Q. 1. (a) Define multiset and power set.

Ans. Multiset : Multisets are sets where an element can occur as a member more than o example :

$$A = \{a, a, a, b, b, c\}$$

$$B = \{a, a, a, a, b, b, b, d, d\}$$

are multisets. The multisets A and B can also be written as :

$$A = \{3.a, 2.b, 1.c\} \text{ and}$$

$$B = \{4.a, 3.b, 2.d\}$$

Power Set : If S is any set, then the family of all the subsets of S is called the power set o power set of S is denoted by $P(S)$. Symbolically

$$P(S) = \{T : T \in S\}$$

Thus, ϕ and S are both element of $P(S)$. If set S is finite and contain n elements, then the pc of S will contain 2^n elements.

e.g., if $A = \{1,2\}$, then $P(A) = \{\{1\}, \{2\}, \{1,2\}, \{\ \}\}$.

Q. 1. (b) Define Cartesian product of sets and equivalence relation.

Ans. Cartesian Product of Sets : Let A and B be sets. Cartesian product of A and B, den $A \times B$, is defined as :

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

Thus, $A \times B$ is the set of all possible ordered pairs whose first component comes from A anc second component comes from B.

Equilvalence Relation : A relation on a set A is called equivalence relation if it is re symmetric and transitive, i.e., R is an equivalence relation on A if it has the following three proper

(i) $(a, a) \in R$ for all $a \in A$ (reflexive)

(ii) $(a, b) \in R$ implies $(b, a) \in R$ (symmetric)

(iii) (a, b) and $(b, c) \in R$ imply $(a, c) \in R$ (transitive).

Q. 1. (c) If $f(x) = 3x^4 - 5x^2 + 9$, find $f(x-1)$.

Ans. $f(x) = 3x^4 - 5x^2 + 9$

Then $f(x-1) = 3(x-1)^4 - 5(x-1)^2 + 9$

$$= 3(x^4 - 4x^3 + 6x^2 - 4x + 1) + 5(x^2 - 2x + 1) + 9$$

$$= 3x^4 - 12x^3 + 23x^2 - 22x + 17$$

Q. 1. (d) Define partial order relation and lattice.

Ans. Partial Order Relation : A relation R on a set S is called a partial ordering if it is reflexive, antisymmetric and transitive. That is :

(i) aRa for all $a \in S$ (reflexivity)

(ii) aRb and $bRa \Rightarrow a = b$ (antisymmetry)

(iii) aRb and $bRc \Rightarrow aRc$ (transitivity)

Lattice : A poset $(P \leq)$ is called a lattice if every 2-element subset of P has both a least upper bound, a greatest lower bound, i..e.. if lub (x, y) and glb (x, y) exist for every x and y in P. This is denoted as :

$$x \vee y = \text{lub}\{x, y\} \qquad (x \text{ join } y)$$

$$x \wedge y = \text{glb}\{x, y\} \qquad (x \text{ meet } y)$$

Every chain is a lattice. Since any two elements a, b of a chain are comparable. We find

$$x \vee y = \text{lub}(x, y) = y$$

$$x \wedge y = \text{glb}(x, y) = x$$

Q. 1. (f) How many variable names of 8 letters can be formed from the letters a, b, c, d, e, f, g, h, i if no letter is repeated.

Ans. If repetition is not allowed the no. of variable names of 8 letters formed from the letters : $a, b, c, d, e, f, g, h, i$ are :

$$P(n, r) = \frac{n!}{(n-r)!}$$

Given, $\quad n = 9$ and $r = 8$

$$\therefore \quad P(9, 8) = \frac{9!}{(9-8)!} = 362880$$

Q. 1. (g) How many 4-digits numbers can be formed by using the digits 2, 4, 6, 8 when repetition of digits is allowed.

Ans. When repetition of digits is allowed the no. of 4-digits numbers that can be formed using the digits 2, 4, 6, 8 are : n^r.

Given, $\quad n = 4$ and $r = 4$

$$\therefore \quad \text{No. of 4-digit numbers} = n^r = 4^4 = 256$$

Q. 1. (h) Define Monoid and Semigroup with suitable example.

Ans. Monoid : An algebraic structure $(S, *)$ is called a monoid if the following conditions are satisfied :

(i) The binary operation $*$ is a closed operation. (Closure law)

(ii) The binary operation $*$ is an associative operation. (Associative law)

(iii) There exists an identity element, i.e., for some $e \in S$, $e * a = a * e = a$ for all $a \in S$.

Thus, a monoid is a semigroup $(S, *)$ that has an identity element.

e.g., If Z be a set of all integers, then the structure $(Z, +)$ is a monoid with identity element 0 and $(Z, \bullet)$ is a monoid with 1 as the identity element.

Semigroup : An algebraic structure $(S, *)$ is called a semigroup if the following conditions are satisfied :

(i) The binary operation $*$ is a closed operation, i.e., $a * b \in S$ for all $a, b \in S$. (Closure law)

(ii) The binary operation $*$ is an associative operation, i.e., $a * (b * c) = (a * b) * c$ for all $a, b, c \in S$. (Associative-law).

e.g., If Z be a set of all integers, then $(Z, +)$ and $(Z, \bullet)$ are semigroup as these two operations are closed associative in Z.

Q. 1. (i) Define Isomorphic graph and Homeomorphic graph.

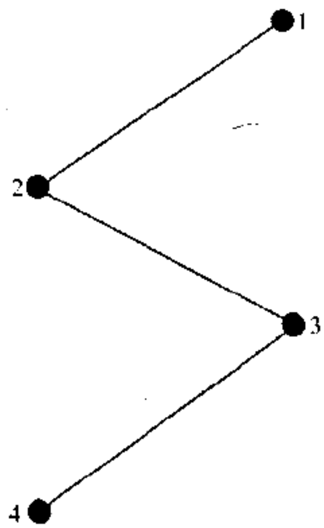
Ans. Isomorphic Graph : Let $G_1 = (v_1, E_1)$ and $G_2 = (v_2, E_2)$ be two graphs. A function $f = v_1 \to v_2$ is called graph isomorphism if :

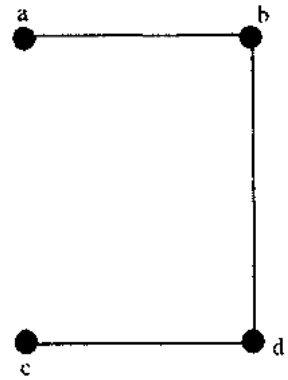
(a) f is one-to-one onto, and

(b) for all $a, b \in V_1$ $\{a, b\} \in E_1$ if and only if $\{f(a), f(b)\} \in E_2$

When such a function exists, G_1 and G_2 are called isomorphic graphs and is written as $G_1 \cong G_2$.

e.g., The following two graphs are isomorphic :





Here, $V(G_1) = \{1, 2, 3, 4\}$, $V(G_2) = \{a, b, c, d\}$, $E(G_1) = \{\{1,2\}, \{2,3\}, \{3,4\}\}$

& $E(G_2) = \{\{a, b\}, \{b, d\}, \{d, c\}\}$

Let function $f : V(G_1) \to V(G_2)$ as

$$f(1) = a, \quad f(2) = b, \quad f(3) = d \quad \text{and} \quad f(4) = c$$

f is clearly one-one and onto, hence an isomorphism.

