HOMEWORK 5

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SECTION 002 (MW 2:30-3:50 PM)

CSE 2320 - Homework 5 NAME: Elsy Fernandes

Total points: 100 Topics: Recurrences , solved with methods: Master Theorem, Tree, Substitution (induction)

Convention: $\lceil\ \rceil$ *means rounded up and* $\lfloor\ \rfloor$ *means rounded down.*

P1. (23 points) Use the tree and table method to compute the **Θ** time complexity for $T(N)=5T(\lfloor N/4 \rfloor)+2N^3$.
Assume T(0) = 1 and T(1) = 1. Fill in the table below and finish the computations outside of it:

Level	Argument/ Problem size	Cost of one node	Nodes per level	Cost of whole level
0	N	$2N^3$	1	$1 \cdot 2N^3$
1	$N/4$	$2(N/4)^3$	5	$5 \cdot 2(N/4)^3 = 2N^3\left(\frac{5}{4^3}\right)^1$
2	$(N/4^2)$	$2(N/4^2)^3$	5^2	$5^2 \cdot 2(N/4^2)^3 = 2N^3\left(\frac{5}{4^3}\right)^2$
i	$(N/4^i)$	$2(N/4^i)^3$	5^i	$5^i \cdot 2(N/4^i)^3 = 2N^3\left(\frac{5}{4^3}\right)^i$
k= $\log_4 n$ *Leaf level. Write k as a function of N.*	$1 = \frac{N}{4^k}$	$2 \cdot \left(\frac{N}{4^k}\right)^3$	5^k	$5^k \cdot 2\left(\frac{N}{4^k}\right)^3 = 2N^3\left(\frac{5}{4^3}\right)^k$

Total tree cost calculation: we showed that $T(N) = 2N^3 \times \frac{64}{59}$

T(N) = Θ(N^3)

Draw the tree. Show **levels 0,1,2** and the **leaves level**. Show the problem size T(...) as a label next to the node and inside the node show the local cost (cost of one node) as done in class. For the leaf level and level 2 it suffices to show a few nodes.

Total Tree Cost Calculation:

$$T(N) = 2N^3 + 2N^3\left(\frac{5}{4^3}\right)^1 + 2N^3\left(\frac{5}{4^3}\right)^2 + \cdots$$

$$+ 2N^3\left(\frac{5}{4^3}\right)^i + \cdots \cdots 2N^3\left(\frac{5}{4^3}\right)^k$$

Take out $2N^3$ common

$$T(N) = 2N^3\left(1 + \left(\frac{5}{4^3}\right)^1 + \left(\frac{5}{4^3}\right)^2 + \cdots + \left(\frac{5}{4^3}\right)^i + \cdots\cdots \left(\frac{5}{4^3}\right)^k\right)$$

Summation of geometric Series $\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}$

when $0 < x < 1$

Similary, $T(N) = 2N^3\left[\sum_{i=0}^{k}\left(\frac{5}{4^3}\right)^i\right]$ — ① $\quad X = \frac{5}{4^3} < 1$

we know that, $\sum_{k=0}^{n} x^k \leq \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

$$\sum_{i=0}^{K}\left(\frac{5}{4^3}\right)^i = \frac{1}{1-\frac{5}{4^3}} = \frac{1}{1-\frac{5}{64}} = \frac{64}{64-5}$$

