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UNIT-3 Partially ordered sets:

Partialally ordered set]

Definition → Consider a relation R on a set S satisfying the following properties:

(1) R is reflexive i.e xRx for every $x \in S$.
(2) R is antisymmetric i.e if xRy and yRx then $x=y$
(3) R is ~~antisymmetric~~ transitive i.e if xRy and yRz then xRz

Then R is called a partially order relation and the set S together with partial order is called a partially order set POSET and is denoted by $(S, \leq)$.

Example Consider a set $A = \{4, 9, 19, 36\}$. Is the relation 'divides' a partial order.

Soln The relation 'divides' is a partial order if it satisfies the property of reflexivity antisymmetry and transitivity

(1) Since for every ~~a~~ 'divides' is $a \in A$ ~~partial~~ we have $a|a$. Hence divides is reflexive.

(2) If $a|b$ and $b|a$, we have $a=b$ for any

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$a, b \in A$. Hence 'divides' is antisymmetric

(3) If $a|b$ and $b|c$, we have $a|c$ for any $a, b, c \in A$. Hence, the relation 'divides' is a partial order and $(A, |)$ is a poset.

Q: Let $A = \{1, 2, 3, 4\}$
and Relation
$R = \{(1,1), (1,2), (1,3), (1,4), (2,2)$
$(2,3), (2,4), (3,3), (3,4),$
$(4,4)\}$
Determine (A, R) is a poset?

Soln (1) $(1,1)$, $(2,2), (3,3), (4,4) \in R$.
So R is Reflexive.

(2) $(1,2) \in R$ but $(2,1) \notin R$
or $(2,3) \in R$ but $(3,2) \notin R$
so R is Antisymmetry.

(3) $(1,2) \in R$, $(2,3) \in R$
$\Rightarrow (1,3) \in R$
so R is Transitive.

Hence (A, R) is a poset.

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Q2. Check whether the relation R defined in the set $\{1, 2, 3, 4, 5, 6\}$ as $R = \{(a, b) : b = a+1\}$ is reflexive, ~~symm~~ Antisymmetry or transitive

Soln Let set $A = \{1, 2, 3, 4, 5, 6\}$

Relation $R = \{(a, b) : b = a+1\}$

$R = \{(1,2)\ (2,3)\ (3,4)\ (4,5)\ (5,6) \dots\}$

(1) Since $(1,1)\ (2,2)\ (3,3) \dots \notin R$
so R is not reflexive

(2) $(1,2) \in R$, $(2,1) \notin R$
so R is Antisymmetry

(3) $(1,2) \in R$, $(2,3) \in R$ but $(1,3) \notin R$
so R is Not Transitive

$\Rightarrow$ R is not a poset.

Complete (total) partial ordering chain :

A set paired with a total order is called totally ordered set or a chain.

* **Chain** → If S is totally ordered under $\leq$ then the following statement hold for all a, b, c in S.

1. If $a \leq b$ and $b \leq a$ then $a = b$ (Anti symmetric)
2. If $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity)
3. $a \leq b$ or $b \leq a$ (totally property)

Example ① In the poset $(Z, \leq)$ $a \leq b$ or $b \leq a$ for all integers a and b hence $(Z, \leq)$ is totally ordered.

Example ② But the poset $(Z \leq)$ is not totally ordered since it contains elements that are incomparable such as 3 and 5.

⇒ Although an ordered set may not be totally ordered. ~~clearly~~ it ~~very~~ is still possible for a subset A of S to be totally ordered. Clearly every subset of totally ordered set S must also be totally ordered.

Example Consider the poset (Z, |) which is not totally ordered but $A = \{2, 6, 12, 36\}$ is a totally ordered subset of Z since 2|6 (2 divides 6), 6|12 (6 divides 12) and 12|36 (12 divides 36)

Note A chain in S is a subset C of S in which each pair of elements is comparable i.e is C is totally ordered.

Lattices :-

A lattice L is a poset in which every pair of elements has a least upper bound (LUB) or supremum and greatest lower bound (GLB) or infimum.

