Test No.3:→ Comparison Test

If two positive terms Σu_n and Σv_n be such that

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \text{finite no. (let say K)}$$, then both series converge or diverge together.

Remember!→ If Σv_n is convergent, then Σu_n is also convergent.
also, If Σv_n is divergent, then Σu_n is also divergent.

Note * we chose Σv_n (p-series) in such a way that

$$\lim_{n\to\infty} \frac{u_n}{v_n} = \text{finite no.}$$

Then the nature of both the series is the same.

"The nature of Σv_n (p-series) is already known, so the nature of Σu_n is also known."

Ex. (1) Test the series $\sum_{n=1}^{\infty} \frac{1}{n+10}$ for convergence or divergence.

Sol:→ Already given $u_n = \frac{1}{n+10}$

let $v_n = \frac{1}{n}$

Thus $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n}{n+10}$

$$= \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\left(1+\frac{10}{n}\right)} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{10}{n}\right)}$$

$= 1$ (finite no.)

According to comparison test both series converge or diverge together, but Σv_n is divergent as $p=1$ $\left\{\because v_n = \frac{1}{n^1}, \text{ where } p=1\right\}$

$\therefore \Sigma u_n$ is also divergent

Ex. (2) Test the convergence of the following series :→

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\dots$$

Sol:→ Here $u_n = \dfrac{1}{\sqrt{n}+\sqrt{n+1}}$

$$= \frac{1}{\sqrt{n}\left(1+\frac{\sqrt{n+1}}{\sqrt{n}}\right)} = \frac{1}{\sqrt{n}\left(1+\sqrt{\frac{n+1}{n}}\right)}$$

$$= \frac{1}{\sqrt{n}\left[1+\sqrt{\frac{\cancel{n}(1+1/n)}{\cancel{n}}}\right]}$$

or $u_n = \dfrac{1}{\sqrt{n}\left[1+\sqrt{1+\frac{1}{n}}\right]}$

and choosing $v_n = \dfrac{1}{\sqrt{n}}$, or $v_n = \dfrac{1}{(n)^{1/2}}$

Now $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \dfrac{\dfrac{1}{\lim\limits_{n\to\infty}\sqrt{n}\left[1+\sqrt{1+\frac{1}{n}}\right]}}{\dfrac{1}{\sqrt{n}}} = \lim\limits_{n\to\infty} \dfrac{1}{\cancel{\sqrt{n}}\left[1+\sqrt{1+\frac{1}{n}}\right]} \times \cancel{\sqrt{n}}$

$$= \lim_{n\to\infty} \frac{1}{1+\sqrt{1+\frac{1}{n}}} = \frac{1}{1+1} = \frac{1}{2} \text{ (finite)}$$

Here $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n}$ = finite,

∴ Σu_n and Σv_n converges or diverges together,

since $\Sigma v_n = \Sigma \dfrac{1}{(n)^{1/2}}$ is of the form $\Sigma \dfrac{1}{n^p}$ (p series)

Here $p = 1/2 < 1$

so according to p-series test Σv_n is divergent

⇒ Σu_n is also divergent.

Ex. (3) Test the convergence and divergence of the following Series:

$$\sum_{n=1}^{\infty} \frac{2n^2+3n}{5+n^5}$$

Sol:→ Here $u_n = \frac{2n^2+3n}{5+n^5} = \frac{n^2\left(2+\frac{3}{n}\right)}{n^5\left(\frac{5}{n^5}+1\right)} = \frac{1}{n^3}\frac{\left(2+\frac{3}{n}\right)}{\left(\frac{5}{n^5}+1\right)}$

Now let $v_n = \frac{1}{n^3}$

By Comparison test $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{(2+3/n)}{n^3\left(\frac{5}{n^5}+1\right)}}{\frac{1}{n^3}}$

$$= \lim_{n\to\infty} \frac{\cancel{n^3}\left(2+\frac{3}{n}\right)}{\cancel{n^3}\left(\frac{5}{n^5}+1\right)} = \lim_{n\to\infty} \frac{\left(2+\frac{3}{n}\right)}{\left(\frac{5}{n^5}+1\right)} = 2 \text{ (finite no.)}$$

According to comparison test both series converge or diverge together but $\sum v_n = \sum \frac{1}{n^3}$ is convergent as $p=3$ (According to p-series test).

Hence the given series $\sum u_n = \sum_{n=1}^{\infty} \frac{2n^2+3n}{5+n^5}$ is also convergent.

Ex. (4) Test for convergence the series: (UTU 2012)

$$\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \ldots \infty$$

Sol:→ Here $u_n = \frac{2n-1}{n(n+1)(n+2)} = \frac{n\left(2-\frac{1}{n}\right)}{n.n\left(1+\frac{1}{n}\right).n\left(1+\frac{2}{n}\right)}$

$$= \frac{\cancel{n}\left(2-\frac{1}{n}\right)}{n^{\cancel{3}}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)} = \frac{(2-1/n)}{n^2\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$$

let $v_n = \frac{1}{n^2}$

By Comparison test $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\frac{(2-1/n)}{\cancel{n^2}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}}{\frac{1}{\cancel{n^2}}} = \lim_{n\to\infty} \frac{2-1/n}{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)}$

$$= 2 \text{ (finite no.)}$$

Acc. to comparison test both series $\sum u_n$ and $\sum v_n$ converge or diverge together, but $\sum v_n = \sum 1/n^2$ is convergent as $p=2$. (Acc. to p-series test)

Hence the given series is also convergent

Ex.5 Test the convergence of the following series:→

$$\frac{1}{1+2^{-1}} + \frac{2}{1+2^{-2}} + \frac{3}{1+2^{-3}} + \ldots$$

Sol:→ Here $u_n = \frac{n}{1+2^{-n}} = \frac{n}{1+1/2^n}$

let $v_n = n$

Now by Comparison test $\lim_{n\to\infty} \frac{u_n}{v_n} = \frac{\frac{n}{\lim_{n\to\infty} 1+\frac{1}{2^n}}}{n} = \lim_{n\to\infty} \frac{\cancel{n}}{\left(1+\frac{1}{2^n}\right)\cancel{n}}$

$$= \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{2^n}\right)} = \frac{1}{1+0} = 1 \text{ (finite)}$$

∴ Σu_n and Σv_n, converges or diverge together since $\Sigma v_n = \Sigma \frac{1}{n}$ is of the form $\Sigma \frac{1}{n^p}$, where $p=1 \Rightarrow \Sigma v_n$ is divergent

Thus given series Σu_n is also divergent.

* Try Yourself:→ (i) Examine the convergence of the series

$$\frac{\sqrt{2}-1}{3^3-1} + \frac{\sqrt{3}-1}{4^3-1} + \frac{\sqrt{4}-1}{5^3-1} + \ldots$$

(ii) Test the series for convergence $1 + \frac{1}{2^2} + \frac{2^2}{3^3} + \frac{3^3}{4^4} + \ldots$