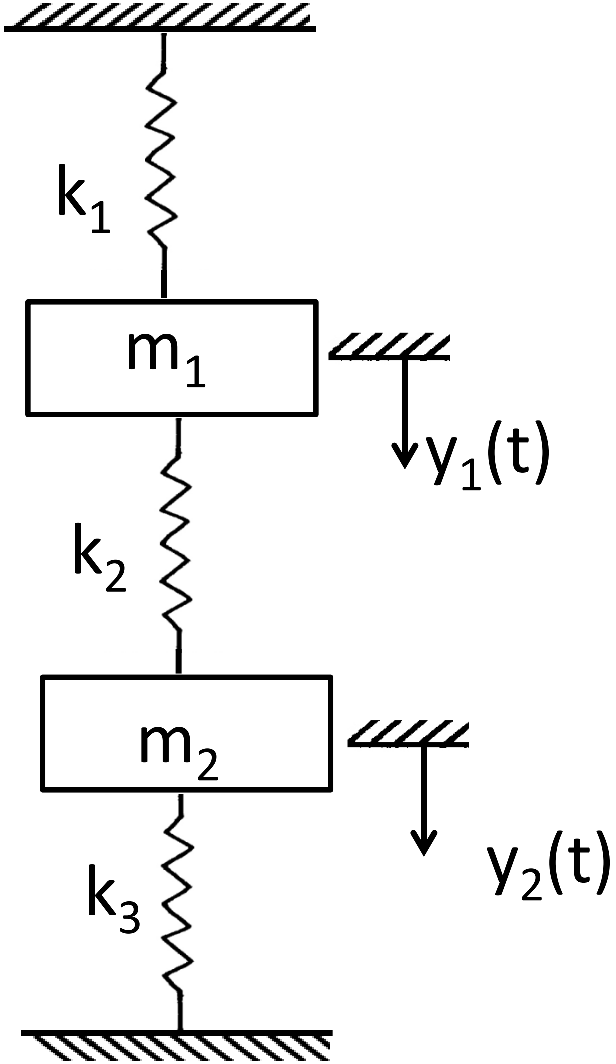
ME 316 KDOM MDOF VIBRATION



Two-degree of freedom system

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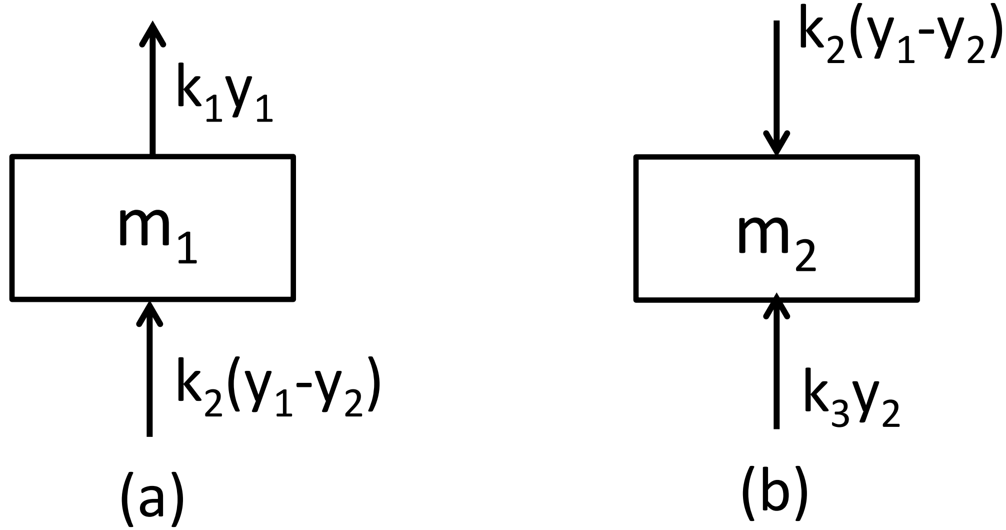

Free body diagram of masses
due to initial deflection in the downward direction (assume y1>y2)

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Equations of motion: undamped system

$$m_1\ddot{y}_1 = -k_1 y_1 - k_2(y_1 - y_2)$$
$$m_2\ddot{y}_2 = k_2(y_1 - y_2) - k_3 y_2$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = 0$$