Join → Consider a poset L under the ordering ≤. Let $a, b \in L$. Then LUB (a,b) or sup (a,b) is denoted by $a \vee b$ or $a \cup b$ and is called the join of a and b i.e $a \vee b = \sup(a,b)$

Meet → Consider a poset L under the ordering ≤. Let $a, b \in L$. Then GLB (a,b) or inf (a,b) is denoted by $a \wedge b$ or $a \cap b$ and is called the meet of a and i.e $a \cap b = \inf(a,b)$

From the above it follows that a lattice L is a Mathematical structure with two binary operation $\vee$ (Joint) and $\wedge$ (meet). It is denoted by $(L, \vee, \wedge)$. The lattice L for any elements a, b and c satisfies the

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following properties :-

(A) Commutative property :-

(i) $a \wedge b = b \wedge a$
(ii) $a \vee b = b \vee a$

(b) Associative Property

(i) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$

(ii) $(a \vee b) \vee c = a \vee (b \vee c)$

(C) Absorption Property :-

(i) $a \wedge (a \vee b) = a$
(ii) $a \vee (a \wedge b) = a$

Theorem → Prove that if L be a lattice then $a \wedge b = a$ if and only if $a \vee b = b$.

proof → Let us first assume that

$a \wedge b = a$.

using absorption property

$b = b \vee (b \wedge a) = b \vee (a \wedge b)$

$= b \vee a = a \vee b$ —— (1)

Conversely, let us assume $a \vee b = b$

Again using the absorption property, we have,

$a = a \wedge (a \vee b) = a \vee b$. —— (2)

from equation (1) and (2)

$a \wedge b = a$ if and only if $a \vee b = b$.

Theorem → Prove that for elements of lattice

i) $a \wedge a = a$ ii) $a \vee a = a$

Proof → (i) $a \wedge a = a \wedge (a \vee (a \wedge b))$

$= a$ [By Absorption law]

(ii) $a \vee a = a \vee (a \wedge (a \vee b))$

$= a$ (By Absorption law)

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Theorem → Consider a lattice L. Prove that the relation $a \le b$ defined by either $a \wedge b = a$ $a \vee b = b$ is a partial ordering on lattice L.

Proof → For any element $a \in L$, we have $a \wedge a = a$
$a \le a$. Therefore, the relation $\le$ is reflexive. Now assume $a \le b$ and $b \le a$. Then we have

$$a \wedge b = a \text{ and } b \wedge a = b$$

Thus $a = a \wedge b = b \wedge a = b$ therefore the relation $\le$ is antisymmetric. At last, we assume $a \le b$ and $b \le c$ so, we have

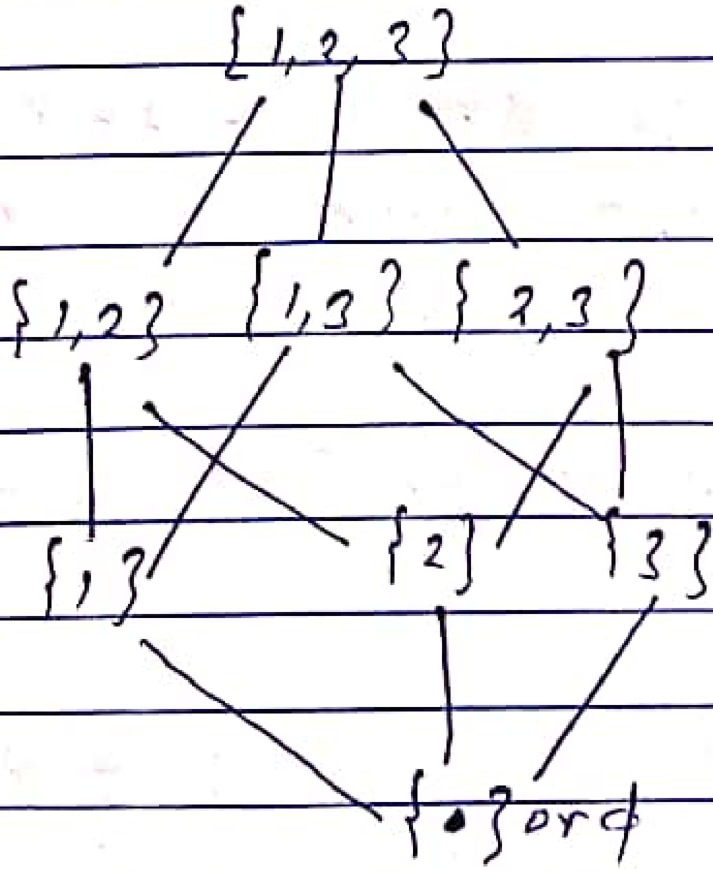
$$a \wedge b = a \text{ and } b \wedge c = b$$

$$a \wedge c = (a \wedge b) \wedge c = (a \wedge (b \wedge c))$$
$$= a \wedge b = a.$$

Therefore $a \le c$. So the relation $\le$ is transitive. From above we can say, $\le$ is a partial order on L.

