we perform the ratio test.
Series y_1:

$$\lim_{n\to\infty} \left|\frac{x}{(n+1)(2n+3)}\right| = 0$$

hence y_1 is valid for all $x > 0$, as the series converges absolutely.
Series y_2:

$$\lim_{n\to\infty} \left|\frac{-x2x}{(2n)(2n+1)(n+1)}\right| = \lim_{n\to\infty} \left|\frac{x}{(2n+1)(n+1)}\right| = 0$$

hence y_2 converges absolutely for $x > 0$.

In the above example the indicial equation had real distinct roots. If the roots are complex then we would proceed in the same way, but we would find complex-valued functions of x for y_1 and y_2, however as with Euler's equation we can obtain real-values solutions by taking real and imaginary parts of the complex solutions.

Repeated Real Roots:

If the indicial equation has a repeated root $r_1 = r_2$ then $(r - r_1)^2 = 0$. We need to determine a_n as a function of r and we arrive at one solution

$$y_1(x) = \sum_{n=0}^{\infty} a_n(r_1)(x - x_0)^{n+r_1}.$$

The second solution is

$$y_2(x) = y_1(x)\ln(x - x_0) + \sum_{n=1}^{\infty} \frac{\partial a_n}{\partial r}\Big|_{r=r_1} (x - x_0)^{n+r_1}.$$

We illustrate how this second solution is found via the following example.

Example 2.7 (Bessel's equation of order 0)**.** Find a series solutions about $x = 0$ for the ODE:

$$L[y] = x^2 y'' + xy' + x^2 y = 0.$$

Solution. $x = 0$ is a singular point and

$$\lim_{x\to 0} x\frac{b(x)}{a(x)} = \lim_{x\to 0} \frac{xx}{x^2} = 1, \text{ and } \lim_{x\to 0} \frac{x^2 c(x)}{a(x)} = \lim_{x\to 0} \frac{x^2 x^2}{x^2} = 0.$$

As both limits are finite $x = 0$ is a regular singular point. Looking for series solutions of the form $y = \sum_{n=0}^{\infty} a_n x^{n+r}$, $a_0 \neq 0$ gives

$$\sum_{n=0}^{\infty} \left[a_n(n+r)(n+r-1)x^{n+r} + a_n(n+r)x^{n+r} + a_n x^{n+r+2}\right] = 0. \tag{2.4}$$

Coefficients of x^r are:

$$r(r-1) + r = r^2 = 0 \quad \boxed{\text{Indicial Equation}}$$

Hence $r = 0$ is a repeated root. The coefficients of x^{r+1} are:

$$a_1(r+1)r + (r+1)a_1 = a_1(r+1)^2 = 0,$$

since $r = 0$ we must have $a_1 = 0$. The coefficients of a general term x^{N+r} are:

$$a_N(N+r)(N+r-1) + a_N(N+r) + a_{N-2} = 0, \quad N \geq 2.$$

Hence the $\boxed{\text{recurrence relation}}$ is

$$a_N = \frac{-a_{N-2}}{(r+N)^2}, \quad N \geq 2$$

First solution y_1**:** Setting $r = 0$ in the recurrence relation gives $a_N = \frac{-a_{N-2}}{N^2}$. Since $a_1 = 0$ it follows that $a_3, a_5, a_7, \ldots$ are all zero. The only a_N which are non-zero correspond to even N. Let $N = 2m$, $m \in \mathbb{N}$ and reformulate the reccurence relation:

$$a_{2m} = \frac{-a_{2(m-1)}}{(2m)^2}.$$

So

$$a_2 = \frac{-a_0}{(2\cdot 1)^2}, \quad a_4 = \frac{-a_2}{(2\cdot 2)^2} = \frac{+a_0}{2^2 2^2 (1\cdot 2)^2}, \quad a_6 = \frac{-a_4}{(2\cdot 3)^2} = \frac{-a_0}{2^2 2^2 2^2 (1\cdot 2\cdot 3)^2}, \cdots$$

and

$$a_{2m} = \frac{(-1)^m \, a_0}{2^{2m} \, (m!)^2}$$

Hence

$$y_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m \, x^{2m}}{2^{2m} \, (m!)^2}$$

.

