Test No. 5 Raabe's Test (Higher Ratio Test) :→

If $\sum u_n$ is a positive term series such that $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=K$, then (i) the series is convergent if $K>1$

(ii) the series is divergent if $K<1$

Example:→ Test the convergence for the series

$$\frac{x}{1\cdot2}+\frac{x^2}{3\cdot4}+\frac{x^3}{5\cdot6}+\frac{x^4}{7\cdot8}+\cdots$$

Solution:→ Here $u_n=\frac{x^n}{(2n-1)2n}$ and $u_{n+1}=\frac{x^{n+1}}{(2n+1)(2n+2)}$

By D'Alembert's Ratio Test :→

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{\frac{x^{n+1}}{(2n+1)(2n+2)}}{\frac{x^n}{(2n-1)2n}}=\lim_{n\to\infty}\frac{x^{n+1}}{(2n+1)(2n+2)}\times\frac{(2n-1)2n}{x^n}$$

$$\lim_{n\to\infty}\frac{\cancel{x^n}\cdot x\times 2\cancel{n}\left(1-\frac{1}{2n}\right)2\cancel{n}}{2\cancel{n}\left(1+\frac{1}{2n}\right)2\cancel{n}\left(1+\frac{2}{2n}\right)\cancel{x^n}}=\lim_{n\to\infty}\frac{x\left(1-\frac{1}{2n}\right)}{\left(1+\frac{1}{2n}\right)\left(1+\frac{2}{2n}\right)}=x$$

$$\Rightarrow \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=x$$

(i) If $x<1$, $\sum u_n$ is convergent.

(ii) and if $x>1$, $\sum u_n$ is divergent.

(iii) If $x=1$, Test fails.

Now let us apply Raabe's Test, when $x=1$

So Now $u_n=\frac{1}{(2n-1)2n}$ and $u_{n+1}=\frac{1}{(2n+1)(2n+2)}$

thus $$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim_{n\to\infty} n\left[\frac{\frac{1}{(2n-1)2n}}{\frac{1}{(2n+1)(2n+2)}}-1\right]$$

$$=\lim_{n\to\infty} n\left[\frac{(2n+1)(2n+2)}{(2n-1)2n}-1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{(2n+1)(2n+2)-(2n-1)2n}{(2n-1)2n}\right] = \lim_{n\to\infty} n\left[\frac{4n^2+4n+2n+2-4n^2+2n}{(2n-1)2n}\right]$$

$$= \lim_{n\to\infty} \cancel{n}\left[\frac{8n+2}{(2n-1)2\cancel{n}}\right] = \lim_{n\to\infty} \frac{8n\left[1+\frac{2}{8n}\right]}{\cancel{2}n\left[1-\frac{1}{2n}\right]\cancel{2}} = \underline{2}$$

Here $K = 2 > 1$, So According to Raabe's test, the series is convergent. (if $x=1$).

$\Rightarrow$ The given series is convergent if $x \leq 1$ and divergent if $x > 1$.

Example (2) Test the following series for convergence $\sum \frac{1}{\sqrt{n+1}-1}$.

Solution: $\rightarrow$ Let $u_n = \frac{1}{\sqrt{n+1}-1}$, $\therefore u_{n+1} = \frac{1}{\sqrt{n+2}-1}$

By D'Alembert Ratio Test:$\rightarrow$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+2}-1}}{\frac{1}{\sqrt{n+1}-1}} = \lim_{n\to\infty} \frac{\sqrt{n+1}-1}{\sqrt{n+2}-1}$$

$$\Rightarrow \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}\right)-1}{\sqrt{n}\left(\sqrt{1+\frac{2}{n}}\right)-1}$$

$$= \lim_{n\to\infty} \frac{\cancel{\sqrt{n}}\left[\sqrt{1+\frac{1}{n}}-\frac{1}{\sqrt{n}}\right]}{\cancel{\sqrt{n}}\left[\sqrt{1+\frac{2}{n}}-\frac{1}{\sqrt{n}}\right]} = 1$$

D'Alembert's Ratio Test fails.

Now Applying Raabe's Test:$\rightarrow$ $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)$

$$= \lim_{n\to\infty} n\left[\frac{\sqrt{1+2/n}-1/\sqrt{n}}{\sqrt{1+1/n}-1/\sqrt{n}}-1\right]$$

$$= \lim_{n\to\infty} n\left[\frac{\sqrt{1+2/n}-\cancel{1/\sqrt{n}}-\sqrt{1+1/n}+\cancel{1/\sqrt{n}}}{\sqrt{1+1/n}-1/\sqrt{n}}\right]$$

$$= 0 < 1$$

$\longrightarrow$ Hence given series is divergent.