Q. Let P(S) be the power set of the set S = {1,2,3}. Construct the Hasse diagram of the partial order induced on P(S) by the lattice (P(S), ∧, ∨).

Soln The Hasse diagram obtain by lattice is same as obtain under the partial ordering of set inclusion. In the lattice, $a \le b$ whenever $a \wedge b = a$. Thus in the above case $a \le b$ whenever $a \wedge b = a$



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Complete lattice → A lattice P L is called complete iff every non-empty subset of L has greatest lower bound and lower uper bound.

Theorem If L is a complete lattice then it is a bounded lattice.

proof → Let us consider $(L, \leq)$ be a complete lattice, every non empty subset of L has its supremum and Infimum in L.
i.e every non empty subset of L has the least upper bound and greatest lower bound in L.
In particular L is a non empty subset of itself.
∴ L has the least upper bound and greatest lower bound in it

Theorem 2 → Every finite lattice is complete.

Let S be any subset of L
i.e $S \subseteq L$
∵ L is finite &
∴ S is a finite set

Let $S = \{x_1, x_2, \dots x_n\}$
$\therefore$ L is a lattice
$x_1, x_2 \in S \Rightarrow x_1, x_2 \in L$
According to the definition of lattice.
$Inf\{x_1, x_2\}$ and $sup\{x_1, x_2\}$.
exist in L i.e $x_1 \wedge x_2 \in L$
and $x_1 \vee x_2 \in L$
Again $x_1 \wedge x_2 \in L$ and $x_3 \in L$

$\Rightarrow (x_1 \wedge x_2) \wedge x_3 \in L$
$\Rightarrow x_1 \wedge x_2 \wedge x_3 \in L$
Similarly we may show that

$x_1 \wedge x_2 \wedge x_3 \wedge \dots \wedge x_n \in L$.
i.e $Inf(S)$ exits.
Similarly we may show that
$sup(S)$ exists.
$\therefore$ S is an arbitary subset of L
$\therefore$ every subset of L has inf and Sup in L
$\therefore (L, \leq, \wedge, \vee)$ is a complete lattice proved.

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Q A bounded lattice need not be a complete lattice

Proof → We shall prove this with the help of particular example

Let us consider
$L = \{x : 0 \leq x \leq 2$ and x is a rational Number $\}$
The relation '$\leq$' is the usual "less than or equal to" relation defined on L then it is obvious that $(L, \leq)$ is a lattice
The least element of L is 0 and greatest element is 2.

$\therefore (L, \leq)$ is a bounded lattice
Now we shall show that $(L, \leq)$ is not a complete lattice
Let us $S \subseteq L$ such that ~~$(L, \leq)$ is not a complete lattice~~
$S = \{x : 0 \leq x^2 \leq 2$ and x is a Rational Number $\}$
i) Let $x = 0$. since $1 \in S$
$\therefore 0 < 1$ in S
$\therefore 0$ can not be an upper bound

of S.
i.e 0 is not a lower upper bound of S.

(ii) Let $x > 0$ and $x^2 < 2$

$\Rightarrow x^2 - 2 < 0$ and $2 - x^2 > 0$ } — (1)

Now let $y = \frac{4+3x}{3+2x}$ — (2)

$\Rightarrow y^2 - 2 = \frac{x^2 - 2}{(3+2x)^2}$ — (3)

and $y - x = \frac{2(2-x^2)}{3+2x}$ — (4)

using (1) $y - x > 0$
$\Rightarrow y > x$ — (5)

$\because$ x is a positive rational Number
By (2) y is a positive rational Number.
from (1) and (2)
$y^2 - 2 < 0$
$\Rightarrow y^2 < 2$
$\Rightarrow y \in S$
from (5) it is clear that x is not an upper bound in S.
i.e S is not lub in S.

DISTRIBUTIVE LATTICE :-

A lattice L is called distributive lattice if for any elements a, b and c of L, it satisfies following distributive properties :-

(i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

If the lattice L does not satisfies the above properties, it is called a non - distributive lattice.