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = 0$$

Rules for stiffness matrix

The stiffness matrix can be formed based on the following rule:

Sum of all the stiffnesses connected to a degree of freedom form the diagonal terms

Negative of the stiffness connecting one degree of freedom to the other degrees of freedom form the off-diagonal elements

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$

Solution: natural frequencies

$$\mathbf{y} = \mathbf{Y}\cos\omega t$$

$$\ddot{\mathbf{y}} = -\omega^2\mathbf{Y}\cos\omega t$$

$$\left[\mathbf{K} - \omega^2\mathbf{M}\right]\mathbf{Y} = 0$$

$$\left|\mathbf{K} - \omega^2\mathbf{M}\right| = 0$$

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$$y_1 = Y_1 \cos \omega t$$

$$y_2 = Y_2 \cos \omega t$$

$$\ddot{y}_1 = -\omega^2 Y_1 \cos \omega t$$

$$\ddot{y}_2 = -\omega^2 Y_2 \cos \omega t$$

$$\begin{bmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 + k_3 - m_2\omega^2 \end{bmatrix} \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = 0$$

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$$m_1 m_2 \omega^4 - \left[m_1 (k_2 + k_3) + m_2 (k_1 + k_2) \right] \omega^2$$

$$+ k_1 k_2 + k_2 k_3 + k_1 k_3 = 0$$

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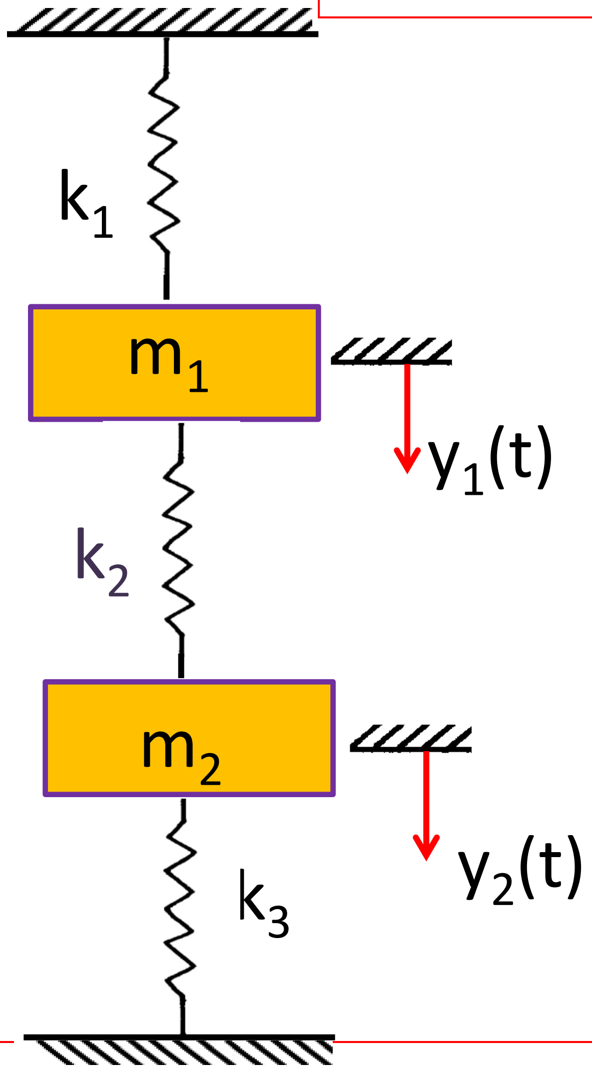
Example



$k_1=k$
$k_2=2k$
$k_3=3k$

$m_1=m$
$m_2=2m$

Determine natural frequencies and mode shapes

Example

Mass matrix
$$\mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} = m\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Stiffness matrix
$$\mathbf{K} = \begin{bmatrix} 3k & -2k \\ -2k & 5k \end{bmatrix} = k\begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix}$$

$$\left| k\begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix} - \omega^2 m\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \right| = 0$$

