Assignment -2 Vivekshome

Q4) Evaluate the Contour integration $\int_0^{2\pi} \frac{1}{a+b\sin\theta}\, d\theta$ where $a>b$

Sol→ $d\theta = \frac{dz}{iz}$ $\qquad \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$

$$= \int_c \frac{1}{a+\frac{b}{2}\left(z+\frac{1}{z}\right) iz}$$

$$\frac{1}{i}\int \frac{1}{2az+bz^2+b} \Rightarrow \frac{1}{bi}\int \frac{1}{\frac{2a}{b}z+z^2+1}$$

Now $z^2 + \frac{2az}{2} + 1 = 0$

$$D = \frac{-b \pm \sqrt{b^2-4ab}}{2a}$$

$$= \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2}{b^2}-4}}{2}$$

$$\Rightarrow \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2-4b^2}{b^2}}}{2}$$

$$\alpha \Rightarrow -\frac{a}{b} + \frac{\sqrt{a^2-b^2}}{b} \quad —(1)$$

$$\beta = -\frac{a}{b} - \frac{\sqrt{a^2-b^2}}{b} \quad —(2)$$

$$\Rightarrow \frac{1}{bi}\int \frac{dz}{(z-\alpha)(z-\beta)} = \frac{1}{bi}\, 2\pi i\,(R_1 + R_2)$$

We know here $|c| = 1$

$\therefore$ β lies outside R_2,
$\therefore R_2 = 0$

$$\lim_{z\to\alpha} (z-\alpha)\,\frac{1}{(z-\alpha)(z-\beta)}$$

$$\Rightarrow \frac{1}{z-\beta}$$

$$\Rightarrow \frac{1}{\alpha-\beta} \Rightarrow \frac{b}{2\sqrt{a^2-\beta^2}}$$

[putting the value of α & β from eq.

Now $$\int_c \frac{dz}{(z-\alpha)(z-\beta)} = \frac{1}{bi}\, 2\pi i\, \frac{b}{2\sqrt{a^2-b^2}}$$

$$\Rightarrow \frac{\pi}{\sqrt{a^2-b^2}}$$

Q6) Using Cauchy residue theorem to evaluate the integral

$$\int_c \frac{z^2-2z}{(z+1)^2(z^2+4)}\,dz$$ where C is the circle $|z|=$

Sol $\Rightarrow$ Poles $\Rightarrow$ $(z+1)^2 = ~~-1, 4~~ \; 1, 1$
$z^2+4 = 2i, -2i$

$$\lim_{z\to -1} \frac{1}{(2+1)!}\left[\frac{d}{dz}\cancel{(z+1)^2}\frac{(z^2-2z)}{\cancel{(z+1)^2}(z+2i)(z-2i)}\right]$$

$$\lim_{z\to 1} \Rightarrow \frac{1}{3!}\left[\frac{d}{dz}\frac{z^2-2z}{(z+2i)(z-2i)}\right]$$

$$\lim_{z\to 1} \Rightarrow \frac{1}{9}\left[\frac{2z-2\ (z^2+4) - 2z(z^2-2z)}{(z^2+4)^2}\right]$$

$$\frac{1}{9}\left[\frac{2-2\ (4+4) - 2(1-2)}{(5)^2}\right]$$

$$\frac{1}{9}\left[\frac{2-10+2}{25}\right] \Rightarrow \frac{6}{9\times 25} \Rightarrow \frac{2}{75}$$

~~Res~~ for

Res$(z=2i)$

$$\lim_{z=2i} = \left[(z-2i)\times\frac{z^2-2z}{(z-2i)(z+2i)\ (z+1)^2}\right]$$

$$\lim_{z\to 2i} \Rightarrow \left[\frac{z^2-2z}{(z+2i)(z+1)^2}\right]$$

$$\lim \Rightarrow \frac{-4-2}{(4i)(2i+1)^2} \Rightarrow -\frac{\cancel{6}}{4\times 9} \Rightarrow -\frac{1}{6}$$

Res$(z=-2i)$

$$\lim_{z=-2i} = [\cancel{z+2i}]\times\frac{z^2-2z}{(z-2i)\cancel{(z+2i)}(z+1)^2}$$

$$\Rightarrow -4- \quad = 0$$

$I = \cancel{\frac{1}{2\pi i} \times 2\pi i \left[\frac{2}{75} - \frac{1}{6} + 0\right]}$

$\Rightarrow \quad I = 2\pi i \left[\frac{2}{75} - \frac{1}{6}\right] \Rightarrow 2\pi i$

Q3) using residue calculus to evaluate the following integral

$$\int_0^{2\pi} \frac{1}{5-4\sin\theta}\, d\theta$$

Sol $\Rightarrow$ $\displaystyle\int_0^{2\pi} \frac{dz/iz}{5-4\left[\frac{z-\frac{1}{z}}{2i}\right]} \Rightarrow \int_0^{2\pi} \frac{\frac{dz}{iz}}{\frac{10i-4z^2-4}{2iz}}$

Lim $\Rightarrow$ $\displaystyle\int_0^{2\pi} \frac{\frac{dz}{z}}{5i-2z^2-2}$

$$\Rightarrow \frac{1}{i}\int_0^{2\pi} \frac{dz}{5z-2z^2-2}$$

$$\Rightarrow -\frac{1}{i}\int_0^{2\pi} \frac{dz}{2z^2-5z+2} \Rightarrow \frac{-1}{i}\, 2\pi i(R_1+R_2)$$

