BACHELOR OF SCIENCE IN INFORMATION TECHNOLOGY

KIBABII UNIVERSITY

STA 205

Probability and Statistics

Lecture notes

Course Outline

1. Introduction to probability
   (a) Definition of terms, addition and multiplication of probabilities
   (b) Conditional probabilities, independent events, Mutually exclusive events
   (c) Bayes Theorem
2. Probability distributions
   (a) Normal distribution
   (b) Binomial distribution
   (c) Poisson distribution
3. Statistical tests
   (a) z test
   (b) t test
   (c) Chi-square test
   (d) F test
4. Confidence intervals
5. Estimation
6. Hypothesis testing

References

1. George G. Roussas (1997) A Course in Mathematical Statistics Second Edition University of California Davis, California
2. Robert V. Hogg and Allen T. Craig (2004) Introduction to Mathematical Statistics 5th Edition Pearson Education Asia limited and Higher Education Press.
3. Dass H. K (2013) Advanced Engineering Mathematics S. Chand and Company Limited New Delhi.

Introduction to probability

Probability is a branch of mathematics that deals with calculating the chance of a given event's occurrence.

It is expressed as a number between 0 and 1 where:

- a number close to 0 means not likely
- a number close to 1 means quite likely

If the probability of an event is exactly 0, then the event cannot occur. For example, if today is Monday the probability that tomorrow is Sunday is 0 - an uncertain occurrence.

If the probability of an event is exactly 1, then the event will definitely occur. If today is Monday the probability that tomorrow is Tuesday is 1 - a certain occurrence.

Hence we now know that probability is a number between 0 and 1.

Approaches to Probability

Our problem is then: how does an event get assigned a particular probability value?

There are three approaches to (interpretations of) probability namely:

1. The subjective approach
2. The relative frequency approach
3. The classical approach

The subjective approach

This approach is the simplest in practice, but therefore it also the least reliable. Think of it as the 'whatever it is to you' approach. Here are some examples:

- 'I think there is an 80% chance of rain today.'
- 'I think there is a 50% chance that the world's oil reserves will be depleted by the year 2100.'
- 'I think there is a 1% chance that the Harambee Stars will end up in the World Cup.'

Example 1

At which end of the probability scale would you put the probability that:

1. One day you will die?
2. You can swim around the world in 30 hours?
3. You will win the lottery some day?
4. A randomly selected student will get an A in this course?
5. You will get an A in this course?

The relative frequency approach

Here probability is interpreted to mean the relative frequency with which the outcome will be obtained if the process were repeated a large number of times under similar conditions.
The relative frequency approach involves taking the following three steps in order to determine $p(A)$, the probability of an event A:

1. Perform an experiment a large number of times, n, say.
2. Count the number of times the event A of interest occurs, call the number $N(A)$, say.
3. Then, the probability of event A equals:

$$p(A) = \frac{N(A)}{n}$$

Arguments against the relative frequency approach

✓ There is no definite indication of an actual number that would be considered large enough.

✓ If you toss your coin 1,000,000 times, do you get 500,000 heads? (No limit is specified for the permissible variation from 0.5).

✓ Besides, if a process is repeated under similar conditions – you should expect the same outcome

✓ Not all processes can be repeated a large number of times e.g. Tom and Mary are having an affair; what is the probability that they will get married in two years?

Example 2

Some trees in a forest were showing signs of disease. A random sample of 200 trees of various sizes was examined yielding the following results:

Type	Disease free	Doubtful	Diseased	Total
Large	35	18	15	68
Medium	46	32	14	92
Small	24	8	8	40
Total	105	58	37	200

a) What is the probability that one tree selected at random is large?

b) What is the probability that one tree selected at random is diseased?

c) What is the probability that one tree selected at random is both small and diseased?

d) What is the probability that one tree selected at random is either small or disease-free?

e) What is the probability that one tree selected at random from the population of medium trees is doubtful of disease?

The classical approach

Here the concept of probability is based on the concept of equally likely outcomes e.g. when a coin is tossed, there are two (2) possible outcomes – a Head and a Tail.
If the two outcomes are equally likely, then they must have the same probability of occurrence.
Since the total probability is 1 then

$$p(\text{Head}) = p(\text{Tail}) = \frac{1}{2}$$

As long as the possible outcomes are equally likely (!!!), the probability of event A is:

$$p(A) = \frac{p(N(A))}{p(N(S))}$$

where

$N(A)$ - the number of elements in the event A, and

$N(S)$ - the number of all possible elements.

