Probability Distributions

Binomial Distribution

Suppossed that we have an experiment such as tossing a coin or die repeatedly or choosing a marble from an urn repeatedly. In any single trial there will be a probability associated with a particular event. Such trials are then said to be independent and are often called Bernoulli trials.

Let p be the probability that an event will happen in any single Bernoulli trial (called the prob of success). Then $q = 1-p$ is the prob that the event will fail to happen in any single trial (called the prob of failure)

The probability that the event will happen exactly x times in n trials (ie x successes and $n-x$ failure will occur) is given by the probability function.

$$f(x) = P(X=x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x!\,(n-x)!} p^x q^{n-x} \text{ --- eq } ①$$

where the random variable X denotes the number of successes in n trials and $x = 0, 1, \dots, n$.

Ex

The probability for getting exactly 2 heads in 6 tosses of a fair coin is

Soln

$$P(X=2) = \binom{6}{2}\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^{6-2} = \frac{6!}{2!\,4!}\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^{6-2} = \frac{15}{64}$$

The discrete prob fxn eqtn ① is often called binomial dist since for $x = 0, 1, 2, \dots, n$, it corresponds to successive term in the binomial expansion

$$(q+p)^n = q^n + \binom{n}{1} q^{n-1} p + \binom{n}{2} q^{n-2} p^2 + \dots + p^n = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}$$

The special case of a binomial distribution with $n = 1$ is also called the Bernoulli distribution

Some properties of the Binomial distribution

Mean	$\mu = np$
Variance	$\sigma^2 = npq$
Standard deviation	$\sigma = \sqrt{npq}$
Coefficient of skewness	$\alpha_3 = \frac{q-p}{\sqrt{npq}}$
Coefficient of kurtosis	$\alpha_4 = 3 + \frac{1-6pq}{npq}$
Moment generating function	$M(t) = (q+pe^t)^n$
Characteristic function	$\phi(\omega) = (q+pe^{i\omega})^n$

Ex

What is the mean & standard deviation of a fair coin tossed 100 times

Soln

$\mu = np$

$\mu = 100(1/2)$

$= 50$

$\sigma = \sqrt{npq}$

$\sigma = \sqrt{100(1/2)(1/2)}$

$\sigma = 5$

Ex

Find the probability that in tossing a fair coin 3 times, there will appear

a. 3 heads
b. 2 tails and 1 head
c. at least 1 head
d. not more than 1 tail.

Soln

a. $P(3 \text{ heads}) = \binom{3}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0 = \frac{1}{8}$

b. $P(2 \text{ tails and } 1 \text{ head}) = \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1 = \frac{3}{8}$

c. $P(\text{at least } 1 \text{ head}) = P(1, 2, \text{ or } 3 \text{ head})$

$$= P(1 \text{ heads}) + P(2 \text{ heads}) + P(3 \text{ heads})$$

$$= \binom{3}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^2 + \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1 + \binom{3}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0 = \frac{7}{8}$$

or

$P(\text{at least } 1 \text{ head}) = 1 - P(\text{no head})$

$$= 1 - \binom{3}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^3 = \frac{7}{8}$$

d. $P(\text{not more than } 1 \text{ tail}) = P(0 \text{ tails or } 1 \text{ tail})$

$$= P(0 \text{ tails}) + P(1 \text{ tail})$$

$$= \binom{3}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^0 + \binom{3}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) = \frac{1}{2}$$

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