9.5 LAGRANGE'S LINEAR EQUATION IS AN EQUATION OF THE TYPE

$$Pp + Qq = R$$

where P, Q, R are the function of x, y, z and $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$

Solution. $\qquad Pp + Qq = R$

This form of the equation is obtained by eliminating an arbitrary function f from

$$f(u, v) = 0$$

where u, v are functions of x, y, z.

Differentiating (2) partially w.r.t. to x and y.

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0$$

and

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right) = 0$$

Let us eliminate $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ from (3) and (4).

From (3), $\qquad \dfrac{\partial f}{\partial u}\left[\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial z}p\right] = -\dfrac{\partial f}{\partial v}\left[\dfrac{\partial v}{\partial x} + \dfrac{\partial v}{\partial z}p\right]$

From (4), $\qquad \dfrac{\partial f}{}\left[\dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z}q\right] = -\dfrac{\partial f}{\partial v}\left[\dfrac{\partial v}{\partial y} + \dfrac{\partial v}{\partial z}q\right]$

Dividing (5) by (6), we get

$$\frac{\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot p}{\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot q}=\frac{\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot p}{\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot q}$$

$$\Rightarrow \quad \left[\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot p\right]\left[\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}\cdot q\right]=\left[\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\cdot q\right]\left[\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot p\right]$$

$$\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial y}+\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial z}\cdot q+\frac{\partial u}{\partial z}\times p\times\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial z}\cdot pq$$

$$=\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial z}p+\frac{\partial u}{\partial z}\cdot q\times\frac{\partial v}{\partial x}+\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial z}\cdot pq$$

$$\Rightarrow \quad \left[\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial y}\right]p+\left[\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial z}\right]q$$

$$=\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial x} \qquad \ldots (7)$$

If (1) and (7) are the same, then the coefficients of p, q are equal.

$$P=\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial y}$$

$$Q=\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial z} \qquad \ldots (8)$$

$$R=\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial x}$$

Now suppose $u = c_1$ and $v = c_2$ are two solutions, where c_1, c_2 are constants.

Differentiating $u = c_1$ and $v = c_2$

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=0 \qquad \ldots (9)$$

and

$$\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz=0 \qquad \ldots (10)$$

Solving (9) and (10), we get

$$\frac{dx}{\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial y}}=\frac{dy}{\frac{\partial u}{\partial z}\times\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial z}}=\frac{dz}{\frac{\partial u}{\partial x}\times\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\times\frac{\partial v}{\partial x}} \qquad \ldots (11)$$

From (8) and (11), we have

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

Solutions of these equations are

$$u = c_1 \quad \text{and} \quad v = c_2$$

$\therefore$ $f(u, v) = 0$ is the required solution of (1).

9.6. WORKING RULE TO SOLVE $Pp + Qq = R$

Step 1. Write down the auxiliary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Step 2. Solve the above auxiliary equations.

Let the two solutions be $u = c_1$ and $v = c_2$.

Step 3. Then $f(u, v) = 0$ or $u = \phi(v)$ is the required solution of

$$Pp + Qq = R.$$

Example 6. *Solve the following partial differential equation*

$$yq - xp = z, \qquad \text{where } p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y}.$$

Solution. We have, $yq - xp = z$

Here the auxiliary equations are

$$\frac{dx}{-x} = \frac{dy}{y} = \frac{dz}{z}$$

$$-\log x = \log y - \log a \quad \text{(From first two equations)}$$

$$xy = a \qquad \text{... (1)}$$

$$\log y = \log z + \log b \quad \text{(From last two equations)}$$

$$\frac{y}{z} = b \qquad \text{... (2)}$$

From (1) and (2) we get the solution

$$f\left(xy, \frac{y}{z}\right) = 0.$$ Ans.