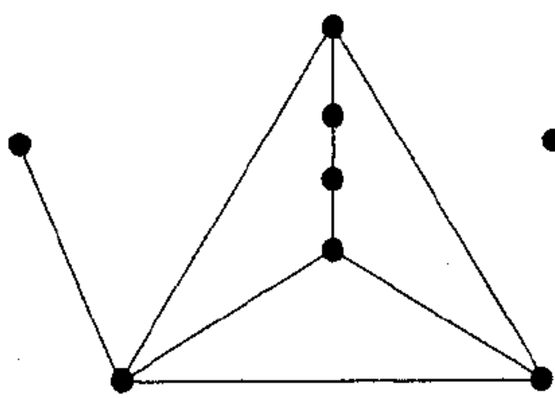
Further, $\{1,2\} \in E(G_1)$ and $\{f(1), f(2)\} = \{a, b\} \in E(G_2)$

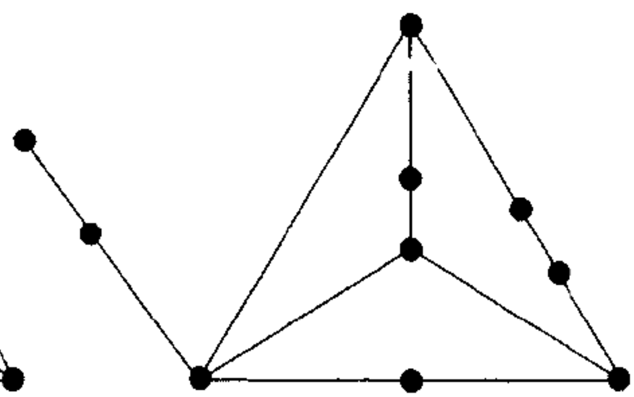
$\{2,3\} \in E(G_1)$ and $\{f(2), f(3)\} = \{b, d\} \in E(G_2)$

$\{3,4\} \in E(G_1)$ and $\{f(3), f(4)\} = \{d, c\} \in E(G_2)$

Hence, G_1 and G_2 are isomorphic.

Homeomorphic Graph : Two graphs are said to be homeomorphic if both can be obtained from the same graph by inserting new vertices of degree 2 into its edges or by the merger of edges in series. Such an operation is called an elementary subdivision. For example, any two cycle graphs are homeomorphic, as are the graphs of fig. :





Q. 2. (a) Prove :

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Ans.

(i) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Let x be any arbitrary element of the set $A \cap (B \cup C)$. Then,

$$\begin{aligned} x \in A \cap (B \cup C) &\Rightarrow x \in A \text{ and } x \in (B \cup C) \\ &\Rightarrow x \in A \text{ and } (x \in B \text{ or } x \in C) \\ &\Rightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\ &\Rightarrow (x \in A \cap B) \text{ or } (x \in A \cap C) \\ &\Rightarrow x \in (A \cap B) \cup (A \cap C) \end{aligned}$$

Thus, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

Conversely, let x be any arbitrary element of the set $(A \cap B) \cup (A \cap C)$. Then,

$$\begin{aligned} x \in (A \cap B) \cup (A \cap C) &\Rightarrow x \in (A \cap B) \text{ or } x \in (A \cap C) \\ &\Rightarrow (x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \in C) \\ &\Rightarrow x \in A \text{ and } (x \in B \text{ or } x \in C) \\ &\Rightarrow x \in A \cap (B \cup C) \end{aligned}$$

or $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

From equations (i) and (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Let x be any arbitrary element of the set $A \cup (B \cap C)$. Then,

$$\begin{aligned} x \in A \cup (B \cap C) &\Rightarrow x \in A \text{ or } x \in (B \cap C) \\ &\Rightarrow x \in A \text{ or } (x \in B \text{ and } x \in C) \\ &\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C) \\ &\Rightarrow x \in (A \cup B) \text{ and } x \in (A \cup C) \\ &\Rightarrow x \in (A \cup B) \cap (A \cup C) \end{aligned}$$

Thus, $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Conversely, let x be any arbitrary element of the set $(A \cup B) \cap (A \cup C)$. Then,

$$\begin{aligned} x \in (A \cup B) \cap (A \cup C) &\Rightarrow x \in (A \cup B) \text{ and } x \in (A \cup C) \\ &\Rightarrow (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C) \\ &\Rightarrow x \in A \text{ or } (x \in B \text{ and } x \in C) \\ &\Rightarrow x \in A \cup (x \in (B \cap C)) \\ &\Rightarrow x \in A \cup (B \cap C) \end{aligned}$$

or $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

From equations (i) and (ii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Q. 2. (b) Let $A = \{1, 2, 3, 4\}$ and

$$R = \{(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (4, 2), (4, 4)\}$$

Show that R is an equivalence relation.

Ans. Let $A = \{1, 2, 3, 4\}$ and

$$R = \{(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (4, 2), (4, 4)\}$$

Let $a \in A$, then R is reflexive since $(a, a) \in R \; \forall \; a \in R$

R is reflexive since, $(1, 1), (2, 2), (3, 3), (4, 4) \in R$.

R is symmetric because whenever, $(a, b) \in R$

$\Rightarrow$ $(b, a) \in R$

Here, $(1, 3) \in R$ so is $(3, 1)$

$(2, 4) \in R$ so is $4, 2)$

Therefore, R is a symmetric relation.

R is transitive if and only if (a, b) and $(b, c) \in R$ imply $(a, c) \in R$

Here, $(1, 3)$ and $(3, 1) \in R \Rightarrow (1, 1) \in R$

& $(2, 4)$ and $(4, 2) \in R \Rightarrow (2, 2) \in R$ and so on.

Thus, R is a transitive relation.

Since, R is reflexive, symmetric and transitive, it is an equivalence relation.

Q. 2. (c) Let $f(x) = x^2 + 3x + 1$, $g(x) = 2x - 3$.

Find :

(i) ***fof*** **(ii)** ***fog*** **(iii)** ***gof***

Ans. (i)

$$fof = f[f(x)] = f(x^2 + 3x + 1)$$
$$= (x^2 + 3x + 1)^2 + 3(x^2 + 3x + 1) + 1$$
$$= x^4 + 6x^3 + 11x^2 + 6x + 1 + 3x^2 + 9x + 3 + 1$$
$$= x^4 + 6x^3 + 14x^2 + 15x + 5$$

(ii)

$$fog = f[g(x)] = f(2x - 3)$$
$$= (2x - 3)^2 + 3(2x - 3) + 1$$
$$= 4x^2 - 12x + 9 + 6x - 9 + 1$$
$$= 4x^2 - 3x + 1$$

(iii)

$$gof = g[f(x)] = g[x^2 + 3x - 1]$$
$$= 2(x^2 + 3x + 1) - 3 = \mathbf{2x^2 + 6x - 1}$$

Q. 3. (a) Prove that the statement :

$(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$ **is a tautology.**

Ans. The truth table of the given proposition is shown below :

p	q	$\sim p$	$\sim q$	$(p \to q)$	$(\sim q \to \sim p)$	$(p \to q) \leftrightarrow (\sim q \to \sim p)$
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

Since, the truth value is TRUE for all possible values of the propositional variables which can be seen in the last column of the table, the given proposition is a tautology.

