Test No. 4 :→

D'Alembert's Ratio Test

Statement :→ If $\sum u_n$ is positive term series such that

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = K, \text{ then}$$

(i) the series is convergent if $K<1$

(ii) the series is divergent if $K>1$

(iii) Test fails $K=1$

↓ Explaination

let consider the series. nth term $= 1/n$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}(1+1/n)} = 1 \quad \text{——— (1)}$$

~~Again~~, let consider the another series with nth term $= 1/n^2$

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1 \quad \text{——— (2)}$$

Thus from ① and ② in both cases $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$

But we know that first series is divergent as $p=1$

and the second series is convergent as as $p=2$

Hence, when $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$, then series may be convergent or divergent $\Rightarrow$ The Ratio test fails, when $k=1$.

Ex. ① Test for convergence of the series whose nth term is $n^2/2^n$

Solution:→ Here, we have $u_n = \frac{n^2}{2^n}$

$$\therefore u_{n+1} = \frac{(n+1)^2}{2^{(n+1)}}$$

By D-Alembert's test $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^2}{2^{n+1}} \cdot \frac{2^n}{n^2}$

$$= \lim_{n\to\infty} \frac{1}{2}\left(1+\frac{1}{n}\right)^2 = \frac{1}{2} < 1 \quad (\because k<1)$$

Hence the series is convergent by D'Alembert's ratio test.

Ex. ② Test for convergence the series whose nth term is $\frac{2^n}{n^3}$

Here, we have $u_n = \frac{2^n}{n^3}$

$$\therefore u_{n+1} = \frac{2^{n+1}}{(n+1)^3}$$

By D-Alembert's test: $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \frac{\frac{2^{n+1}}{(n+1)^3}}{\frac{2^n}{n^3}}$

$$\Rightarrow \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)^3} \times \frac{n^3}{2^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{2^n} \cdot 2 \times \cancel{n^3}}{\cancel{n^3}\left(1+\frac{1}{n}\right)^3 \times \cancel{2^n}} = \lim_{n\to\infty} \frac{2}{\left(1+\frac{1}{n}\right)^3}$$

$$= \lim_{n\to\infty} \frac{2}{(1+1/n)^3} = 2 > 1$$

Hence $k>1$, then the series is divergent.

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Ex. (3) Discuss the convergence of the series:

$$\sum \frac{\sqrt{n}}{\sqrt{n^2+1}} x^n \quad (x>0)$$

Solution:→ Here, we have $u_n = \sqrt{\frac{n}{n^2+1}}\, x^n$

$$\therefore u_{n+1} = \sqrt{\frac{(n+1)}{(n+1)^2+1}}\, x^{n+1}$$

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\sqrt{\frac{(n+1)}{(n+1)^2+1}}\, x^{n+1}}{\frac{\sqrt{n}}{\sqrt{n^2+1}}\, x^n} = \lim_{n\to\infty} \frac{\sqrt{n+1}}{\sqrt{(n+1)^2+1}} x^{n+1} \times \frac{\sqrt{n^2+1}}{x^n}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}\sqrt{1+\frac{1}{n}}}{\sqrt{n^2+1+2n+1}}\, x^{n+1} \times \frac{n\sqrt{1+\frac{1}{n^2}}}{x^n}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}\sqrt{1+1/n}}{\sqrt{n^2+2n+2}} \times \frac{\cancel{x^n}\cdot x \times n\sqrt{1+1/n^2}}{\cancel{x^n}}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\sqrt{n}\sqrt{1+1/n}}{\cancel{n}\sqrt{1+\frac{2}{n}+\frac{2}{n^2}}} \times x \times \cancel{n}\sqrt{1+1/n^2}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\sqrt{1+1/n} \times \sqrt{1+1/n^2}}{\sqrt{1+\frac{2}{n}+\frac{2}{n^2}}} \times x$$

$$= x$$

$\therefore$ By D'Alembert test, $\sum u_n$ converges if $x<1$ and diverges if $x>1$.

When $x=1$, the Ratio Test fails:→

When $x=1$, $u_n = \sqrt{\frac{n}{n^2+1}} = \sqrt{\frac{n}{n^2\left(1+\frac{1}{n^2}\right)}} = \frac{1}{\sqrt{n}} \times \frac{1}{\sqrt{1+1/n^2}}$

Thus, $v_n = 1/\sqrt{n}$

By Comparison test $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}\sqrt{1+1/n^2}}}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+1/n^2}} = 1$ (finite and non zero)

$\therefore$ By comparison test $\sum u_n$ and $\sum v_n$ converges or diverges together.

Since $\sum v_n = \sum \frac{1}{\sqrt{n}}$, where $p = 1/2 < 1$.

$\Rightarrow \sum v_n$ is divergent $\Rightarrow \sum u_n$ is divergent, if $x=1$.

Ex. (4) By D'-Alembert's ratio test, discuss the convergence of the series:→

$$\frac{1^2 \cdot 2^2}{1!} + \frac{2^2 \cdot 3^2}{2!} + \frac{3^2 \cdot 4^2}{3!} + \dots$$

Solution:→ Here $\sum u_n = \frac{1^2 \cdot 2^2}{1!} + \frac{2^2 \cdot 3^2}{2!} + \frac{3^2 \cdot 4^2}{3!} + \dots + \frac{n^2 \cdot (n+1)^2}{n!}$

Therefore $u_n = \frac{n^2(n+1)^2}{n!}$

$\therefore u_{n+1} = \frac{(n+1)^2(n+2)^2}{(n+1)!}$

By D'-Alembert's Ratio Test:→

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\frac{(n+1)^2(n+2)^2}{(n+1)!}}{\frac{n^2(n+1)^2}{n!}} = \lim_{n\to\infty} \frac{\cancel{(n+1)^2}(n+2)^2}{(n+1)!} \times \frac{n!}{n^2\cancel{(n+1)^2}}$$

$$= \lim_{n\to\infty} \frac{(n+2)^2 \times \cancel{n!}}{(n+1)\cancel{n!} \times n^2}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^2}(1+2/n)^2}{(n+1) \times \cancel{n^2}}$$

$$= \lim_{n\to\infty} \frac{(1+2/n)^2}{(n+1)} = \lim_{n\to\infty} \frac{\left(1+\frac{2}{n}\right)^2}{(n+1)} = \underline{0}$$

Hence, the series is convergent by D'-Alembert ratio test. $\Rightarrow \underline{0 < 1}$