Example 7. *Solve $y^2p - xyq = x(z - 2y)$.* *(A.M.I.E., Summer 2001)*

Solution. We have, $y^2p - xyq = x(z - 2y)$

The auxiliary equations are

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)} \qquad \text{... (1)}$$

Considering first two members of the equations

$$\frac{dx}{y} = \frac{dy}{-x} \quad \Rightarrow \quad x\,dx = -y\,dy$$

Integrating $\quad \frac{x^2}{2} = -\frac{y^2}{2} + \frac{C_1}{2}$

$$\Rightarrow \quad x^2 + y^2 = C_1 \qquad \text{... (2)}$$

From last two equations of (1), we have

$$-\frac{dy}{y} = \frac{dz}{z-2y}$$

$$\Rightarrow \quad -z\,dy + 2y\,dy = y\,dz \quad \Rightarrow \quad 2y\,dy = y\,dz + z\,dy$$

On integration, we get

$$y^2 = yz + C_2 \qquad \text{... (3)}$$

$$y^2 - yz = C_2$$

From (2) and (3), we have

$$x^2 + y^2 = f(y^2 - yz)$$ Ans.

Example 8. *Solve* $(x^2 - yz)\, p + (y^2 - zx)\, q = z^2 - xy$

(A.M.I.E., Summer 2001)

Solution. $(x^2 - y\,z)\, p + (y^2 - z\,x)\, q = z^2 - x\,y$ ... (1)

The auxiliary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$$

$$\frac{dx - dy}{x^2 - yz - y^2 + zx} = \frac{dy - dz}{y^2 - zx - z^2 + xy} = \frac{dz - dx}{z^2 - xy - x^2 + yz}$$

$$\frac{dx - dy}{(x-y)(x+y+z)} = \frac{dy - dz}{(y-z)(x+y+z)} = \frac{dz - dx}{(z-x)(x+y+z)}$$

$$\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z} = \frac{dz - dx}{z - x} \qquad \text{... (2)}$$

Integrating first two members of (2), we have

$$\log (x - y) = \log (y - z) + \log c_1$$

$$\log \frac{x - y}{y - z} = \log c_1 \qquad \Rightarrow \qquad \frac{x - y}{y - z} = c_1 \qquad \text{... (3)}$$

Similarly from last two members of (2), we have

$$\frac{y - z}{z - x} = c_2 \qquad \text{.... (4)}$$

From (3) and (4), the required solution is

$$f\left[\frac{x - y}{y - z}, \frac{y - z}{z - x}\right] = 0$$

Ans.

9.7 METHOD OF MULTIPLIERS

Let the auxiliary equations be

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

l, m, n may be constants or functions of x, y, z then, we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

l, m, n are chosen in such a way that

$$lP + mQ + nR = 0$$

Thus
$$l\,dx + m\,dy + n\,dz = 0$$

Solve this differential equation, if the solution is $u = c_1$.

Similarly, choose another set of multipliers (l_1, m_1, n_1) and if the second solution is $v = c_2$.

$\therefore$ Required solution is $f(u, v) = 0$.

Example 9. *Solve*

$$(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = l\,y - mx.$$

(A.M.I.E. Winter 2001)

Solution. We have, $(mz - ny)\dfrac{\partial z}{\partial x} + (nx - lz)\dfrac{\partial z}{\partial y} = l\,y - mx$

Here, the auxiliary equations, are

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}$$

Using multipliers $x, y, z,$ we get

$$\text{each fraction} = \frac{x\,dx+y\,dy+z\,dz}{x(mz-ny)+y(nx-lz)+z(ly-mx)}=\frac{x\,dx+y\,dy+z\,dz}{0}$$

$\therefore$ $x\,dx + y\,dy + z\,dz = 0$

which on integration gives

$$x^2+y^2+z^2=c_1 \quad \ldots(1)$$

Again using multipliers, $l, m, n,$ we get

$$\text{each fraction} = \frac{l\,dx+m\,dy+n\,dz}{l(mz-ny)+m(nx-lz)+n(ly-mx)}=\frac{l\,dx+m\,dy+n\,dz}{0}$$

$\therefore$ $l\,dx + m\,dy + n\,dz = 0$

which on integration gives

$$lx+my+nz=c_2 \quad \ldots(2)$$

Hence from (1) and (2), the required solution is

$$x^2+y^2+z^2=f(lx+my+nz)$$

Ans.