Q. 3. (b) Prove that $p \leftrightarrow q$ is equivalent to $(p \to q) \wedge (q \to p)$.

Ans. The truth table given below shows that these two expressions are logically equivalent, the two columns corresponding to the given two expressions have identical truth values.

p	q	$p \Leftrightarrow q$	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

Q. 3. (c) Let P and Q be the relations on set

$A = \{1, 2, 3, 4\}$ defined by

$P = \{(1, 2), (2, 2), (2, 3), (2, 4), (3, 2), (4, 2), (4, 3)\}$ and

$Q = \{(2, 2), (2, 3), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2)\}$

Find :

(i) POP **(ii) POQ** **(iii) POPOQ**

Ans. Given : $A = \{1, 2, 3, 4\}$ and

$P = \{(1, 2), (2, 2), (2, 3), (2, 4), (3, 2), (4, 2), (4, 3)\}$

$Q = \{(2, 2), (2, 3), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2)\}$

(i) POP = $\{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (2, 3), (3, 2), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)\}$

(ii) POQ = $\{(1, 2), (1, 3), (2, 2), (2, 3), (2, 4), (2, 1), (3, 2), (3, 3), (4, 2), (4, 3), (4, 4)\}$

(iii) POPOQ = PO(POQ)

$\Rightarrow \quad P = \{(1, 2), (2, 2), (2, 3), (2, 4), (3, 2), (4, 2), (4, 3)\}$

$(POQ) = \{(1, 2), (1, 3), (2, 2), (2, 3), (2, 4), (2, 1), (3, 2), (3, 3), (4, 2), (4, 3), (4, 4)\}$

$\therefore \quad PO(POQ) = \{(1, 2), (1, 3), (1, 4), (1, 1), (2, 2), (2, 3), (2, 4), (2, 1), (3, 2), (3, 3), (3, 4), (3, 1), (4, 2), (4, 3), (4, 4) (4, 1)\}$

Q. 3. (d) Prove that if L be a lattice then $a \wedge b = a$ if and only if $a \vee b = b$.

Ans. Given that if L be a lattice then $a \wedge b = a$ if and only if $a \vee b =.b$.

If L be a lattice then for every a and b in L.

To prove this property, we first have to prove the following two properties :

(i) $a \vee b = b$ if and only if $a \leq b$ and

(ii) $a \wedge b = a$ if and only if $a \leq b$.

(i) Let $a \vee b = b$. Since, $a \leq a \vee b = b$, we get $a \leq b$. Conversely if $a \leq b$, then, since $b \leq b$, b is an upper bound of a and b. So, by definition of lub; we have $a \vee b \leq b$. Again since $a \vee b$ is an upper bound, $b \leq a \vee b$, so $a \vee b = b$. Hence proved.

(ii) Similarly, $a \wedge b = a$

By combining equations (i) and (ii) we get :

$$a \wedge b = a \quad \text{iff} \quad a \vee b = b$$

Q. 4. (a) Determine the number of permutations that can be made out of the letters of the word 'PROGRAMMING'.

Ans. The word 'PROGRAMMING' consists of 11 letters which can be arranged in :

Total no. of letters in the word 'PROGRAMMING' is 11, out of which R occurs twice, G occurs twice, M occurs twice and the rest are all different. Since some of the letters are repeated we need to apply multimonial theorem. Hence the no. of arrangements are :

$$= \frac{11!}{2!2!2!} = 4989600$$

Q. 4. (b) How many 2-digits even number can be formed by using the digits 1, 3, 4, 6, 8 when repetition of digits is allowed ?

Ans. To form 2-digits even numbers using digits : 1, 3, 4, 6, 8 when repetition of digits is allowed, the no. must end with an even digit, the even digits here are – 4, 6 and 8.

Therefore, by product rule :

The first digit can be chosen from the five digits : 1, 3, 4, 6 and 8.

The second digit can be any one of the three digits : 4, 6 and 8.

Hence, the no. of 2-digits even no. are $5 \bullet 3 = 15$.

Q. 4. (c) How many ways can we select a software development group of 1 project leader, 5 programmers and 6 data entry operators from a group of 5 project leaders, 20 programmers and 25 data entry operator's ?

Ans. No. of ways to select 1 project leader from a group of 5 project leaders

$$= C(5, 1) = \frac{5!}{1!(5-1)!} = \frac{5 \times 4!}{4!} = 5$$

No. of ways to select 5 programmers from a group of 20 programmers

$$= C(20\,5) = \frac{20!}{5!(20-5!)} = \frac{20 \times 19 \times 18 \times 17 \times 16 \times 15!}{5 \times 4 \times 3 \times 2 \times 15!}$$

$$= 15504$$

No. of ways to select 6 data entry operators from a group of 25 data entry operators

$$C(26, 6) = \frac{25!}{6!(25-6)!}$$

$$= \frac{25 \times 24 \times 23 \times 22 \times 21 \times 20 \times 19!}{6 \times 5 \times 4 \times 3 \times 2 \times 19!}$$

$$= 177100$$

$\therefore$ Total no. of ways of select a group of 1 project leader, 5 programmer and 6 data entry operators are :

$$= C(5, 1) \times C(20, 5) \times C(25, 6)$$
$$= 5 \times 15504 \times 177100$$

Q. 5. (a) Solve the recurrence relation :

$$a_{r+2} - 3a_{r+1} + 2a_r = 0$$

by the method of generating functions with the initial conditions $a_0 = 2, a_1 = 3$.

Ans. Recurrence relation is

$$a_{r+2} - 3a_{r+1} + 2a_r = 0 \text{ with } a_0 = 2, a_1 = 3$$

The characteristic equations of the recurrence relation is

$$\alpha^2 - 3\alpha + 2 = 0 \Rightarrow (\alpha - 1)(\alpha - 2) = 0 \Rightarrow \alpha = 1, 2$$

The general homogeneous solution of the recurrence relation

$$\alpha^*_{nh} = A_1 (1)^n + A_2 (2)^n \quad \ldots(i)$$

Where A_1 & A_2 are constants.

Putting $n = 0, 1$ in equation (i)

$$a_0 = A_1 + A_2 \quad \Rightarrow \quad A_1 + A_2 = 2 \quad \Rightarrow \quad A_1 = 1, \quad A_2 = 1$$
$$a_1 = A_1 + 2A_2 \quad\quad A_1 + 2A_2 = 3$$

Hence solution is

$$\alpha^*_{np} = 1 + (2)^n$$

Q. 5. (b) Solve the recurrence relation

$$2a_r - 5a_{r-1} + 2a_{r-2} = 0 \text{ and find}$$

particular solution such that $a_0 = 0, a_1 = 1$.

