GAME THEORY

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1 Game Theory Intro

1.1 Self-interested Agents and Utility Theory

An agent is **self-interested** mean that agent has its own description of states of the world. And each such agent has a utility function which represents their potential responses for the impact of uncertainty.

1.2 Defining Games

The **Players** of the Game are decision makers. And the **Actions** in the Game are things that players can do.
There are two standard representations of Games:
- Normal Form, which lists what payoffs get as a function of their actions.
- Extensive Form, which includes timing of moves.

The Normal Form of Games is:
Assuming we have n-person normal form: (N, A, u):
- Players: $N = \{1, ..., n\}$ is a finite set, indexed by i
- Actions for player is A_i, and the action profile is $a = (a_1, a_2, ..., a_n) \in A = A_1 \times ... \times A_n$.
- Utility function for player i: $u_i : A \to R$. And $u = (u_1, ..., u_n)$ is a profile of utility functions.

1.3 Kinds of Games

Pure Competition Games.In pure competition games, for all action profile, the sum of payoffs is a constant number. Special case: zero sum.

Cooperation Games.Players have exactly the same interests. There is no conflict: all players want the same things. And they always get the same payoffs.

General Games.The most interesting games combine elements of **cooperation** and **competition**. The sum of the payoffs are always positive and players will never get "fail". They just compete for limited payoffs.

1.4 Nash Equilibrium Intro

Nash Equilibrium refers a situation where each player in a game is assumed to know the equilibrium strategies of the other players, and no player has anything to gain by changing their own strategy.
And if you know what everyone else was going to do, it would be easy to pick your **Best Response**.
Let $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$
now $a = (a_{-i}, a_i)$
Your Best Response could be definite as:

$$a^* \in BR(a_{-i}) iff \forall a_i \in A_i, u_i(a_i^*, a_{-i}) \geq u_i(a_i, a_{-i})$$

Then the Nash Equilibrium could be definite as:
$a = (a_1, \ldots, a_n)$ is a ("pure strategy")Nash Equilibrium if $\forall i, a_i \in BR(a_{-i})$

1.5 Dominant Strategies

Let s_i and s_i' be two strategies for player i, and let S_{-i} be the set of all possible strategy profiles for the other players.

Definition:
s_i strictly dominates s_i' if $\forall s_{-i} \in S_{-i}, u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$
s_i very weakly dominates s_i' if $\forall s_{-i} \in S_{-i}, u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$
If one strategy dominates all others, we say it is **dominant**.

1.6 Analyzing Games

Previous notes, we only talk about the Best Response of an Agent. Sometimes, one outcome o is at least as good for every agent as another outcome o', and there is some agent who strictly prefers o to o'. We say that o Pareto-dominates o'. An outcome o^* is **Pareto-Optimal** if there is no other outcome that Pareto-dominates it.

2 Mixed Strategies and Nash Equilibrium

It would be a pretty bad idea to play any deterministic strategy in matching pennies. So we can confuse the opponent by playing randomly.
Define a strategy s_i for agent i as any probability distribution over the actions A_i. We have:

- **pure strategy**: only one action is played with positive probability.
- **mixed strategy**: more than one action is played with positive probability. These actions are called the **support** of the mixed strategy.

Let the set of all strategies for i be S_i
Let the set of all strategy profiles be $S = S_1 \times ... \times S_n$

2.1 Utility under Mixed Strategies

So what is your payoff if all the players follow mixed strategy profile $s \in S$?
We use the idea of **expected utility** from decision theory:

$$u_i(s) = \sum_{a \in A} u_i(a) Pr(a|s)$$

$$Pr(a|s) = \prod_{j \in N} s_j(a_j)$$

And our definitions of best response and Nash equilibrium generalize from actions to strategies. And every finite game has a Nash equilibrium(not pure Nash equilibrium).

2.2 Computing Mixed Nash Equilibrium

Make battle of the sexes as an example, let player2 play B with p, F with $1-p$. If player 1 best-responds with a mixed strategy, player 2 must make him indifferent between F and B.

Some interpretations to play a mixed strategy: Randomize to **confuse** your opponent. Randomize when **uncertain** about the other's action.

2.3 Hardness beyond 2 $\times 2 games$

Two example algorithms for finding NE:
- LCP(Linear Complementary) formulation
- Support Enumeration Method

Still, finding even a single Nash Equilibrium seems hard, for the complexity of the Nash Equilibrium, we have **theorem**: Computing a Nash Equilibrium is PPAD-complete("Polynomial Parity Arguments on Directed graphs")