PARTIAL DIFFERENTIAL EQUATIONS

4.8. INTRODUCTION

A differential equation containing partial derivatives of a function of two or more independent variables is called a partial differential equation. *e.g.*,

$$y\frac{\partial z}{\partial x}+x\frac{\partial z}{\partial y}=0$$

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$$

are the partial differential equations.

When we have a function z of two independent variables x and y, we use the alphabets p, q, r, s, t to denote the partial derivatives as follows :

$$p=\frac{\partial z}{\partial x}$$

$$q=\frac{\partial z}{\partial y}$$

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial p}{\partial x}$$

$$s=\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}=\frac{\partial q}{\partial x}=\frac{\partial p}{\partial y}$$

and

$$t=\frac{\partial^2 z}{\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial q}{\partial y}$$

Partial differential equations generally occur in the problems of Physics and Engineering. Some of the important partial differential equations are

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0 \quad ...(1)$$

$$\frac{\partial^2 u}{\partial t^2}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \quad ...(2)$$

$$\frac{\partial u}{\partial t}=c^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \quad ...(3)$$

Equations (1), (2) and (3) are respectively known as Laplace's equation, wave equation and heat conduction equation.

Sometimes for brevity, the partial differentiation with regard to a variable is denoted by a suffix. *e.g.*, Laplace's equation may be rewritten as $u_{xx}+u_{yy}+u_{zz}=0$.

If u does not depend on z, then we get two dimensional Laplace's equation as $u_{xx}+u_{yy}=0$.

4.9. ORDER AND DEGREE OF PARTIAL DIFFERENTIAL EQUATION

Order of a partial differential equation is the order of the highest ordered derivative present in the equation.

Degree of a partial differential equation is the greatest exponent (power) of the highest ordered derivative present in the equation when it has been made free from radical signs and fractional powers.

4.10. SOLUTION OF PARTIAL DIFFERENTIAL EQUATION

Solution is one which satisfies.

The solution of a partial differential equation in a region D is a function having partial derivatives which satisfy the differential equation at every point in D.

The general solution of a p.d.e. contains arbitrary constants or arbitrary functions or both. Consequently, we can say that by the elimination of arbitrary constants or arbitrary functions, partial differential equations can be formed.

4.11. FORMATION OF PARTIAL DIFFERENTIAL EQUATION

(1) By the elimination of arbitrary constants

Let

$$f(x, y, z, a, b)=0 \quad ...(1)$$

be the given function, where a, b are arbitrary constants. x and y are independent variables and z is a dependent variable. Differentiating eqn. (1) partially w.r.t. x, we get

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}=0 \;\Rightarrow\; \frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}=0 \quad ...(2)$$

Again differentiating eqn. (1) partially w.r.t. y, we get

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y}=0 \;\Rightarrow\; \frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}=0 \quad ...(3)$$

Eliminating a, b from equations (1), (2) and (3), we get

$$F(x, y, z, p, q) = 0$$

which is a partial differential equation of first order.

Note. If the number of arbitrary constants is more than the number of independent variables, then the order of the partial differential equation obtained will be greater than 1.

(2) By the elimination of arbitrary functions

Let u, v be two known functions of x, y, z connected by the relation

$$\phi(u, v) = 0 \quad \ldots(1)$$

where ϕ is an arbitrary function.

Differentiating eqn. (1) partially w.r.t. x, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} \cdot \frac{\partial z}{\partial x}\right) = 0$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

$$\Rightarrow \quad \frac{\partial \phi/\partial u}{\partial \phi/\partial v} = -\left(\frac{\dfrac{\partial v}{\partial x} + p\dfrac{\partial v}{\partial z}}{\dfrac{\partial u}{\partial x} + p\dfrac{\partial u}{\partial z}}\right) \quad \ldots(2)$$

Diff. (1) partially w.r.t. y, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right) = 0$$

$$\Rightarrow \quad \frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0$$

$$\Rightarrow \quad \frac{\partial \phi/\partial u}{\partial \phi/\partial v} = -\left(\frac{\dfrac{\partial v}{\partial y} + q\dfrac{\partial v}{\partial z}}{\dfrac{\partial u}{\partial y} + q\dfrac{\partial u}{\partial z}}\right) \quad \ldots(3)$$

