Positive Term Series :→ If all terms after few negative terms in an infinite series are positive, such series is a positive term series

Ex:- $-10, -6, -1, +5, +12, 20 + \dots$ is a positive term series.

Necessary Condition for convergent series:→

for every convergent series $\sum u_n$

$\lim_{n\to\infty} u_n = 0$, but converse is not true.

Cauchy's fundamental Test for divergence (Positive Series)

Test for divergence:→ " If $\lim_{n\to\infty} u_n \neq 0$, the series $\sum u_n$ must be divergent."

Ex. ① Test for convergence of the series $1 + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \dots \frac{n}{n+1} + \dots \infty$

Sol:→ Here $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n+1}$

$$= \lim_{n\to\infty} \frac{1}{1 + 1/n}$$

$$= 1 \neq 0$$

Hence, according to Cauchy's fundamental test for divergence, the series is divergent.

* To determine the nature of a series we have to find S_n, since it is not possible to find S_n for every series, we have to devise test for convergence without ~~having~~ involving S_n

Ex. ② Test for convergence the series $1 + \frac{3}{5} + \frac{8}{10} + \frac{15}{17} + \dots \frac{2^n - 1}{2^n + 1} + \dots + \infty$

Sol:→ Applying Cauchy's fundamental test

$$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{2^n - 1}{2^n + 1} = \lim_{n\to\infty} \frac{1 - 1/2^n}{1 + 1/2^n} = 1 \neq 0$$

So, According to Cauchy's fundamental test for divergence the series is divergent.

Ex (3) Test the convergence of the following series:

$$\sqrt{\frac{1}{4}}+\sqrt{\frac{2}{6}}+\sqrt{\frac{3}{8}}+\dots+\sqrt{\frac{n}{2(n+1)}}+\dots$$

Here $u_n=\sqrt{\frac{n}{2(n+1)}}=\sqrt{\frac{n}{2n(1+1/n)}}=\sqrt{\frac{1}{2\left(1+\frac{1}{n}\right)}}$

Thus $\lim_{n\to\infty} u_n=\lim_{n\to\infty}\frac{1}{\sqrt{2\left(1+\frac{1}{n}\right)}}=\frac{1}{\sqrt{2}}\neq 0$

$\Rightarrow \sum u_n$ does not converge., Hence by Cauchy fundamental test for divergence, the series is divergent.

Ex. (4) Test the convergence of $\sum_{n=1}^{\infty}\cos(1/n)$

Sol:→ Here, it is clear that series is positive term series, and

$$u_n=\cos\left(\frac{1}{n}\right)$$

$$\Rightarrow u_n=\left[1-\frac{1}{2!\,n^2}+\frac{1}{4!\,n^4}-\frac{1}{6!\,n^6}+\dots\right] \quad \left(\because \cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots\right)$$

Now $\lim_{n\to\infty}u_n=\lim_{n\to\infty}\left[1-\frac{1}{2!\,n^2}+\frac{1}{24n^4}-\frac{1}{720n^6}+\dots\right]$

$$=1\neq 0$$

So, According to Cauchy's fundamental test for divergence, the series is divergent.

Ex. (5) Test whether the following series is convergent or divergent

$$\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{1}{n}\right)$$

Sol:→ Given $u_n=\frac{1}{n}\sin\left(\frac{1}{n}\right)$

$$=\frac{1}{n}\left[\frac{1}{n}-\frac{1}{3!\,n^3}+\frac{1}{5!\,n^5}-\frac{1}{7!\,n^7}+\dots\right] \quad \left\{\because \sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}\right\}$$

$$=\frac{1}{n}\left[\frac{1}{n}-\frac{1}{6n^3}+\frac{1}{120n^5}-\frac{1}{5040n^7}\right]$$

$$=\frac{1}{n^2}-\frac{1}{6n^4}+\frac{1}{120n^6}-\frac{1}{5040n^8}+\dots$$

Now $\lim_{n\to\infty}u_n=\lim_{n\to\infty}\left[\frac{1}{n^2}-\frac{1}{6n^4}+\frac{1}{120n^6}-\frac{1}{5040n^8}+\dots\right]$

$$=0$$

According to Cauchy's fundamental test for divergence, the series is convergent.

p-series :→ The series $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\dots+\infty$ is

(i) Convergent if $p>1$

(ii) Divergent if $p\leq 1$

{ Inshort $\sum_{n=1}^{\infty}\frac{1}{n^p}$ { Convergent, if $p>1$; divergent, if $p\leq 1$ } }

Ex. (1) Test the convergence of $\sum_{n=1}^{\infty}\frac{1}{n}$

Sol:→ Given series is $\sum_{n=1}^{\infty}\frac{1}{n}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots+\infty$

Here $p=1$, then according to p-series test the series is **divergent**.

Ex. (2) Test the convergence of $\sum_{n=1}^{\infty}\frac{1}{n^2}$

Sol:→ Without open the series, we know that $p=2$ here, so series is Convergent ($\because p>1$).

Ex. (3) Test whether the following series is convergent or divergent $\sum_{n=1}^{\infty}\frac{n^{-\pi}}{1}$.

Sol:→ Given series is $\sum_{n=1}^{\infty}\frac{n^{-\pi}}{1}=\sum_{n=1}^{\infty}\frac{1}{n^{\pi}}$, here $p=\pi\approx 3.14>1$

It means that series is convergent.

Ex. (4) Test the convergence of $\sum_{n=1}^{\infty}\sqrt[5]{n^2}$

Sol:→ Here $\sum_{n=1}^{\infty}\sqrt[5]{n^2}=\sum_{n=1}^{\infty}\frac{(n)^{2/5}}{1}=\sum_{n=1}^{\infty}\frac{1}{n^{-2/5}}$.

Now it is clearly seen that $p=-2/5$ (≤ 1), so according to p-series test, the series is divergent.

Ex. (5) Test the convergence $\sum_{n=1}^{\infty}(n^{-2.4}+8n^{-1.6})$

Sol:→ $\Rightarrow \sum_{n=1}^{\infty} n^{-2.4} + 8\sum_{n=1}^{\infty} n^{-1.6} = \sum_{n=1}^{\infty}\frac{1}{n^{2.4}} + 8\sum_{n=1}^{\infty}\frac{1}{n^{1.6}}$

Here $p=2.4$ Clearly $p\geq 1$ | Here $p=1.6$ clearly $p\geq 1$

Thus, according to p-series test, the series is convergent.

Ex. (6) Test the convergence of $\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\ldots\infty = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$

Sol:→ $\sum_{n=1}^{\infty}\frac{1}{(n)^{1/2}}$, since $p=1/2\ (\leq 1)$, thus series is divergent.

Try Yourself:→ Test the convergence of :→

(a) $\frac{1}{\sqrt[3]{1}}+\frac{1}{\sqrt[3]{4}}+\frac{1}{\sqrt[3]{9}}+\frac{1}{\sqrt[3]{16}}+\ldots+\infty$

(b) $1+\frac{1}{2.\sqrt[3]{2}}+\frac{1}{3.\sqrt[3]{3}}+\frac{1}{4.\sqrt[3]{4}}+\ldots+\infty$