Test No. 6 :→

Cauchy's Root Test:→ If $\sum u_n$ is positive term series such that $\lim (u_n)^{1/n} = k$, then :→

(i) if $k<1$, the series converges (ii) if $k>1$, the series diverges

(iii) if $k=1$, Test fails.

Ex. 1 Examine the convergence of the series $\sum \left(\frac{n}{n+1}\right)^{n^2}$. (UTU 2010)

Sol:→ Here $u_n = \left(\frac{n}{n+1}\right)^{n^2}$

Applying Cauchy's Root test $\lim\limits_{n\to\infty} (u_n)^{1/n} = \lim\limits_{n\to\infty} \left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{1/n}$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{\cancel{n^2}/\cancel{n}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n} = \lim_{n\to\infty} \left[\frac{\cancel{n}}{\cancel{n}(1+1/n)}\right]^n$$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{(1+1/n)^n} = \frac{1}{e} < 1$$

$\Rightarrow$ The given series is convergent.

Ex. 2 Test the following series for convergence $\sum\limits_{n=1}^{\infty} \frac{(n+1)^n x^n}{n^{n+1}}$

Sol:→ Here $u_n = \frac{(n+1)^n x^n}{n^{n+1}} = \frac{(n+1)^n x^n}{n^{n+1}} = \left[\frac{(n+1)x}{n}\right]^n \cdot \frac{1}{n}$

Applying Cauchy's Root test $\lim\limits_{n\to\infty} (u_n)^{1/n} = \lim\limits_{n\to\infty} \left[\left[\frac{(n+1)x}{n}\right]^n\right]^{1/n} \cdot \left(\frac{1}{n}\right)^{1/n}$

$$= \lim_{n\to\infty} \frac{(n+1)x}{n} \cdot \frac{1}{(n)^{1/n}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(1+1/n)x}{\cancel{n}} \times \lim_{n\to\infty} \frac{1}{(n)^{1/n}}$$

$$= (1+0)x \cdot \frac{1}{1} = x$$

By Cauchy's Root test $\sum u_n$ is convergent if $x<1$ and divergent $x>1$.

The test fails when $x=1$

So, when $x=1$, $u_n = \frac{(n+1)^n}{n^{n+1}} = \frac{1}{n}\frac{(n+1)^n}{n^n} = \frac{1}{n}\left(1+\frac{1}{n}\right)^n$

let $v_n = 1/n$

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Now using comparison test

$$\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\frac{1}{\cancel{n}}\left(1+\frac{1}{n}\right)^n}{1/\cancel{n}}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e\ \text{(finite) (nonzero)}$$

$\therefore$ By comparison test $\sum u_n$ and $\sum v_n$ converges or diverges together.

Since $\sum v_n=\sum\frac{1}{n}$ is of the form $\sum\frac{1}{n^p}$ with $p=1$

$\Rightarrow \sum v_n$ is divergent $\Rightarrow \sum u_n$ is also divergent.

Ex. (3) Discuss the convergence of the following series

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\ldots\infty$$

Sol:→ Here $u_n=\left[\frac{(n+1)^{n+1}}{n^{n+1}}-\frac{(n+1)}{n}\right]^{-n}$

Applying Cauchy's Root test

$$\lim_{n\to\infty}(u_n)^{1/n}=\lim_{n\to\infty}\left[\left[\frac{(n+1)^{n+1}}{n^{n+1}}-\frac{(n+1)}{n}\right]^{-n}\right]^{1/n}$$

$$=\lim_{n\to\infty}\left[\frac{(n+1)^{n+1}}{n^{n+1}}-\frac{n+1}{n}\right]^{-1}$$

$$=\lim_{n\to\infty}\left[\frac{\cancel{n^{n+1}}\left[1+\frac{1}{n}\right]^{n+1}}{\cancel{n^{n+1}}}-\frac{\cancel{n}(1+1/n)}{\cancel{n}}\right]^{-1}$$

$$=\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n+1}-\left(1+\frac{1}{n}\right)\right]^{-1}$$

$$=\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1}-\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\right]^{-1}$$

$$=(e-1)^{-1}=\frac{1}{e-1}<1$$

$\Rightarrow$ The given series is convergent ($\because K<1$)

Ex. (4) Discuss the convergence of the series

$$\frac{1}{2}+\frac{2}{3}x+\left(\frac{3}{4}\right)^2x^2+\left(\frac{4}{5}\right)^3x^3+\ldots\infty$$

Sol:→ Here $u_n=\left(\frac{n+1}{n+2}\right)^n x^n$

Using Cauchy's Root test $\lim_{n\to\infty}(u_n)^{1/n}=\lim_{n\to\infty}\left[\left(\frac{n+1}{n+2}\right)^n x^n\right]^{1/n}=\lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)x$

$$\Rightarrow \lim_{n\to\infty} \frac{\cancel{n}(1+1/n)}{\cancel{n}(1+2/n)} x = x$$

Hence, the series converges if $x<1$ and diverges if $x>1$

If $x=1$, then $u_n = \left(\frac{n+1}{n+2}\right)^n = \frac{(1+1/n)^n}{(1+2/n)^n}$

$$\Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{(1+1/n)^n}{\left[(1+2/n)^{n/2}\right]^2} = \frac{e}{e^2} = \frac{1}{e} \neq 0$$

$\Rightarrow$ So According to Cauchy's fundamental test for divergent test, series is divergent.

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