From eqns. (2) and (3), we get

$$\frac{\dfrac{\partial v}{\partial x} + p\dfrac{\partial v}{\partial z}}{\dfrac{\partial u}{\partial x} + p\dfrac{\partial u}{\partial z}} = \frac{\dfrac{\partial v}{\partial y} + q\dfrac{\partial v}{\partial z}}{\dfrac{\partial u}{\partial y} + q\dfrac{\partial u}{\partial z}}$$

$$\Rightarrow \quad \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p + \left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

$$\Rightarrow \quad \boxed{Pp + Qq = R} \quad \ldots(4)$$

Eqn. (4) is a partial diff. equation of first degree (*i.e.* linear) in p and q.

Note. If the given relation between x, y, z contains two arbitrary functions, then the p.d.e. derived therefrom will be, in general of order greater than one.

ILLUSTRATIVE EXAMPLES

Example 1. *Form partial differential equations from the following equations by eliminating the arbitrary constants :*

(i) $z = ax + by + ab$ (ii) $z = ax + a^2y^2 + b$

(iii) $z = (x + a)(y + b)$ (iv) $az + b = a^2x + y$

(v) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$

Sol. (i) Differentiating z partially w.r.t. x and y,

$$p = \frac{\partial z}{\partial x} = a, \quad q = \frac{\partial z}{\partial y} = b$$

Substituting for a and b in the given equation, we get

$$z = px + qy + pq$$

which is a partial differential equation.

(ii) Differentiating z partially w.r.t. x and y

$$p = \frac{\partial z}{\partial x} = a, \quad q = \frac{\partial z}{\partial y} = 2a^2y$$

Eliminating a between these results, we get $q = 2p^2y$

which is a partial differential equation.

(iii) Differentiating z partially w.r.t. x, we get

$$\frac{\partial z}{\partial x} = p = y + b \qquad \text{...(1)}$$

Differentiating z partially w.r.t. y, we get

$$\frac{\partial z}{\partial y} = q = x + a \qquad \text{...(2)}$$

Multiplying eqns. (1) and (2), we get

$$pq = (y + b)(x + a)$$

$$\Rightarrow \quad pq = z$$

which is a partial differential equation.

(iv) Differentiating the given relation w.r.t. x partially, we get

$$a\frac{\partial z}{\partial x} = a^2$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = p = a \qquad \text{...(1)}$$

Again differentiating the given relation w.r.t. y partially, we get

$$a\frac{\partial z}{\partial y} = 1$$

$$\Rightarrow \quad \frac{\partial z}{\partial y} = q = \frac{1}{a} \qquad \text{...(2)}$$

Multiplying eqns.(1) and (2), we get $pq = 1$

which is a partial differential equation.

(*v*) Here the number of arbitrary constants (a, b, c) is greater than the number of independent variables (x, y).

Differentiating partially w.r.t. x and y, we have

$$\frac{2x}{a^2} + \frac{2z}{c^2} \cdot \frac{\partial z}{\partial x} = 0 \quad \text{or} \quad c^2 x + a^2 z \frac{\partial z}{\partial x} = 0 \qquad \ldots(1)$$

and
$$\frac{2y}{b^2} + \frac{2z}{c^2} \cdot \frac{\partial z}{\partial y} = 0 \quad \text{or} \quad c^2 y + b^2 z \frac{\partial z}{\partial y} = 0 \qquad \ldots(2)$$

Again differentiating (1) partially w.r.t. x, we have

$$c^2 + a^2 \left(\frac{\partial z}{\partial x}\right)^2 + a^2 z \frac{\partial^2 z}{\partial x^2} = 0 \quad \text{or} \quad \frac{c^2}{a^2} + \left(\frac{\partial z}{\partial x}\right)^2 + z \frac{\partial^2 z}{\partial x^2} = 0$$

Substituting $\frac{c^2}{a^2} = -\frac{z}{x}\frac{\partial z}{\partial x}$ from (1), we have

$$-\frac{z}{x}\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + z \frac{\partial^2 z}{\partial x^2} = 0 \quad \text{or} \quad xz \frac{\partial^2 z}{\partial x^2} + x \left(\frac{\partial z}{\partial x}\right)^2 - z \frac{\partial z}{\partial x} = 0$$

which is a partial differential equation of the second order.