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Example (3) Discuss the convergence of the series

$$\frac{1}{2}+\frac{1\cdot3}{2\cdot4}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}+\dots$$

Sol:→ Here $\sum u_n=\frac{1}{2}+\frac{1\cdot3}{2\cdot4}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}$

Thus $u_n=\dfrac{1\cdot3\cdot5\dots(2n-1)}{2\cdot4\cdot6\dots2n}$

$\therefore\ u_{n+1}=\dfrac{1\cdot3\cdot5\dots(2n-1)\cdot(2n+1)}{2\cdot4\cdot6\dots(2n)(2n+2)}$

Now By D'Alembert's test $\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n}$

$$\Rightarrow \lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{\cancel{1\cdot3\cdot5\dots(2n-1)}(2n+1)}{\cancel{2\cdot4\cdot6\dots(2n)}(2n+2)}\times\frac{\cancel{2\cdot4\cdot6\dots2n}}{\cancel{1\cdot3\cdot5\dots(2n-1)}}$$

$$=\lim_{n\to\infty}\frac{(2n+1)}{(2n+2)}=\lim_{n\to\infty}\frac{\cancel{2n}(1+1/2n)}{\cancel{2n}(1+2/2n)}=1.$$

D'Alembert Ratio Test fails.

Now let us apply Raabe's Test:→

$$\lim_{n\to\infty}n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim_{n\to\infty}n\left[\frac{2n+2}{2n+1}-1\right]=\lim_{n\to\infty}n\left[\frac{\cancel{2n}+2-\cancel{2n}-1}{2n+1}\right]$$

$$=\lim_{n\to\infty}n\left[\frac{1}{2n+1}\right]=\lim_{n\to\infty}\frac{\cancel{n}}{2\cancel{n}[1+1/2n]}$$

$$=\frac{1}{2}<1$$

Hence the series is convergent by Raabe's test.

Example (4) Discuss the convergence of the series

$$\frac{x}{1}+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{x^5}{5}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{x^7}{7}+\dots\quad(x>0)$$

Sol:→ Neglecting the first term, we have, then

$u_n=\dfrac{1\cdot3\cdot5\dots(2n-1)\,x^{2n+1}}{2\cdot4\cdot5\dots(2n)\,(2n+1)}$

and $u_{n+1}=\dfrac{1\cdot3\cdot5\dots(2n-1)(2n+1)\cdot x^{2n+3}}{2\cdot4\cdot5\dots(2n)(2n+2)\,(2n+3)}$

Now by Ratio test (D'Alembert's Ratio test):→

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{1.3.5\ldots(2n-1)(2n+1)x^{2n+3}}{2.4.5\ldots(2n)(2n+2)(2n+3)}\times\frac{2.4.5\ldots(2n)(2n+1)}{1.3.5\ldots(2n-1)x^{2n+1}}$$

$$=\lim_{n\to\infty}\frac{(2n+1)(2n+1)x^{2n+3}}{(2n+2)(2n+3)x^{2n+1}}$$

$$=\lim_{n\to\infty}\frac{\cancel{2n}\left(1+\frac{1}{2n}\right)\times\cancel{2n}\left(1+\frac{1}{2n}\right)}{\cancel{2n}\left(1+\frac{2}{2n}\right)\cancel{2n}\left(1+\frac{3}{2n}\right)}\times x^2$$

$$=\lim_{n\to\infty}\frac{(1+1/2n)^2}{\left(1+\frac{1}{n}\right)\left(1+\frac{3}{2n}\right)}x^2=\underline{x^2}$$

∴ According to D'Alembert's Ratio Test, if $x^2>1\Rightarrow$ Series divergent

if $x^2<1\Rightarrow$ Series convergent.

$x^2=1\Rightarrow$ Test fails.

Now using Raabe's test:→

When $x^2=1$, We have $\frac{u_n}{u_{n+1}}=\frac{(2n+2)(2n+3)}{(2n+1)^2}=\frac{4n^2+10n+6}{4n^2+4n+1}$.

$$\lim_{n\to\infty}n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim_{n\to\infty}n\left(\frac{4n^2+10n+6}{4n^2+4n+1}-1\right)$$

$$=\lim_{n\to\infty}n\left(\frac{\cancel{4n^2}+10n+6-\cancel{4n^2}-4n-1}{4n^2+4n+1}\right)$$

$$=\lim_{n\to\infty}n\left(\frac{6n+5}{4n^2+4n+1}\right)=\lim_{n\to\infty}\frac{6n^2+5n}{4n^2+4n+1}$$

$$=\lim_{n\to\infty}\frac{\cancel{n^2}(6+5/n)}{\cancel{n^2}(4+4/n+1/n^2)}=\frac{6}{4}=\frac{3}{2}>1$$

⇒ By Raabe's Test, the series converges.

Hence $\sum u_n$ is convergent if $x^2\leq 1$ and divergent if $x^2>1$.