BACHELOR OF SCIENCE IN INFORMATION TECHNOLOGY

KIBABII UNIVERSITY

STA 205

Probability and Statistics

Lecture notes

AXIOMS OF PROBABILITY

Previously, we defined probability informally. We now consider a formal definition using the axioms of probability.

An axiom is simply a rule that has to be satisfied.

Definition

Probability is a (real-valued) set function p that assigns to each event A in the sample space S a number $p(A)$, called the probability of the event A which must satisfy the following three axioms:

1. The probability of any event A must be nonnegative, that is $p(A) \geqslant 0$.

2. The probability of the sample space is 1, that is, $p(S) = 1$ implying $0 \leqslant p(A) \leqslant 1$.

3. Given mutually exclusive (disjoint) events $A_1, A_2, A_3, \dots$ that is where $A_i \cap A_j = \phi, \text{for} i \neq j$,

   A. The probability of a finite union of the events is the sum of the probabilities of the individual events, that is:

$$\begin{aligned} p(A_1 \cup A_2 \cup \cdots \cup A_k) &= p(A_1) + p(A_2) + p(A_3) + \cdots + p(A_k) \\ &= \sum_{i=1}^{k} p(A_i) \end{aligned}$$

   B. The probability of a countably infinite union of the events is the sum of the probabilities of the individual events, that is:

$$\begin{aligned} p(A_1 \cup A_2 \cup \cdots) &= p(A_1) + p(A_2) + p(A_3) + \cdots \\ &= \sum_{i=1}^{\infty} p(A_i) \end{aligned}$$

These conditions are known as the axioms of the theory of probability.

1. The first axiom states that all the probabilities are nonnegative real numbers.

2. The second axiom attributes a probability of unity to the universal event S, thus providing a normalization of the probability measure.

3. The third axiom states that the probability function must be additive, consistently with the intuitive idea of how probabilities behave.

All probabilistic results are based directly or indirectly on the axioms and only the axioms. Any assignment of probability to an event must satisfy the THREE axioms stated above regardless of your interpretation of probability.

Class Exercise

Suppose that your class has 43 students, such that 1 is Physics, 4 are Mathematics, 20 are Business, 9 are Computer, and 9 are Geography major students. Randomly select a student from the class. Define the following events:

1. P - the event that a Physics major is selected.
2. M - the event that a Mathematics major is selected.
3. B - the event that a Business major is selected.
4. C - the event that a Computer major is selected.
5. G - the event that a Geography major is selected.

The sample space is $S = \{P, M, B, C, G\}$.
Using the relative frequency approach to assigning probability to events, assign probabilities to the events $p(P)$, $p(M)$,$p(B)$,$p(C)$ and $p(G)$. Show that the three axioms of probability are satisfied.

Probability rules derived from the axioms of probability

Theorem 1: $p(A) = 1 - p(A^{'})$
Theorem 2: $p(\phi) = 0$
Theorem 3: If events A and B are such that $A \subseteq B$, then $p(A) \leqslant p(B)$.
Theorem 4: $p(A) \leqslant 1$.
Theorem 5: For any two events A and B, $p(A \cup B) = p(A) + p(B) - p(A \cap B)$.

Class Exercise

1. Prove the five theorems derived from the three axioms of probability.
2. A company has bid on two large construction projects. The company president believes that the probability of winning the first contract is 0.6, the probability of winning the second contract is 0.4, and the probability of winning both contracts is 0.2. What is the probability that the company wins:
   a) at least one contract?
   b) the first contract but not the second contract?

c) neither contract?

d) exactly one contract?

3. If it is known that $A \subseteq B$, what can be definitively said about $p(A \cap B)$?

4. If 7% of the population smokes cigars, 28% of the population smokes cigarettes, and 5% of the population smokes both, what percentage of the population smokes neither cigars nor cigarettes?

Conditional Probability

A conditional probability is a probability of an event given that another event has occurred. For example, rather than being interested in knowing the probability that a randomly selected students fails the probability exam, we might instead be interested in knowing the probability that a randomly selected student fails the probability exam given that the student is an education major.

We are finding the probability of event B occurring given that event A has occurred denoted $p(B/A)$.

Event A is termed the prior event while event B is the subsequent event.

Conditional probabilities have the effect of shrinking the sample space to the prior event.

Example 1

A researcher is interested in evaluating how well a diagnostic test works for detecting renal disease in patients with high blood pressure. She performs the diagnostic test on 137 patients, 67 with known renal disease and 70 who are known to be healthy. The diagnostic test comes back either positive (the patient has renal disease) or negative (the patient does not have renal disease). Here are the results of her experiment:

Test Result

Truth	Positive	Negative	Total
Renal disease	44	23	67
Healthy	10	60	70
Total	54	83	137

a) What is the probability that the patient tests positive?

b) What is the probability that a patient has renal disease?

c) What is the probability that a patient tests positive given that they have renal disease?

