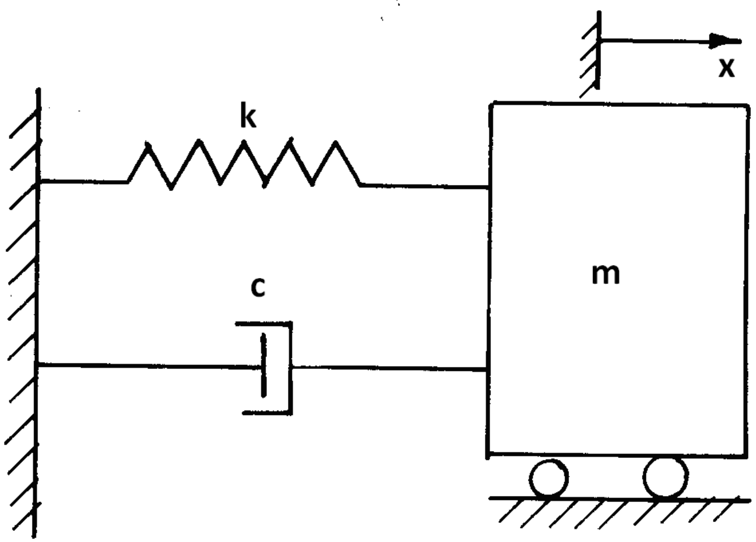
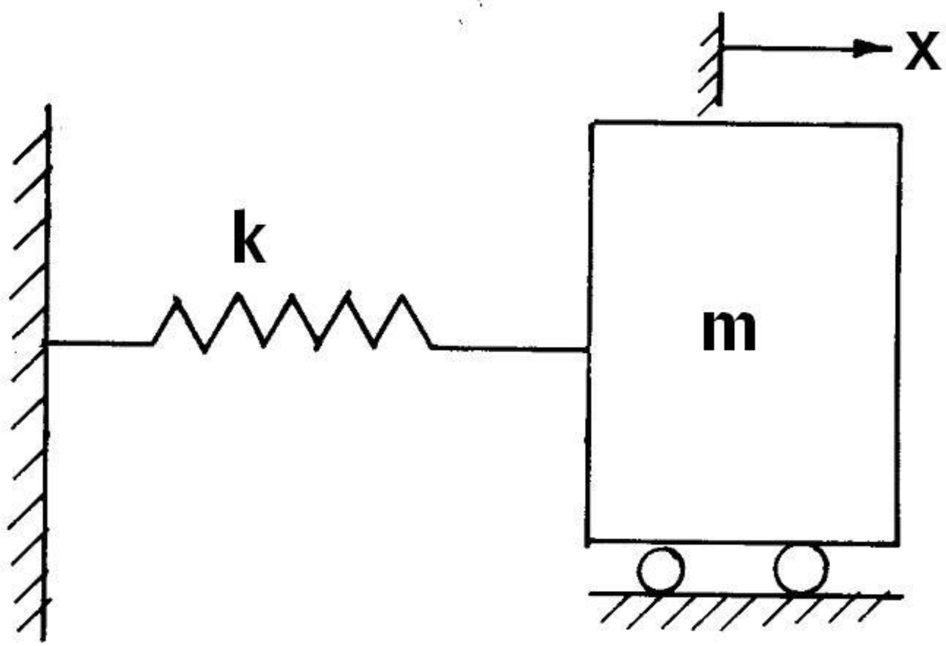
# ME 316 KDOM SDOF Vibration




Single degree of freedom system

# ME 316 KDOM SDOF Vibration




**Undamped single degree of freedom (SDOF) system**

# ME 316 KDOM SDOF Vibration

## Free undamped vibration

$$f_r = -kx$$

$$f_r = -kx = m\frac{d^2x}{dt^2}$$

$$m\ddot{x} + kx = 0,$$

$$\ddot{x} + \frac{k}{m}x = 0$$

# ME 316 KDOM SDOF Vibration

## Free undamped vibration

$$\ddot{x} + \frac{k}{m}x = 0$$

$$\left(D^2 + \frac{k}{m}\right)x = 0 \quad \boxed{D \equiv d/dt}$$

$$D^2 + \frac{k}{m} = 0$$

$$D = \pm j\sqrt{\frac{k}{m}}$$

# ME 316 KDOM SDOF Vibration

## Free undamped vibration

$$x(t) = A\sin\sqrt{\frac{k}{m}}t + B\cos\sqrt{\frac{k}{m}}t$$

$$\omega_n = \sqrt{\frac{k}{m}} \text{ rad/s} \qquad \boxed{f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \text{ Hz}}$$

Hertz shortened as Hz is same as cycles/s
Hertz-Armstrong-Marconi are the three scientists who have made significant contribution to inventions related to radio and the first letter of each of their last names is given the name HAM for amateur radio

# ME 316 KDOM SDOF Vibration

## Free undamped vibration

$$x(t) = A \sin \omega_n t + B \cos \omega_n t$$

$$x(t) = C \cos(\omega_n t + \theta)$$

$$\boxed{x(0) = x_0} \quad \boxed{\dot{x}(0) = 0}$$ **Initial conditions**

$$A = 0, B = x_0$$

$$\boxed{x(t) = x_0 \cos \omega_n t}$$

# ME 316 KDOM SDOF Vibration

## Free undamped vibration | Example

A single degree of freedom system with no damping consists of a mass of 0.1 kg attached to a spring of stiffness 1000 N/m attached to a fixed support. (a) Compute the undamped natural frequency in rad/s and Hz and period of free oscillation (b) If an initial displacement of 10 mm is given to the mass by pulling the mass down, determine the equation of motion of the mass (c) What is the maximum velocity of the mass and when does it occur (d) Draw the displacement and velocity response due to the above initial condition for four cycles

# ME 316 KDOM SDOF Vibration

**Free undamped vibration** **Example**

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{0.1}} = 100 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = \frac{100}{2\pi} = 15.92\,\text{Hz}\ \left(\text{cycles/s}\right)$$

$$T = \frac{1}{f_n} = \frac{1}{15.92} = 63\ ms$$

# ME 316 KDOM SDOF Vibration

**Free undamped vibration** **Example**

$$x(t) = x_o \cos(\omega_n t + \phi)$$

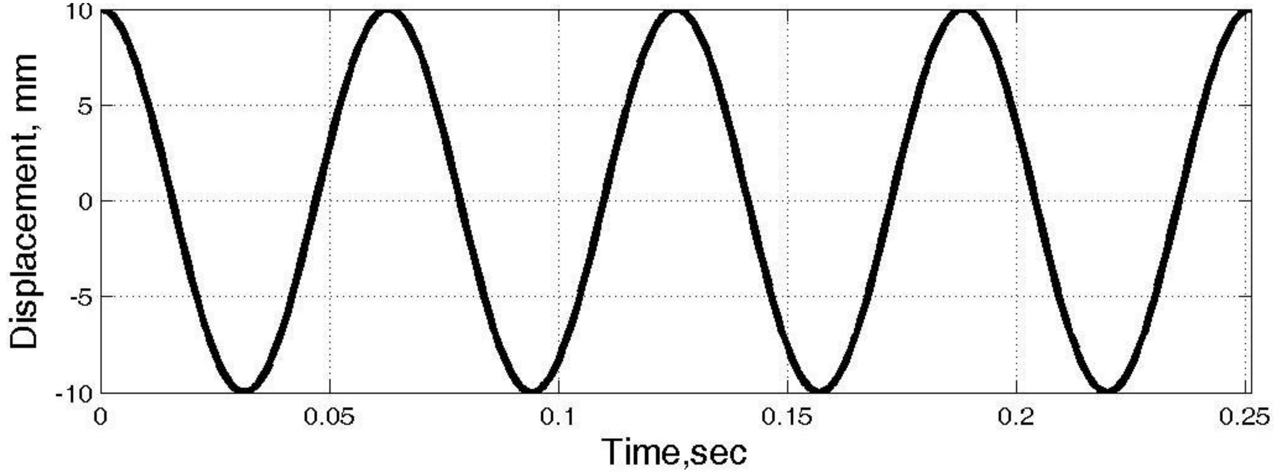
$$\dot{x}(t) = v(t) = -x_o \omega_n \sin(\omega_n t + \phi)$$