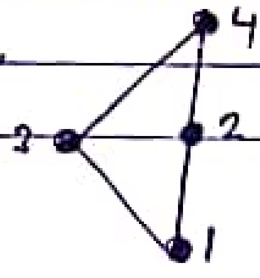
Example 1.

① The power set $P(S)$ of the set S under the operations of intersection and union is a distributive function. Since

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$$

and also $a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$ for any sets a, b, and c of $P(S)$

② The lattice of given figure is distributive. Since it satisfies the distributive properties for all ordered triples which are taken from 1, 2, 3 and 4.



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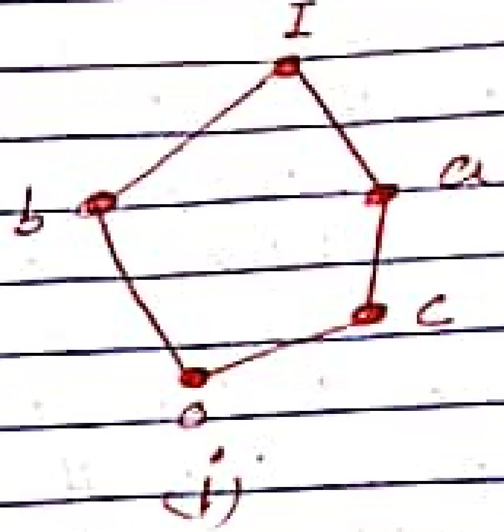
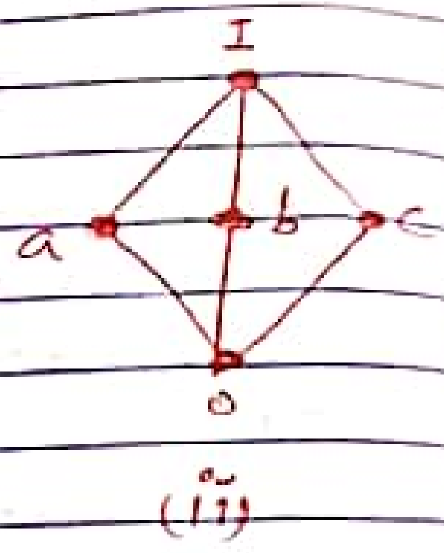
Q. Show that the lattice shown in fig are non-distributive





Sol/n (i) $a \wedge (b \vee c) = a \wedge I = a$

But $(a \wedge b) \vee (a \wedge c) = 0 \vee c = c$

Since $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$

Hence, the lattice is not distributive.

(ii) Again $a \wedge (b \vee c) = a \wedge I = a$

But $(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$

Since $a \wedge (b \vee c) \neq (a \wedge b) \vee (a \wedge c)$

Hence, the lattice is not distributive.

Theorem → Prove that in a distributive lattice $(L, \wedge, \vee)$, $(a\wedge b)\vee(b\wedge c)\vee(c\wedge a) = (a\vee b)\wedge(b\vee c)\wedge(c\vee a)$ holds for all $a, b, c \in L$.

Proof → We have given that L is distributive, so using distributive property we have.

$(a\wedge b)\vee(b\wedge c)\vee(c\wedge a)$

$= \{\{(a\vee b)\vee b\}\wedge\{(a\wedge b)\vee c\}\}\vee(c\wedge a)$

$= \{b\wedge\{(a\vee c)\wedge(b\vee c)\}\}\vee(c\wedge a)$

$= \{(a\vee c)\wedge\{b\wedge(b\vee c)\}\}\vee(c\wedge a)$

$= \{(a\vee c)\wedge b\}\vee(c\wedge a)$

$= \{(a\vee c)\vee(c\wedge a)\}\wedge\{b\vee(c\wedge a)\}$

$= \{\{(a\vee c)\vee c\}\wedge\{(a\vee c)\vee a\}\}\wedge\{(b\vee c)\wedge(b\vee a)\}$

$= (a\vee c)\wedge(a\vee c)\wedge(b\vee c)\wedge(a\vee b)$

$= (a\vee c)\wedge(b\vee c)\wedge(a\vee b) = R.H.S$

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MODULAR LATTICE

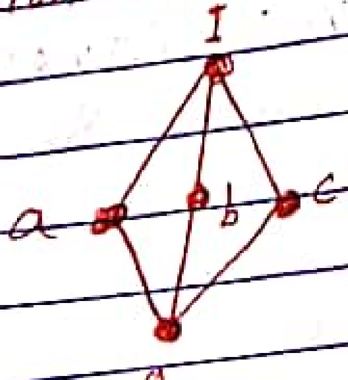
A lattice $(L, \wedge, \vee)$ is called a modular lattice if $a \vee (b \wedge c) = (a \vee b) \wedge c$ whenever $a \leq c$.