$\omega_1 = 1.15\sqrt{\frac{k}{m}}$ **rad/s** $\omega_2 = 2.05\sqrt{\frac{k}{m}}$ **rad/s**

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Example

$$\left[k\begin{bmatrix}3 & -2\\ -2 & 5\end{bmatrix}-\omega_1^2 m\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\right]\begin{Bmatrix}Y_{11}\\ Y_{21}\end{Bmatrix}=0$$

First mode

$$Y_{21}=0.84Y_{11}$$

$$\left[k\begin{bmatrix}3 & -2\\ -2 & 5\end{bmatrix}-\omega_2^2 m\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\right]\begin{Bmatrix}Y_{12}\\ Y_{22}\end{Bmatrix}=0$$

Second mode

$$Y_{22}=-0.6Y_{12}$$

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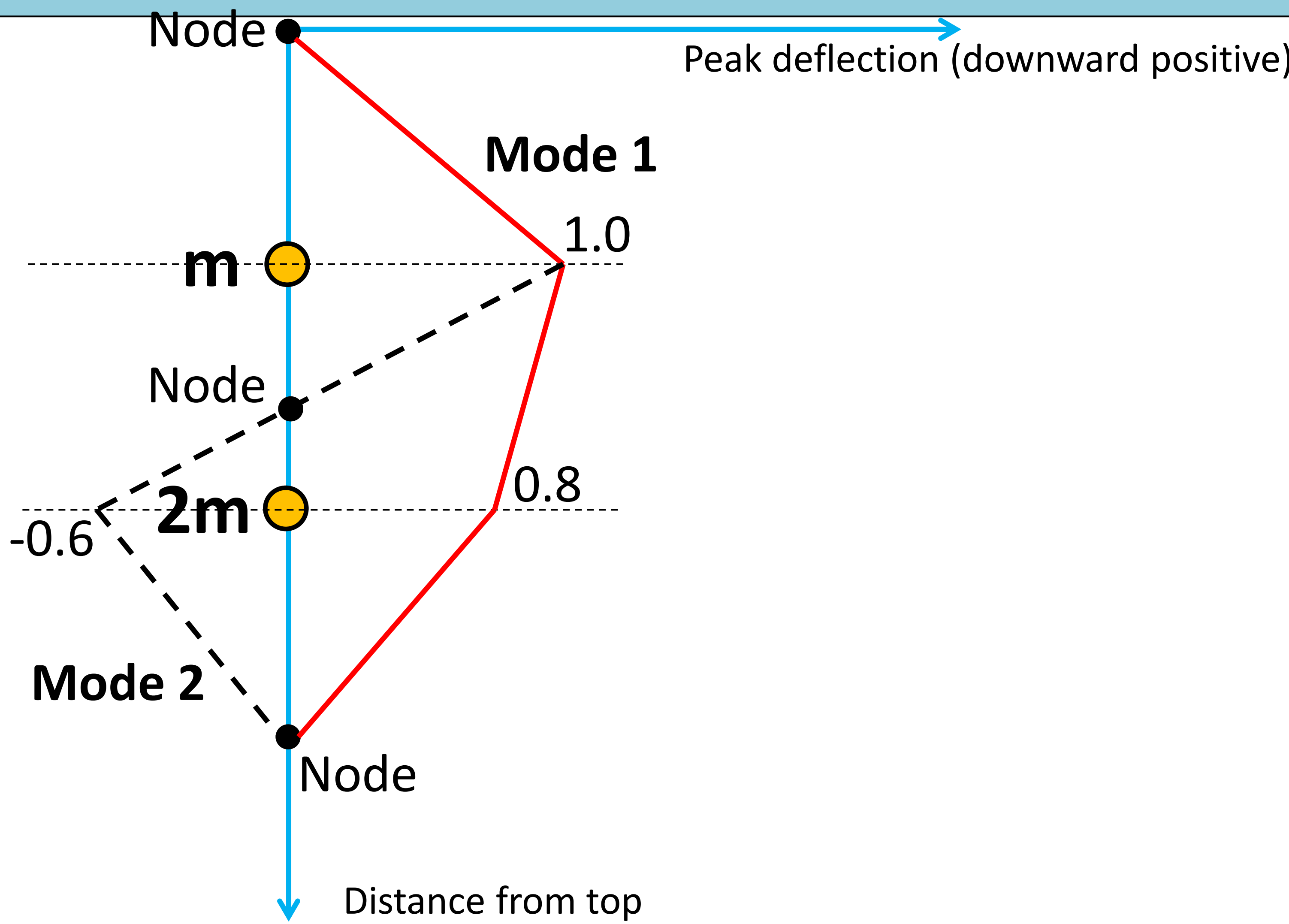