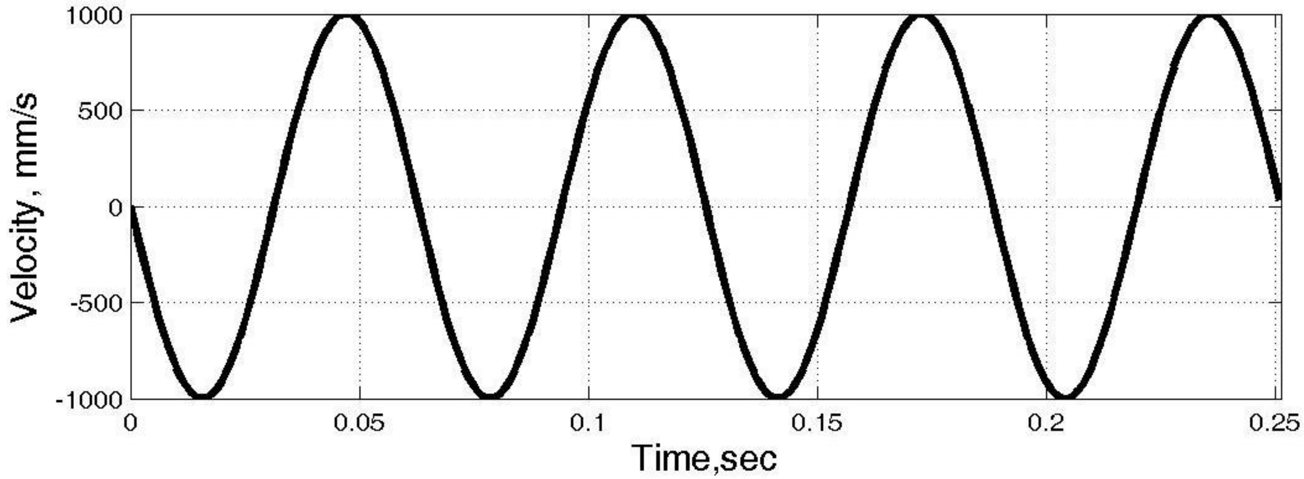
$$x(t) = 10 \cos \omega_n t \text{ , } mm$$

$$\dot{x}(t) = v(t) = -10 \omega_n \sin \omega_n t \text{ , } mm/s$$

# ME 316 KDOM SDOF Vibration

## Free undamped vibration | Example



# ME 316 KDOM SDOF Vibration

## Free damped vibration

$$f_r = -kx$$

$$f_d = -c\dot{x}$$

$$f_r + f_d = -kx - c\dot{x} = m\frac{d^2x}{dt^2}$$

$$m\ddot{x} + c\dot{x} + kx = 0$$

# ME 316 KDOM SDOF Vibration

## Free damped vibration

$$\ddot{x}+\frac{c}{m}\dot{x}+\frac{k}{m}x=0$$

$$\omega_n^2=\frac{k}{m}$$

$$\zeta=\frac{c}{2m\omega_n}=\frac{c}{2\sqrt{km}}$$

# ME 316 KDOM SDOF Vibration

## Free damped vibration

$$\frac{c}{m} = 2\zeta\omega_n$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

$$\left(D^2 + 2\zeta\omega_n D + \omega_n^2\right)x = 0$$

$$D^2 + 2\zeta\omega_n D + \omega_n^2 = 0$$

# ME 316 KDOM SDOF Vibration

## Free damped vibration

$$D = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Damping factor $\zeta > 1$

$$x(t) = e^{-\zeta\omega_n t}\left(Ae^{\left(\omega_n\sqrt{\zeta^2-1}\right)t} + Be^{\left(-\omega_n\sqrt{\zeta^2-1}\right)t}\right)$$

**Critical damping** $\zeta = 1$

$$x(t) = \left(Ae^{-\zeta\omega_n t} + Bte^{-\zeta\omega_n t}\right)$$

# ME 316 KDOM SDOF Vibration

## Free damped vibration

**Underdamping** $\zeta < 1$

$$D = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$D = -\zeta\omega_n \pm j\,\omega_d$$

$$x(t) = e^{-\zeta\omega_n t}\left(A\sin\omega_d t + B\cos\omega_d t\right)$$

# ME 316 KDOM SDOF Vibration

## Free damped vibration

$$x(t) = e^{-\zeta\omega_n t}\left(A\sin\omega_d t + B\cos\omega_d t\right)$$

$$x(t) = Ae^{-\zeta\omega_n t}\cos\left(\omega_d t + \varphi\right)$$

$$\dot{x}(t) = -Ae^{-\zeta\omega_n t}\left[\zeta\omega_n\cos\left(\omega_d t + \varphi\right) + \omega_d\sin\left(\omega_d t + \varphi\right)\right]$$

# ME 316 KDOM SDOF Vibration

## Free damped vibration

$$\dot{x}(t) = -Ae^{-\zeta\omega_n t}\left[\zeta\omega_n \cos\left(\omega_d t + \varphi\right) + \omega_d \sin\left(\omega_d t + \varphi\right)\right]$$

If the initial conditions are $x(0)$ and $\dot{x}(0)$,

$$.\phi = -\tan^{-1}\left(\frac{\zeta\omega_n x(0) + \dot{x}(0)}{\omega_d x(0)}\right)$$

$$.A = \sqrt{\left[x(0)\right]^2 + \left[\frac{\zeta\omega_n x(0) + \dot{x}(0)}{\omega_d}\right]^2}$$

# ME 316 KDOM SDOF Vibration

**Free damped vibration** **Example**

A single degree of freedom system (SDOF) has a mass of 0.1 kg, stiffness 1000 N/m, and a damping coefficient of 5 N-s/m is subjected to an initial condition by moving it by 10 mm from its equilibrium position. (a) Determine the damping factor and damped natural frequency of oscillations (b) Determine the constants from initial conditions and obtain the equation for displacement and velocity response (c) Plot the displacement and velocity response for four cycles

## Free damped vibration

## Example

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{0.1}} = 100 \text{ rad/s}$$

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_n} = \frac{5}{2\times 0.1\times 100} = 0.25$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 100\sqrt{1-0.25^2} = 96.82 \text{ rad/s}$$

