Working Rule to finding P.I.

Case (III) When $f(x,y) = \phi(ax+by)$

i.e. $F(D,D')z = \phi(ax+by)$, $F(a,b) \neq 0$

Working Rule to finding P.I :-

* **Step ①** Replace D by a, D' by b in $F(D,D')$ to get $F(a,b)$

* **Step ②** put $ax+by = u$ and integrate $\phi(u)$ n-times w.r.t. u

Then

$$P.I = \frac{1}{F(a,b)} \underbrace{\iiint \cdots \int}_{(n\text{-times})} \phi(u)\, \underbrace{du\,du\,du \ldots du}_{(n\text{-times})}$$

Step ③ Replace u by $ax+by$ atlast.

Note:- If $F(a,b) = 0$, then method fails.

then differentiate $F(D,D')$ partially w.r.t D and multiply with x in numerator, and again check $F'(a,b) \neq 0$

i.e. $$\frac{x \cdot \phi(ax+by)}{\frac{\partial}{\partial D}[F(D,D')]}, \text{ provided } F'(a,b) \neq 0$$

Example (1)

Solve $\frac{\partial^2 z}{\partial x^2} + 3\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = x+y$

$\Rightarrow (D^2 + 3DD' + 2D'^2)z = x+y$

A.E. $m^2 + 3m + 2 = 0$

or $(m+1)(m+2) = 0$

$\Rightarrow m = -1, -2$

then $C.F. = f_1(y-x) + f_2(y-2x)$

Now $P.I. = \frac{(x+y)}{D^2+3DD'+2D'^2}$ $\left\{ \because a=1, b=1 \right\}$

put $D=1$ and $D'=1$

We get

$P.I. = \frac{1}{(1)^2+3(1)(1)+2(1)^2} \iint u \, du \, du$

(Where $x+y = u$)

$P.I. = \frac{1}{6} \cdot \int \frac{u^2}{2} du$

or $P.I. = \frac{1}{6} \times \frac{u^3}{3\times 2} = \frac{u^3}{36}$

or $P.I. = \frac{u^3}{36}$

or $P.I. = \frac{(x+y)^3}{36}$

Then complete sol:-

$z = C.F. + P.I. = f_1(y-x) + f_2(y-2x) + \frac{(x+y)^3}{36}$

Ans.

Ex. (2) Solve $(D^2 - D'^2)z = x - y$

A.E. $m^2 - 1 = 0$

$\Rightarrow m = 1, -1$

C.F. $\Rightarrow f_1(y+x) + f_2(y-x)$

$$P.I. = \frac{1}{D^2 - D'^2}(x-y) \quad \left\{ \begin{array}{l} \text{Here } a = 1 \\ \text{and } b = -1 \end{array} \right\}$$

put $D = 1$ and $D' = -1$

We get

$$P.I. = \frac{1}{1-(-1)^2}(x-y)$$

denominator becomes zero, which is not possible

then $P.I. = \frac{x}{2D}(x-y)$

$$= \frac{x}{2}\int u\,du \quad \left\{ \begin{array}{l} \text{where} \\ u = x-y \end{array} \right\}$$

$$= \frac{x}{2} \times \frac{u^2}{2}$$

$$\Rightarrow \frac{xu^2}{4}$$

$$P.I. = \frac{x(x-y)^2}{4}$$

Thus Complete sol^n $\rightarrow$

$z = C.F. + P.I.$

$= f_1(y+x) + f_2(y-x) + \frac{x}{4}(x-y)^2$ Ans.

Ex. (3) Solve $(D^2+2DD'-8D'^2)z = \sqrt{2x+3y}$

A.E. $m^2+2m-8=0$

$\Rightarrow (m+4)(m-2)=0$

$\Rightarrow m = 2, -4$

C.F. $= f_1(y+2x) + f_2(y-4x)$

$$P.I. = \frac{1}{D^2+2DD'-8D'^2}\sqrt{2x+3y}$$

$$= \frac{1}{(2)^2+2(2)(3)-8(3)^2}(2x+3y)^{1/2} \quad \begin{bmatrix} \because a=2 \\ b=3 \end{bmatrix}$$

$$P.I. = \frac{1}{4+12-72}\iint u^{1/2}\,du\,du$$

$\left\{\begin{array}{l} \text{where} \\ u = 2x+3y \end{array}\right\}$

$$= \frac{1}{-56}\int \frac{2}{3}u^{3/2}\,du$$

$$= \frac{2}{-56\times 3}\times\frac{2}{5}u^{5/2}$$

$$P.I. = \frac{-4}{840}u^{5/2}$$

or $$P.I. = \frac{-1}{210}(2x+3y)^{5/2}$$

Complete sol:→

$$z = f_1(y+2x) + f_2(y-4x) - \frac{1}{210}(2x+3y)^{5/2}$$

Solve

Ex. 4 $(D^3 - 3D^2D' - 4DD'^2 + 12D'^3)z = \sin(y+2x)$

A.E. $m^3 - 3m^2 - 4m + 12 = 0$

$\Rightarrow (m^2-4)(m-3) = 0$

$\Rightarrow m = 2, -2, 3$

C.F. $= f_1(y+2x) + f_2(y-2x) + f_3(y+3x)$

$$P.I. = \frac{1}{D^3 - 3D^2D' - 4DD'^2 + 12D'^3} \sin(y+2x)$$

put $D = 2$ and $D' = 1$ $\left(\because \phi(x,y) = y+2x \atop \phi(x,y) = ax+by\right)$