Ans. The given recurrence relation is

$$2a_r - 5a_{r-1} + 2a_{r-2} = 0 \qquad \& \qquad a_0 = 0,\ a_1 = 1$$

The given characteristic equation of given recurrence relation is

$$2\alpha^2 - 5\alpha + 2 = 0$$

$$\Rightarrow \quad (\alpha - 2)(2\alpha - 1) = 0 \qquad \Rightarrow \qquad \alpha = 2, \frac{1}{2}$$

The general homogeneous solution of recurrence relation is

$$\alpha_{nh}^{n} = A_1 2^n + A_2 \left(\frac{1}{2}\right)^n$$

Putting $n = 0, 1$

$$a_0 = A_1 + A_2 \qquad \Rightarrow \qquad A_1 + A_2 = 0 \qquad \Rightarrow \qquad A_1 = \frac{2}{3}, \qquad A_2 = -\frac{2}{3}$$

$$a_1 = 2A_1 + \frac{1}{2}A_2 \quad \Rightarrow \quad 2A_1 + \frac{1}{2}A_2 = 1$$

Hence solution is

$$\alpha_{nh}^{n} = \frac{2}{3}(2)^n - \frac{2}{3}\left(\frac{1}{2}\right)^n$$

Q. 6. Define the following with suitable example :

(i) Rings

(ii) Field

(iii) Integral domain

(iv) Normal subgroup

(v) Homomorphism

Ans. (i) Rings : A ring $(R, +, \cdot)$ is a set R together with two binary operations + (addition) and $\cdot$ (multiplication) defined on R such that the following axioms are satisfied :

(R_1) $(a+b)+c = a+(b+c)$ for all $a, b, c \in R$

(R_2) $a+b = b+a$ for all $a, b \in R$

(R_3) There exists an element 0 in R such that $a+0 = a$ for all $a \in R$.

(R_4) For all $a \in R$, there exists an element $-a \in R$ such that $a+(-a) = 0$

(R_5) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in R$.

(R_6) $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$ for all $a, b, c \in R$

(R_7) $(b+c) \cdot a = (b \cdot a) + (c \cdot a)$ for all $a, b, c \in R$.

We call 0, the zero element of the ring $(R, +, \cdot)$

e.g.,

(i) The set Z of integers under ordinary addition and multiplication is a commutative ring with unity 1. The unit elements of Z are 1 and -1.

(ii) The set $Zn = \{0, 1, 2, \ldots\ldots, n-1\}$ under addition and multiplication modulo n is a commutative ring with unity 1.

(ii) Field : A ring containing at least two elements is called a field if it :

(a) is commutative

(b) has unity and

(c) is such that every non-zero element has multiplicative inverse in R.

That is – a system $(R, +, \cdot)$ is a field if

(i) $(R, +)$ is an abelian group.

(ii) $(R', \cdot)$ is a commutative group, where $R' = R - \{0\}$.

(iii) The distributive laws :

$$a(b+c) = ab + ac$$

$$(b+c)a = ba + ca \qquad \text{hold for all } a, b, c \in R.$$

e.g.,

(i) The ring of rational numbers $(Q, +, *)$ is a field since it is a commutative ring with unity and each non-zero element has multiplicative inverse.

(ii) The set R of all real numbers is a field.

(iii) Integral Domain : A ring containing at least two elements is called an Integral domain if it :

(a) is commutative

(b) has unit element and

(c) is without zero divisors.

Thus, in an integral domain a product is 0 only when one of the factors is 0; i.e., $ab = 0$ only when $a = 0$ or $b = 0$.

e.g.,

(i) The ring of integers $(Z, +, \cdot)$ is an integral domain since it is commutative ring with unity and for any two integers a, b, $ab = 0 \Rightarrow a = 0$ or $b = 0$.

(iv) Normal Subgroup : A subgroup H of a group G is said to be normal subgroup of G if $Ha = aH$ for all $a \in G$. Clearly every subgroup of an Abelian group is a normal subgroup.

Thus, a subgroup H of a group G can be defined to be a normal subgroup if

$$g^{-1}hg \in H \;\forall\; h \in H, g \in G$$

e.g., If H is a subgroup of G such that $x^2 \in H$ for every $x \in G$, then H is a normal subgroup of G.

(v) Homomorphism : Let $(G, *)$ and $(G_1, *_1)$ be two groups and f is a function from G into G_1. Then f is called a homomorphism of G into G_1 if for all $a, b \in G$.

$$f(a * b) = f(a) *_1 f(b)$$

e.g.,

Let $G = Z$ and $G' = \{1, -1\}$ the multiplicative group. The mapping $f : G \rightarrow G'$ defined by $f(n) = 1$ if n is even and $f(n) = -1$ if n is odd is a group homomorphism, as $f(m+n) = f(m)f(n)$ for all $m, n \in Z$.

Q. 7. (a) State and prove Lagrange's theorem.

Ans. Lagrange's Theorem : The order of each sub-group of a finite group G is a divisor of the order of the group G.

Proof : Let H be any sub-group of order m of a finite group G of order n. We consider the left coset decomposition of G relative to H.

We first show that each coset a H consists of m different elements.

Let $\quad H = \{h_1, h_2, \ldots, h_m\}$

Then $ah_1, ah_a, \ldots, ah_m$ are the m members of aH, all distinct.

For, we have

$$ah_i = ah_j \Rightarrow h_i = h_j, \text{ by cancellation law in } G.$$

Since, G is a finite group, the number of distinct left cosets will also be finite, say k. Hence the total no. of elements of all cosets is km which is equal to the total no. of elements of G. Hence,

$$n = mk$$

This shows that m, the order of H, is a divisor of n, the order of the group G.

Q. 7. (b) Define the following with suitable example :

(i) Automorphism

(ii) Cyclic group

(iii) Cosets

Ans. (i) Automorphism : A homomorphism f of a group G into a group G_1 is called a isomorphism of G onto G_1 if f is one-one onto G_1. G and G_1 are said to be isomorphic and denoted by $G \cong G_1$. An isomorphism of a group G onto G is called an automorphism.

(ii) Cyclic Group : A group G is called a cyclic group if, for some $a \in G$, every element of G is of the form a^n, where n is some integer. The element a is then called a generator of G.

If G is a cyclic group generated by a, it is denoted by $G = < a >$. The elements of G are in the form :

$$\ldots\ldots a^{-2}, a^{-1}, a^0, 0, a, a^2, a^3 \ldots\ldots$$

There may be more than one generator of a cyclic group.

e.g., The set of integers with respect to + i.e., $(Z, +)$ is a cyclic group, a generator being 1. i.e., each element of G can be expressed as some integral power of 1.

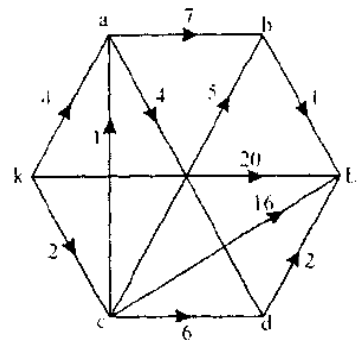
(iii) Cosets : Let H be a subgroup of a group G and let $a \in G$. Then the set $\{a * h : h \in H\}$ is called the left coset generated by a and H and is denoted by aH.