# ME 316 KDOM SDOF Vibration

## Free damped vibration

## Example

The given initial conditions are $x(0)=20\,\text{mm};\ \dot{x}(0)=0$.

$$.\phi=-\tan^{-1}\left(\frac{\zeta\omega_n x(0)+\dot{x}(0)}{\omega_d x(0)}\right)=-\tan^{-1}\left(\frac{0.25\times100\times20)}{96.82\times20}\right)=-14.47^0$$

$$.A=\sqrt{\left[x(0)\right]^2+\left[\frac{\zeta\omega_n x(0)+\dot{x}(0)}{\omega_d}\right]^2}=\sqrt{20^2+\left[\frac{0.25\times100\times20}{96.82}\right]^2}=20.65\,\text{mm}$$

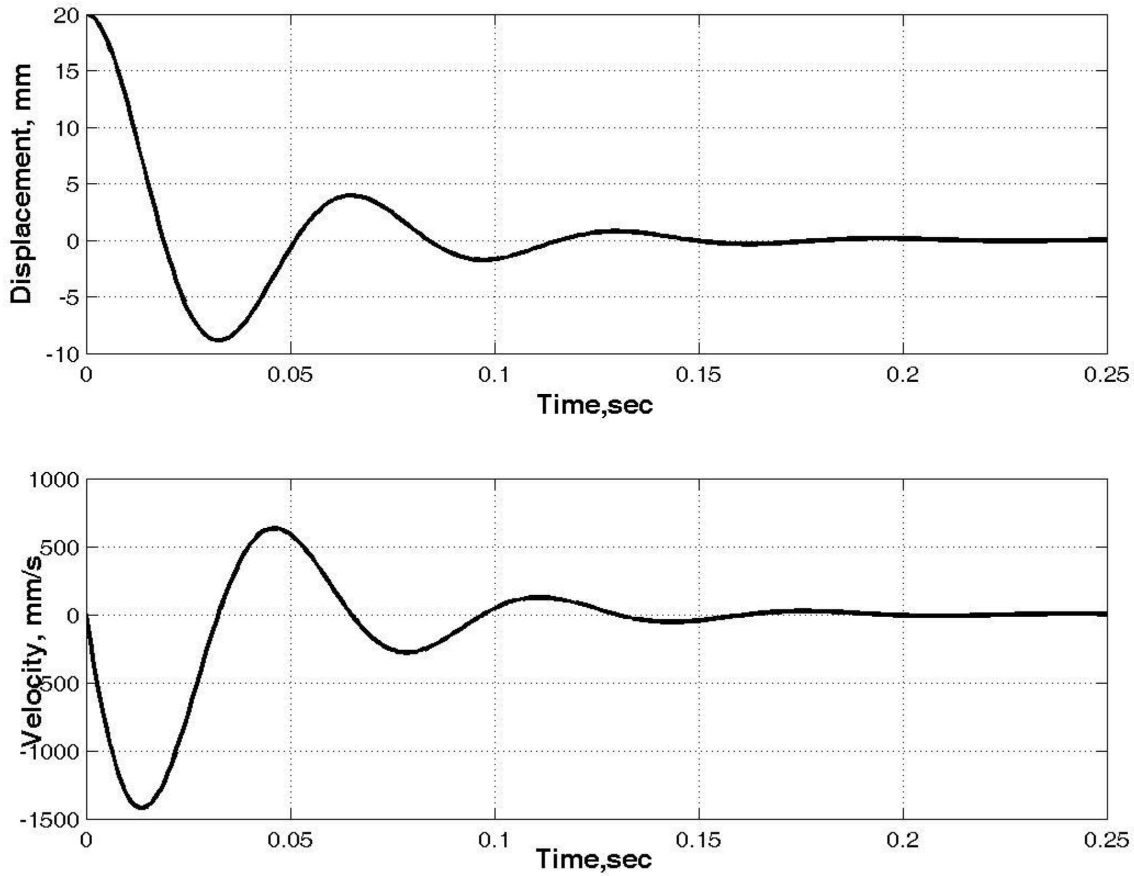
$$x(t)=20.65e^{-25t}\cos\left(\omega_d t-14.14^0\right)\text{mm}$$

$$\dot{x}(t)=-20.65e^{-25t}\left[25\cos\left(\omega_d t-14.14^0\right)+96.82\sin\left(\omega_d t-14.14^0\right)\right]\text{mm/s}$$