Poles $\Rightarrow$ $2z^2-5z+2=0$

$2z^2-4z-z+2=0$

$2z(z-2)-1(z-2)=0$

$z=\frac{1}{2}, \; z=2$

we know that $|c|=1$

$z=2 \Rightarrow R_1=0$

$\Rightarrow \lim_{z \to \frac{1}{2}} \left(z-\frac{1}{2}\right) \frac{\cancel{dz}\, 1}{(z-2)(2z-1)}$

$\Rightarrow \lim_{z \to \frac{1}{2}} \left[\frac{\cancel{2z-1}}{2}\right] \frac{\cancel{dz}\, 1}{(2z-1)(z-2)}$

$\Rightarrow \lim_{z \to \frac{1}{2}} \frac{\cancel{dz}\, 1}{2z-4} \Rightarrow \frac{1}{1-4} \Rightarrow -\frac{1}{3}$

Now $\Rightarrow \int_c \frac{dz}{(2z-1)(z-2)} \Rightarrow \frac{-1}{4} \times 2\pi i \times \frac{-1}{3} \Rightarrow \frac{2\pi}{3}$

Q2) By Cauchy reside theorem

$\oint_c \frac{3z^2+z+1}{(z^2-1)(z+3)} dz$ where $|z|=2$

Sol $\Rightarrow$ $(z^2-1)(z+3)=0$

$z = \pm 1, -3$ simple poles

R $\Rightarrow$ at

-3 does not exist as it is outer ~~part~~ part of circle as lies outside the circle

$R(\text{at } z=1) = \lim_{z \to 1} \cancel{3z^2}(z-1) \cdot f(z)$

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$$\Rightarrow \lim_{z\to 1} (z-1) \frac{(3z^2+z+1)i}{(z-1)(z+1)(z+3)}$$

$$\Rightarrow \lim_{z\to 1} \frac{(3z^2+z+1)i}{(z+1)(z+3)} \Rightarrow \frac{(3+1+1)i}{2\times 4}$$

$$\Rightarrow \frac{5}{8}i$$

$$R(\text{at } z=-1) = \lim_{z\to -1} (z+1) f(z)$$

$$\Rightarrow \lim_{z\to -1} (z+1) \frac{(3z^2+z+1)i}{(z-1)(z+1)(z+3)}$$

$$\Rightarrow \frac{3-1+1}{-2\times 2} \Rightarrow \frac{-3}{4}i$$

$$\int_c f(z)dz = 2\pi i \left[R_1 + R_2\right]$$

$$\Rightarrow 2\pi i\left[\frac{5}{8}i - \frac{3}{4}i\right] = 2\pi i\left[\frac{5-6}{8}\right]$$

$$\Rightarrow \frac{2\pi}{8} = \frac{\pi}{4}$$

Q1) Apply calcus of residue to evaluate

$$\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta$$

Sol ⇒ $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$e^{i\theta} = z$

$d\theta = \frac{dz}{iz}$

$$\Rightarrow \int_0^{2\pi} \left[\frac{1 + \cancel{2}\left(\frac{e^{i\theta} + e^{-i\theta}}{\cancel{2}}\right)}{5 + 4\left[\frac{e^{i\theta} + e^{-i\theta}}{2}\right]} \right] \frac{dz}{iz}$$

$$\Rightarrow \int_0^{2\pi} \left[\frac{1 + z + \frac{1}{z}}{5 + 2\left[z + \frac{1}{z}\right]} \right] \frac{dz}{iz}$$

$$\Rightarrow \int_0^{2\pi} \frac{z + z^2 + 1}{5z + 2z^2 + 2} \times \frac{dz}{iz}$$

$$\Rightarrow \frac{1}{i} \int_0^{2\pi} \frac{z + z^2 + 1}{5z^2 + 2z^3 + 2z} \, dz$$

$5z^2 + 2z^3 + z = 0$

$z\left[2z^2 + 5z + 2\right] = 0$

$z = 0, \; z = 2, 1/2$

only pole $1/2$, 0, lies inside the circle $|z| = 1$

Res at $z = 0$ $\lim_{z \to 0} (z - 0)\, f(z)$

$$\lim_{z \to 0} \cancel{(z-0)} \frac{(z + z^2 + 1)\, i}{\cancel{z}[z - 2][z + 1/2]} \Rightarrow \frac{(-1)\, i}{1}$$

$\Rightarrow -1\, i$

R_2 at $z = -\frac{1}{2}$ $\lim_{z \to -\frac{1}{2}} \frac{\cancel{(z+\frac{1}{2})} (z + z^2 + 1) i}{\cancel{(z+\frac{1}{2})}(z-2)(z)}$

$\Rightarrow \frac{\left[\frac{1}{4} - \frac{1}{2} + 1\right] i}{\left(-\frac{1}{2} - 2\right)\left(-\frac{1}{2}\right)}$ $\Rightarrow \left[\frac{1 - 2 + 4}{5}\right] i$

$\Rightarrow \frac{3}{5} i$

Caushy Residues theorem $\Rightarrow \int_0^{2\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta} d\theta$

$\Rightarrow \oint 2\pi i \left[\frac{3}{5} i - i\right]$

$\Rightarrow -\frac{4\pi i}{5} \times i = \frac{4\pi}{5}$