then

$$P.I. = \frac{1}{(2)^3 - 3(2)^2(1) - 4(2)(1) + 12(1)^3} \iiint \sin u \, du\,du\,du$$

(where $u = y+2x$)

or

$$P.I. = \frac{1}{8-12-8+12} \iiint \sin u \, du\,du\,du$$

Not possible (since denominator can't be zero)

So

$$P.I. = \frac{x}{3D^2 - 6DD' - 4D'^2} \iint \sin u \, du\,du$$ (Hence integrate two times)

$$= \frac{x}{3(2)^2 - 6(2)(1) - 4(1)^2}(-\sin u)$$

$$= \frac{x}{4} \sin(y+2x)$$

* Case (IV)

When $f(x,y) = x^m y^n$

Then

$$P.I. = \frac{x^m y^n}{F(D,D')} = [F(D,D')]^{-1} x^m y^n$$

but cases aries:-

(a) If $m > n$, then $\frac{1}{F(D,D')}$ is expanded in the power of $\frac{D'}{D}$.

(b) If $m < n$, then $\frac{1}{F(D,D')}$ is expanded in the power of $\frac{D}{D'}$.

dono powers equal hain

Ex. (1) Solve $(D^2 + D'^2) z = x^2 y^2$ {Here powers $m = n = 2$ (equal)}

A.E. $\Rightarrow m^2 + 1 = 0$

$\Rightarrow m = \pm i$

then $C.F. = f_1(y + ix) + f_2(y - ix)$

$$P.I. = \frac{1}{D^2 + D'^2}(x^2 y^2)$$

$$= \frac{1}{D^2\left(1 + \frac{D'^2}{D^2}\right)}(x^2 y^2)$$

$$= \frac{1}{D^2}\left(1+\frac{D'^2}{D^2}\right)^{-1}(x^2y^2)$$

$$= \frac{1}{D^2}\left[1-\frac{D'^2}{D^2}\right](x^2y^2)$$

$$= \frac{1}{D^2}(x^2y^2) - \frac{D'^2}{D^4}(x^2y^2)$$

$$= \frac{x^4}{12}y^2 - \frac{1}{D^4}(2x^2) \qquad \left[\because \frac{1}{D} = \int f(x)\,dx \quad \frac{1}{D'} = \int f\,dy\right.$$

$$\Rightarrow \frac{x^4y^2}{12} - \frac{\cancel{2}\,x^6}{3\times4\times5\times\cancel{6}_3}$$

$$= \frac{x^4y^2}{12} - \frac{x^6}{180}$$

$$= \frac{1}{180}(15x^4y^2 - x^6)$$

Complete sol $\Rightarrow$

$$z = f_1(y+ix) + f_2(y-ix) + \frac{1}{180}(15x^4y^2 - x^6)$$

Ans.

Ex. (2) Solve

$$(D^2+4D'^2)z = x^2y^4$$

$m=2$
$n=4$

Yahan y ki power jyada hai x se, isliye $4D'^2$ ko common lenge P.I. nikalne ke liye

A.E. $\Rightarrow m^2+4=0$

$m = 2i, -2i$

C.F. $= f_1(y+2ix) + f_2(y-2ix)$

$$P.I. = \frac{1}{D^2+4D'^2}x^2y^4$$

$$= \frac{1}{4D'^2\left[1+\frac{D^2}{4D'^2}\right]}x^2y^4$$

$$= \frac{1}{4D'^2}\left[1+\frac{D^2}{4D'^2}\right]^{-1}x^2y^4$$

$$= \frac{1}{4D'^2}\left[1-\frac{D^2}{4D'^2}+\dots\right]x^2y^4$$

$$= \frac{1}{4D'^2}\left[x^2y^4-\frac{D^2}{4D'^2}(x^2y^4)+\dots\right]$$

$$= \frac{1}{4D'^2}\left[x^2y^4-\frac{1}{2}\frac{y^6}{5\times6}\right]$$

$$= \frac{1}{4}\left[\frac{x^2y^6}{5\times6}-\frac{1}{2}\frac{y^8}{5\times6\times7\times8}\right]$$

$$= \frac{1}{13440}\left[112x^2y^6-y^8\right]$$

Then complete sol^n $z = C.F. + P.I.$

or $z = f_1(y+2ix)+f_2(y-2ix)+\frac{1}{13440}\left[112x^2y^6-y^8\right]$ Ans.