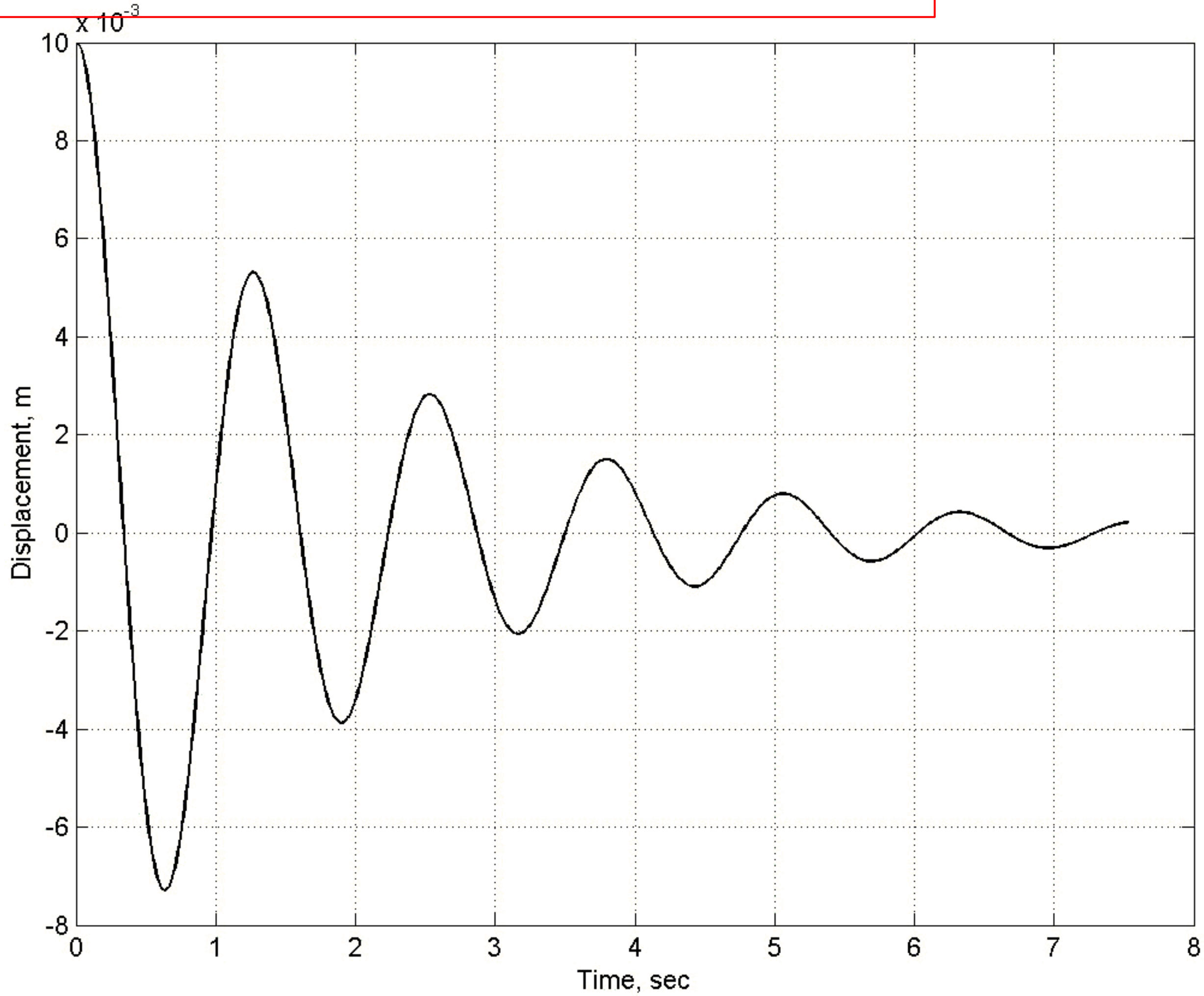
# ME 316 KDOM SDOF Vibration

## Free damped vibration

## Example



# Logarithmic decrement



# ME 316 KDOM SDOF Vibration

## Logarithmic decrement

$$x = x_0 e^{-\zeta\omega_n t} \cos(\omega_d t)$$

$$x_1 = x_0 e^{-\zeta\omega_n (2\pi/\omega_d)} \cos 2\pi$$

$$x_2 = x_0 e^{-\zeta\omega_n (4\pi/\omega_d)} \cos 4\pi$$

$$\Delta = \ln \frac{x_0}{x_1} = \ln \frac{x_1}{x_2} = \frac{2\pi\zeta\omega_n}{\omega_d} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

# ME 316 KDOM SDOF Vibration

## Impulse response function

$$x(0) = 0 \qquad m\dot{x}(0) = 1$$

$$x = Ae^{-\zeta\omega_n t}\cos(\omega_d t + \phi)$$

$$\dot{x} = -Ae^{-\zeta\omega_n t}\left[\zeta\omega_n \cos(\omega_d t + \varphi) + \omega_d \sin(\omega_d t + \varphi)\right]$$

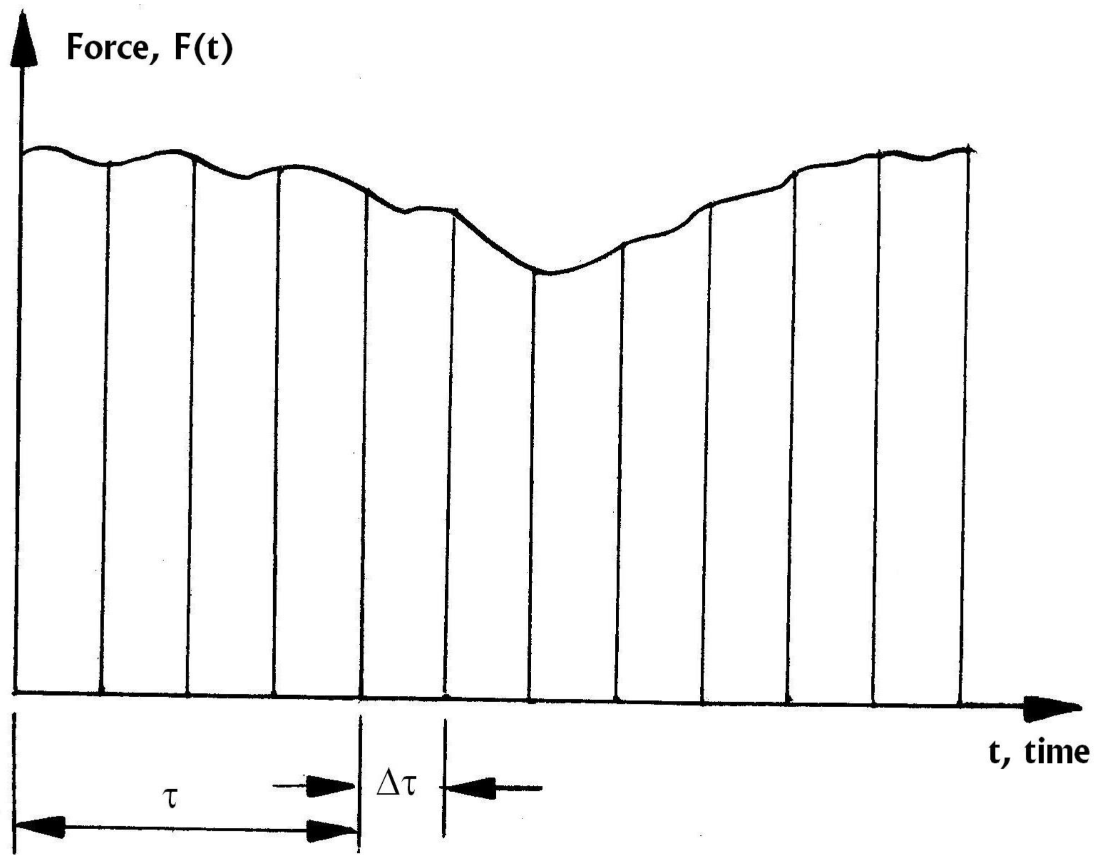
$$\phi = \frac{\pi}{2} \qquad A = -\frac{1}{m\omega_d \sin\phi}$$

## Impulse response function

$$x(t) = \frac{e^{-\zeta\omega_n t}\sin(\omega_d t)}{m\omega_d}$$

$$h(t) = \frac{e^{-\zeta\omega_n t}\sin(\omega_d t)}{m\omega_d}$$

## Convolution Integral



## Convolution Integral

$$x(t)\Big|_{F(\tau)\Delta\tau} = \frac{F(\tau)\Delta\tau e^{-\zeta\omega_n(t-\tau)}\sin\left[\omega_d\left(t-\tau\right)\right]}{m\omega_d}$$

$$x(t) = \int_0^t \frac{F(\tau)e^{-\zeta\omega_n(t-\tau)}\sin\left[\omega_d\left(t-\tau\right)\right]d\tau}{m\omega_d}$$