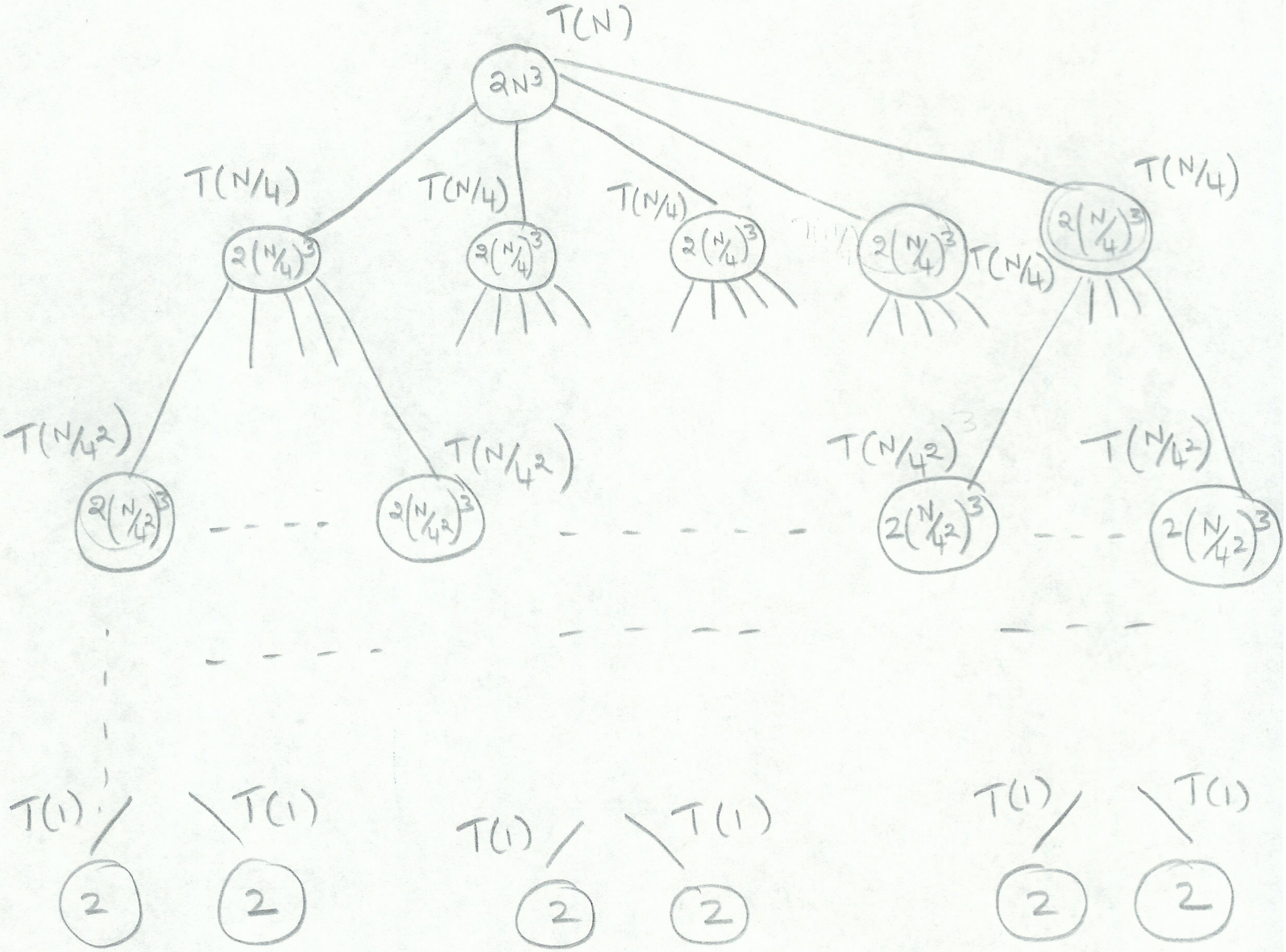
$$= \frac{64}{59}$$ Substitute Back in ①

$T(N) = 2N^3 \times \frac{64}{59}$ So it is $O(N^3)$

we showed that it is $O(N^3)$ but sum $\neq 0$ so it is $\Omega(N^3)$

Hence, $\Theta(N^3)$

Tree structure



P2. (23 **points**) Use the tree and table method to compute the Θ time complexity for $T(N) = 4T(N-5) + 7$. Assume $T(N) = 1$ for all $0 \le N \le 4$. Assume N is a convenient value for your computations.