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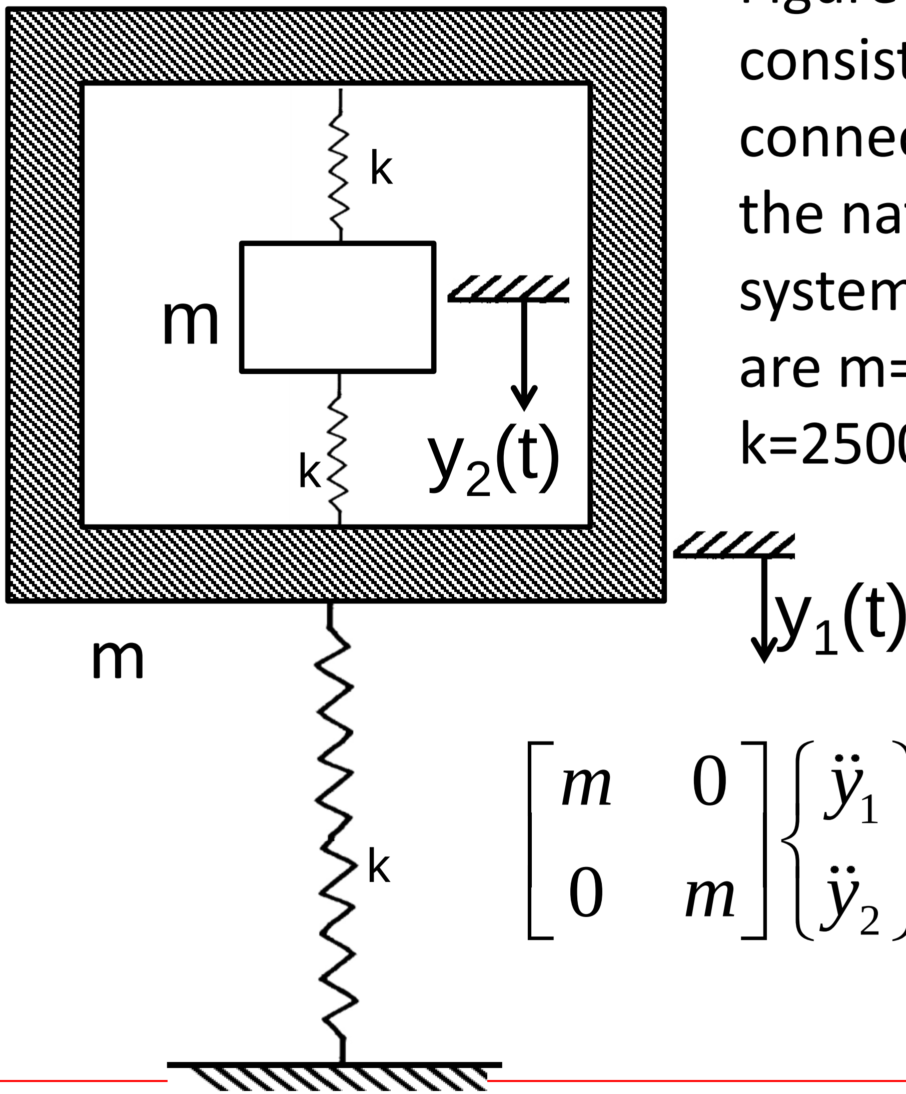

Figure shows a packaging system consisting of masses and springs connected as shown. Determine the natural frequencies of the system. Assume that all the masses are m=2 kg and all the stiffness are k=2500 N/m

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} 3k & -2k \\ -2k & 2k \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = 0$$