$$x(t) = \int_0^t F(\tau)h\left(t-\tau\right)d\tau$$

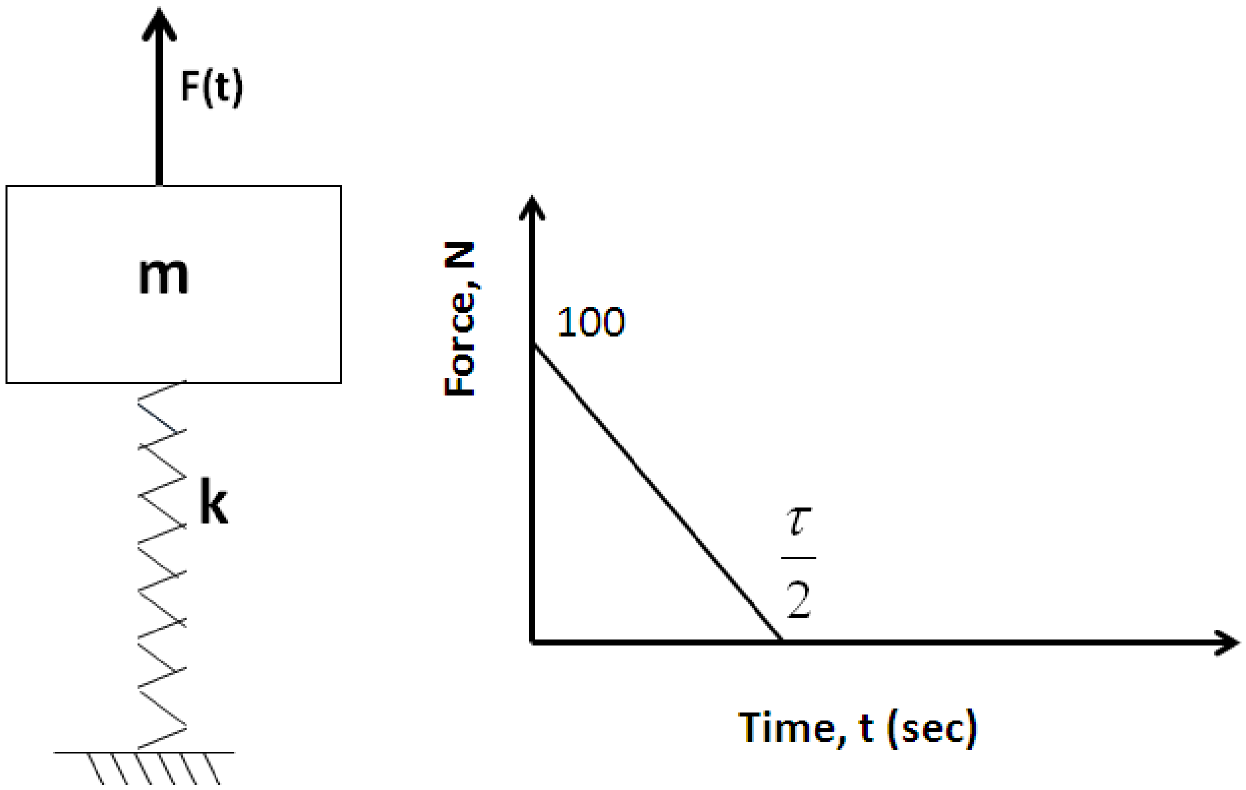
## Convolution Integral

**Example**

A spring-mass system that is initially at rest is subjected to a pulse that linearly drops from 100 N to 0 within half of the time period of natural oscillation of the spring mass system. Determine the displacement response of the system during and after removing the force . Assume the mass m=10 kg and stiffness k=100 kN.

# Convolution Integral

**Example**



## Convolution Integral

**Example**

The **natural frequency** is given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{100000}{10}} = 100\,\text{rad/s}$$

The **time period** is given by $\tau = \dfrac{2\pi}{\omega_n} = \dfrac{2\pi}{100} = 0.0628\,s$

The given force can be represented by

$$F(t) = 100 - 200\frac{t}{\tau} \qquad 0 \le t \le \frac{\tau}{2}$$

# Convolution Integral

**Example**

$$\tau = \frac{2\pi}{\omega_n}$$

$$h(t) = \frac{1}{m\omega_n} \sin \omega_n t$$

$$y(t) = \frac{1}{m\omega_n} \int_0^t \sin \omega_n (t - \eta) \left( 100 - 200 \frac{\eta}{\tau} \right) d\eta$$

$$0 \le t \le \frac{\tau}{2}$$

## Convolution Integral

**Example**

$$y(t)=\frac{100}{m\omega_n}\int_0^t \sin\omega_n(t-\eta)d\eta-\frac{200}{\tau m\omega_n}\int_0^t \eta\sin\omega_n(t-\eta)d\eta$$

$$=\frac{100}{m\omega_n}\left[\left.\frac{\cos\omega_n(t-\eta)}{\omega_n}\right|_0^t\right]-\frac{200}{\tau m\omega_n}\left[\left.\frac{\eta\cos\omega_n(t-\eta)}{\omega_n}\right|_0^t+\left.\frac{1}{\omega_n^2}\sin\omega_n(t-\eta)\right|_0^t\right]$$

$$0\le t\le\frac{\tau}{2}$$

$$y(t)=\frac{100}{k}\left(1-\cos\omega_n t\right)-\frac{200}{k}\left(\frac{t}{\tau}-\frac{\sin\omega_n t}{2\pi}\right)$$

$$0\le t\le\frac{\tau}{2}$$

# ME 316 KDOM SDOF Vibration

## Convolution Integral

**Example**

$$v(t) = \frac{100\omega_n}{k}\sin\omega_n t - \frac{200}{k\tau}\left(1 - \cos\omega_n t\right)$$

$$y(t = \tau / 2) = 1\,\text{mm}$$

$$v(t = \tau / 2) = 68.2\,\text{mm/s}$$

Time can be reset to zero at the end of forced response and the above values of displacement and velocity can be taken as initial conditions at t=0 for free vibration. With the reset time, the new initial conditions are

## Convolution Integral

**Example**

$$y(t=0)=1\,\text{mm}$$

$$v(t=0)=68.2\,\text{mm/s}$$

$$y(t)=A\sin(\omega_n t+\varphi)$$

$$v(t)=A\omega_n\cos(\omega_n t+\varphi)$$

$$\varphi=a\tan\left\{\omega_n\frac{y(0)}{v(0)}\right\}=0.97\,\text{rad}$$

$$A=\frac{y(0)}{\sin(\varphi)}=1.2\,\text{mm}$$

$$y(t)=1.2\times10^{-3}\sin(\omega_n t+0.97)$$

m

$$t\ge\frac{\tau}{2}$$

# ME 316 KDOM SDOF Vibration

## Sinusoidal Excitation

$$m\ddot{x} + c\dot{x} + kx = F\sin\omega t$$

$$x_p = A\sin\left(\omega t - \phi\right)$$

$$\dot{x}_p = A\omega\cos\left(\omega t - \phi\right)$$

$$\ddot{x}_p = -A\omega^2\sin\left(\omega t - \phi\right)$$

$$-mA\omega^2\sin\left(\omega t - \phi\right) + cA\omega\cos\left(\omega t - \phi\right)$$
$$+kA\sin\left(\omega t - \phi\right) = F\sin\omega t$$

# ME 316 KDOM SDOF Vibration

## Sinusoidal Excitation

$$-mA\omega^2\left[\sin\omega t\cos\phi-\cos\omega t\sin\phi\right]+cA\omega\left[\cos\omega t\cos\phi+\sin\omega t\sin\phi\right]$$
$$+kA\left[\sin\omega t\cos\phi-\cos\omega t\sin\phi\right]=F\sin\omega t$$

$$\sin\omega t\left[-mA\omega^2\cos\phi+kA\cos\phi+cA\omega\sin\phi\right]+$$
$$\cos\omega t\left[mA\omega^2\sin\phi+cA\omega\cos\phi-kA\sin\phi\right]=F\sin\omega t$$