Note. Instead of differentiating (1) partially w.r.t. x, if we differentiate (2) partially w.r.t. y and substitute for $\frac{c^2}{b^2}$ from (2), we shall get $yz \frac{\partial^2 z}{\partial y^2} + y \left(\frac{\partial z}{\partial y}\right)^2 - z \frac{\partial z}{\partial y} = 0.$

Example 2. *Form the partial differential equation by eliminating the arbitrary function (s) from the following :*

(*i*) $z = f(x^2 - y^2)$ (*ii*) $z = \phi(x) \,.\, \psi(y)$

(*iii*) $z = x + y + f(xy)$ (*iv*) $z = f(x + it) + g(x - it)$.

Sol. (*i*) Differentiating z partially w.r.t. x, we get

$$\frac{\partial z}{\partial x} = p = f'(x^2 - y^2) \,.\, 2x \qquad \ldots(1)$$

Differentiating z partially w.r.t. y, we get

$$\frac{\partial z}{\partial y} = q = f'(x^2 - y^2) \,.\, (-2y) \qquad \ldots(2)$$

Dividing eqn. (1) by eqn. (2), we get

$$\frac{p}{q} = \frac{x}{(-y)} \quad \Rightarrow \quad py + qx = 0$$

which is a partial differential equation.

(*ii*) Differentiating z w.r.t. x, partially, we get

$$\frac{\partial z}{\partial x} = p = \phi'(x)\, \psi(y) \qquad \ldots(1)$$

Differentiating z w.r.t. y partially, we get

$$\frac{\partial z}{\partial y} = q = \phi(x)\, \psi'(y) \qquad \ldots(2)$$

Differentiating eqn. (1) partially w.r.t. y, we get

$$\frac{\partial^2 z}{\partial y \partial x} = s = \phi'(x)\, \psi'(y) \qquad \ldots(3)$$

Multiplying eqns. (1) and (2), we get

$$pq = \phi(x)\,\psi(y)\,\phi'(x)\,\psi'(y) = zs \qquad | \text{ Using (3)}$$

$$\Rightarrow \qquad pq - zs = 0$$

which is the required partial differential equation

(*iii*) Differentiating z w.r.t. x, we get

$$\frac{\partial z}{\partial x} = p = 1 + f'(xy)\,.\,y \;\Rightarrow\; p - 1 = yf'(xy) \qquad ...(1)$$

Differentiating z w.r.t. y, we get

$$\frac{\partial z}{\partial y} = q = 1 + f'(xy)\,.\,x \;\Rightarrow\; q - 1 = x\,f'(xy) \qquad ...(2)$$

Dividing eqn. (1) by eqn. (2), we get

$$\frac{p-1}{q-1} = \frac{y}{x} \;\Rightarrow\; px - qy = x - y$$

which is the reqd. partial differential equation.

(*iv*) Given $z = f(x + it) + g(x - it)$

Differentiating z twice partially w.r.t. x and t, we have

$$\frac{\partial z}{\partial x} = f'(x + it) + g'(x - it)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x + it) + g''(x - it) \qquad ...(1)$$

$$\frac{\partial z}{\partial t} = i\,f'(x + it) - ig'(x - it)$$

$$\frac{\partial^2 z}{\partial t^2} = i^2 f''(x + it) + i^2 g''(x - it)$$

or

$$\frac{\partial^2 z}{\partial t^2} = -f''(x + it) - g''(x - it) \qquad ...(2)$$

Adding (1) and (2), we have $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial t^2} = 0$

which is a partial differential equation of the second order.