This is the Bessel function of the first kind of order zero and is denoted $J_0(x)$.

Ex. Check the series converges absolutely.

Second solutions y_2**:** As we are dealing with power series it would be difficult to use the Wronskian to find the second linearly independent solution. We use and alternative approach. Rewriting equation (2.4), $L[y]$ gives

$$L[y] = a_0 r^2 x^r + \underbrace{a_1 (r+1)^2 x^{r+1}}_{a_1=0} + \sum_{n=2}^{\infty} \underbrace{\Big[a_n(n+r)^2 + a_{n-2}\Big]}_{=0 \text{ by the reccurence relation}} x^{n+r}$$

Hence $L[y] = a_0 r^2 x^r$. Differentiating with respect to r gives

$$L\left[\frac{\partial y}{\partial r}\right] = 2a_0 r x^r + a_0 r^2 x^r \ln x$$

Observer $L\left[\frac{\partial y}{\partial r}\Big|_{r=0}\right] = 0$ so $\frac{\partial y}{\partial r}\Big|_{r=0}$ is a solution to the ODE $L[y] = 0$. This is the second solution we were looking for.

$$y_2(x) = \frac{\partial y}{\partial r}\Big|_{r=0} = \frac{\partial}{\partial r}\left[\sum_{n=0}^{\infty} a_n(r) x^{n+r}\right]\Bigg|_{r=0}$$

Using the product rule

$$\frac{\partial y}{\partial r}\Big|_{r=0} = \frac{\partial}{\partial r}\left[x^r \ln x \left(\sum_{n=0}^{\infty} a_n(r) x^n\right) + x^r \sum_{n=1}^{\infty} \frac{\partial a_n(r)}{\partial r} x^n\right]\Bigg|_{r=0}$$

Hence

$$y_2(x) = \ln x y_1(x) + \left[\sum_{n=1}^{\infty} \frac{\partial a_n(r)}{\partial r}\Big|_{r=0} x^n\right].$$

We now need to find $\frac{\partial a_n(r)}{\partial r}\Big|_{r=0}$ using the recurrence relation

$$a_N = \frac{-a_{N-2}}{(r+N)^2}.$$

Hence,

$$a_2 = \frac{-a_0}{(r+2)^2} \implies \frac{\partial a_2}{\partial r} = \frac{2a_0}{(r+2)^3} \text{ and } \frac{\partial a_2}{\partial r}\Big|_{r=0} = \frac{a_0}{4}.$$

$$a_4 = \frac{-a_2}{(r+4)^2} = \frac{a_0}{\big[(r+4)(r+2)\big]^2} \implies \frac{\partial a_4}{\partial r} = \frac{-2a_0(2r+6)}{\big[(r+4)(r+2)\big]^3} \text{ and } \frac{\partial a_4}{\partial r}\Big|_{r=0} = \frac{-3a_0}{128}.$$

Hence the first few terms of y_2 are:

$$y_2(x) = \ln(x) y_1(x) + \left[\frac{1}{4}x^2 - \frac{3}{128}x^4 + \cdots\right], \quad x > 0$$

Note $y_1(x) \to 1$ as $x \to 0$ so $y_2(x)$ has logarithmic singularity at $x = 0$. The gernal solution is $y = c_1 y_1 + c_2 y_2,\ c_1, c_2 \in \mathbb{R}$.

3 Inhomogeneous Equations

Theorem 3.1. *Every solution y of the inhomogeneous equation*

$$L[y] = f$$

with $L[y] = y'' + py' + qy$, where p, q and f are continuous ffunctions, is given by

$$y = c_1 y_1 + c_2 y_2 + y_p$$

where the function y_1 and y_2 are fundamental solutions of the homogeneous equation $L[y] = 0$ and y_p is any solution of the inhomogeneous equation $L[y] = f$.