# ME 316 KDOM SDOF Vibration

## Sinusoidal Excitation

$$\left[\left(k - m\omega^2\right)\cos\phi + c\omega\sin\phi\right]A = F$$

$$\left[\left(k - m\omega^2\right)\sin\phi - c\omega\cos\phi\right]A = 0$$

$$\tan\phi = \frac{c\omega}{k - m\omega^2}$$

# ME 316 KDOM SDOF Vibration

## Sinusoidal Excitation

$$\sin\phi = \frac{c\omega}{\sqrt{\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2}}$$

$$\cos\phi = \frac{k - m\omega^2}{\sqrt{\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2}}$$

## Sinusoidal Excitation

$$A = \frac{F}{\left[\left(k - m\omega^2\right)\cos\phi + c\omega\sin\phi\right]}$$

$$= \frac{F}{\left[\dfrac{\left(k - m\omega^2\right)^2}{\sqrt{\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2}} + \dfrac{\left(c\omega\right)^2}{\sqrt{\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2}}\right]}$$

$$= \frac{F}{\sqrt{\left(k - m\omega^2\right)^2 + \left(c\omega\right)^2}}$$

## Sinusoidal Excitation

Frequency ratio, $r = \dfrac{\omega}{\omega_n}$

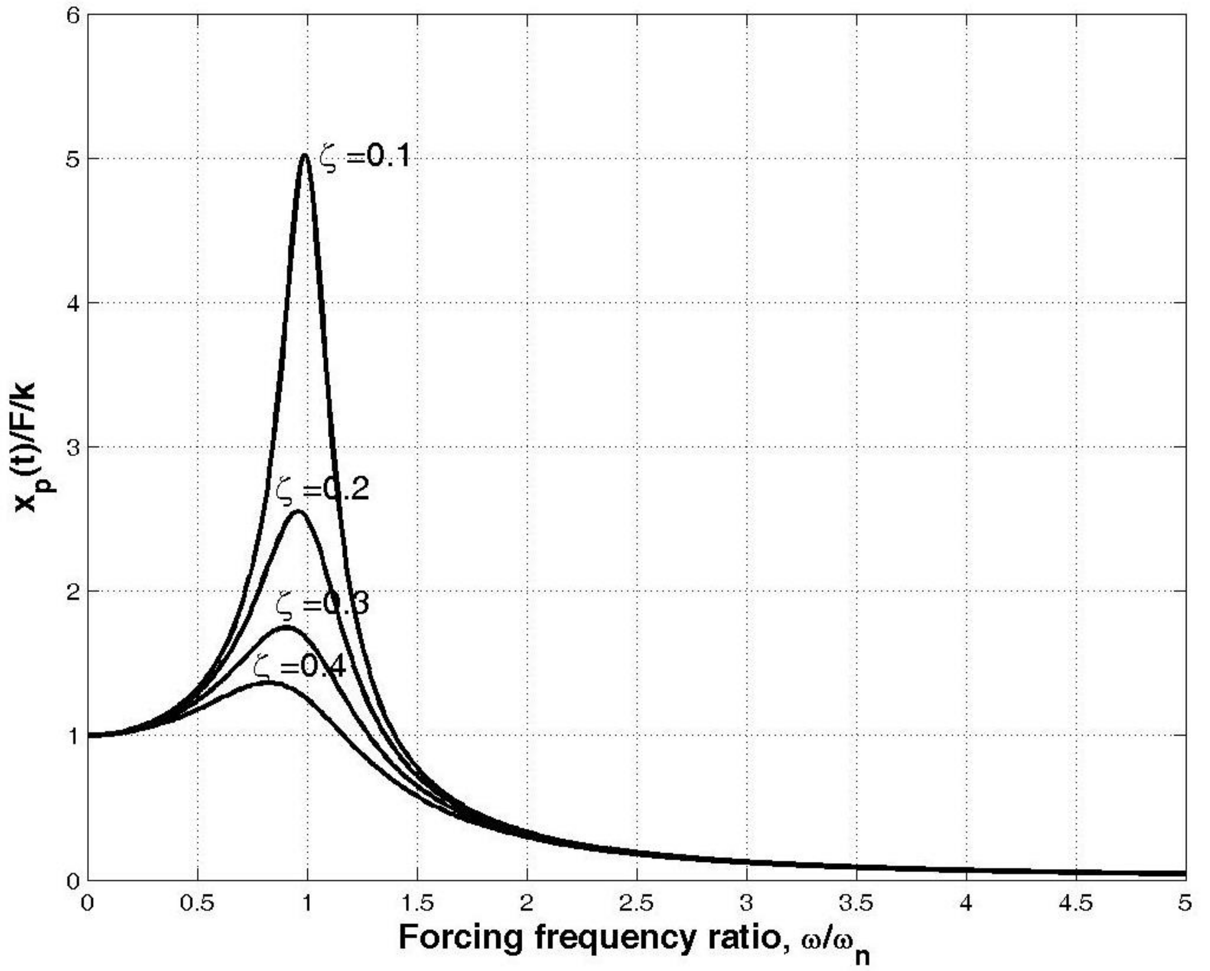
$$A = \frac{F}{k\sqrt{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}} \qquad \phi = \tan^{-1}\frac{2\zeta r}{1-r^2}$$

$$x_p = \frac{F}{k\sqrt{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}}\sin\left(\omega t - \phi\right)$$

## Sinusoidal Excitation

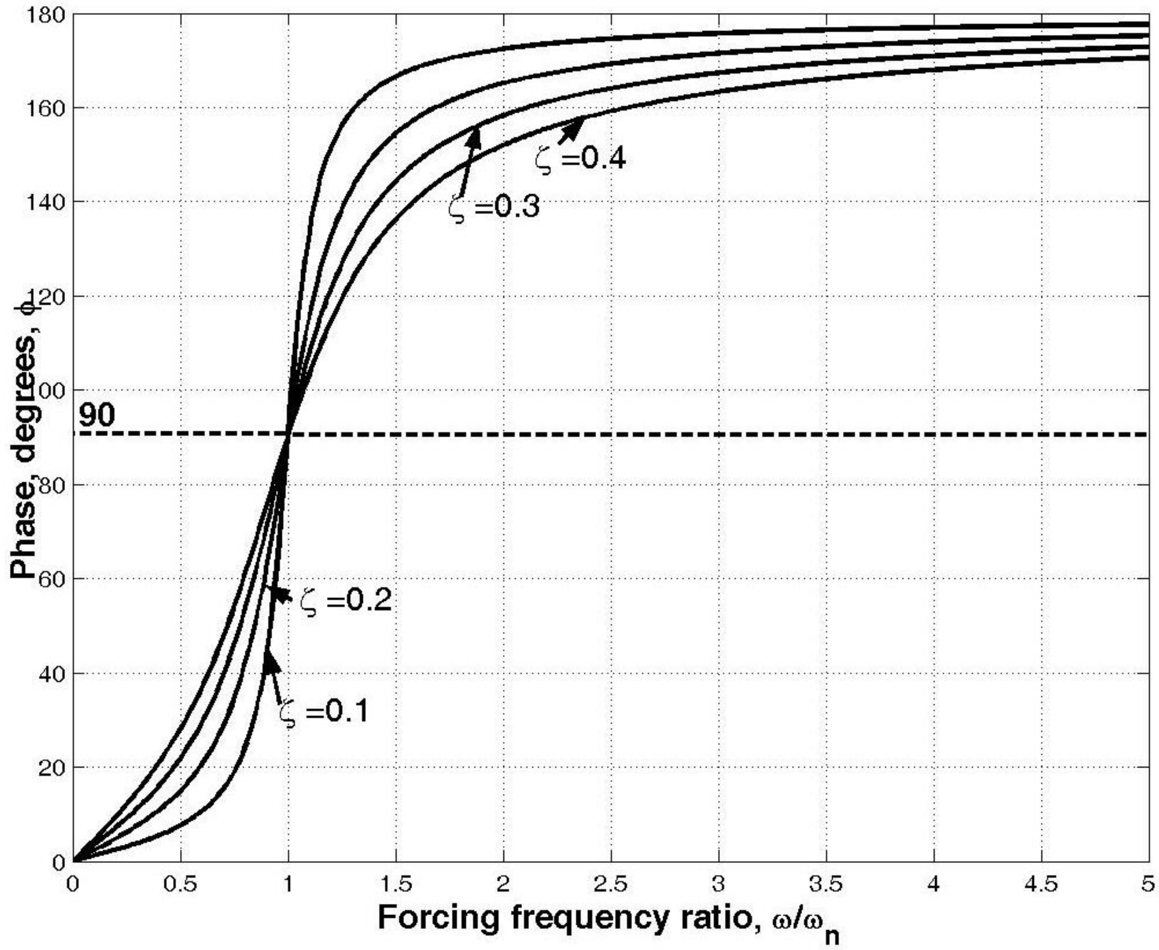
$$\frac{x_p}{F/k} = \frac{1}{\sqrt{\left(1-r^2\right)^2 + \left(2\zeta r\right)^2}}$$