Fill in the table below and finish the computations outside of it:

Level	Argument/ Problem size	Cost of one node	Nodes per level	Cost of whole level
0	N	7	1	$1 \cdot 7 = 7$
1	$(N-5)$	7	4	$4 \cdot 7$
2	$(N-10)$	7	4^2	$4^2 \cdot 7$
i	$(N-5i)$	7	4^i	$4^i \cdot 7$
k= $N/5$ *Leaf level. Write k as a function of N.*	$(N-5k)$	7	4^k	$4^k \cdot 7$

Total tree cost calculation: $\frac{7}{3}\left(4^{N/5} \cdot 4 - 1\right)$

$T(N) = \Theta(4^{N/5}) = (4^N)$

Draw the tree. Show **levels 0,1,2** and the **leaves level**. Show the problem size T(...) as a label next to the node and inside the node show the local cost (cost of one node) as done in class. For the leaf level and level 2 it suffices to show a few nodes.

$T(N) = 1 \quad 0 \le N \le 4$

$T(N-5k) = 1$

$0 \le N-5k \le 4$

$0 \le N-5k \qquad N-5k \le 4$

$5k \le N \qquad N-4 \le 5k$

$k \le N/5 \qquad k \ge \frac{N-4}{5}$

Total Tree Cost calculation

$$T(N) = 7 + 4\cdot 7 + 4^2\cdot 7 \cdots\cdots 4^i\cdot 7 + \cdots\cdots + 4^k\cdot 7$$

$$= 7\left(4^0 + 4^1 + 4^2 + \cdots\cdots 4^k\right)$$

$$= 7\sum_{i=0}^{k} 4^{i}$$

$$= 7\sum_{i=0}^{k} 4^i \quad -① \quad x \text{ here is } 4 > 1$$

when $x > 1$ we have Summation formula

$$\sum_{k=0}^{n} x^k = \frac{x^{n+1} - 1}{x - 1}$$

$$\sum_{i=0}^{n} 4^i = \frac{4^{k+1} - 1}{4 - 1} = \frac{4^{k+1} - 1}{3}$$

Substitute in ①

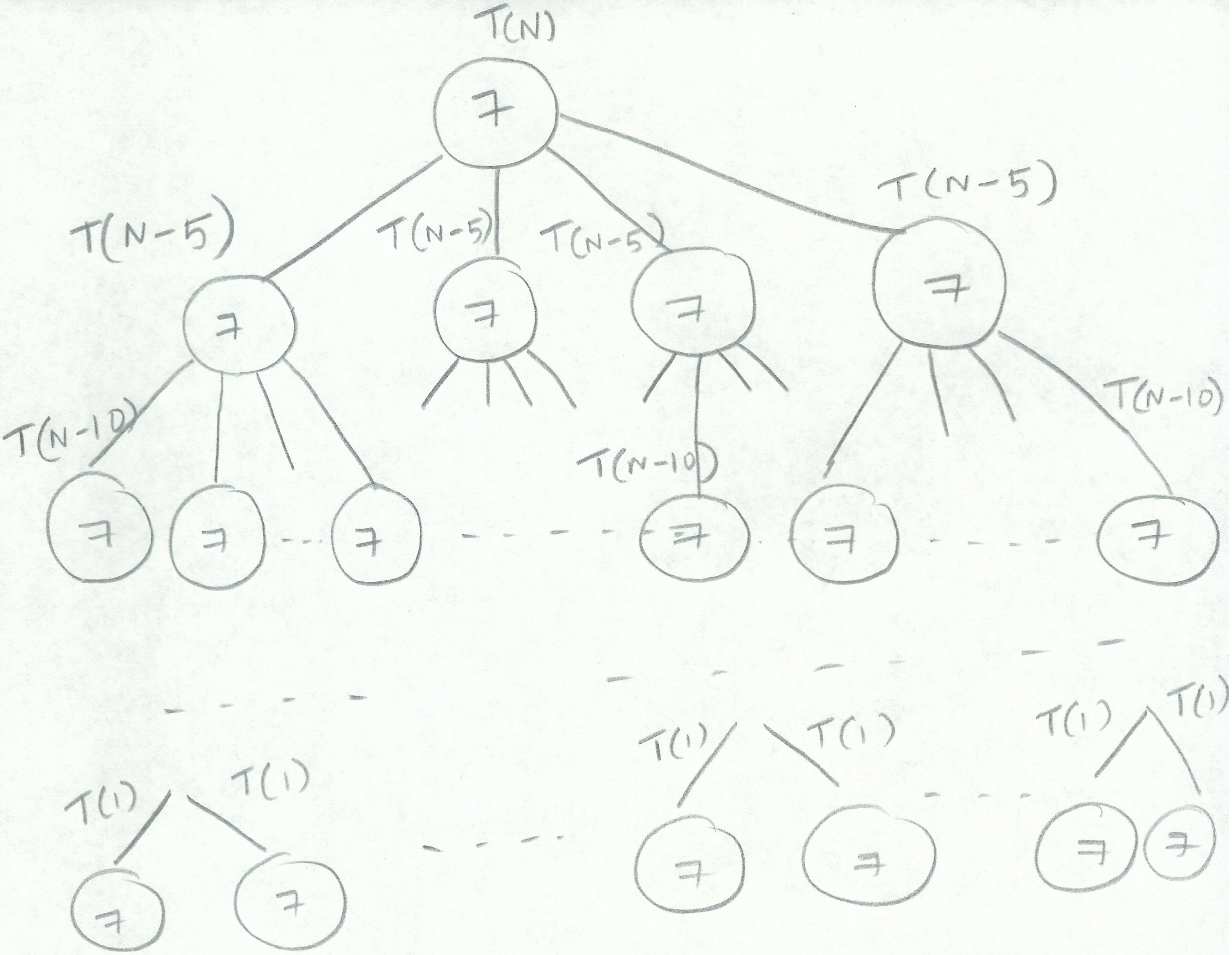
$$= 7 \times \frac{4^{k+1} - 1}{3} \Rightarrow \frac{7}{3}\left(4^{k+1} - 1\right) \qquad k = N/5$$