Example 10. *Solve the partial differential equation* $x(y^2+z)\,p-y(x^2+z)\,q=z(x^2-y^2)$

where, $p=\dfrac{\partial z}{\partial x}$ and $q=\dfrac{\partial z}{\partial y}$. *(U.P., II Semester, 2005)*

Solution. Lagrange's subsidiary equations are

$$\frac{dx}{x(y^2+z)}=\frac{dy}{-y(x^2+z)}=\frac{dz}{z(x^2-y^2)} \quad \ldots(1)$$

Using $x, y, -1$ as multipliers, we get

$$\text{each fraction} = \frac{x\,dx+y\,dy-dz}{x^2(y^2+z)-y^2(x^2+z)-z(x^2-y^2)}$$

$$=\frac{x\,dx+y\,dy-dz}{0}$$

$\therefore$ $x\,dx+y\,dy-dz=0$

Integrating, we get

$$\frac{x^2}{2}+\frac{y^2}{2}-z=\frac{C_1}{2}$$

$$\Rightarrow \quad x^2+y^2-2z=C_1 \quad \ldots(2)$$

Again, using $\dfrac{1}{x}, \dfrac{1}{y}$ and $\dfrac{1}{z}$ as multipliers, we get

$$\text{each fraction} = \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{y^2+z-x^2-z+x^2-y^2}$$

$$=\frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{0}$$

$$\therefore \quad \frac{1}{x}\,dx + \frac{1}{y}\,dy + \frac{1}{z}\,dz = 0$$

Integrating, we get

$$\log x + \log y + \log z = \log C_2$$

$$\Rightarrow \quad xyz = C_2 \qquad ...(3)$$

Hence the general solution is

$$\phi(x^2 + y^2 - 2z, xyz) = 0 \qquad \textbf{Ans.}$$

Example 11. *Find the general solution of*

$$x(z^2 - y^2)\frac{\partial z}{\partial x} + y(x^2 - z^2)\frac{\partial z}{\partial y} = z(y^2 - x^2)$$

Solution. $x(z^2 - y^2)\dfrac{\partial z}{\partial x} + y(x^2 - z^2)\dfrac{\partial z}{\partial y} = z(y^2 - x^2)$ ... (1)

$\therefore$ The auxiliary simultaneous equations are

$$\frac{dx}{x(z^2 - y^2)} = \frac{dy}{y(x^2 - z^2)} = \frac{dz}{z(y^2 - x^2)} \qquad ... (2)$$

Using multipliers x, y, z, we get

$$\text{Each term of (2)} = \frac{x\,dx + y\,dy + z\,dz}{x^2(z^2 - y^2) + y^2(x^2 - z^2) + z^2(y^2 - x^2)} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$\therefore \quad x\,dx + y\,dy + z\,dz = 0 \qquad ... (3)$$

On integration $x^2 + y^2 + z^2 = C_1$

Again (2) can be written as

$$\frac{\frac{dx}{x}}{z^2 - y^2} = \frac{\frac{dy}{y}}{x^2 - z^2} = \frac{\frac{dz}{z}}{y^2 - x^2} = \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{(z^2 - y^2) + (x^2 - z^2) + (y^2 - x^2)} \qquad ... (4)$$

$$= \frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{0} \quad \Rightarrow \quad \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