## Sinusoidal Excitation



# ME 316 KDOM SDOF Vibration

## Sinusoidal Excitation



# ME 316 KDOM SDOF Vibration

## Sinusoidal Excitation

**Example**

A spring-mass-damper system has a mass of 0.1 kg, a spring of stiffness constant 1000 N/m and damping coefficient 0.15 N-s/m. It is subjected to a driving force that has two excitation frequencies, represented by $F = 2\cos(85t) + 4\sin(120t)$ N. (a) Determine the time-domain displacement response of the above system (b) Why the phase angles are different by an order of magnitude?

# ME 316 KDOM SDOF Vibration

## Sinusoidal Excitation

**Example**

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{0.1}} = 100\,\text{rad/s}$$

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{0.15}{2\sqrt{1000 \times 0.1}} = 0.0075$$

## Sinusoidal Excitation

**Example**

Amplitude of the first force F1=2 N

Amplitude of the second force F2=4 N

Excitation frequency of the first force $\omega_1$=85 rad/s

Excitation frequency of the second force $\omega_2$=120 rad/s

Frequency ratio of the first force $r_1 = \frac{\omega_1}{\omega_n} = \frac{85}{100} = 0.85$

Frequency ratio of the second force $r_2 = \frac{\omega_2}{\omega_n} = \frac{120}{100} = 1.2$

## Sinusoidal Excitation

**Example**

$$y(t) = \frac{F1}{k\sqrt{(1-r_1^2)^2+(2\zeta r_1)^2}}\cos(\omega_1 t - \phi_1)$$
$$+\frac{F2}{k\sqrt{(1-r_2^2)^2+(2\zeta r_2)^2}}\sin(\omega_2 t - \phi_2)$$

$$\phi_1 = \tan^{-1}\frac{2\zeta r_1}{1-r_1^2} = \tan^{-1}\frac{2\times 0.0075\times 0.85}{1-0.85^2} = 2.63^0$$

## Sinusoidal Excitation

**Example**

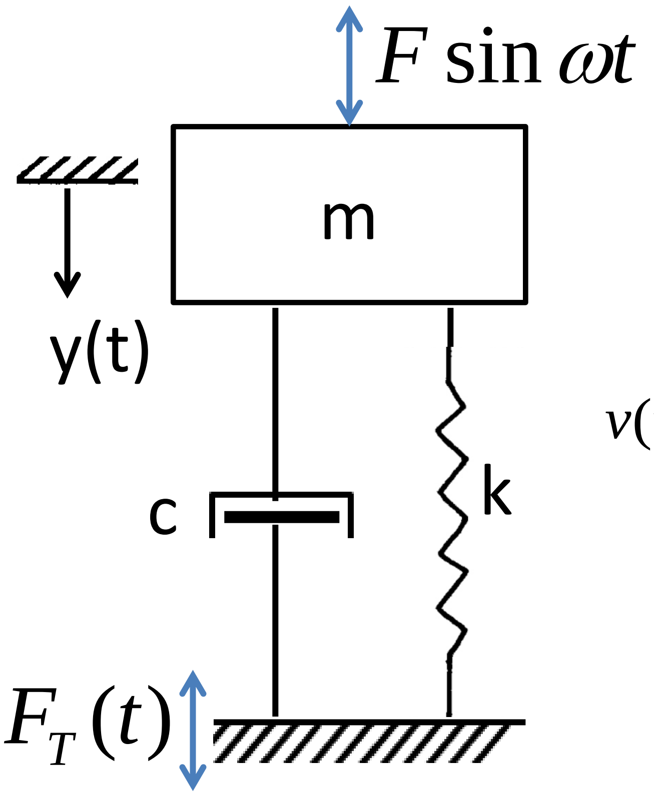
$$\phi_2 = \tan^{-1}\frac{2\zeta r_2}{1-r_2^2} = \tan^{-1}\frac{2\times 0.0075\times 1.2}{1-1.2^2} = 177.65^0$$

$$y(t) = \frac{2}{1000\sqrt{(1-0.85^2)^2+(2\times 0.0075\times 0.85)^2}}\cos(85t-2.63^0)+$$

$$\frac{4}{1000\sqrt{(1-1.2^2)^2+(2\times 0.0075\times 1.2)^2}}\sin(120t-177.65^0)$$

$$y(t) = 0.0072\cos(85t-2.63^0)+0.0091\sin(120t-177.65^0)$$

# Vibration Isolation




Transmitted force

$$y(t) = \frac{F}{k\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\omega t - \phi)$$

$$v(t) = \dot{y}(t) = \frac{F\omega}{k\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \cos(\omega t - \phi)$$