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$$\begin{vmatrix} 3k - m\omega^2 & -2k \\ -2k & 2k - m\omega^2 \end{vmatrix} = 0$$

$$\omega^4 - \frac{5k}{m}\omega^2 + 2\left(\frac{k}{m}\right)^2 = 0$$

$$\boxed{\omega_1 = 23.4\,\text{rad/s};\, \omega_2 = 75.5\,\text{rad/s}}$$

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$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f}(t)$$

Response

General equation of a MDOF system subjected to force excitation

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{K}\mathbf{y} = 0$$

For free vibrations without damping

$$\mathbf{y} = \mathbf{\Psi}\mathbf{q}$$

Coordinate transformation using the modal matrix

$$\ddot{\mathbf{y}} = \mathbf{\Psi}\ddot{\mathbf{q}}$$

$$\mathbf{M}\mathbf{\Psi}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{\Psi}\mathbf{q} = 0$$

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Response

$$\mathbf{M\Psi\ddot{q}} + \mathbf{K\Psi q} = 0$$

Pre-multiplying throughout by $\mathbf{\Psi}^T$

$$\mathbf{\Psi}^T\mathbf{M\Psi\ddot{q}} + \mathbf{\Psi}^T\mathbf{K\Psi q} = 0$$

$$\mathbf{\Psi}^T\mathbf{M\Psi} = \begin{bmatrix} \nwarrow & 0 & 0 \\ 0 & m_i^g & 0 \\ 0 & 0 & \searrow \end{bmatrix}$$

$$\mathbf{\Psi}^T\mathbf{K\Psi} = \begin{bmatrix} \nwarrow & 0 & 0 \\ 0 & k_i^g & 0 \\ 0 & 0 & \searrow \end{bmatrix}$$

$$m_i^g \ddot{q}_i + k_i^g q_i = 0$$

System of uncoupled differential equations

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$$\ddot{q}_i + \omega_i^2 q_i = 0$$

Response

$$\omega_i^2 = \frac{k_i^g}{m_i^g}$$

Same as the undamped natural frequencies of the original system

Now the following general equation can also be decoupled similarly using the modal matrix

$$\boxed{\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f}(t)}$$

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$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{f}(t)$$

Response

$$\mathbf{y} = \mathbf{\Psi}\mathbf{q} \qquad \dot{\mathbf{y}} = \mathbf{\Psi}\dot{\mathbf{q}} \qquad \ddot{\mathbf{y}} = \mathbf{\Psi}\ddot{\mathbf{q}}$$

$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$ Proportional damping

$$\mathbf{M}\mathbf{\Psi}\ddot{\mathbf{q}} + \left\{\alpha\mathbf{M} + \beta\mathbf{K}\right\}\mathbf{\Psi}\dot{\mathbf{q}}$$

$$+\mathbf{K}\mathbf{\Psi}\mathbf{q} = \mathbf{f}(t)$$

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Response

Pre-multiplying throughout by $\boldsymbol{\Psi}^T$

$$\boldsymbol{\Psi}^T \mathbf{M} \boldsymbol{\Psi} \ddot{\mathbf{q}} + \boldsymbol{\Psi}^T \left\{ \alpha \mathbf{M} + \beta \mathbf{K} \right\} \boldsymbol{\Psi} \dot{\mathbf{q}} + \boldsymbol{\Psi}^T \mathbf{K} \boldsymbol{\Psi} \mathbf{q} = \boldsymbol{\Psi}^T \mathbf{f}(t)$$

$$\boldsymbol{\Psi}^T \mathbf{M} \boldsymbol{\Psi} = \begin{bmatrix} \nwarrow & 0 & 0 \\ 0 & m_i^g & 0 \\ 0 & 0 & \searrow \end{bmatrix}$$

$$\boldsymbol{\Psi}^T \mathbf{K} \boldsymbol{\Psi} = \begin{bmatrix} \nwarrow & 0 & 0 \\ 0 & k_i^g & 0 \\ 0 & 0 & \searrow \end{bmatrix}$$