On integration, we get

$$\Rightarrow \quad \log x + \log y + \log z = \log C_2 \qquad .... (5)$$

$$\log x\,y\,z = \log C_2 \quad \Rightarrow \quad x\,y\,z = C_2$$

From (3) and (5), the general solution is

$$xyz = f(x^2 + y^2 + z^2) \qquad \textbf{Ans.}$$

Example 12. *Solve the partial differential equation*

$$\frac{y - z}{yz}\,p + \frac{z - x}{zx}\,q = \frac{x - y}{xy}$$

Solution. We have, $\dfrac{y - z}{yz}\,p + \dfrac{z - x}{zx}\,q = \dfrac{x - y}{xy}$

Multiplying by xyz, we get

$$x(y - z)\,p + y(z - x)\,q = z(x - y)$$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} = \frac{dx+dy+dz}{x(y-z)+y(z-x)+z(x-y)} \quad \text{...(1)}$$

$$= \frac{dx+dy+dz}{0}$$

$\therefore$ $$dx + dy + dz = 0$$

Which on integration gives

$$x + y + z = a \quad \text{...(2)}$$

Again (1) can be written as

$$\frac{\frac{dx}{x}}{y-z} = \frac{\frac{dy}{y}}{z-x} = \frac{\frac{dz}{z}}{x-y} = \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{(y-z)+(z-x)+(x-y)} = \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$$

$\Rightarrow$ $$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z} = 0$$

On integration, we get

$$\log x + \log y + \log z = \log b \Rightarrow \log xyz = \log b \Rightarrow xyz = b \quad \text{...(3)}$$

From (2) and (3) the general solution is

$$xyz = f(x+y+z)$$

Ans.

Example 13. *Solve* $(x^2 - y^2 - z^2)\, p + 2xy\, q = 2xz.$ *(A.M.I.E., Summer 2004, 2000)*

Solution. We have, $(x^2 - y^2 - z^2)\, p + 2xy\, q = 2xz$ ...(1)

Here the auxiliary equations are

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} \quad \text{...(2)}$$

From the last two members of (2), we have

$$\frac{dy}{y} = \frac{dz}{z}$$

which on integration gives

$$\log y = \log z + \log a \Rightarrow \log \frac{y}{z} = \log a$$

$\Rightarrow$ $$\frac{y}{z} = a \quad \text{...(3)}$$

Using multipliers x, y, z, we have

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2+y^2+z^2)}$$

$$\frac{2x\,dx + 2y\,dy + 2z\,dz}{(x^2+y^2+z^2)} = \frac{dz}{z}$$

which on integration gives

$$\log (x^2 + y^2 + z^2) = \log z + \log b \quad \text{...(4)}$$

$$\frac{x^2+y^2+z^2}{z} = b$$

Hence from (3) and (4), the required solution is

$$x^2 + y^2 + z^2 = z\, f\left(\frac{y}{z}\right)$$

Ans.

Example 14. *Solve the differential equation*

$$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=(x+y)z$$

Solution. We have, $$x^2\frac{\partial z}{\partial x}+y^2\frac{\partial z}{\partial y}=(x+y)z \quad ...(1)$$

The auxiliary equations of (1) are

$$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{(x+y)z} \quad ...(2)$$

Take first two members of (2) and integrate them

$$-\frac{1}{x}=-\frac{1}{y}+c$$

$$\frac{1}{x}-\frac{1}{y}=c_1 \quad ...(3)$$

(2) can be written as

$$\frac{\frac{dx}{x}}{x}=\frac{\frac{dy}{y}}{y}=\frac{\frac{dz}{z}}{x+y}=\frac{\frac{dx}{x}+\frac{dy}{y}-\frac{dz}{z}}{(x+y)-(x+y)}$$

$$\Rightarrow \quad \frac{dx}{x}+\frac{dy}{y}-\frac{dz}{z}=0$$

On integration, we get

$$\Rightarrow \quad \log x+\log y-\log z=\log c_2$$

$$\Rightarrow \quad \log\frac{xy}{z}=\log c_2 \quad \Rightarrow \quad \frac{xy}{z}=c_2 \quad ...(4)$$