Similarly, the set $Ha = \{h * a : h \in H\}$ is called the right coset and is denoted by Ha. The element a is called a representative of aH and Ha. If the group operation be addition, then the right coset of H in G generated by a is defined as

$$H + a = \{h + a : h \in H\}$$

Similarly, left coset $a + H = \{a + h : h \in H\}$.

Q. 8. (a) Find the shortest path from K to L.



Ans. Dijkshtra Algorithm to find the shortest path from K to L :

The initial labelling is given by :

Vertex V	k	b	c	d	a	L
$L(v)$	0	∞	∞	∞	∞	∞
T	$\{k$	b	c	d	a	$L\}$

Iteration 1 : $u = k$ has $L(u) = 0$. T becomes $T - \{k\}$. There are three edges incident with k : a, c and L where $a, c\ L \in T$.

$$L(a) = \min(\text{old } L(a), L(k) + w(ka))$$
$$= \min(\infty, 0+4) = 4$$
$$L(c) = \min(\text{old } L(c), L(k) + w(kc))$$
$$= \min(\infty, 0+2) = 2$$
$$L(L) = \min(\text{old } L(L), L(k) + w(kL))$$
$$= \min(\infty, 0+20) = 20$$

Hence, min. label is $L(c) = 2$.

↓

Vertex	k	b	c	d	a	L
$L(v)$	0	∞	2	$\propto$	4	20
T	$\{b$	c	d	a	$L\}$	

Iteration 2 : $u = c$, the permanent label of c is 2. T becomes $T - \{c\}$. There are four edges incident with c. a, b, L, d where a, b, L, $d \in T$.

$$L(a) = \min(4, L(c) + w(ca))$$

$$= \min(4, 2 + 1) = 3$$

$$L(b) = \min(\infty, 2 + 5) = 7$$

$$L(L) = \min(20, 2 + 16) = 18$$

$$L(d) = \min(\infty, 2 + 6) = 8$$

Hence, min. label is $L(a) = 3$.

Vertex	k	b	c	d	↓ a	L
$L(v)$	0	7	2	8	3	18
T	{	b,		d,	a,	L}

Iteration 3 : $u = a$, the permanent label of a is 3. T becomes $T - \{a\}$. There are two edges incident with a: b, d where $b, d \in T$.

$$L(b) = \min(7, 3 + 7) = 7$$

$$L(d) = \min(8, 3 + 4) = 7$$

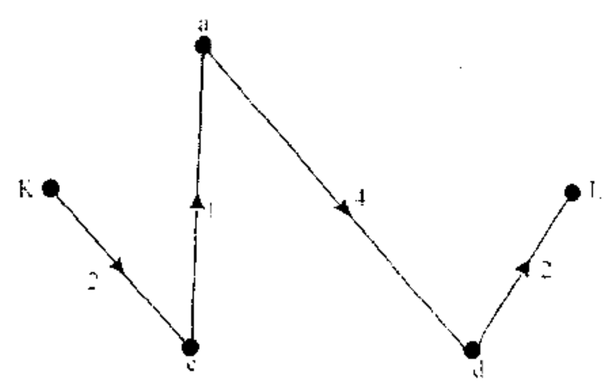
Hence, min. label is $L(d) = 7$.

Vertex	k	b	c	d	a	L
$L(v)$	0	7	2	7	3	18
T	{	b,		d,	,	L}

Iteration 4 : $u = d$, the permanent label of d is 7. T becomes $T - \{d\}$. There is only one edge incident with d : L where $L \in T$.

$$L(L) = \min(18, 7 + 2) = 9$$

Now iteration stops. Thus, the shortest distance between k to L is 9 and shortest path is (k, c, a, d, L).

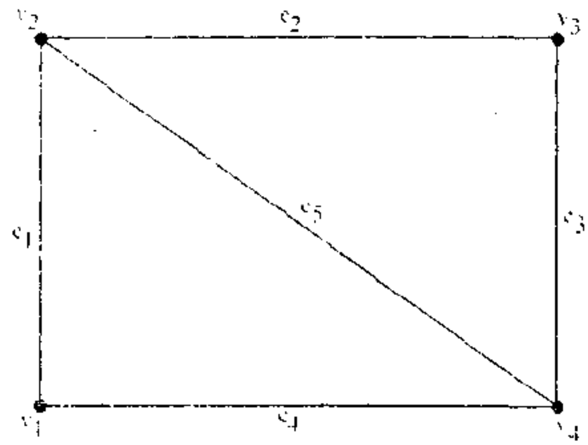


Q. (b) (i) Define Hamilton path and circuit with example.

Ans. Hamilton Path : A Hamiltonian path is a simple path that contains all vertices of G where the end points may be distinct.

Hamilton Circuit : A circuit in a graph, G that contains each vertex in G exactly once, except for the starting and ending vertex that appears twice is known as Hamiltonian circuit.

e.g., The graph shown in fig. has Hamiltonian circuit given by $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_1$.



It has Hamiltonian path given by $v_1 - v_2 - v_3 - v_4$.

Q. 8. (b) (ii) Define sub-graph.

Ans. Sub-graph : A sub-graph is a graph obtained by removing some of the vertices of a graph G as well as all edges incident with a vertex that was removed, are also removed from the graph.

If G and H are two graphs with vertex sets $V(H)$, $V(G)$ and edge sets $E(H)$ and $E(G)$ respectively such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ then we call H as a subgraph of G.

Q. 8. (b) (iii) What properties should graph possess to qualify as tree ?

Ans. A graph G with n vertices is called a tree if :

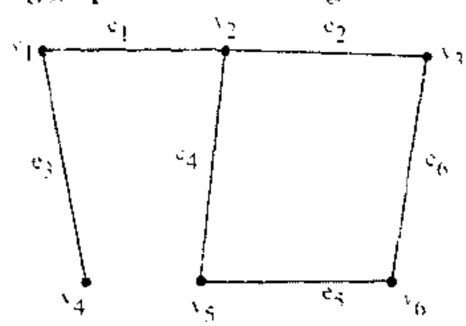
(i) G is connected and has no cycles (acyclic).

(ii) G is connected and has $n - 1$ edges.

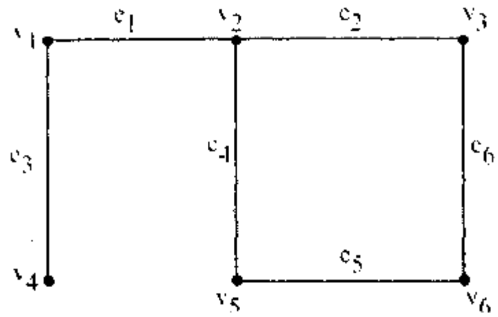
(iii) G is acyclic and has $n - 1$ edges.

(iv) There is exactly one path between every pair of vertices in G.