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Response

$$\mathbf{\Psi}^T\left\{\alpha\mathbf{M}+\beta\mathbf{K}\right\}\mathbf{\Psi}=\alpha\begin{bmatrix}\nwarrow & 0 & 0\\ 0 & m_i^g & 0\\ 0 & 0 & \searrow\end{bmatrix}+\beta\begin{bmatrix}\nwarrow & 0 & 0\\ 0 & k_i^g & 0\\ 0 & 0 & \searrow\end{bmatrix}=\begin{bmatrix}\nwarrow & 0 & 0\\ 0 & c_i^g & 0\\ 0 & 0 & \searrow\end{bmatrix}$$

$$\boxed{m_i^g\ddot{q}_i+c_i^g\dot{q}_i+k_i^g q_i=Q_i}$$

Uncoupled equations

i=1,2..n

$$\boxed{\mathbf{Q}=\mathbf{\Psi}^T\mathbf{f}(t)}$$

Generalized force

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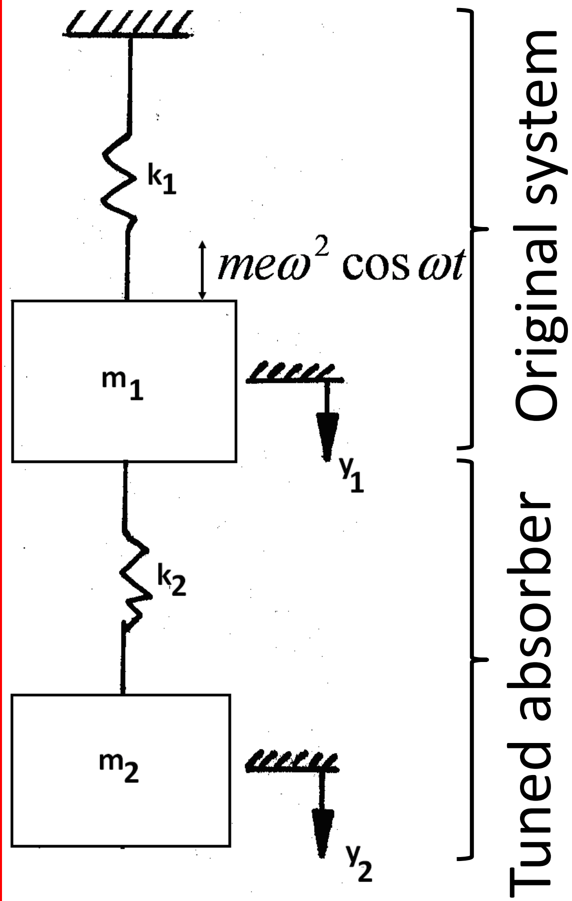
DYNAMIC VIBRATION ABSORBERS



$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} F_o \cos\omega t \\ 0 \end{Bmatrix}$$

$$y_1 = Y_1 \cos\omega t$$

$$y_2 = Y_2 \cos\omega t$$

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DYNAMIC VIBRATION ABSORBERS

$$\begin{bmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 - m_2\omega^2 \end{bmatrix} \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \begin{Bmatrix} F_o \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \frac{1}{D} \begin{bmatrix} k_2 - m_2\omega^2 & k_2 \\ k_2 & k_1 + k_2 - m_1\omega^2 \end{bmatrix} \begin{Bmatrix} F_o \\ 0 \end{Bmatrix}$$

$$D = m_1 m_2 \omega^4 - (k_1 m_2 + k_2 m_2 + k_2 m_1)\omega^2 + k_1 k_2$$

DYNAMIC VIBRATION ABSORBERS

$$Y_1 = \frac{1}{D}(k_2 - m_2\omega^2)F_0$$

$$Y_2 = \frac{k_2}{D}F_0$$

To make the amplitude Y_1=0,

$$(k_2 - m_2\omega^2) = 0$$

$$\omega = \omega_{n2} = \sqrt{\frac{k_2}{m_2}}$$

$$D = m_1 m_2 \left(\frac{k_2}{m_2}\right)^2 - (k_1 m_2 + k_2 m_2 + k_2 m_1)\left(\frac{k_2}{m_2}\right) + k_1 k_2$$

$$= -k_2^2$$

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DYNAMIC VIBRATION ABSORBERS

$$k_2 = \frac{F_0}{Y_2} = \frac{me\omega^2}{Y_2}$$

$$m_2 = \frac{k_2}{\omega_{n2}^2}$$

Assuming that the maximum peak amplitude of the tuned mass is given

$$D = m_1 m_2 \left(\frac{k_2}{m_2}\right)^2 - (k_1 m_2 + k_2 m_2 + k_2 m_1)\left(\frac{k_2}{m_2}\right) + k_1 k_2$$