$$F_T = ky(t) + cv(t)$$

# ME 316 KDOM SDOF Vibration

## Vibration Isolation

Force transmissibility

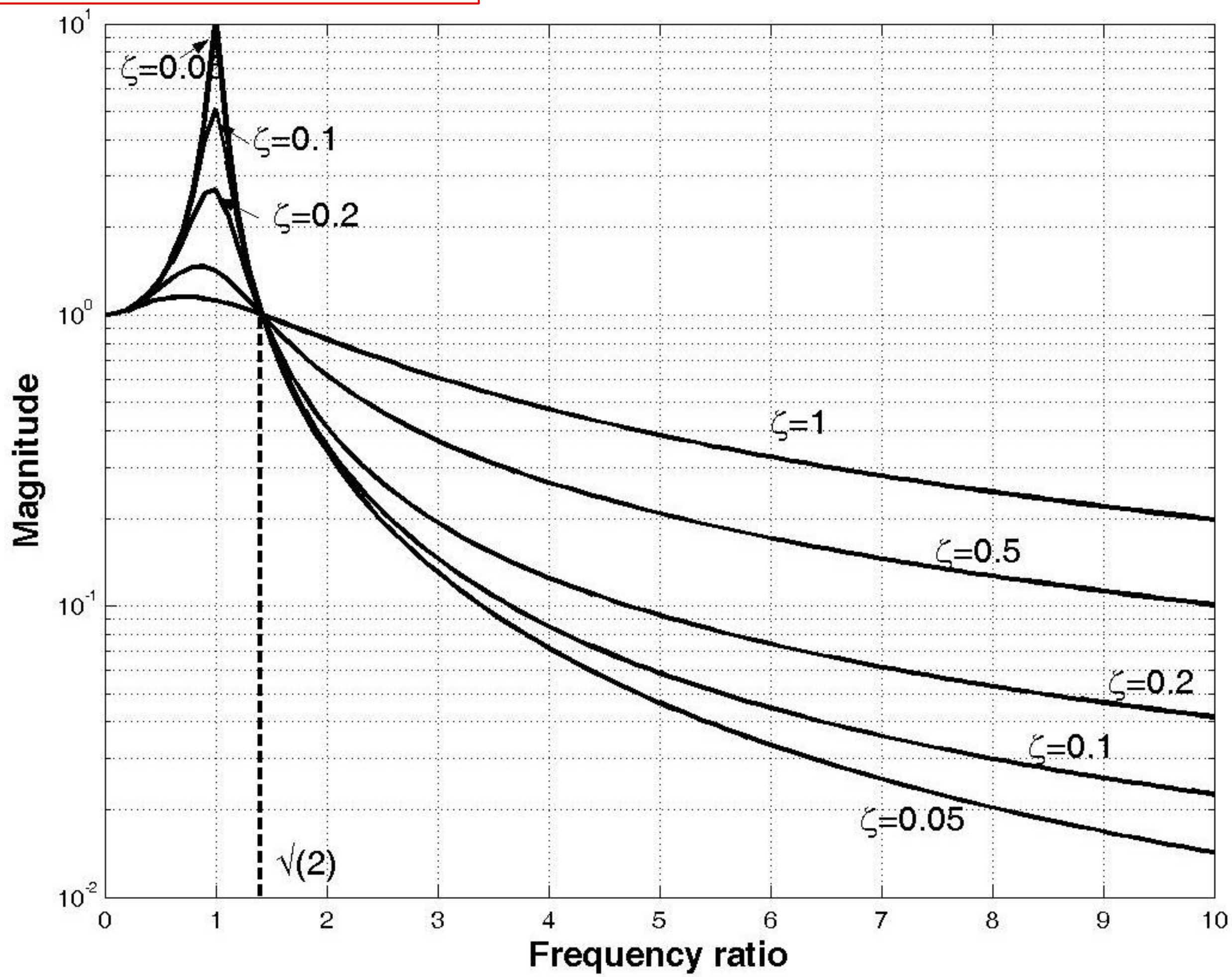
$$F_T = \frac{kF}{k\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(\omega t-\phi) + \frac{cF\omega}{k\sqrt{(1-r^2)^2+(2\zeta r)^2}}\cos(\omega t-\phi)$$

$$\frac{F_T}{F} = \frac{\sqrt{1+\left(2\zeta r\right)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(\omega t+\theta-\phi)$$

# Vibration Isolation

$$\theta = \tan^{-1} 2\zeta r; \quad \phi = \tan^{-1} \frac{2\zeta r}{1-r^2}$$

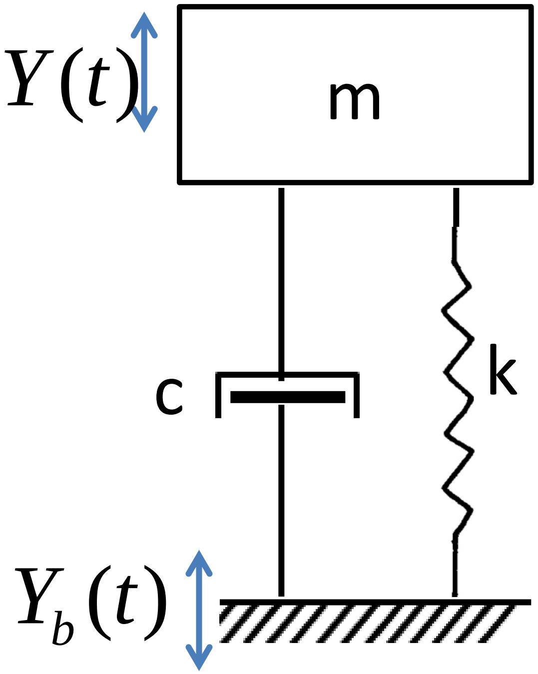
# Vibration Isolation



# Vibration Isolation

1. The transmissibility decreases with increase in damping at the resonance.
2. The family of curves for different values of damping intersect at *T=1* and $r = \sqrt{2}$ .
3. For frequency ratio *r* greater than $\sqrt{2}$ , the transmissibility decreases.
4. The design of vibration isolators is based on the transmissibility curves for $r > \sqrt{2}$ .

# Vibration Isolation




Displacement transmissibility

$$\frac{Y}{Y_b} = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \sin(\omega t + \theta - \phi)$$

# Vibration Isolation

**Example**

A measuring instrument of mass 10 kg has to be installed on a floor that is subject to vibration excitation by other machines that are installed on the same floor, which vibrates with amplitude of 5 mm at a frequency of 3000 RPM. In order that the electronic components of the instrument satisfactorily operate, the instrument requires that the maximum displacement amplitude is limited to 100 microns.

# Vibration Isolation

**Example**

(a) What is the equivalent stiffness of an isolator with a damping ratio of 0.01 to limit the transmitted displacement to an acceptable level?

(b) What is the maximum acceleration of the measuring instrument?

(c) What is the maximum deformation of the isolator?

# Vibration Isolation

**Example**

$$T = \frac{Y}{Y_b} = \frac{100 \times 10^{-6}}{5 \times 10^{-3}} = 0.02$$

$$T = \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

$$T^2 \left[ (1 - r^2)^2 + (2\zeta r)^2 \right] = 1 + (2\zeta r)^2$$

$$T^2 r^4 + \left[ (2\zeta)^2 (T^2 - 1) - 2T^2 \right] r^2 + T^2 - 1 = 0$$

# ME 316 KDOM SDOF Vibration

## Vibration Isolation

**Example**

$$4\times10^{-4}r^4-1.199\times10^{-3}r^2-0.9996=0$$

$$r=\sqrt{\frac{0.001199\pm\sqrt{0.001199^2+4\times4\times10^{-4}\times0.9996}}{2\times4\times10^{-4}}}=7.18$$

$$\omega_n=\frac{\omega}{r}=\frac{314.16}{7.18}=43.75 \text{ rad/s}$$

# Vibration Isolation

**Example**

$$A = Y\omega^2$$
$$= 100\times10^{-6}\times314.16^2$$
$$= 9.86\ \text{m/s}^2$$

$$k = m\omega_n^2 = 10\times43.75^2 = 19.14\,kN/m$$

$$\delta = \left(Y_b - Y\right)$$
$$= 5 - 0.1 = 4.9\,mm$$