Q. 9. (a) For the following graph find all the bridges.



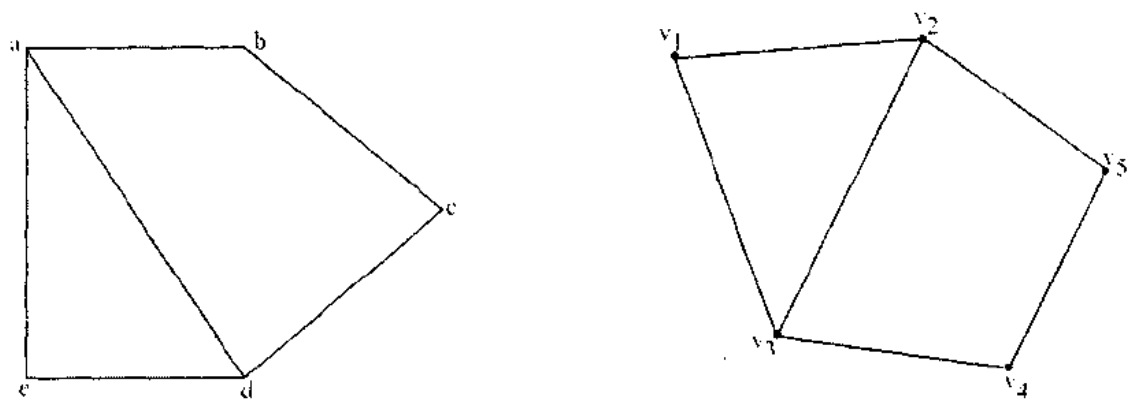
Ans. Given graph is :



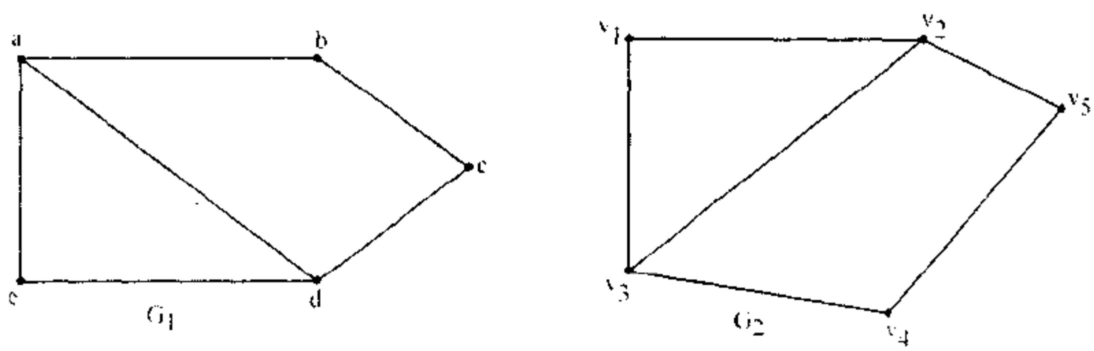
An edge whose removal produces a graph with more connected components than the original graph is a cut edge or bridge. The given fig. has following bridges :

$\{e_3, e_1\}$

Q. 9. (b) Show that the following graphs are isomorphic.



Ans. Given graphs are :



If G_1 and G_2 are isomorphic graphs, then G_1 and G_2 have :

(i) same no. of vertices

(ii) same no. of edges

(iii) same degree sequences.

[redacted] Show that for any a, b, c in A if $a * b = a * c$ then $b = c$.

Ans. Given that $a, b, c \in A$ also $a * b = a * c$

We have, $$b = eb = \left(a^{-1}a\right)b = a^{-1}(ab) = a^{-1}(ac) \quad [\because a * b = a * c]$$

$$= \left(a^{-1}ac\right) = ec = c$$

Hence $$a * b = a * c \Rightarrow b = c.$$

~~Q.5. (b) Show that (A, *)~~ is a group by showing that e is an identity element.

Ans. Consider that there are two identity elements e and e'

Since $e \in A$ and e' is an identity we have,

$$e'e = ee' = e$$

Also, $e' \in A$ and e is an identity we have,

$$e'e = ee' = e'$$

$$\therefore \quad e = e'$$

Hence e is an identity element.

Q.2 (a) Suppose G is a finite cycle free graph with at least one edge. Show that G has at least two vertices of degree 1.

Ans. Let G be the graph having n vertices, now of which is isolated and $n-1$ edges where $n \geq 2$. By using induction on n; the case n = 1 is trivial. Thus, G can be connected on $n \geq 2$ vertices. Choose arbitrary vertex V of G and consider graph H=G/V. H is not necessarily connected. Suppose H has connected component Z having n_i vertices that is $n_1 + ... n_k = n - 1$ edges. More over, v must be connected in G with each of the components Z_i by atleast one edge. Thus, G contains atleast $(n_1 - 1) + ... + (n_k - 1) + k = n - 1$ edge. Hence proved.

Q. (b) Show that a connected graph G with n vertices must have at least n-1 edges.

Ans. Suppose G is a graph with n vertices. It is a connected graph. WE prove that (n-1) edges by induction on n. If n = 1 it is tree. Assume that it is free for n = 1, 2, ... (n-1). Since every edge is a bridge, the subgraph G' obtained from G after deleting an edge will have two components G_1 and G_2 with n_1 and n_2 vertices respect. Where $n_1 + n_2 = n$. By induction hypothesis, the number of edges in both the component together is $(n_1 - 1) + (n_2 - 1) = (n - 2)$. Thus, the number of edges in G will be $(n - 2) + 1 = (n - 1)$ which is true.

Ans. Eulter path :

An Euler path through a graph is a path whose edge list contains each edge of the graph exactly once.

Euler circuit :

An euler circuit is a path through a graph, in which the initial vertex appears second time as the terminal vertex.

Euler graph :

An euler graph is a graph that possesses an euler circuit. An euler circuit uses every edge exactly once but vertices may be repeated.

(b) Planner graphs and binary trees :

A set is defined as a collection of distinct objects of same-type or classes of objects. The objects of a set are called elements or member of sets. The set can be of following types :

(a) Finite Set :

If a set consists of specific number of different elements then that set is called finite sets.

(b) Infinite Set : If a set consists of infinite number of different elements or if the counting of different elements of the set does not come to an end. the set is called infinite set.

(c) Disjoint set : Two sets A and B are said to be disjoint if no element of A is in b and no element of B is A e.g.,

$$R = \{a, b, c\}, \ S = \{k, p, m\}$$

R and S are disjoint sets.

(d) Null set or empty set :

The set that contains no element is called null set or the empty set and is denoted by of.

(e) Power set :

The power set of any given set A is the set of all subsets of A And is denoted by P(A). If A has n elements then P(A) has 2^n elements.

(c) Planar graph :

A graph is said to be planar if it can be drawn is a place so that number edge cross.

Binary Tree :

If the out degree of every node is less than or equal to 2, in a directed tree than the tree is called binary tree. A tree consisting of no nodes is also a binary tree.

Q. 5. (a) What do you mean by Eulerian Circuit and a Hamiltonian circuit?

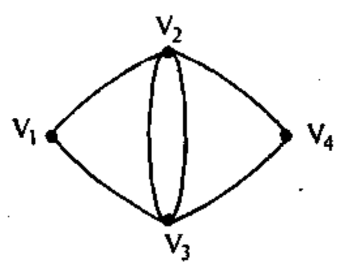
Ans. Hamiltonian Circuit : A Hamiltonian circuit is a path in which the initial vertex appears a second time as the terminal vertex.