Arguments against the classical approach

- ✓ What if the outcomes are not equally likely, how do we assign probabilities? e.g. a student sits for an exam, what are the chances that they pass? – Passing and Failing are not equally likely outcomes. We therefore require other methods to assign probability.

- ✓ Besides the concept of equally likely outcomes is essentially based on the concept of probability that we are trying to define – the statement that the two possible outcomes are equally likely is the same as the statement that the two outcomes have the same probability.

Example 2

Suppose you draw one card at random from a standard deck of 52 cards. Assume the cards were manufactured to ensure that each outcome is equally likely.

- ☐ Let A be the event that the card drawn is a 2, 3, or 7.

- ☐ Let B be the event that the card is a 2 of hearts (H), 3 of diamonds (D), 8 of spades (S) or king of clubs (C).

a) What is the probability that a 2, 3, or 7 is drawn?

b) What is the probability that the card is a 2 of hearts, 3 of diamonds, 8 of spades or king of clubs?

c) What is the probability that the card is either a 2, 3, or 7 or a 2 of hearts, 3 of diamonds, 8 of spades or king of clubs?

d) What is the probability that event A and B occurs?

Note however that:

- ✓ probability will be the way we quantify how likely something is to occur.

- ✓ the theory of probability does not depend on the interpretation or approach to probability.

The Mathematical Theory of Probability

Once probabilities have been assigned to the outcome of a process, the mathematical theory of probability provides the appropriate methodology for the further studies of these probabilities namely:

1. Methods of determining the probabilities of certain events from the specified probability of each possible outcome of an experiment e.g. given $p(A)$ and $p(B)$, find $p(A \quad \text{and} \quad B)$ and
2. Methods of revising the probabilities of events when additional relevant information is obtained.

The problem is twofold.

Basic Concepts in Probability Theory

To understand the language of probability, we begin by defining basic concepts used in probability theory.

Definition of Terms

Experiment: A repeatable process which gives rise to a number of known possible outcomes e.g.

- ✓ Tossing a coin is an experiment whose outcomes are Head (H) or Tail (T);
- ✓ Tossing two coins – HH, HT, TH, TT;
- ✓ Throwing a die – 1, 2, 3, 4, 5, 6;
- ✓ Throwing two die – $(1, 1)$, $(1, 2)$, $(1, 3)$, $\ldots$, $(6, 6)$; 36 possible outcomes;
- ✓ Throwing a die and tossing a coin – $1H$, $2H$, $3H$, $4H$, $5H$, $6H$, $1T$, $\cdots$, $6T$; 12 possible outcomes;
- ✓ Picking a student from a MAS103 class – Male or Female.

Random Experiment: In probability theory we start off with an experiment whose result cannot be determined beforehand but we know all the possible outcome.
An experiment is said to be random if its result cannot be determined beforehand.
Sample Space: The sample space is denoted by S or Ω. It refers to the set (or a collection) of all the possible outcomes of an experiment e.g.

- ✓ Tossing a coin, $S = \{\text{Head}, \text{Tail}\}$ or $S = \{H, T\}$;
- ✓ Tossing two coins $S = \{HH, HT, TH, TT\}$;

Each possible outcome of a random experiment is called a **sample point**.

Since we have random experiments of many types, the set Ω may consist of:

i) A finite number of elements

ii) Countably infinite number of elements

iii) Uncountably many elements

The elements could be numerical or non-numerical.

Example 1

Consider the following random experiments denoted by E. Find the sample space and state whether they have numerical/non-numerical, finite/countably infinite/uncountably many elements.

✓ E: Tossing a coin;

✓ E: Rolling a dice;

✓ E: Number of on-going calls in a particular telephone exchange;

✓ E: Temperature of a given city;

An Event is denoted with capital letters A, B, C ... is a subset of the sample space S. That is, $\mathsf{A} \subset \mathsf{S}$, where '$\subset$' denotes 'is a subset of' – meaning is contained in.

Consider the experiment of throwing two dice. The sample space is:

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

An event A - getting a sum of 7; or the event B - getting a sum of 5; or the event C - getting a sum of 2;

Simple and Compound Events: An event is said to be a simple event if it consists of one outcome otherwise it is a compound event e.g. when tossing two die event C – getting a sum of two is a simple event while event A above is a compound event.