$$= \frac{7}{3}\left(4^k\cdot 4 - 1\right)$$

$$= \frac{7}{3}\left(4^{N/5}\cdot 4 - 1\right)$$

Hence

$$T(N) = \Theta\left(4^{N/5}\right)$$



T(N)
7
T(N-5)
T(N-5)
T(N-5)
T(N-5)
7
7
7
7
T(N-10)
T(N-10)
T(N-10)
7
7
7
7
7
7
T(1)
T(1)
T(1)
T(1)
T(1)
T(1)
7
7
7
7
7
7

P 3) a) $T(N) = 5T(\lfloor N/4 \rfloor) + 2N^3$. Assume $T(0)=1$, $T(1)=1$

This Equation of The form $T(N) = aT(N/b) + f(n)$

we have $a=5$ $b=4$ $f(n) = 2N^3$

$$\left.\begin{array}{l} a=5 \\ b=4 \end{array}\right\} n^{\log_b a} \Rightarrow n^{\log_4 5} \Rightarrow n^{1.12}$$

[Note $N = n$]

$f(n) = 2N^3$ $g(n) = 1.12$

$f(n) = \_\_\_ \; g(n)$

$2N^3 = \underline{\Omega} \; n^{1.12}$

<u>Case 3</u>: $f(n) = \Omega(n^{\log_b a} + \epsilon)$ for Some Constant $\epsilon > 0$ and if $af(n/b) \leq kf(n)$ for some constant $k<1$ and all sufficiently large n.

then $T(n) = \Theta f(n)$

$f(n) = \Omega(n^{(\log_b a)} + \epsilon)$

$2N^3 = \Omega(n^{1.12+\epsilon})$ $\epsilon = 0.88$

$2N^3 = \Omega(n^{1.12+0.88})$

$2N^3 = \Omega(n^2)$ $\rightarrow$ satisfy the equation if $af(n/b) \leq kf(n)$ for some constant $k < 1$

$a = 5 \quad b = 4$

$5 \times f(n/4) \leq kf(n)$

$5 \times 2(n/4)^3 \leq k(2n^3)$

$\frac{5}{4^3} \times 2(\cancel{n^3}) \leq k(\cancel{2n^3})$

$\frac{5}{64} \leq k$ $\quad$ $k = \frac{5}{64}$ which is less than $\underline{1}$

Hence, we have $T(n) = \theta f(n)$

$T(n) = \theta(2N^3)$

$= \underline{\underline{\theta(N^3)}}$

P3. b) $T(N) = 4T(\lceil N/4 \rceil) + d$, for some Constant $d > 0$.

