New topic (In Unit 3)

Non-linear equations of the first order

Definition: → A partial differential equation (which involves first order partial derivatives p and q with degree higher than one and the products of p and q is called a non-linear partial differential equation.

Example: →
(i) $x^2p^2 + y^2q^2 = z^2$ [$\because$ p and q have degree → 2]

(ii) $pq = p+q$ [p and q are in the form of product]

(iii) $z^2 = pqxy$ [p and q are in the form of product]

(iv) $z = p^2x + q^2y$

(v) $p^2+q^2 = 2$

Remember: → $p = \partial z/\partial x$

and $q = \partial z/\partial y$

{We are dealing with first order PDE's.}

Usually these equations (PDEs) are solved by "**Charpit's Method.**", before starting the charpit's method, let's discuss about some special type of such equations which can be solved easily by methods other than General method. (Charpit method)

Type ① $f(p,q)=0$ i.e. equation involving only p and q. (No x, y, z)

They complete solution is

$$\boxed{z = ax+by+c}$$

(where a and b are connected by the relation $f(a,b)=0$ — ②

from ②, we can find b in terms of a

let $b=\phi(a)$.

put the value of b in (i), the complete sol. is $z = ax+\phi(a)y+c$

((where a and c are arbitrary const.)

Let's take an example:→

(i) Solve $\sqrt{p}+\sqrt{q}=1$

Sol:- The equation is the form $f(p,q)=0$

The complete sol. is $z=ax+by+c$ —①

our task to find the value of b.

The relation $f(a,b)=0$

$\Rightarrow \sqrt{a}+\sqrt{b}=1$ {Nothing, just put $p=a$ and $q=b$ in given equation}

then $\sqrt{b}=1-\sqrt{a}$

OR $b=(1-\sqrt{a})^2$

put the value of b in equation ①, we get:→

Complete Sol $\quad z=ax+(1-\sqrt{a})^2y+c$ — Answer

(ii) Solve $pq=p+q$ —①

Sol.→ The equation is the form $f(p,q)=0$

Then complete sol:- $z=ax+by+c$ —(II)

Now put $p=a$, $q=b$ in Eq. ①, we get

$$ab=a+b$$

OR

$$ab-b=a$$

$$b(a-1)=a$$

$$b=\frac{a}{a-1}$$

put the value of b in Eq. (ii), we get complete sol^n →

$$z = ax + \frac{a}{a-1}y + c$$

Type ② Equations of the form :→ $z = px + qy + f(p,q)$

⇒ The complete sol is $\boxed{z = ax + by + f(a,b)}$

Ex. ① Solve $z = px + qy + \sqrt{1+p^2+q^2}$

Sol:- Equation is the form $z = px + qy + f(p,q)$

Then complete sol^n → $\boxed{z = ax + by + \sqrt{1+a^2+b^2}}$

Type ③ Equations of the form $f(z, p, q) = 0$

i.e. equations not containing x and y.

Working Rule to finding sol:→

(i) Assume $u = x + ay$, so that $p = \frac{dz}{du}$ and

$$q = a\frac{dz}{du}$$

(ii) Substitute these values of p and q in the given equation.

(iii) Solve the resulting ordinary differential equation in z and u.

(iv) Replace u by $x+ay$.

Ex. ① Solve $z^2(p^2+q^2+1)=a^2$ —— ①

The given equation is of the form

$$f(z,p,q)=0$$

Step ① let $u=x+by$

So that $p=\frac{dz}{du}$ and $q=b\frac{dz}{du}$

Substitute the values of p and q in Eq ①, we get

$$z^2\left[\left(\frac{dz}{du}\right)^2+b^2\left(\frac{dz}{du}\right)^2+1\right]=a^2$$

$$z^2\left[\left(\frac{dz}{du}\right)^2(1+b^2)+1\right]=a^2$$

or $$z^2\left(\frac{dz}{du}\right)^2(1+b^2)+z^2=a^2$$

$$\Rightarrow \quad z^2\left(\frac{dz}{du}\right)^2(1+b^2)=a^2-z^2$$

$$\Rightarrow \quad z\left(\frac{dz}{du}\right)\sqrt{1+b^2}=\pm\sqrt{a^2-z^2}$$

Integrating $$\pm\sqrt{1+b^2}\sqrt{a^2-z^2}=u+c$$

or $$\boxed{(1+b^2)(a^2-z^2)=(x+by+c)^2}$$

Type ④ Equations of the form
$$f_1(x,p) = f_2(y,q)$$

(Working Rule to find solution:→

Step ① let $f_1(x,p) = f_2(y,q) = a$

Solving these equations for p and q,

Step ② let $p = F_1(x)$ and $q = F_2(y)$

Since $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$

Step ③ OR $dz = p\,dx + q\,dy$

$\therefore dz = F_1(x)dx + F_2(y)dy$

Integrating ↦

Complete Solution → $\boxed{z = \int F_1(x)dx + \int F_2(y)dy + c}$

Example ↦ Solve $p^2 - q^2 = x - y$

Sol :→ Can be written as $p^2 - x = q^2 - y$

(Which is $f_1(x,p) = f_2(y,q)$ type.

So let :→ $p^2 - x = q^2 - y = a$ —— ①

Hence from Eq ① $\Rightarrow p^2 = x + a \Rightarrow p = \sqrt{x+a}$

$\Rightarrow q^2 = y + a \Rightarrow q = \sqrt{y+a}$

Step 2

Substitute the value of p and q in

$$dz = p\,dx + q\,dy$$

$$dz = \sqrt{x+a}\,dx + \sqrt{y+a}\,dy$$

Integrating $\int dz = \int \sqrt{x+a}\,dx + \int \sqrt{y+a}\,dy$

$$\Rightarrow \boxed{z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y+a)^{3/2} + c}$$

Ex. 2 Solve $yp = 2yx + \log q$ —— (1)

Eq. (1) can be written as:

$$p = 2x + \frac{1}{y}\log q$$

OR $p - 2x = \frac{1}{y}\log q$

(Which is the form :- $f(x,p) = f(y,q)$)

let $p - 2x = \frac{1}{y}\log q = a$ —— (2)

then from (2) $p - 2x = a \Rightarrow p = 2x + a$

$$\frac{1}{y}\log q = a \Rightarrow \log q = ay$$

OR. $q = e^{ay}$

Substitute the values of p and q in

$$dz = p\,dx + q\,dy$$

$$\Rightarrow dz = (2x+a)\,dx + e^{ay}\,dy$$

$$\Rightarrow \int dz = \int (2x+a)\,dx + \int e^{ay}\,dy$$

$$\Rightarrow \boxed{z = x^2 + ax + \frac{1}{a}e^{ay} + c} \rightarrow \text{Complete Sol}^n$$