NB: The probability of an event A occurring is denoted $p(A)$

Equally Likely Events: These are events that have the same theoretical probability (or likelihood) of occurring. For example: Each numeral on an unbiased die is equally likely to occur when the die is tossed. Sample space of throwing a die: $\{1, 2, 3, 4, 5, 6\}$.

Note:

- ✓ We say that a particular event A has occurred if the outcome of the experiment is a member of (i.e. contained in) A.
- ✓ The sample space of an experiment is an event which always occurs when the experiment is performed. This implies $p(S) = 1$.

ALGEBRA OF SETS

Since events and sample spaces are just sets, let us review the algebra of sets to enable us apply them when calculating the probability of events.

What is a set?

A set is a well-defined collection of objects called the elements or members of the set. They are denoted using capital letters just like events.

The algebra of sets defines the properties and laws of sets, the set-theoretic operations of union, intersection, and complementation and the relations of set equality and set inclusion,(*Wikipedia*).

The following laws will help us answer probability questions.

1. ϕ denotes the empty or null set - the set that has no elements.
2. $C \cup D$ read C union D defines the elements in C or D or both; It is the event consisting of all outcomes that are in C or in D or in both.
3. $A \cap B$ read A intersection B defines the elements in A and B; It is the event consisting of all outcomes that are in both A and B.
4. If $A \cap B = \phi$, then A and B are called mutually exclusive events (or disjoint events) – A and B have no outcomes in common.
5. The complement, D' of D also denoted D^c is the event consisting of all outcomes that are not in D but are in the sample space.

6. If $E \cap F \cap G \cap \ldots = S$, then E, F, G, and so on are called exhaustive events.

7. The difference $A–B$ is defined as $A \cap B^c$ - the event that both A and B^c occur.

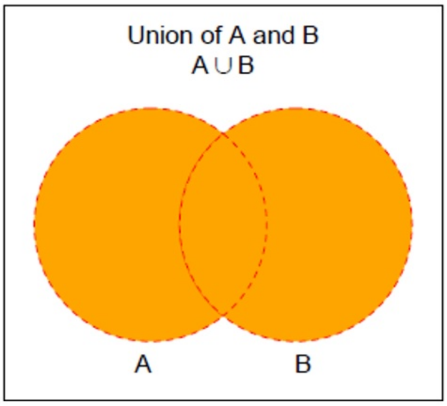
8. A is a subset of B, denoted $A \subset B$, if $e \in A$ implies $e \in B$.

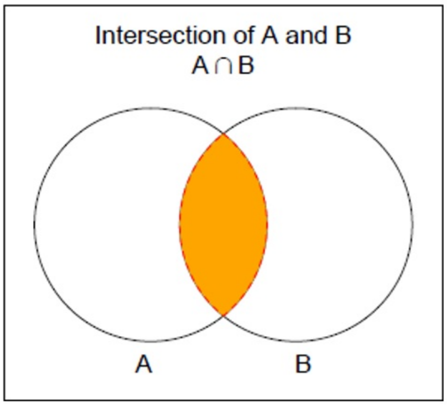
9. Two sets are equal, $A = B$, if $A \subset B$ and $B \subset A$.

VENN DIAGRAMS

Venn Diagrams provided a pictorial representation of sets hence events.

For Union and Intersection

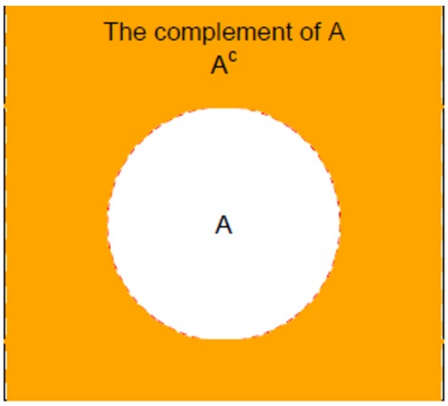


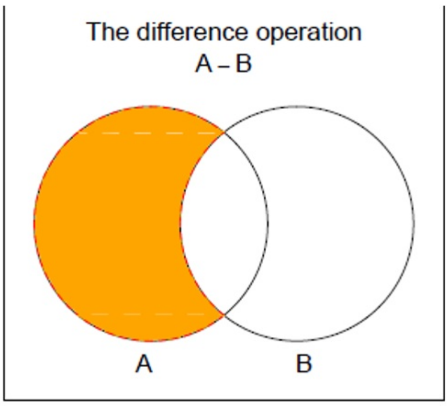


The event $A \cup B$ denotes the event that either A or B or both occur.

