Moment Generating Functions

Definition: $E(X^r)$ defines the r^{th} moment of the random variable X about the origin.

Moment Generating Functions (mgfs) are used to generate moments of random variables about the origin.

Definition: The Moment Generating Function of the random variable X is defined as

$$M_X t = E(e^{tx}) = \begin{cases} \sum_{\forall x} e^{tx} p(x), & \text{for } x \text{ discrete} \\ \int_{-\infty}^{\infty} e^{tx} f(x), & \text{for } x \text{ continuous} \end{cases}$$

where t is a real number, if it exists.

NB: Moment generating functions do not exist for all random variables.

Recall: (from the special results provided earlier)

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Hence

$$e^{tx} = 1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + ... = \sum_{k=0}^{\infty} \frac{t^kx^k}{k!}$$

$$\begin{aligned} M_X(t) &= E(e^{tx}) \\ &= E\left[1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + ...\right] \\ &= 1 + tE(x) + \frac{t^2}{2}E(x^2) + \frac{t^3}{6}E(x^3) + ... \end{aligned}$$

Let us define $M_X'(t)$ as the first derivative of $M_X(t)$ with respect to t.

$$M_X'(t) = E(x) + tE(x^2) + \frac{t^2}{2}E(x^3) + \textit{terms with higher powers ot } t \tag{1}$$

Let $t = 0$ in Equation (1)

$$M_X'(0) = E(x) \tag{2}$$

Let us define $M_X''(t)$ as the second derivative of $M_X(t)$ with respect to t. We get this by differentiating Equation (1) with respect to t to get

$$M_X''(t) = E(x^2) + tE(x^3) + \textit{terms with higher powers ot } t \tag{3}$$

Let $t = 0$ in Equation (3)

$$M_X''(0) = E(x^2) \tag{4}$$

In general, finding the n^{th} derivative of $M_X(t)$ with respect to t and letting $t = 0$ gives the n^{th} moment of the random variable X about the origin i.e.

$$M_X^{(n)}(0) = E(x^n) \tag{5}$$

Hence, we are able to use moment generating functions of random variables if they exist to find the mean and variance of a random variable since

$$\begin{aligned} E(X) &= M_X'(0) \\ Var(X) &= E(X^2) - [E(X)]^2 \\ &= M_X''(0) - [M_X'(0)]^2 \end{aligned}$$

Example 1

The probability mass function of a random variable X is given by

$$p(x) = \begin{cases} \binom{n}{x} p^x q^{n-x}, & x = 0, 1, 2, ..., n \quad \text{where } p + q = 1 \\ 0, & \text{otherwise} \end{cases}$$

Determine the moment generating function of X. Hence determine $E(X)$ and $Var(X)$.

Solution

$$\begin{aligned} M_X(t) &= E(e^t x) \\ &= \sum_{\forall x} e^{tx} p(x) \\ &= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x} \\ &= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x} \\ &= (pe^t + q)^n \end{aligned}$$

$$\begin{aligned} M_X'(t) &= npe^t(pe^t + q)^{n-1} \\ M_X''(t) &= n(n-1)(pe^t)^2(pe^t + q)^{n-2} \end{aligned}$$

$$
\begin{aligned}
E(X) &= M'_X(0) \\
&= np \\
Var(X) &= M''_X(0) - [M'_X(0)]^2 \\
&= n(n-1)p^2 - (np)^2 \\
&= np(1-p)
\end{aligned}
$$

Theorems on Moment Generating Functions

Theorem 1

$M_{cX}(t) = M_X(ct)$, c being a constant.

Proof By definition:

$L.H.S = M_{cX}(t) = E(e^{tcX})$

$R.H.S = M_X(ct) = E(e^{ctX}) = L.H.S$

Theorem 2

The moment generating function of the sum of a number of independent random variables is equal to the product of their respective moment generating functions. Symbolically, if $X_1, X_2, \dots, X_n$ are independent random variables, then the moment generating function of their sum $X_1 + X_2 + \dots + X_n$ is given by

$$M_{(X_1+X_2+\dots+X_n)}(t) = M_{X_1}(t)M_{X_2}(t)\dots M_{X_n}(t)$$

Proof By definition:

$$
\begin{aligned}
M_{(X_1+X_2+\dots+X_n)}(t) &= E[e^{t(X_1+X_2+\dots+X_n)}] \\
&= E[e^{t(X_1)}e^{t(X_2)}...e^{t(X_n)}] \\
&= E[e^{t(X_1)}]E[e^{t(X_2)}]...E[e^{t(X_n)}] \quad \text{because the } X_i'\text{s are independent} \\
&= M_{X_1}(t)M_{X_2}(t)\dots M_{X_n}(t)
\end{aligned}
$$

Hence the theorem.

Theorem 3

Effect of change of origin and scale on the MGF. Let us transform X to the new variable U by changing both the origin and scale in X as follows:

$$U = \frac{X - a}{h}$$

where a and h are constants.

The moment generating function of U (about origin) is given by

$$\begin{aligned} M_U(t) &= E(e^{tU}) \\ &= E\left(e^{t\left(\frac{X-a}{h}\right)}\right) \\ &= E\left[e^{\frac{tX}{h}} \times e^{-\frac{at}{h}}\right] \\ &= e^{-\frac{at}{h}} E\left[e^{\frac{tX}{h}}\right] \\ &= e^{-\frac{at}{h}} M_X\left(\frac{t}{h}\right) \end{aligned}$$

where $M_X(t)$ is the moment generating function of X about the origin.

Theorem 4

The moment generating function of a distribution, if it exists, uniquely determines the distribution. This implies that corresponding to a given probability distribution, there is only one moment generating function (provided it exists) and corresponding to a given moment generating function, there is only one probability distribution. Hence $M_X(t) = M_Y(t) \Rightarrow X$ and Y are identically distributed.

Limitation of the Moment Generating Function

1. A random variable X may have no moments although its moment generation function exists.

2. A random variable X can have a moment generating function and some (or all) moments, yet the moment generation function does not generate the moments.

3. A random variable X can have all or some moments; but the moment generation function does not exist except perhaps at one point.

Class Exercise

1. Define the moment generating function of a random variable. Hence or otherwise find the moment generating function of

   a) $Y = aX + b$

   b) $Y = \dfrac{X - m}{\sigma}$

2. The random variable X takes the value n with probability $\dfrac{1}{2^n}, n = 1, 2, 3, \ldots$. Find the moment generating function of X and hence find the mean and variance of X.

3. Show that if $\bar{X}$ is the mean of n independent random variables, then

$$M_{\bar{X}}(t) = \left[M_X\left(\frac{t}{n}\right)\right]^n$$

4. Show that the moment generating function of the random variable X having the probability density function

$$f(x) = \begin{cases} \frac{1}{3}, & -1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

   is

$$M_X(t) = \begin{cases} \dfrac{e^{2t} - e^{-t}}{3t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

5. X is a random variable and $p(x) = ab^x$ where a and b are positive, $a + b = 1$ with x taking the values $0, 1, 2, \ldots$. Find the moment generating function of X. Hence show that $m_2 = m_1(2m_1 + 1)$, m_1 and m_2 being the first two moments.

Reference

1. Morris H DeGroot & Mark J Schervish (2012) Probability and Statistics; 4^{th} Edition, Pearson Education, Inc.

2. Robert V Hogg, Elliot A. Tanis & Dale L. Zimmerman (2015) Probability & Statistical Inference; 9^{th} Edition, Pearson Education, Inc.