Solution

Let T denote the event that a patient test positive and D the event that a patient has renal disease

a) $p(T) = \dfrac{n(T)}{n(S)} = \dfrac{54}{137}$.

b) $p(D) = \dfrac{n(D)}{n(S)} = \dfrac{67}{137}$.

c) $p(T/D) = \dfrac{n(T \cap D)}{n(D)} = \dfrac{44}{67}$. Here you are considering only the proportion of those who are diseased (sample space shrinks); how many of them test positive.

Therefore, from part c) in the example we can write down a formula for conditional probability as

$$p(A/B) = \frac{p(A \cap B)}{p(B)}; \quad p(B) > 0$$

Multiplying both sides by $p(B)$ we get

$$p(A \cap B) = p(B) \times p(A/B) - \text{the Multiplication Rule}$$

or

$$p(B/A) = \frac{p(A \cap B)}{p(A)}; \quad p(A) > 0$$

Multiplying both sides by $p(A)$ we get

$$p(A \cap B) = p(A) \times p(B/A) - \text{the Multiplication Rule}$$

Class Exercise 1

1. Two fair spinners each have faces numbered 1 – 4. The two spinners are thrown together and the sum of the faces shown on the spinners is recorded. Given that at least one spinner lands on a 3, find the probability of the spinners indicating a sum of exactly 5.

2. C and D are events such that $p(C) = 0.2$, $p(D) = 0.1$ and $p(C/D) = 0.3$. Find $p(D/C)$, $p(C' \cap D')$ and $p(C' \cap D)$.

3. The turnout of spectators at a motor rally is dependent on the weather. On a rainy day, the probability of a big turnout is 0.4 but if it does not rain the probability of a big turn increases to 0.9. The weather forecast gives a probability of 0.75 that it will rain on the day of the race.

   a) Draw a tree diagram to represent this information.

   b) Find the probability that

      i) there is a big turnout and it rains.

      ii) there is a big turnout.

Properties of Conditional Probability

Because conditional probability is just a probability, it satisfies the three axioms of probability. That is, as long as $p(B) > 0$:

1. $p(A/B) \geqslant 0$.

2. $p(B/B) = 1$.

3. If $A_1, A_2, \ldots, A_k$ are mutually exclusive events, then $p(A_1 \cup A_2 \cup ... \cup A_k/B) = p(A_1/B) + p(A_2/B) + ... + P(A_k/B)$ and likewise for infinite unions.

Class Exercise 2

Show that conditional probabilities satisfy the three axioms of probability.

MUTUALLY EXCLUSIVE AND INDEPENDENT EVENTS

Mutually Exclusive Events

Two or more events are said to be mutually exclusive if they cannot occur at the same time i.e. events that have no outcomes in common are said to be mutually exclusive events or disjoint events.

If A and B are mutually exclusive then $A \cap B = \phi$. This implies $p(A \cap B) = 0$. Recall: for any two events A and B

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

If A and B are mutually exclusive then

$$p(A \cup B) = p(A) + p(B)$$

Independent Events

Two events are independent, if the occurrence of one does not affect the occurrence of the other. Therefore if A and B are independent, the probability of A happening is the same whether or not B has happened i.e. $p(A/B) = p(A)$; similarly $p(B/A) = p(B)$.
Recall for any two events A and B

$$p(A \cap B) = p(A) \times p(B/A)$$

If A and B are independent then

$$p(A \cap B) = p(A) \times p(B)$$

Example 1

Events A and B are mutually exclusive and $p(A) = 0.2$ and $p(B) = 0.4$. Determine $p(A \cup B)$, $p(A \cap B^{'})$ and $p(A^{'} \cap B^{'})$.

Solution Example 1

$p(A \cup B) = p(A) + p(B) = 0.2 + 0.4 = 0.6$
$p(A \cap B^{'}) = p(A) + p(B^{'}) - p(A \cup B^{'}) = 0.2 + 0.6 - 0.6 = 0.2$
$(A^{'} \cap B^{'}) = p(A^{'}) + p(B^{'}) - p(A^{'} \cup B^{'}) = 0.8 + 0.6 - 1 = 0.4$

Tree Diagrams to answer probability questions

When the experiment of interest consists of a sequence of several stages, it is convenient to represent these with a tree diagram. Tree diagrams are used to display the sample space. The tiers in a tree diagram represent events while the branches represent the outcomes of the event. Once we have an appropriate tree diagram, probabilities and conditional probabilities can be entered on the various branches; this will make repeated use of the multiplication rule quite straightforward.

Exercise 1

1. A chain of video stores sells three different brands of DVD players. Of its DVD player sales, 50% are brand 1 (the least expensive), 30% are brand 2, and 20% are brand 3. Each manufacturer offers a 1-year warranty on parts and labor. It is known that 25% of brand 1's DVD players require warranty repair work, whereas the corresponding percentages for brands 2 and 3 are 20% and 10%, respectively.

a) What is the probability that a randomly selected purchaser has bought a brand 1 DVD player that will need repair while under warranty?

b) What is the probability that a randomly selected purchaser has a DVD player that will need repair while under warranty?

c) If a customer returns to the store with a DVD player that needs warranty repair work, what is the probability that it is a brand 1 DVD player? A brand 2 DVD player? A brand 3 DVD player?

