Last topic in Unit (3)

"Linear partial differential equation with constant coefficient of nth order"

Definition:→ An equation of the type

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \dots a_n \frac{\partial^n z}{\partial y^n} = F(x,y) \quad \text{—(1)}$$

Put $\partial/\partial x = D$ and $\frac{\partial}{\partial y} = D'$ in (1)

We get:→ $(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'' + \dots a_n D'^n) = F(x,y)$

$$\Rightarrow f(D, D')z = F(x,y)$$

*** Working Rule to finding Complementary fun^n:→**

let's start with an example:→

Solve:– $(D^3 - 4D^2D' + 3DD'^2)z = 0$

Remember

$D = \partial/\partial x$

and $D' = \partial/\partial y$

Hence A.E. $\Rightarrow$ [by putting $D = m$, $D' = 1$]

We get $m^3 - 4m^2 + 3m = 0$

$\Rightarrow m(m^2 - 4m + 3) = 0$

$\Rightarrow m(m-1)(m-3) = 0$

$\Rightarrow \underline{m = 0, 1, 3}$ (All roots are different) (Also it is real roots)

The required sol:$\rightarrow$

$$\underline{z} = f_1(y + 0x) + f_2(y + x) + f_3(y + 3x)$$

Solve

Ex. 2 $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x \partial y} + 4\dfrac{\partial^2 z}{\partial y^2} = 0$

Write it in operator form:$\rightarrow$

Step 1 $(D^2 - 4DD' + 4D'^2)z = 0$ $\left\{ \because D = \partial/\partial x, \; D' = \partial/\partial y \right\}$

Step 2 Auxiliary Equation$\rightarrow$

$(m^2 - 4m + 4) = 0$ ($\because$ put $D = m$ and $D' = 1$)

Now solve $\rightarrow$ $m^2 - 4m + 4 = 0$

$\Rightarrow (m-2)^2 = 0$

$\Rightarrow m = \underline{\underline{2, 2}}$ (Roots are repeated and real)

then required sol:$\rightarrow$

$$z = f_1(y + 2x) + x f_2(y + 2x)$$

Ex. (3) Solve PDE → $\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} = 0$

Step (1) :→ operator form :→ $(D^4 + D'^4)z = 0$

$(\because \partial/\partial x = D$ and $\partial/\partial y = D')$

Step (2) :→ Auxiliary equation :→ $m^4 + 1 = 0$

(put $D = m, D' = 1$)

Now solve A.E. ⇒ $m^4 + 1 = 0$

$\Rightarrow m^4 + 1 + 2m^2 - 2m^2 = 0$

$\Rightarrow (m^2+1)^2 - (m\sqrt{2})^2 = 0$

$\Rightarrow (m^2 + 1 + m\sqrt{2})(m^2 + 1 - m\sqrt{2}) = 0$

$\Rightarrow m^2 + 1 + m\sqrt{2} = 0$

$\Rightarrow m^2 + \sqrt{2}m + 1 = 0$

$m = \frac{-\sqrt{2} \pm \sqrt{2-4}}{2}$

$m = -\frac{\sqrt{2} \pm \sqrt{2}i}{2}$

$\Rightarrow m = -\frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$

$\Rightarrow m = \frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, \frac{-1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

(I) (II)

other root

$m^2 + 1 - m\sqrt{2} = 0$

$\Rightarrow m^2 - \sqrt{2}m + 1 = 0$

$m = \frac{\sqrt{2} \pm \sqrt{2-4}}{2}$

$m = \frac{1}{\sqrt{2}} \pm \frac{\sqrt{2}i}{2}$

$m = \frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}$

$m = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}$

(III) (IV)

Roots are complex :→ So solution ⇒

$$z = f_1\left[y + \left(\frac{-1+i}{\sqrt{2}}\right)x\right] + f_2\left[y + \left(\frac{1+i}{\sqrt{2}}\right)x\right]$$
$$+ f_3\left[y + \left(\frac{-1-i}{2}\right)x\right] + f_4\left[y + \left(\frac{1-i}{\sqrt{2}}\right)x\right]$$

Ans.

Summary

Roots of A.E.	C.F.
(1) m_1, m_2, m_3 (different)	$f_1(y+m_1x) + f_2(y+m_2x) + f_3(y+m_3x)$
(2) $\underbrace{m_1, m_1}_{\text{(Same)}}, \underbrace{m_2}_{\text{(diff.)}}$	$f_1(y+m_1x) + x f_2(y+m_1x) + f_3(y+m_2x)$
(3) m_1, m_2, m_3 $\Rightarrow m_1 = m_2 = m_3$ (All are identical)	$f_1(y+m_1x) + x f_2(y+m_1x) + x^2 f_3(y+m_1x)$

Ex.

If $m = 3, 3, 3$

Then I.F.

$$\Rightarrow f_1(y+3x) + x f_2(y+3x) + x^2 f_3(y+3x)$$

Side-by-side comparison

Linear differential equation of nth order with constant coefficients

Solve

Ex:-

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$$

Step ① operator form:-

$\Rightarrow (D^2+5D+6)y = 0$

Step ② Auxiliary Equation

$\Rightarrow (m^2+5m+6) = 0$

Now solve (After putting $D=m, y=1$)

$\Rightarrow m^2+3m+2m+6=0$

$m(m+3)+2(m+3)=0$

$\Rightarrow (m+3)(m+2)=0$

$\Rightarrow m = -3, -2$

(Real and distinct roots)

Then

C.F. $= c_1 e^{-3x} + c_2 e^{-2x}$

or

$y = c_1 e^{-3x} + c_2 e^{-2x}$

Linear Partial differential equation of nth order with constant coefficients

Solve

Ex.

$$\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 6\frac{\partial^2 z}{\partial y^2} = 0$$

Step ① Operator form

$\Rightarrow (D^2 + 5DD' + 6D'^2)z = 0$

Step ② Auxiliary Equation :-

$\Rightarrow (m^2+5m+6) = 0$

Now Solve $\left\{ \text{but here put } D=m,\ D'=1 \right\}$

$\Rightarrow m^2+3m+2m+6=0$

$\Rightarrow m(m+3)+2(m+3)=0$

$\Rightarrow (m+3)(m+2)=0$

$\Rightarrow m = -3, -2$

(Real and distinct roots)

then

C.F. $\Rightarrow f_1(y-3x) + f_2(y-2x)$

OR

$z = f_1(y-3x) + f_2(y-2x)$