Theorem – Show that $a \vee (b \wedge c) = (a \vee b) \wedge c$ whenever $a \leq c$.

Proof → If $a \leq c$, then $a \vee c = c$.
If L is distributive, then we have
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$
$$= (a \vee b) \wedge c$$

Note (i) Every distributive lattice is modular.
(ii) Every modular lattice is not distributive.

Q. The lattice shown in figure is a Non distributive modular lattice.



$$a \wedge (b \vee c) = a \wedge 1 = a$$
$$(a \wedge b) \vee (a \wedge c) = 0 \vee 0 = 0$$

Complemented lattices :-

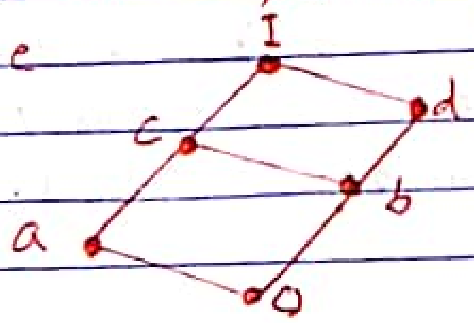
Consider a bounded lattice L with greatest element I and the least element 0. An element $x \in L$ is called a complement of x if $x \vee x' = 1$ and $x' \wedge x = 0$

From the definition of complement if x' is a complement of x then x is a complement of x'. It is not necessary that an element x has a complement. Also the complements need not be unique i.e an element have more than one complement.

Note. That $1' = 0$ and $0' = 1$

Definition → A lattice L is called a complemented lattice if L is bounded and every element in L has a complement.

Q. Determine the complement of a and c in figure



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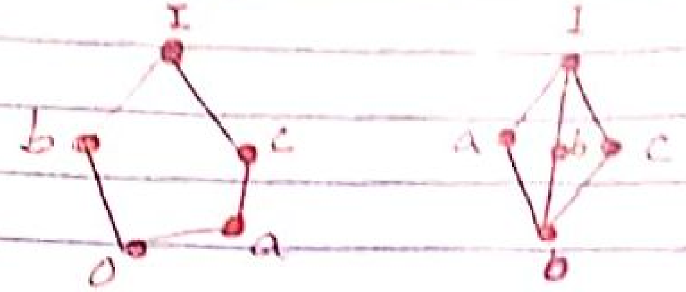
Sol: The complement of a is d. Since, $a \vee d = 1$ and $a \wedge d = 0$
The complement of c does not exist. Since, there does not exist any element c such that
$c \vee c' = 1$ and $c \wedge c' = 0$

Theorem → Prove that 0 and 1 are complement of each other.

Proof: To show that 1 is the only complement of ~~each other~~ 0.
Consider that $c \neq 1$ is a complement of 0 and $c \in L$.
Then, $0 \wedge c = 0$ and $0 \vee c = 1$
But $0 \vee c = c$ (By bounded lattice)
and $c \neq 1$. leads to a contradiction.
Similarly, we can show that 0 is the only complement of 1

Example → The power set P(S) of the set S under the operation of intersection and union is a complemented lattice L since each element of L has a unique complement.

Example The Lattice shown below are complemented lattice



But the complements of some of the elements are not unique e.g. b has two complements a and c in both the cases

Theorem Prove that for a bounded distributive lattice L, the complement are unique if they exist

Proof :- Consider a_1 and a_2 be complements of some elements $a \in L$. Then, we have $a \vee a_1 = 1$ and $a_1 \vee a_2 = 1$

Also $a \wedge a_1 = 0$ and $a \wedge a_2 = 0$

Now using the distributive law We have

$a_1 = a_1 \vee 0 = a_1 \vee (a \wedge a_2)$

$= (a_1 \vee a) \wedge (a_1 \vee a_2) = (a \vee a_1) \wedge (a_1 \vee a_2)$

$= 1 \wedge (a_1 \vee a_2) = a_1 \vee a_2$

Similarly, $a_2 = a_2 \vee 0 = a_2 \vee (a \wedge a_1) = (a_2 \vee a) \wedge (a_2 \vee a_1)$

$= (a \vee a_2) \wedge (a_1 \vee a_2) = 1 \wedge (a_1 \vee a_2) = a_1 \vee a_2$

$a_1 = a_2$ Hence proved.