2. It is known that 30% of a certain company's washing machines require service while under warranty, whereas only 10% of its dryers need such service. If someone purchases both a washer and a dryer made by this company, what is the probability that

   a) both machines need warranty service?

   b) neither machine needs service?

3. Each day, Monday through Friday, a batch of components sent by a first supplier arrives at a certain inspection facility. Two days a week, a batch also arrives from a second supplier. Eighty percent of all supplier 1's batches pass inspection, and 90% of supplier 2's do likewise. What is the probability that, on a randomly selected day, two batches pass inspection? Assume that on days when two batches are tested, whether the first batch passes is independent of whether the second batch does so.

Bayes Theorem

The Bayes' theorem enables us compute a posterior probability $p(A_j/B)$ from given prior probabilities $p(A_i)$ and conditional probabilities $p(B/A_i)$. In other words, if a subsequent event has occurred, we use the Bayes' theorem to find the probability that a given prior event occurred.

To state the Bayes' theorem, we first need another result, the Law of Total Probability. Recall that events $A_1, ..., A_k$ are mutually exclusive if no two have any common outcomes.

The events are exhaustive if one A_i must occur, so that $A_1 \cup A_2 \cup \cdots \cup A_k = S$.

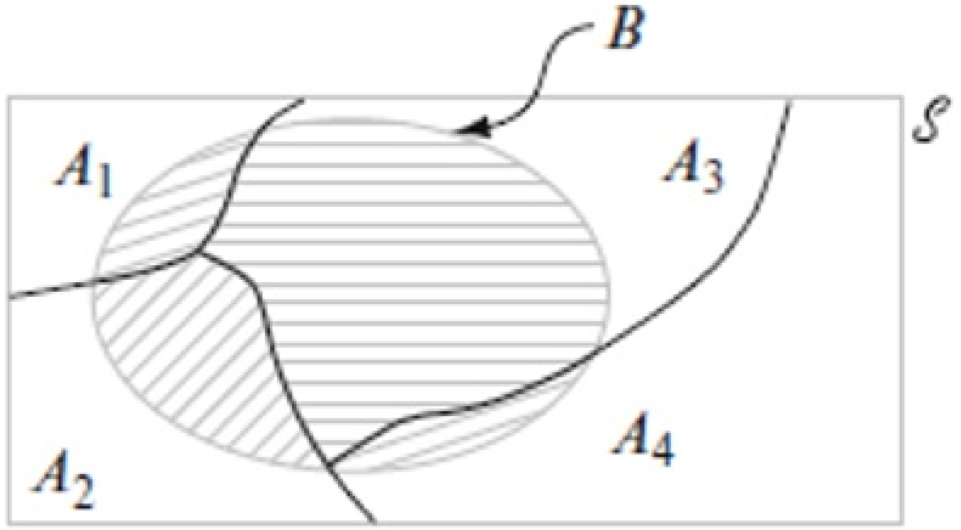
Law of Total Probability

Let $A_1, ..., A_k$ be mutually exclusive and exhaustive events. Then for any other event B,

$$p(B) = p(B/A_1)p(A_1) + \cdots + p(B/A_k)p(A_k) = \sum_{i=1}^{k} p(B/A_i)p(A_i)$$

Proof

Because the $A_i's$ are mutually exclusive and exhaustive, if B occurs it must be in conjunction with exactly one of the $A_i's$. That is, $B = (A_1 \quad \text{and} \quad B)$ or $\cdots$ or $(A_k \quad \text{and} \quad B) = (A_1 \cap B) \cup \cdots \cup (A_k \cap B)$, where the events $(A_i \cap B)$ are mutually exclusive. This 'partitioning of B' is illustrated in the figure below.



Thus

$$p(B) = p(B/A_1)p(A_1) + \cdots + p(B/A_k)p(A_k) = \sum_{i=1}^{k} p(B/A_i)p(A_i)$$

as desired.

The Bayes' Theorem

Let $A_1, ..., A_k$ be a collection of mutually exclusive and exhaustive events with $p(A_i) > 0$ for $i = 1, \cdots, k$. Then for any other event B, for which $p(B) > 0$

$$p(A_j/B) = \frac{p(A_j \cap B)}{p(B)} = \frac{p(B/A_j)p(A_j)}{\sum_{i=1}^{k} p(B/A_i).p(A_i)}, j = 1, \cdots, k.$$

The transition from the second to the third expression in the Bayes' Theorem rests on using the multiplication rule in the numerator and the law of total probability in the denominator. The proliferation of events and subscripts in the Bayes' Theorem can be a bit intimidating to probability newcomers. As long as there are relatively few events in the partition, a tree diagram can be used as a basis for calculating posterior probabilities without ever referring explicitly to Bayes' theorem.

Example 2

In a factory machines A, B and C produce electronic components. Machine A produces 16%, Machine B produces 50% and Machine C produces 34%. Some of the components are defective. Machine A produces 4% defective, Machine B produces 3% defective and Machine C produces 7% defective.

a) Draw a tree diagram to represent this information.

b) Find the probability that a randomly selected component is

   i) produced by Machine A and is defective.

   ii) defective.

c) Given that a randomly selected component is defective, find the probability that it was produced by Machine B.