Test No. 7

Logarithmic Test :-

If Σu_n is positive term series such that $\lim_{n\to\infty}\left(n\log\frac{u_n}{u_{n+1}}\right)=k$

(i) If $k>1$, then the series is convergent.

(ii) If $k<1$, then the series is divergent.

(iii) Test fails, if $k=1$.

Ex. ① Test the convergence of the series $x+\frac{2^2 x^2}{2!}+\frac{3^3 x^3}{3!}+\frac{4^4 x^4}{4!}+\ldots\infty$

After Ignoring first term, We have $u_n=\frac{n^n x^n}{n!}$

and $u_{n+1}=\frac{(n+1)^{n+1} x^{n+1}}{(n+1)!}$

Now
$$\frac{u_n}{u_{n+1}}=\frac{n^n x^n}{n!}\times\frac{(n+1)!}{(n+1)^{n+1}x^{n+1}}=\frac{n^n \cancel{x^n}(n+1)\cancel{n!}}{\cancel{n!}(n+1)^{n+1}\cancel{x^n}\cdot x}$$
$$=\frac{n^n}{(n+1)^n}\cdot\frac{1}{x}=\frac{\cancel{n^n}}{\cancel{n^n}\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{x}=\frac{1}{(1+1/n)^n}\cdot\frac{1}{x}$$

Also $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{1}{(1+1/n)^n}\cdot\frac{1}{x}=\frac{1}{e}\cdot\frac{1}{x}$

If $\frac{1}{ex}>1$ or $x<\frac{1}{e}\Rightarrow$ Series is convergent.

and if $\frac{1}{ex}<1$ or $\frac{1}{e}<x\Rightarrow$ the series is divergent.

also if $\frac{1}{ex}=1$ or $x=\frac{1}{e}$, the test fails.

$$\log\left(\frac{u_n}{u_{n+1}}\right)=\log\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot e=\log e-\log\left(1+\frac{1}{n}\right)^n$$
$$=1-n\log(1+1/n)$$
$$=1-n\left[\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}+\ldots\right]$$

$\left(\because \log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\right)$

$$= 1 - 1 + \frac{1}{2n} - \frac{1}{3n^2} + \ldots = \frac{1}{2n} - \frac{1}{3n^2}$$

So $$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} n\left[\frac{1}{2n} - \frac{1}{3n^2}\right] = \lim_{n\to\infty}\left[\frac{1}{2} - \frac{1}{3n}\right] = \frac{1}{2} < 1$$

$\Rightarrow$ Thus according to Logarithmic test, the series is divergent

Ex. (2) Discuss the convergence of the series:→

$$1 + \frac{x}{2} + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \frac{4!}{5^4}x^4 + \ldots \infty \quad (x>0)$$

Solution:→ Neglecting first term from series, we have

$$u_n = \frac{n!}{(n+1)^n}x^n, \quad \therefore \quad u_{n+1} = \frac{(n+1)!}{(n+2)^{n+1}}x^{n+1}$$

Now applying D'Alembert's root test:→

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty} \frac{(n+1)!\, x^{n+1}}{(n+2)^{n+1}} \times \frac{(n+1)^n}{n!\, x^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)\cancel{n!}\,\cancel{x^n}\, x\,(n+1)^n}{(n+2)^n(n+2)\times \cancel{n!}\,\cancel{x^n}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(1+1/n)\,\cancel{n^n}(1+1/n)^n}{\cancel{n^n}\left(1+\frac{2}{n}\right)^n \cancel{n}(1+2/n)} \cdot x$$

$$= \lim_{n\to\infty} \frac{(1+1/n)(1+1/n)^n}{\left(1+\frac{2}{n}\right)^n(1+2/n)} \cdot x$$

$$= \frac{e \cdot x}{e^2 \cdot 1} \cdot x = \frac{x}{e}$$

$\left\{ \because \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e \right.$ Also $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n$ $= \lim_{n\to\infty}\left[\left(1+\frac{2}{n}\right)^{n/2}\right]^2 = e^2 \}$

$\therefore$ By D'Alembert's ratio test, the series converges, if $\frac{x}{e} < 1 \Rightarrow x < e$

and diverges, if $x/e > 1 \Rightarrow x > e$

If $x = e$, the ratio test fails ($\because K = 1$)

Now when $x = e$, Applying logarithmic test $\{\because u_{n+1}/u_n$ involves the no. $e\}$

$$\Rightarrow \log\left(\frac{u_n}{u_{n+1}}\right) = (n+1)\log(1+2/n) - (n+1)\log(1+1/n) - \log e$$

$$= (n+1)\left[\log(1+2/n) - \log(1+1/n)\right] - 1$$

$$=(n+1)\left[\left(\frac{2}{n}-\frac{1}{2}\cdot\frac{4}{n^2}+\frac{1}{3}\cdot\frac{8}{n^3}\cdots\right)-\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}+\cdots\right)\right]-1$$

$$=(n+1)\left[\frac{1}{n}-\frac{3}{2n^2}+\frac{7}{3n^3}-\cdots\right]-1$$

$$=\left(1-\frac{3}{2n}\right)+\left(\frac{1}{n}-\frac{3}{n^2}+\cdots\right)-1=\frac{1}{n}-\frac{3}{n^2}-\frac{3}{2n^2}$$

$$=1-\frac{1}{2n}-\frac{3}{2n^2}+\cdots-1=\frac{-1}{2n}-\frac{3}{2n^2}$$

$\therefore \lim_{n\to\infty} n\log\left(\frac{u_n}{u_{n+1}}\right)=\lim_{n\to\infty} n\left[\frac{-1}{2n}-\frac{3}{2n^2}+\cdots\right]=\lim_{n\to\infty}\left(\frac{-1}{2}-\frac{3}{2n}+\cdots\right)=\frac{-1}{2}<1$

$\therefore$ By logarithmic test, the series is divergent.

Hence the given series $\sum u_n$ converges if $x<e$ and diverges if $x\geq e$.