From (3) and (4), we have

$$f\left[\frac{1}{x}-\frac{1}{y},\ \frac{xy}{z}\right]=0. \qquad \textbf{Ans.}$$

Example 15. *Find the general solution of*

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}+t\frac{\partial z}{\partial t}=xyt$$

Solution. The auxiliary equations are

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dt}{t}=\frac{dz}{xyt} \quad ...(1)$$

Taking the first two members and integrating, we get

$$\log x=\log y+\log a=\log ay$$

$$x=ay, \quad i.e. \quad x/y=a \quad ...(2)$$

Similarly, from the 2nd and 3rd members

$$\frac{t}{y}=b \quad ...(3)$$

Multiplying the equation (1) by xyt, we get

$$dz=\frac{ty\,dx}{1}=\frac{tx\,dy}{1}=\frac{xy\,dt}{1}=\frac{ty\,dx+tx\,dy+xy\,dt}{3}$$

Integrating, we get

$$z = \frac{1}{3}xyt + c \Rightarrow z - \frac{1}{3}xyt = c$$

From (2), (3) and (4) the solution is

$$z - \frac{1}{3}xyt = f\left(\frac{y}{x}\right) + \phi\left(\frac{t}{y}\right)$$

Example 16. *Solve* $(y + z)\,p - (x + z)\,q = x - y$

Solution. $(y + z)\,p - (x + z)\,q = x - y$

$\therefore$ The auxiliary equations are

$$\frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y}$$

$$\Rightarrow \frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y} = \frac{dx+dy+dz}{y+z-(x+z)+x-y}$$

$$\Rightarrow \frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y} = \frac{dx+dy+dz}{0}$$

Thus, we have $dx + dy + dz = 0$

Which on integration gives $x + y + z = c_1$

Using multipliers $x, y, -z$ for (2), we get

$$\frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y} = \frac{x\,dx + y\,dy - z\,dz}{x(y+z) - y(x+z) - z(x-y)}$$

$$\Rightarrow \frac{dx}{y+z} = \frac{dy}{-(x+z)} = \frac{dz}{x-y} = \frac{x\,dx + y\,dy - z\,dz}{0}$$

Integrating $x\,dx + y\,dy - z\,dz = 0$, we get

$$\frac{x^2}{2} + \frac{y^2}{2} - \frac{z^2}{2} = c_2$$

$$\Rightarrow x^2 + y^2 - z^2 = 2c_2$$

From (3) and (4), we get the required solution

$$f(x + y + z,\ x^2 + y^2 - z^2) = 0$$

Example 17. *Solve* $z\,p + yq = x$.

Solution. The auxiliary equations are

$$\underset{(i)}{\frac{dx}{z}} = \underset{(ii)}{\frac{dy}{y}} = \underset{(iii)}{\frac{dz}{x}}$$

From (*i*) and (*iii*)

$$\frac{dx}{z} = \frac{dz}{x} \Rightarrow x\,.\,dx = z\,.\,dz$$

$$\Rightarrow \frac{x^2}{2} = \frac{z^2}{2} - \frac{c_1}{2} \Rightarrow x^2 = z^2 - c_1$$

$$z = \sqrt{x^2 + c_1}$$

Putting the value of z in (1), we get

$$\frac{dx}{\sqrt{x^2 + c_1}} = \frac{dy}{y}$$

$$\sinh^{-1}\frac{x}{\sqrt{c_1}} = \log y + c_2 \qquad \dots (3)$$

From (2) and (3), the required solution is

$$f(z^2 - x^2) = \sinh^{-1}\frac{x}{\sqrt{c_1}} - \log y$$

Ans.

Example 18. *Solve* $px\,(z - 2y^2) = (z - qy)\,(z - y^2 - 2x^3)$.