Proof. Let y be any solution of $L[y] = f$. We already have y_p such that $L[y_p] = f$, so $L[y] - L[y_p] = f - f = 0$ hence $L[y - y_p] = 0$ by linearity of L.

Hence $y - y_p$ is a solution of the homogenenous equation, but all solution to the homogenenous equation can be written as a linear combination of a pair of fundamental solutions y_1, y_2. So there exists $c_1, c_2 \in \mathbb{R}$ such that $y - y_p = c_1 y_1 + c_2 y_2$.

Since for every y, a solution of $L[y] = f$ we can find a c_1, c_2 we have $y = c_1 y_1 + c_2 y_2 + y_p$ for every solution. $\square$

3.1 Exact Second Order Equations

The general second order ODE

$$a(x)y'' + b(x)y' + c(x)y = f(x) \tag{3.1}$$

can be writen as

$$\big(a(x)y'' - a'(x)y\big)' + \big(b(x)y\big)' + \big(a''(x) - b'(x) + c(x)\big)y = f(x) \tag{3.2}$$

We check by exapanding (3.2):

$$a'y' + ay'' - a''y + by' + b'y + a''y - b'y + cy = f \implies ay'' + by' + cy = f \text{ as required.}$$

The differential equation is exact is

$$a'' - b' + c = 0 \tag{3.3}$$

When (3.2) is exact we can integrate to obtain

$$a(x)y' - a'(x)y + b(x)y = \int f(x)dx + c_1$$

where $c_1 \in \mathbb{R}$. This is a first order ODE that we can solve using integrating factors.

Example 3.1. Find the general solution to

$$\frac{1}{x}y'' + \left(\frac{1}{x} - \frac{2}{x^2}\right)y' - \left(\frac{1}{x^2} - \frac{2}{x^3}\right)y = e^x.$$

Solution.

$$a(x) = \frac{1}{x}, \quad b(x) = \frac{1}{x} - \frac{2}{x^2}, \quad c(x) = \frac{-1}{x^2} + \frac{2}{x^3}, \quad f(x) = e^x$$

So (3.3) gives: $2x^{-3} + x^{-2} - 4x^{-3} - x^{-2} + 2x^{-3} = 0$ and hence the ODE is exact. Writing the ODE in the form of (3.2) gives:

$$\left(x^{-1}y' + x^{-2}y\right)' + \left(\left(x^{-1} - 2x^{-2}\right)y\right)' = e^x.$$

Integrating with respect to x gives:

$$x^{-1}y' + x^{-2}y + (x^{-1} - 2x^{-2})y = e^x + c_1, \quad c_1 \in \mathbb{R}$$

Hence,

$$y' + (1 - x^{-1})y = xe^x + c_1 x.$$

Solving the ODE using the integrating factor $\mu(x) = \exp\left(\int 1 - \frac{1}{x}dx\right)$ gives the general solution

$$y(x) = \frac{1}{2}xe^x + c_1 x + c_2 xe^{-x},$$

c_1, c_2 are arbitrary constants.

3.2 The Adjoint Equation and Integrating Factors

If the differential equation (3.1) can be multiplied by a function $z(x)$ so that the resulting ODE is exact then $z(x)$ is called an integrating factor for equation (3.1).

$z(x)$ times equation (3.1) gives:

$$zay'' + zby' + zcy = zf. \tag{3.4}$$

This equation is exact if the LHS can be written as

$$zay'' + zby'' + zay = \frac{d}{dx}\left(U(x)y' + V(x)y\right) \tag{3.5}$$

If such a $z(x)$ exists then equation (3.5) becomes

$$\frac{d}{dx}\left(U(x)y' + V(x)y\right) = z(x)f(x) \tag{3.6}$$

which can be integrated to obtain

$$U(x)y' + V(x)y = \int z(x)f(x)dx$$

This is a first order linear ODE in y which we can sovle. So it remains to find $z(x), U(x)$ and $V(x)$. From equation (3.5) we have

$$\begin{aligned} zay'' + zby' + zcy &= \frac{d}{dx}\left(U(x)y' + V(x)y\right) \\ &= Uy'' + (U' + V)y' + V'y \end{aligned}$$