Euler Circuit : Consider any connected planar graph G = (V, E) having R region-, V vertices and E edges,

$$V + R - E = 2$$

Q. 5. (b) Show a graph that has both an Eulerian Circuit and Hamiltonian circuit.

Ans.

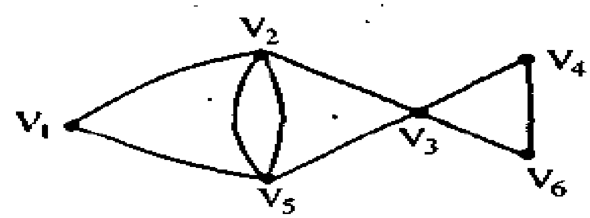


The euler circuit is $V_1, V_5, V_2, V_5, V_4 V_2, V_1$

The Hamiltonian circuit is V_1, V_2, V_4, V_5, V_1

Q. 5. (c) Show a graph that has an Eulerian circuit but has no Hamiltonian circuit.

Ans.

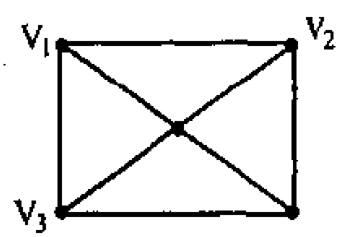


"Euler circuit is $V_1, V_5, V_2, V_5, V_4, V_6, V_3, V_2, V_1$

There is Hamiltonian circuit..

Q. 5. (d) Show a graph that has no Eulerian circuit but has a Hamiltonian circuit.

Ans.



Hamiltonian circuit is V_1, V_2, V_4, V_3, V_1

No Euler circuit.

Q. 6. (a) Draw all possible non similar trees T where :

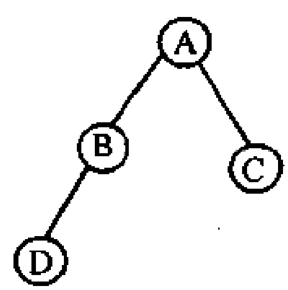
(i) T is a binary tree with 3 nodes.

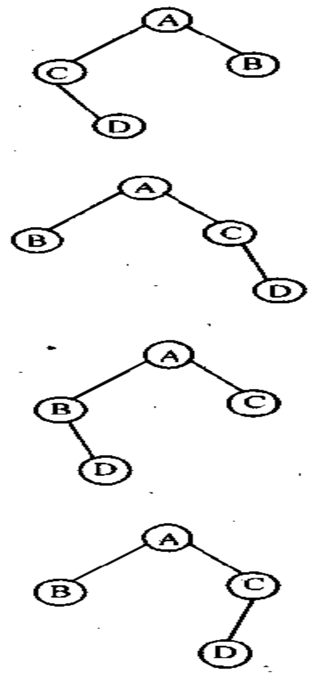
(ii) T is a 2-tree with 4 internal nodes.

Ans. (i) T is a binary tree with 3 node.

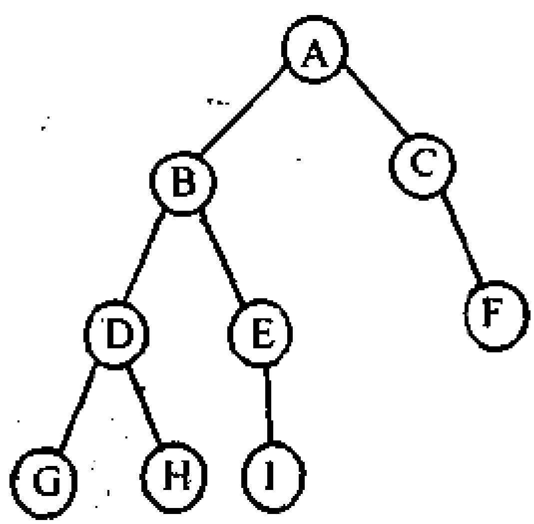
Suppose the tree T has one root and three childrens.

Say A is root and B, C, D are children of A.





(ii) A binary tree with 4 internal nodes.



Q. 7. Write short notes on :

(a) Isomorphism and Automorphism

(b) Homomorphic and Isomorphic graphs

(c) Subgroups and Normal subgroups

(d) Equivalence relations and partitions.

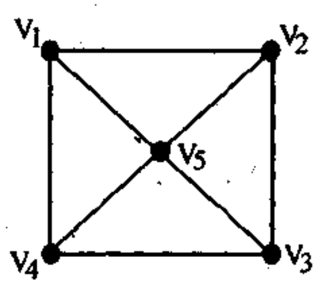
Ans. (a) Isomorphism : Let $(G_1,*)$ and $(G_2,*)$ be two algebraic system where * and 0 both are binary operations. The systems $(G_1,*)$ and $(G_2,0)$ are sand to be isomorphic if there exists an isomorphic mapping $f:G_1 \to G_2$. When two algebraic systems are isomorphic, the system are structbrally equivalent and one can be obtained from another by simply remaining the elements and the operation.

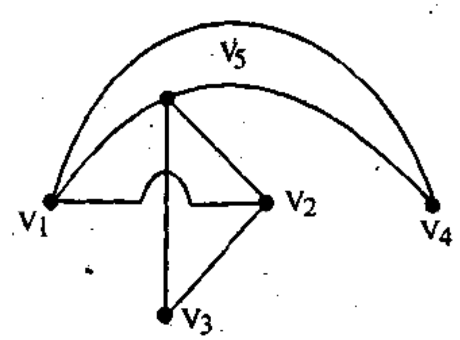
Automorphism : Let $(G_1,*)$ and $(G_2,*)$ be two algebraic systems, where * and 0 both are binary operations on G_1 and G_2 respectively. Then an isomorphism from $(G_1,*)$ and $(G_2,*)$ is called an automorphism if $G_1 = G_2$.

(b) Homomorphic : Let $(G_1,*)$ and $(G_2,*)$ be two algebraric systems, where * and 0 both are binary operations. Then the mapping $f:G_1 \to G_2$ is said to be homomorphism from $(a_1,*)$ to $(G_2,*)$ such that every a, $b \varepsilon G$ we have

$$f(a * b) = f(a).f(b)$$

Isomorphic Graphs : Two graph G_1 and G_2 are called isomorphic graphs if there is one to one correspondence between their vertices and between their edges.





Q. 3. (b) Prove that each of the following is a tautology :

(i) $((p \Rightarrow q)\wedge(q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$

(ii) $\sim(p \Rightarrow q) \Rightarrow p$

Ans. (i) $((p \Rightarrow q)\wedge(q \Rightarrow r)) \Rightarrow (p \Rightarrow r)$

p	q	$p \to q$	q	r	$q \to r$	$(p \to q)\wedge(q \to r)$	$p \to r$
T	T	T	T	T	T	T	T
T	F	F	T	F	F	F	F
F	T	T	F	T	T	T	T
F	F	T	F	F	T	T	F

Yes, the above statement is tautology.

(ii) $\sim(p \Rightarrow q) \Rightarrow p$

p	q	$p \to q$	$\sim(p \to q)$	p
T	T	T	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	F	F

The above statement is not tautology.