(*A.M.I.E., Summer 2000*)

Solution. $px\,(z - 2y^2) = (z - qy)\,(z - y^2 - 2x^3)$

$$px\,(z - 2y^2) + qy\,(z - y^2 - 2x^3) = z\,(z - y^2 - 2x^3) \qquad \dots (1)$$

Here the auxiliary equations are

$$\frac{dx}{x(z-2y^2)} = \frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)} \qquad \dots (2)$$

From the last two members of (2), we have

$$\frac{dy}{y} = \frac{dz}{z}$$

which gives on integration

$$\log y = \log z + \log a \quad \Rightarrow \quad y = a\,z \qquad \dots (3)$$

From the first and third members of (2), we have

$$\frac{dx}{x(z-2y^2)} = \frac{dz}{z(z-y^2-2x^3)} \qquad \text{[Using (3), } y = az]$$

$$\frac{dx}{x(z-2a^2z^2)} = \frac{dz}{z(z-a^2z^2-2x^3)}$$

$$\frac{dx}{x(1-2a^2z)} = \frac{dz}{z-a^2z^2-2x^3}$$

$$z\,dx - a^2z^2dx - 2x^3dx = x\,dz - 2a^2\,x\,z\,dz$$

$$(x\,dz - z\,dx) - a^2\,(2\,x\,z\,dz - z^2dx) + 2x^3\,dx = 0$$

$$\frac{x\,dz - z\,dx}{x^2} - a^2\frac{(2xz\,dz - z^2\,dx)}{x^2} + 2x\,dx = 0$$

On integrating,

$$\frac{z}{x} - a^2\frac{z^2}{x} + x^2 = b \qquad \dots (4)$$

From (3) and (4), we have the required solution:

$$\frac{y}{z} = f\left(\frac{z}{x} - \frac{a^2z^2}{x} + x^2\right)$$

Ans.

EXERCISE 9.3

Solve the following partial differential equations

1. $p \tan x + q \tan y = \tan z$ **Ans.** $f\left(\dfrac{\sin x}{\sin y}, \dfrac{\sin y}{\sin z}\right) = 0$

2. $(y - z)\, p + (x - y)\, q = z - x$ **Ans.** $f(x + y + z, x^2 + 2yz)$

3. $(y + zx)\, p - (x + yz)\, q = x^2 - y^2$ **Ans.** $f(x^2 + y^2 - z^2) = (x - y)^2 - (z ...$

4. $zx\dfrac{\partial z}{\partial x} - zy\dfrac{\partial z}{\partial y} = y^2 - x^2$ **Ans.** $f(x^2 + y^2 + z^2, xy) = 0$

5. $pz - qz = z^2 + (x + y)^2$ **Ans.** $[z^2 + (x + y)^2]\, e^{-2x} = f(x + y)$

6. $p + q + 2xz = 0$ **Ans.** $f(x - y) = x^2 + \log z$

7. $x^2 p + y^2 q + z^2 = 0$ **Ans.** $f\left(\dfrac{1}{y} - \dfrac{1}{x}, \dfrac{1}{y} + \dfrac{1}{z}\right) = 0$

8. $(x^2 + y^2)\, p + 2xyq = (x + y)\, z$ *(A.M.I.E., Summer 2000)* **Ans.** $f\left(\dfrac{x + y}{z}, \dfrac{2y}{x^2 - y^2}\right) = 0$

9. $\dfrac{\partial z}{\partial x} - 2\dfrac{\partial z}{\partial y} = 2x - e^y + 1$ **Ans.** $f(2x + y) = z - \dfrac{(2x+1)^2}{4} - \dfrac{e^y}{2}$

10. $p + 3q = 5z + \tan(y - 3x)$ **Ans.** $f(y - 3x) = \dfrac{e^{5x}}{5z + \tan(y - 3x)}$

11. $xp - yq + x^2 - y^2 = 0$ **Ans.** $f(xy) = \dfrac{x^2}{2} + \dfrac{y^2}{2} + z$

12. $(x + y)\left(\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y}\right) = z - 1$ **Ans.** $f(x - y) = \dfrac{x + y}{(z - 1)^2}$.