The event $A \cap B$ denotes the event that both A and B occur.

For complement and difference

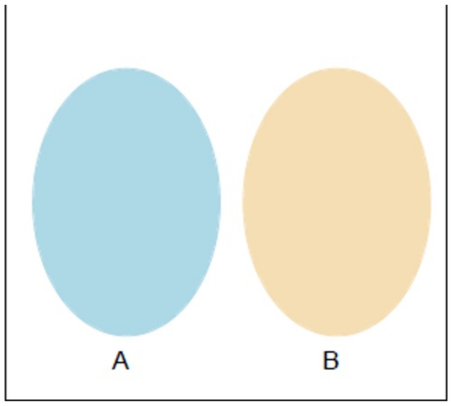


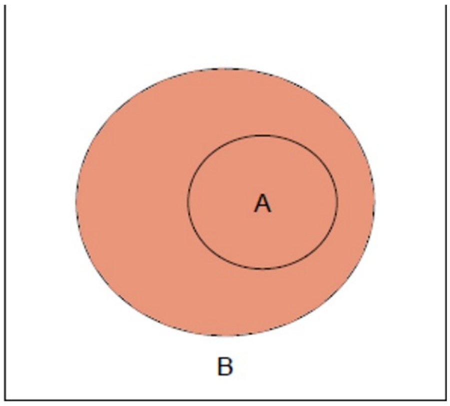


The event A^c denotes the event that A does not occur.

The event $A - B$ denotes the event $A \cap B^c$, the event that both A and B^c occur.

For A, B disjoint and A subset of B





For A and B disjoint, $A \cap B = \phi$ meaning both events cannot occur at the same time. These are referred to as mutually exclusive events.

Class Exercise

1. A card is selected at random from a pack of 52 playing cards. Let A be the event that the card is an ace and D the event that it is a diamond.
   Find

   (a) $p(A \cap D)$

   (b) $p(A \cup D)$

   (c) $p(A')$

   (d) $p(A' \cap D)$

2. A group of 275 people at a music festival were asked if they play guitar, piano or drums:

   - 35 play drums only
   - 20 play guitar only
   - 15 play piano only
   - 30 play guitar and drums
   - 10 play piano and drums
   - 65 people play guitar and piano
   - 1 person plays all three instruments

(a) Draw a Venn diagram to represent the information

(b) A festival goer is chosen at random from the group. Find the probability that the person chosen

i) plays piano.

ii) plays at least two of guitar, piano or drums.

iii) plays exactly one of the instruments.

iv) plays none of the instruments.

Using formulae to solve probability questions

Instead of always drawing Venn diagrams to answer probability questions (in any case they break down when the events are more than three), we can derive formulas to enable us solve probability questions.

Let $p(A) = a$, $p(B) = b$ and $p(A \cap B) = i$ then $p(A \cup B) = (a - i) + i + (b - i) = a + b - i$.

Hence

$$p(A \cup B) = p(A) + p(B) - p(A \cap B) - \text{the Addition Rule.}$$

Rewriting the Addition Rule gives

$$p(A \cap B) = p(A) + p(B) - p(A \cup B).$$

Class Exercise

Use formulae and what you have learned about the algebra of sets to answer the following questions.

1) Let A and B be two events such that $p(A) = 0.6$, $p(B) = 0.7$ and $p(A \cup B) = 0.9$. Determine

(a) $p(A \cap B)$;

(b) $p(A')$;

(c) $p(A' \cap B)$;

(d) $p(A' \cup B)$.

2) Consider two events T and Q where $p(T) = p(Q) = 3p(T \cap Q)$ and $p(T \cup Q) = 0.75$. Determine

(a) $p(T \cap Q)$;

(b) $p(T)$;

(c) $p(Q^{'})$;

(d) $p(T^{'} \cap Q^{'})$;

(e) $p(T^{'} \cap Q)$.

COUNTING TECHNIQUES

The classical approach to probability defines the probability that an event A occurs as

$$p(A) = \frac{\text{Number of required outcomes}}{\text{Total number of possible outcomes}} = \frac{n(A)}{n(S)} \tag{1}$$

It therefore requires you to be able to count the total number of outcomes in the event and in the sample space i.e. we should be able to list down all the possible outcomes of an experiment. There are many situations in which it would be too difficult and/or too tedious to list all of the possible outcomes in a sample space.