Assume $T(0) = 1$ $T(1) = 1$.

$\rightarrow$ This Equation of the form $T(N) = aT(N/b) + f(n)$

we have $a = 4$ $b = 4$ $f(n) = d$, $d > 0$

$$\left.\begin{matrix} a = 4 \\ b = 4 \end{matrix}\right\} n^{\log_4 4} \Rightarrow n^1$$

$f(n) = d$ $d > 0$, Some Constant

$g(n) = n^1$

$f(n) = \_\_\_\_ \, g(n)$

$f(n) = \_\_\_\_ \, g(n)$

$d = \underline{O} \; n^1$ d can be any constant > 0.

Case 1: If $f(n) = O(n^{(\log_b a) - \epsilon})$ for Some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$

$f(n) = O(n^{(\log_b a) - \epsilon})$ $\epsilon > 0$ let $\epsilon = 0.5$

$d = O(n^{1 - 0.5})$

$d = O(n^{0.5})$ which satisfy the $\rightarrow$ Big O

Hence $T(n) = \Theta(n^{\log_b a}) \Rightarrow \Theta(n^{\log_4 4}) = \Theta(n^1)$

$\underline{T(n) = \Theta(n)}$

P3) c) $T(N) = 6T(N/6) + 5N$, Assume $T(0)=1$, $T(1)=1$

This Equation of the form $T(N) = aT(N/b) + f(n)$

We have $a=6$, $b=6$ $f(n) = 5N$

$$\left.\begin{matrix} a=6 \\ b=6 \end{matrix}\right\} n^{\log_b a} \Rightarrow n^{\log_6 6} \Rightarrow n^1 \qquad g(n) = n^1$$

$f(n) =$ ____ $g(n)$

$f(n) =$ ____ $g(n)$

$5N =$ ____ n^1

$5n =$ ____ n^1

$5n = (\Theta)n$

Case 2: If $f(n) = \Theta(n^{\log_b a})$ then $T(n) = \Theta(n^{\log_b a} \lg n)$

$\downarrow$

$5n = \Theta(n^{\log_6 6})$

$5n = \Theta(n^1)$ → still satisfy the condition

Hence, $T(n) = \Theta(n \lg n)$

P3) d) $T(N) = 8T(N/2) + CN^3 \lg N$, Assume $T(0)=1$ $T(1)=1$

→ This Equation of the form $T(N) = aT(N/b) + f(n)$

we have $a=8$ $b=2$ $f(n) = CN^3 \lg N$

$\left.\begin{matrix} a=8 \\ b=2 \end{matrix}\right\} n^{\log_b a} \Rightarrow n^{\log_2 8} \Rightarrow \underline{n^3}$ $g(n) = n^3$

$f(n) = \_\_\_ g(n)$

$f(n) = \_\_\_ g(n)$

$CN^3 \lg N = \underline{\Omega}\ n^3$

<u>lets</u> <u>see</u> <u>Case</u> 3: $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$ and if $af(n/b) \leq kf(n)$ for some constant $k > 1$ and sufficiently large n then $T(n) = \Theta f(n)$

→ $f(n) = \Omega(n^{\log_b a + \epsilon})$

$CN^3 \lg N = \Omega(n^{3+\epsilon})$

$CN^3 \lg N = \Omega(n^3 \cdot n^\epsilon)$ which is not true.

→ log always grows slower than any polynomial. Even if we choose any small value for ϵ, say suppose $\epsilon = 0.5$, The Equation will not hold good.

Hence, $Cn^3 \lg N = O(n^3 \cdot n^\epsilon)$ for any $\epsilon > 0$ we cannot apply Case 3. or any other case. Hence cannot solve this Equation using Masters Theorom.

P(4) go to section "Inadmissible equations" and list the reason why it does not satisfy Master Theorem requirement.

