CHARPIT'S METHOD

This is a general method for finding the complete solution of non-linear partial diff. equation of the first order

Let the given equation be $f(x, y, z, p, q) = 0$ — ①

If we can find another relation $F(x, y, z, p, q) = 0$ — ②

involving x, y, z, p and q, then we can solve Eq ① and ② for p and q and substitute in

$$\boxed{dz = p\,dx + q\,dy}$$

∴ The auxiliary equations are :→

$$\boxed{\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0}}$$

Take the simplest of the integrals so that it becomes easier to solve for p and q.

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Ex. 1 Solve $2zx - px^2 - 2qxy + pq = 0$

Sol: Here $f = 2zx - px^2 - 2qxy + pq = 0$ —— (1)

Charpit's A.E. ⇒

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0} \quad \text{—— (2)}$$

So: $\frac{\partial f}{\partial x} = 2z - 2px - 2qy$, $\frac{\partial f}{\partial y} = -2qx$, $\frac{\partial f}{\partial z} = 2x$, $\frac{\partial f}{\partial p} = -x^2 + q$, $\frac{\partial f}{\partial q} = -2xy + p$

Put these values in Eq. (2), we get:

$$\frac{dp}{(2z - 2px - 2qy) + p(2x)} = \frac{dq}{-2qx + q(2x)} = \frac{dz}{-p(-x^2+q) - q(-2xy+p)} = \frac{dx}{-(-x^2+q)} = \frac{dy}{-(-2xy+p)} = \frac{dF}{0}$$

$$\Rightarrow \frac{dp}{2z - 2qy} = \frac{dq}{0} = \frac{dz}{px^2 - 2pq + 2qxy} = \frac{dx}{x^2 - q} = \frac{dy}{2xy - p} = \frac{dF}{0}$$

Take simplest one (or choose):

$\therefore dq = 0$

$\int dq = c_1$

$\boxed{q = c_1}$

put $q = c$ in Eq. (1) we get $2zx - px^2 - 2cxy + pc = 0$

$$\Rightarrow \boxed{p = \frac{2x(z - c_1 y)}{x^2 - c_1}}$$

$$\therefore \quad dz = p\,dx + q\,dy$$

$$\Rightarrow dz = \frac{2x(z - c_1 y)}{x^2 - c_1}\,dx + c_1\,dy$$

OR $$\frac{dz - c_1\,dy}{z - c_1 y} = \frac{2x}{x^2 - c_1}\,dx$$

~~Integrating~~ rating both side :-

$$\log(z - c_1 y) = \log(x^2 - c_1) + \log c_2$$

$$\Rightarrow \quad z - c_1 y = c_2(x^2 - c_1)$$

OR $$\boxed{z = c_1 y + c_2(x^2 - c_1)}$$

Ex. (2) Solve $(p^2+q^2)y = qz$

Here $f = (p^2+q^2)y - qz = 0$ —— (1)

Then Charpit's A.E. :–

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0} \quad \text{—— (2)}$$

from (1), $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = p^2+q^2$, $\frac{\partial f}{\partial z} = -q$, $\frac{\partial f}{\partial p} = 2py$ and $\frac{\partial f}{\partial q} = 2qy - z$

Put these values in Eq. (2), we get →

$$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-qz} = \frac{dx}{-2py} = \frac{dy}{-2qy+z} = \frac{dF}{0}$$

taking first two members

$$\Rightarrow \frac{dp}{-pq} = \frac{dq}{p^2}$$

$$\Rightarrow p\,dp + q\,dq = 0$$

$$\Rightarrow \int p\,dp + \int q\,dq = c_1$$

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$$\Rightarrow \frac{p^2}{2} + \frac{q^2}{2} = c_1$$

$$\text{OR} \quad p^2 + q^2 = 2c_1$$

$$\text{OR} \quad p^2 + q^2 = a^2 \quad \text{--- (3)}$$

Put the value of (p^2+q^2) in Eq. (1), we get $a^2 y - qz = 0$

$$\Rightarrow \boxed{q = \frac{a^2 y}{z}}$$

from (3) $p = \sqrt{a^2 - q^2}$

$$\text{OR} \quad p = \sqrt{a^2 - \frac{a^4 y^2}{z}} = \frac{a}{z}\sqrt{z^2 - a^2 y^2}$$

$$\text{OR} \quad \boxed{p = \frac{a}{z}\sqrt{z^2 - a^2 y^2}}$$

$$\therefore \quad dz = p\,dx + q\,dy$$

$$dz = \frac{a}{z}\sqrt{z^2 - a^2 y^2}\,dx + \frac{a^2 y}{z}\,dy$$

$$\Rightarrow z\,dz - a^2 y\,dy = a\sqrt{z^2 - a^2 y^2}\,dx$$

$$\Rightarrow \frac{\frac{1}{2}\,d(z^2 - a^2 y^2)}{\sqrt{z^2 - a^2 y^2}} = a\,dx \xrightarrow{\text{Integrating}} \sqrt{z^2 - a^2 y^2} = ax + b$$

$$\Rightarrow \boxed{z^2 = (ax+b)^2 + a^2 y^2}$$

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