In your Basic Mathematics Course, you learned some counting techniques that will enable you count the number of elements in a sample space without actually having to identify the specific outcomes. Reminders of some of the specific counting techniques we will explore include the multiplication rule, permutations and combinations.

The Multiplication Principle

If there are

- n_1 outcomes of a random experiment E_1
- n_2 outcomes of a random experiment E_2
- $\cdots$ and $\cdots$
- n_m outcomes of a random experiment E_m

then there are $n_1 \times n_2 \times ... \times n_m$ outcomes of the composite experiment $E_1 E_2 ... E_m$.

NB: Always take care of whether replications (repetitions) are allowed or if there are any restrictions.

Class Exercise 1

a) How many possible outcomes are there when we toss a coin and roll a die?

b) How many possible license plates could be stamped if each license plate were required to have exactly 3 letters and 4 numbers?

c) How many possible license plates could be stamped if each license plate were required to have 3 unique letters and 4 unique numbers?

Permutations

How many ways can four people fill four executive positions?
$4 \times 3 \times 2 \times 1 = 24$
The counting of the number of permutations can be considered as a generalization of the Multiplication Principle.

A Generalization of the Multiplication Principle

Suppose there are n positions to be filled with n different objects, in which there are:

- n choices for the 1^{st} position
- $n-1$ choices for the 2^{nd} position
- $n-2$ choices for the 3^{rd} position
- $\cdots$ and $\cdots$
- 1 choice for the last position

The Multiplication Principle tells us there are then in general:

$$n \times (n-1) \times (n-2) \times ... \times 1 = n!$$

ways of filling the n positions.
The symbol $n!$ is read as 'n - factorial', and by definition 0! equals 1.
A permutation of n objects is an ordered arrangement of the n objects. We often call such a permutation a 'permutation of n objects taken n at a time', and denote it as ${}^{n}P_{n}$.
That is: ${}^{n}P_{n} = n \times (n-1) \times (n-2) \times ... \times 1 = n!$
NB: The first superscripted n represents the number of objects you want to arrange, while the second subscripted n represents the number of positions you have for the objects to fill.

Class Exercise 2 (sampling with or without replacement)

With 6 names in a bag, randomly select a name.

a) How many ways can the 6 names be assigned to 6 job assignments if we assume that each person can only be assigned to one job? (sampled without replacement)

b) What if the 6 names were sampled with replacement?

Another Generalization of the Multiplication Principle

Suppose there are r positions to be filled with n different objects, in which there are:

- n choices for the 1^{st} position
- $n-1$ choices for the 2^{nd} position
- $n-2$ choices for the 3^{rd} position
- $\cdots$ and $\cdots$
- $n-(r-1)$ choice for the last position

The Multiplication Principle tells us there are in general:

$$n \times (n-1) \times (n-2) \times ... \times n-(r-1)$$

ways of filling the r positions. We can show that in general, this quantity equals

$$\frac{n!}{(n-r)!}$$

A permutation of n objects taken r at a time is an ordered arrangement of n different objects in r positions. The number of such permutations is:

$$^{n}P_{r} = \frac{n!}{(n-r)!} \tag{2}$$

The superscripted n represents the number of objects you want to arrange, while the subscripted r represents the number of positions you have for the objects to fill.

Worked Example

An artist has 9 paintings. How many ways can he hang 4 paintings side-by-side on a gallery wall?

Solution

$$^{9}P_4 = \frac{9!}{(9-4)!} = 3024 \quad \text{arrangements.}$$

Combinations

Example

Maria has three tickets for a concert. She would like to use one of the tickets herself. She could then offer the other two tickets to any of four friends (Ann, Beth, Chris, Dave). How many ways can 2 people be selected from 4 to go to the concert?

Definition: The number of unordered subsets, called a combination of n objects taken r at a time is,

$$^{n}C_r = \frac{n!}{r!(n-r)!} \tag{3}$$

We say 'n choose r'. Hence

$$^{4}C_2 = \frac{4!}{2!(4-2)!} = 12$$

Class Exercise 3

1. Twelve (12) patients are available for use in a research study. Only seven (7) should be assigned to receive the study treatment. How many different subsets of seven patients can be selected?

2. Consider a standard deck of cards containing 13 face values (Ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, and King) and 4 different suits (Clubs, Diamonds, Hearts, and Spades) to play five-card poker.

    i) If you are dealt five cards, what is the probability of getting a 'full-house' hand containing three kings and two aces (KKKAA)?

    ii) If you are dealt five cards, what is the probability of getting any full-house hand?