Ω :Natural frequencies of the combined system

$$m_1 m_2 \Omega^4 - (k_1 m_2 + k_2 m_2 + k_2 m_1)\Omega^2 + k_1 k_2 = 0$$

$$\Omega^4 - (\omega_{n_1}^2 + \mu\omega_{n_2}^2 + \omega_{n_2}^2)\Omega^2 + \omega_{n_1}^2 \omega_{n_2}^2 = 0$$

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DYNAMIC VIBRATION ABSORBERS

A machine of mass 200 kg with a rotating unbalance of 0.1 kg-m operates at a speed of 1200 RPM. It is mounted on a spring of stiffness 700 kN/m.

(a) Determine the vibration due to the unbalance force when no absorber is attached and the machine is running at 1200 RPM.

Design a tuned vibration absorber such that when attached to the above system, the vibrations of the machine will cease at 1200 RPM and the steady state absorber amplitude will be less than 10 mm.

What are the system's natural frequencies with the absorber in place?

After installing the tuned absorber, if the machine is operated between 900 and 1400 RPM, what is the deflection you can expect at these speeds?

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DYNAMIC VIBRATION ABSORBERS

Mass of the machine m_1=200 kg
Unbalance=me=0.1 kg-m
RPM=1200

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 1200}{60} = 125.66$$ rad/s (**excitation frequency)**

$$\omega_{n1} = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{700\times 10^3}{200}} = 59.16\,\text{rad/s}$$

$$y = \frac{me\omega^2}{k - m_1\omega^2} = \frac{0.1\times 125.66^2}{700\times 10^3 - 200\times 125.66^2} = 0.64\,\text{mm}$$

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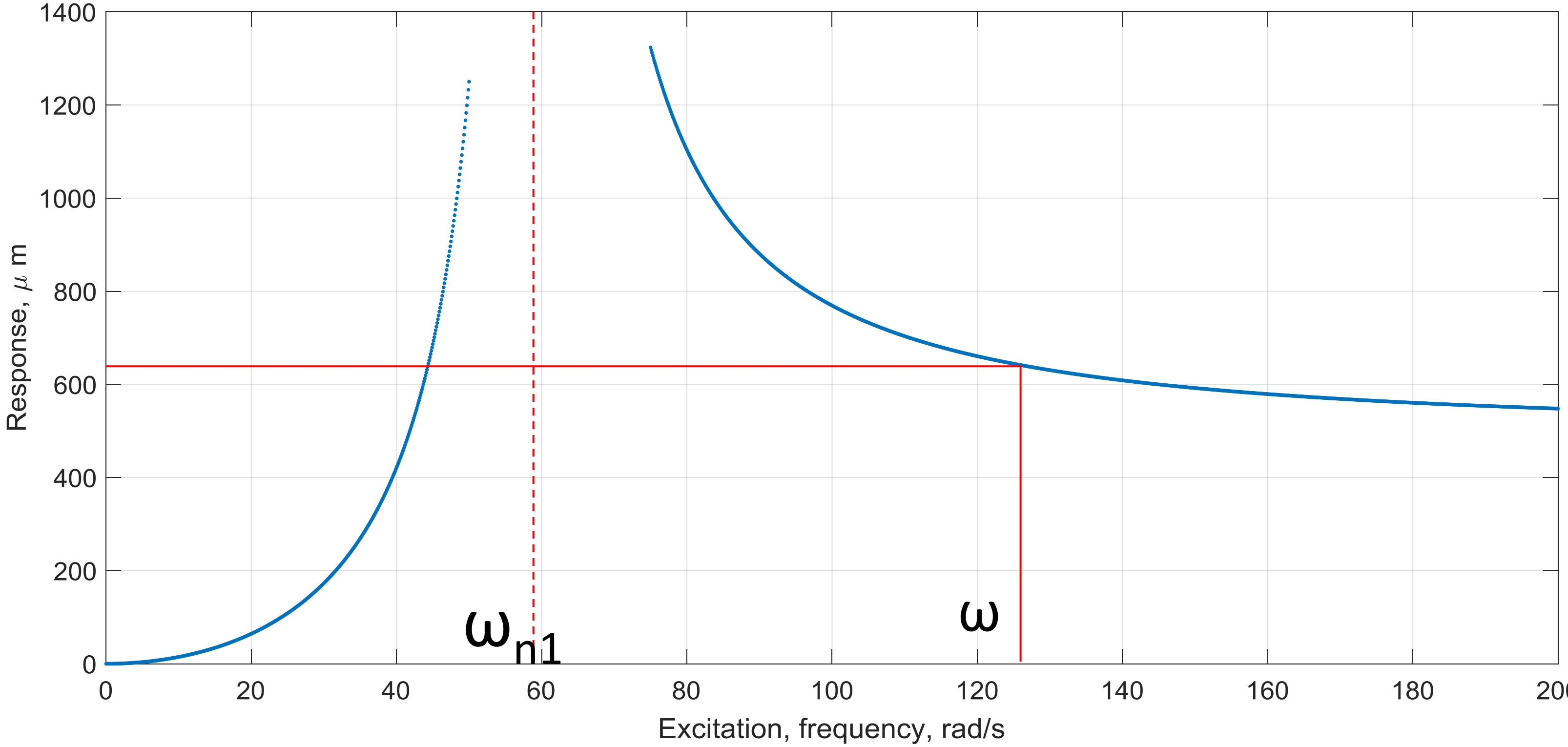
Response without vibration absorbers