1) $T(n) = 2^n T\left(\frac{n}{2}\right) + n^n$

$a = 2^n \quad b = 2 \quad f(n) = n^n$

Here $a = 2^n$, here subproblem ~~nee~~ needs to be fixed. Subproblem is not a constant.

Hence, Master theorem Cannot be applied to this

2) $T(n) = 2T(n/2) + \frac{n}{\log n}$

$a = 2 \quad b = 2 \quad f(n) = \frac{n}{\log n}$

In Master theorem we have a polynomial Relation between $n^{\log_b a}$ and $f(n)$.

Here, $f(n) = \frac{n}{\log n}$ which is non polynomial degree. Hence Master theorem Cannot be applied.

3) $T(n) = 0.5\,T(n/2) + n$

$a = 0.5 \quad b = 2 \quad f(n) = n$

According to the Master theorom definition we have $a \geq 1$. Here $a = 0.5$ which indicates that subproblem is less than 1. Which cannot be accepted by Master Theorom. Hence, Master Theorom cannot be applied to this.

4) $T(n) = 64\,T(n/8) - n^2 \log n$

$a = 64 \quad b = 8 \quad f(n) = (n^2 \log n)$

$f(n)$ which is the combination time here is negative. It should be positive.

as per the Master theorom, Equation of the form

$T(n) = a\,T(n/b) + f(n)$

In this Equation $f(n)$ is clearly negative

Hence, Master theorom cannot be applied to this.

5) $T(n) = T(n/2) + n(2 - \cos n)$

Here, $a = 1$ $b = 2$ $f(n) = n(2 - \cos n)$

This Equation Violates Case 3

→ according to Case 3

we have $af(n/b) \leq kf(n)$

→ But give Equation violates this condition.

Hence cannot apply Master Theorom to this.

P(5) Show that $T(N) = 5T(\lfloor N/4 \rfloor) + 2N^3 = \Theta(N^3)$

by showing that it is $O(N^3)$ and $\Omega(N^3)$

Assume $T(0) = 1$ $T(1) = 1$

a) $\rightarrow T(N) = 5T\left(\lfloor N/4 \rfloor\right) + 2N^3 = O(N^3)$

Use inductive step to show "O" by definition of Constants.

we have $f(n) = O g(n)$ if and only if there exist c, n_0 Such that

$$f(n) \leq c g(n) \qquad \forall n \geq n_0$$

<u>Base Case</u>: $f(n) = 5T\left(\lfloor N/4 \rfloor\right) + 2N^3$ $\quad g(n) = N^3$

We Should Show $f(n) \leq c g(n)$

$T(N) \leq C N^3$

Consider $N=1$ $\quad T(1) = 1$

$$1 \leq cN^3$$

$$1 \leq c$$

$$\underline{\underline{c \geq 1}}$$

Consider $N=2$ $T(2) = 5T\left(\lfloor \frac{2}{4} \rfloor\right) + 2 \times 2^3$

$= 5(T\lfloor 0 \rfloor) + 2 \times 8$

$= 5 \times 1 + 16 = \underline{\underline{21}}$

$T\lfloor \frac{2}{4} \rfloor = \lfloor 0.5 \rfloor$
$= 0$

$21 \leq cn^3$

$21 \leq c8$

$\underline{\underline{c \geq 2.65}}$

$N=3$ $T(3) = 5T\left(\lfloor \frac{3}{4} \rfloor\right) + 2 \times 3^3$

$= 5T\lfloor 0 \rfloor + 54$

$= 5 \times 1 + 54 = \underline{\underline{59}}$

$59 \leq cn^3$

$59 \leq c27$

$\underline{\underline{c \geq 2.18}}$

$N=4$ $T(4) = 5T\lfloor \frac{4}{4} \rfloor + 2 \times 4^3$

$T(4) = 5 \times 1 + 128$

$T(4) = 133$

$133 \leq cn^3$

$133 \leq c64$

$\underline{\underline{c \geq 2.07}}$

$N=5 \quad T(5) = 5T\left\lfloor \frac{5}{4} \right\rfloor + 2 \times 5^3$

$T(5) = 5T\left\lfloor 1.25 \right\rfloor + 2 \times 125$

$T(5) = 5 \times 1 + 250$

$T(5) = 255$

$255 \leq cn^3$

$255 \leq c \times 125$

$\underline{\underline{c \geq 2.04}}$

Inductive step. $\Rightarrow$ Next page

Inductive step

Assume $T(k) \le ck^3 \quad \forall k < n$

$$T(n) = 5T\left(\frac{N}{\lfloor 4 \rfloor}\right) + 2n^3 \qquad \frac{N}{\lfloor 4 \rfloor} < n$$