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II BOOLEAN ALGEBRA (BOOLEAN LATTICE)

Definition I.

A Complemented distributive lattice is called algebra. It is denoted by $(B, \wedge, \vee, ', 0, 1)$, where B is a set on which two binary operation $\wedge$ (*) and $\vee$ (+) and a unary operation ' (complement) are defined. Here 0 and 1 are two distinct elements of B.

Since $(B, \wedge, \vee)$ is a complemented distributive lattice, therefore each elements of B has a unique complement.

ALTERNATE DEFINITION I

Consider a set B on which two binary operation * and + and a uniary operation ' (Complement) are defined. Also let 0 and 1 are two distinct elements of B. Then it is called a Boolean algebra if the following properties are satisfied for any elements a, b and c of the set B by it.

1. Commutative Properties

(i) $a+b = b+a$

(ii) $a*b = b*a$

2. Distributive Properties

(i) $a+(b*c) = (a+b)*(a+c)$

(ii) $a*(b+c) = (a*b)+(a*c)$

3. Identity Properties

(i) $a + 0 = a$

(ii) $a * 1 = a$

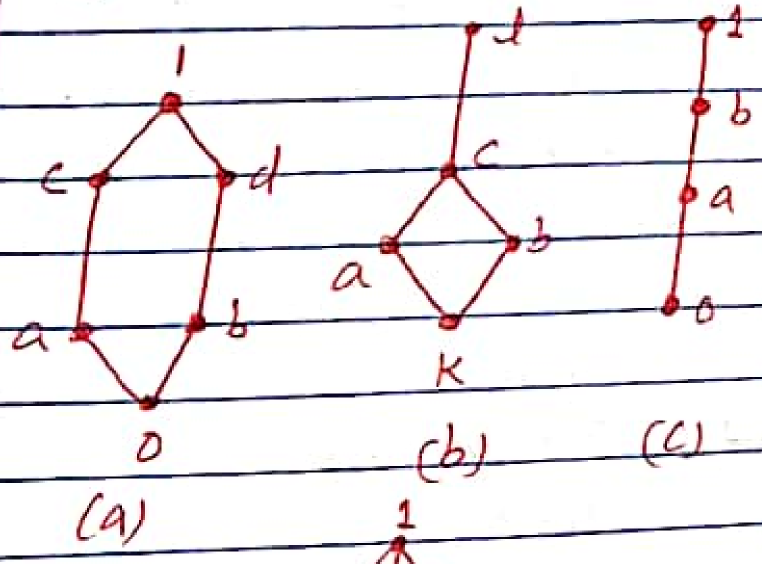
4. Complement Laws

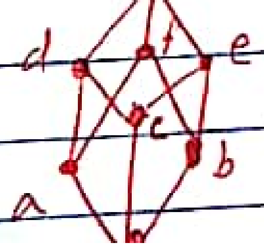
(i) $a + a' = 1$

(ii) $a + a' = 0$

The Boolean algebra is denoted by $(B, +, *, ', 0, 1)$

Example → Determine whether the poset given in figure is a Boolean algebra or not give reason.





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Sol(5) The poset shown in figure (a) is not
(a) Boolean Algebra

$a \wedge d = 0, \quad a \vee d = 1 \Rightarrow a' = d$ and $d' = a$

$0' = 1, \quad 1' = 0$

$a \wedge b = 0, \quad a \vee b = 1 \Rightarrow a' = b$ and $b' = a$

There are every element in L
has a complement
⇒ Complemented lattice
⇒ Not Boolean algebra (lattice)

(b) $l \wedge c = k \quad , l \vee c = l \Rightarrow l' = c$ and $c' = l$
only one complemented in
lattice ⇒ it is distributive
lattice
⇒ Not Boolean Algebra (lattice)

(c)

$0 \wedge 1 = 0 \quad 0 \vee 1 = 1 \Rightarrow 0' = 1$ and $1' = 0$
only one complement
element
⇒ distributive lattice
⇒ Not Boolean Algebra
(lattice)

(d) $d' = e$ and $e' = d \quad f' = c$

$0' = 1, \quad 1' = 0 \quad a' = b$ and $b' = a$

⇒ Boolean Algebra.