Q. 4. (a) Find an explicit formula for the sequence defined by $C_n = 3C_{n-1} - 2C_{n-2}$, $C_1 = 5$ and v.

Ans.

$$C_n = 3C_{n-1} - 2C_{n-2}$$

$$C_1 = 5,\ C_2 = 3$$

$$C_3 = 3C_2 - 2C_1$$

$$= 3*3 - 2*5 = -1$$

$$C_4 = 3C_3 - 2C_2$$

$$= 3*(-1) - 2*3 = -3 - 6 = -9$$

The sequence generated is $2(C_n - C_{n-1})$

Q. 5. (a) Let G be the set of all non-zero real numbers. Define a binary operation * on G as : For $a, b \in G$, $a * b = \frac{ab}{2}$. Show that (G, *) is an abelian group.

Ans. Closure property :

The set G is closed under the operation * since, $a * b = \frac{ab}{2}$ is a real number. Hence, belongs to G.

Associative property :

The operation * is associative. Let a, b, $c \in G$ then we have

$$(a * b) * c = \left(\frac{ab}{4}\right) * c = \frac{(ab)c}{16} = \frac{a(bc)}{16} = \frac{abc}{16}$$

Identity :

To find the identity element, let us assume that e is a +ve real number. Then $e * a = a$ where $a \in G$.

$$\frac{ea}{4} = a \text{ or } e = 4.$$

Thus, the identity element in G is 4.

Inverse : Let us assume that $a \in G$. If $a^{-1} \in Q$ is an inverse of a, then $a * a^{-1} = 4$

$$\therefore \quad \frac{aa^{-1}}{4} = 4 \text{ or } a^{-1} = \frac{16}{a}.$$

Similarly, $\quad a^{-1} * a = 4$

$$\therefore \quad \frac{a^{-1}a}{4} = 4 \text{ or } a^{-1} = 16/a$$

Thus, the inverse of element a in G is 16/a.

Commutative :

The operation * on G is commutative.

Since $\quad a * b = \frac{ab}{4} = b * a$.

Thus, the algebraic system (G, *) is closed, associative, identity element, inverse and commutative. Hence, the system (G, *) is an abelian group.

Q. 6. (a) How many integers are there between 5 and 1004 that are multiple of 3?

Ans.

Q. 6. (b) Find the number of distinguishable words that can be formed from the letters of MISSISSIPPI.

Ans. There are 11 letters in the word.

$\therefore$ The total number of permutations of these letters 11!.

$$= 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$

$$= 39916800 \cdot$$

Q. 6. (c) A box contains six white balls and five red balls. In how many ways, four balls can be drawn from the box if ;

(i) They can be of any colour,

(ii) Two balls are white and two red,

(iii) All the balls are of same colour.

Ans. White ball $= 6$

Red balls $= 5$

(i) There are 11 balls and 4 are to be selected

$$= {}^{11}P_5 = \frac{11!}{(11-4)!} = \frac{11!}{7!}$$

$$= \frac{11 \times 10 \times 9 \times 8 \times 7!}{7!} = 7920.$$

(ii) There are 6 white balls and 5 red balls.

$${}^{6}P_2 + {}^{5}P_2$$

$$= \frac{6!}{(6-2)!} + \frac{5!}{(5-2)!} = \frac{6!}{4!} + \frac{5!}{3!}$$

$$= \frac{6 \times 5 \times 4!}{4!} + \frac{5 \times 4 * 3!}{3!}$$

$$= 30 + 20 = 50$$

(iii) All balls of same colours.

$${}^{6}P_4 + {}^{5}P_4$$

$$= \frac{6!}{(6-4)!} + \frac{5!}{(5-4)!} = \frac{6!}{2!} + \frac{5!}{0}$$

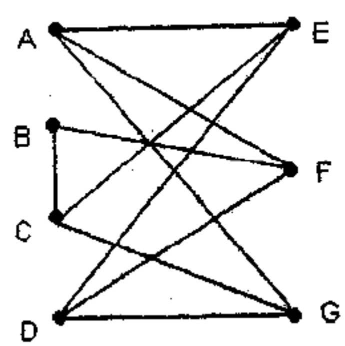
$$= \frac{6 \times 5 \times 4 \times 3 * 2!}{2!} + 5!$$

$$= 360 + 120 = 480 \cdot$$

Q. 8. (b) State Euler formula for connected planar graphs and illustrate it for two such graphs.

Ans. Euler formula : In graph theory, a planar graph is a graph which can be emedded in the plane, i.e., it can be drawn on the plane in such a way that its edges may intersect only at their endpoints. A nonplanar graph is the one which cannot be drawn in the plane without edge intersections. A planar graph already drawn in the plane without edge intersections is called a plane graph or planar embedding of the graph. A plane graph can be defined as a planar graph with a mapping from every node to a point in 2D space and from every edge to a plane curve, such that the extreme points of each curve are the points mapped from its end nodes, and all curves are disjoint except on their extreme points. It is easily sent hat a graph that can be drawn on the plane can be drawn on the sphere as well, and vice-versa. The equivalence class of topologically equivalent drawings on the sphere is called a planar map. Although a plane graph has an external or unbounded face, none of the faces of a planar map have a particular status.

A generalization of planar graphs are graphs which can be drawn on a surface of a given genus. In this terminology, planar graphs have graph genus 0, since the plane (and the sphere) are surfaces of genus 0.



Euler's formula states that if a finite, connected, planar graph is drawn in the plane without any edge intersections and v is the number of vertices, e is the number of edges and f is the number of faces (regions bounded by edges, including the outer, infinitely-large region), then

$$v - e + f = 2$$

i.e. the Euler characteristic is 2. As an illustration, in the first planar graph given above, we have $v = 6$, $e = 7$ and $f = 3$. If the second graph is redrawn without edge intersections, we get $v = 4$, $e = 6$ and $f = 4$. Euler's formula can be proven as follows : If the graph isn't a tree, then remove an edge which completes a cycle. This lowers both e and f by one, leaving $v - e + f$ constant. Repeat until you arrive at a tree; trees have $v = e + 1$ and $f = 1$, yielding $v - e + f = 2$.

In a finite, connected, simple, planar graph, any face (except possibly the outer one) is bounded by at least three edges and every edge touches at most two faces; using Euler's formula; one can then show that these graphs are sparse in the sense that $e \leq 3v - 6$ if $v \geq 3$.

A simple graph is called maximal planar if it is planar but adding any edge would destroy that property. All faces (even the outer one) are then bounded by three edges, explaining the alternative term triangular for these graphs. If a triangular graph has v vertices with $v > 2$, then it has precisely 3v-6 edges and 2v-4 faces.

Euler's formula is also valid for simple polyhedra. This is no coincidence : every simple polyhedron can be turned into a connected, simple, planar graph by using the polyhedron's vertices as vertices of the graph and the polyhedron's edges as edges of the graph. The faces of the resulting planar graph then correspond to the faces of the polyhedron. For example, the second planar graph shown above corresponds to a tetrahedron. Net every connected, simple, planar graph belongs to a simple polyhedron in this fashion; the trees do not, for example. A theorem of Earnest Steinitz says that the planar graphs formed from convex polyhedra (equivalently : those formed from simple polyhedra) are precisely the finite 3-connected simple planar graphs.