$$T\left(\left\lfloor \frac{N}{4} \right\rfloor\right) \le c\left(\left\lfloor \frac{N}{4} \right\rfloor\right)^3$$

$$5T\left(\left\lfloor \frac{N}{4} \right\rfloor\right) + 2N^3$$

$$\le 5\cdot c\left(\left\lfloor \frac{N}{4} \right\rfloor\right)^3 + 2N^3 \le cN^3$$

$$\le 5\cdot c\,\frac{N^3}{4^3} + 2N^3 < cN^3$$

$$\le \cancel{N^3}\left(\frac{5c}{64} + 2\right) \le c\cancel{N^3}$$

$$\frac{5c}{64} + 2 \le c$$

$$5c + 128 \le 64c$$

$$\underline{\underline{c \ge 2.16}}$$

$$5T\left(\frac{4}{4}\right) + 2\times 4^3 = 3(4^3$$

$$5\times 1 + 2\times 64 = 192$$

$$\underline{\underline{133 \le 192}}$$

Hence, $c = 3 \qquad n_0 = 4$

$$T(N) = 5T\left(\left\lfloor \frac{N}{4} \right\rfloor\right) + 2N^3 = \underline{\underline{O(N^3)}}$$

b) To prove $f(n) = \Omega\, g(n)$

We have definition by Constants

There exist positive Constant C_o and n_o

Such that $C_o\, g(n) \leq f(n)$ for all $n \geq n_o$

Here, $g(n) \Rightarrow CN^3$

$f(n) \Rightarrow 5T\left(\lfloor N/4 \rfloor\right) + 2N^3$

$C_o\, g(n) \leq f(n)$

$N = 1$ $\quad CN^3 \leq 5T\left(\lfloor N/4 \rfloor\right) + 2N^3$ $\quad$ → We know $T(1) = 1$ from Subquestion (a)

$CN^3 \leq 1$

$\underline{C \leq 1}$

$N = 2$ $\quad CN^3 \leq 21$ $\quad$ → We know $T(2) = 21$

$C \times 8 \leq 21$

$\underline{C \leq 2.65}$

$N = 3$ $\quad CN^3 \leq 59$ $\quad$ → we know $T(3) = 59$

$C\, 27 \leq 59$

$\underline{C \leq 2.18}$

when $N=4$ $\quad CN^3 \leq 133$ $\quad$ → we know $T(4)=133$

$$C \times 64 \leq 133$$

$$\underline{\underline{C \leq 2.07}}$$

So from the Values we have got from above, I pick $c=1$ by ~~...~~ Considering Base Case when $N=1$

$C=1$, $n_0=4$

$$T(N) = 5T\left(\lfloor N/4 \rfloor\right) + 2N^3 \qquad g(n) = N^3$$

$$C_0 g(n) \leq 5T\left(\lfloor N/4 \rfloor\right) + 2N^3$$

$$C_0 N^3 \leq 5T\left(\lfloor N/4 \rfloor\right) + 2N^3$$

$$1 \times 64 \leq 5T \lfloor 4/4 \rfloor + 2 \times 64$$

$$1 \times 64 \leq 5 \times 1 + 128$$

$$1 \times 64 \leq 133$$ which satisfy the Equation.

Hence we Say $T(N) = 5T\left(\lfloor N/4 \rfloor\right) + 2N^3 = \Omega(N^3)$

We proved that

$$T(n) = 5T\left(\lfloor N/4 \rfloor\right) + 2N^3 = \Theta(N^3)$$

for values of $C=3$, $C=1$ $n_0=4$