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DYNAMIC VIBRATION ABSORBERS

$\omega_{n2} = \omega$=1200 RPM=**125.66 rad/s**

$$k_2 = \frac{F_0}{Y_2} = \frac{me\omega^2}{Y_2} = \frac{0.1\times125.66^2}{10\times10^{-3}}$$

$$= 158\,\text{kN/m}$$

Y_2=maximum amplitude of the tuned mass (given)

$$m_2 = \frac{k_2}{\omega_{n2}^2} = \frac{158000}{125.66^2} = 10\ \text{kg}$$

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Ω_1=58 rad/sec and Ω_2=129.63 rad/s
Natural frequencies of the combined system

Minimum speed is 900 RPM(94.25 rad/s) >Ω_1
Maximum speed is 1400 RPM(146.61 rad/s)>Ω_2

$$
\begin{aligned}
&D(900RPM) \\
&= m_1 m_2 \omega^4 - (k_1 m_2 + k_2 m_2 + k_2 m_1)\omega^2 + k_1 k_2 \\
&= 200 \times 10 \times 94.25^4 - (700 \times 10 + 158 \times 10 + 158 \times 200) \times 10^3 \times 94.25^2 + \\
&\qquad 700 \times 158 \times 10^6 \\
&\qquad = -8.84 \times 10^{10}
\end{aligned}
$$

$$
\begin{aligned}
Y_1(900\,RPM) &= \frac{1}{D}(k_2 - m_2\omega^2)F_0 \\
&= \frac{1}{-8.84\times 10^{10}}(158\times 10^3 - 10\times 94.25^2)\times 0.1\times 94.25^2 \\
&= 694\,\mu m
\end{aligned}
$$

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$$D(1400RPM)$$

$$= m_1 m_2 \omega^4 - (k_1 m_2 + k_2 m_2 + k_2 m_1)\omega^2 + k_1 k_2$$

$$= 200 \times 10 \times 146.61^4 - (700 \times 10 + 158 \times 10 + 158 \times 200) \times 10^3 \times 146.61^2$$

$$+ 700 \times 158 \times 10^6 = 1.7128 \times 10^{11}$$

$$Y_1(1400RPM) = \frac{1}{D}(k_2 - m_2\omega^2)F_0$$

$$= \frac{1}{1.7128 \times 10^{11}}(158 \times 10^3 - 10 \times 146.61^2) \times 0.1 \times 146.61^2$$

$$= 715\,\mu m$$