Comparing coefficients of y'', y' and y we obtain

$$\begin{aligned} za &= U && \text{(i)} \\ zb &= U' + V && \text{(ii)} \\ zc &= V' && \text{(iii)} \end{aligned}$$

The derivative of (ii) gives

$$U'' + v@ = (zb)' \quad \text{(iv)}$$

From equations (i) and (iii), we have

$$U'' + V' = (za)'' + zc \quad \text{(v)}$$

Hence

$$\bar{L}[z] = \frac{d^2}{dx^2}(za) - \frac{d}{dx}(zb) + zc \tag{3.7}$$

is the adjoint of $L[y] = ay'' + by' + cy$.

The adjoint equation is easier to solve. Once $z(x)$ is found we have expressions for U and V and we can solve the exact differential equation (3.6).

Example 3.2. Find the general solution of

$$L[y] = y'' + 4y = x^2$$

Solution. We multiply the differential equation by $z(x)$ yielding

$$zy'' + z4y = zx^2.$$

Reexpressing the LHS as

$$\begin{aligned} zy'' + z4y &= (Uy' + Vy)' \\ &= Uy'' + (U' + V)y' + V'y \end{aligned}$$

and equating coefficients of y'', y', y gives:

$$\begin{aligned} z &= U && (i) \\ 0 &= U + V && (ii) \\ 4z &= V' && (iii) \end{aligned}$$

From (ii):

$$U'' + V' = 0 \quad \text{(iv)}$$

From (i) and (iii):

$$U'' + V' = z'' + 4z \quad \text{(v)}$$

Then (v) minus (iv) gives:

$$z'' + 4z = 0$$

The adjoint of L is therefore:

$$\bar{L} = \frac{d^2}{dx^2} + 4.$$

Solving $\bar{L}[z] = 0$ to find $z(x)$ can be done by looking for solutions of the form $z(x) = e^{mx}$ which gives the auxiliary equation

$$m^2 + 4 = 0.$$

Hence $m = \pm 2i$. Since we only need one solution for $z(x)$ and not the general solution it sufficies to take $z(x) = \sin(2x)$.
From here, $U = z = \sin(2x)$ and $V = -U' = -2\cos(2x)$. Hence the orginal ODE can be expressed as:

$$\frac{d}{dx}\left(\sin(2x)\frac{dy}{dx} - 2\cos(2x)y\right) = x^2 \sin(2x).$$

Integrating gives:

$$\sin(2x)\frac{dy}{dx} - 2\cos(2x)y = \int x^2 \sin(2x)dx.$$

So

$$\frac{dy}{dx} - 2\cot(2x)y = \operatorname{cosec}(2x)\int x^2 \sin(2x)dx.$$

Solving this firsrt order ODE using the integrating factor $\mu(x) = \operatorname{cosec}(2x)$ gives:

$$\frac{d}{dx}\left[\operatorname{cosec}(2x)y\right] = \operatorname{cosec}^2(2x)\int x^2 \sin(2x)dx$$

Then $y(x) = \sin(2x)\int\left(\operatorname{cosec}^2(2x)\int x^2\sin(2x)dx\right)dx$ which yields:

$$y(x) = c_1\sin(2x) + c_2\cos(2x) - \frac{1}{8} + \frac{1}{4}x^2.$$

Example 3.3. Find the adjoint of

$$L[y] = (x^2 - x)y'' + (2x^2 + 4x - 3)y' + 8xy = 1$$

and use this to find the general solution.

3.3 Self-adjoint Operators

4 Boundary Value Problems and Sturm-Liouville Theory

4.1 Inner Product Spaces

4.2 Orthogonality and Reality

4.3 Eigenfunction Expansion Method

5 Green's Functions

5.1 Generalised Functions

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6 First Order Partial Differential Equations

6.1 First Order Constant Coefficient Linear PDEs

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7 Second Order Partial Differential Equations

7.1 Classification of Second Order PDEs

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7.2 Solutions of BVPs by Separation of variables