13. $(x^3 + 3xy^2)\dfrac{\partial z}{\partial x} + (y^3 + 3x^2 y)\dfrac{\partial z}{\partial y} = 2(x^2 + y^2)z$ *(A.M.I.E.T.E., Summer 20..)*

14. $(z^2 - 2yz - y^2)\, P + (xy + zx)\, q = xy - zx$

15. Find the solution of the equation $\dfrac{x\partial z}{\partial y} - \dfrac{y\partial z}{\partial x} = 0$, which passes through the curve $z = 1, x^2 + y^2 = 4$

Ans. $f(x^2 + y^2 - 4, z - 1) = ...$

OBJECTIVE TYPE QUESTIONS

Choose the correct alternative :

1. The partial differential equation from $z = (a + x)^2 + y$ is

(i) $z = \dfrac{1}{4}\left(\dfrac{\partial z}{\partial x}\right)^2 + y$ (ii) $z = \left(\dfrac{\partial z}{\partial x}\right)^2 + y$

(iii) $z = \dfrac{1}{4}\left(\dfrac{\partial z}{\partial y}\right)^2 + y$ (iv) $z = \left(\dfrac{\partial z}{\partial y}\right)^2 + y$ Ans. (...

2. The solution of $xp + yq = z$ is

(i) $f(x^2, y^2) = 0$ (ii) $f(x, y) = 0$

(iii) $f\left(\dfrac{x}{y}, \dfrac{y}{z}\right) = 0$ (iv) $f(xy, yz) = 0$ Ans. (...

3. The solution of $\frac{y-z}{yz}p + \frac{z-x}{zx}q = \frac{x-y}{xy}$ is

(i) $x-y-z = f(x+y+z)$ (ii) $x+y+z = f(x+y+z)$

(iii) $x-y-z = f(x-y-z)$ (iv) $xyz = f(x+y+z)$ **Ans. (iv)**

4. The solution of $(y+z)p-(x+z)q = x-y$ is

(i) $f(x+y+z, x^2 y^2 z^2)$ (ii) $f(x+y+z, x^2+y^2-z^2) = 0$

(iii) $f(x-y-z, x^2+y^2-z^2)$ (iv) $f(xyz, x^2+y^2-z^2) = 0$ **Ans. (ii)**

...icate True or False for the following statements

5. The auxiliary equation of $Pp + Qq = R$ can be written as

$$\frac{dp}{P} = \frac{dq}{Q} = \frac{dz}{R}$$

Ans. True

6. From the equation $z = (x+4)(y+6)$ a partial differential equation $pq = 2$ is formed.

Ans. False

7. The auxiliary equation of

$$(mz-ny)\frac{\partial z}{\partial x} + (nx-lz)\frac{\partial z}{\partial y} = ly - mx \text{ is } \frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly+mx}.$$

Ans. False

8. The auxiliary equation of $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + t\frac{\partial z}{\partial t} = xyt$ is $\frac{dx}{x} = \frac{dy}{y} = \frac{dt}{t}$ **Ans. False**

9. The *A.E.* of $(y+z)p-(x+z)q = x-y$ is $\frac{dx}{y+z} = \frac{dy}{x+z} = \frac{dz}{x-y}$ **Ans. False**

10. With usual symbols, the *P.D.E.* $u_{xx} + u^2 u_{yy} = f(xy)$ is non-linear in 'u' and is of second order.

Ans. True

(U.P., II Semester, 2009)

11. Solution of the *P.D.E.* $\frac{\partial^2 z}{\partial x \partial y} = xy^2$ is $z = \frac{x^2 y^3}{6} + f(y) + \phi(x)$ **Ans. True**