Rutgers Math250 Intro to Linear Algebra **Homework - 7**

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Problem 4.1.11

Let W be the set of all vectors of the form $\begin{bmatrix} 5b+2c \\ b \\ c \end{bmatrix}$, where b and c are arbitrary. Find vectors $\mathbf{u}$ and $\mathbf{v}$ such that $W = \text{Span}\{\mathbf{u}, \mathbf{v}\}$. Why does this show that W is a subspace of $\mathbb{R}^3$?

Solution: To express W as a span of vectors, start by rewriting the vector in W in terms of b and c:

$$\begin{bmatrix} 5b+2c \\ b \\ c \end{bmatrix} = b \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}.$$

This shows that every vector in W can be written as a linear combination of the vectors $\mathbf{u} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. Therefore, $W = \text{Span}\left\{ \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Since W is the span of the vectors $\mathbf{u}$ and $\mathbf{v}$, it is closed under vector addition and scalar multiplication, which are the requirements for a subspace. Therefore, W is a subspace of $\mathbb{R}^3$.

Problem 4.1.23

Mark each statement True or False. Justify each answer.

1. If f is a function in the vector space V of all real-valued functions on $\mathbb{R}$ and if $f(t) = 0$ for some t, then f is the zero vector in V.
2. A vector is an arrow in three-dimensional space.
3. A subset H of a vector space V is a subspace of V if the zero vector is in H.
4. A subspace is also a vector space.
5. Analog signals are used in the major control systems for the space shuttle, mentioned in the introduction to the chapter.

Solution:

1. **False.** Just because $f(t) = 0$ for some value t, it does not mean that f is the zero function (i.e., $f(x) = 0$ for all $x \in \mathbb{R}$). For f to be the zero vector in V, it must be zero at every point in $\mathbb{R}$.
2. **False.** In linear algebra, a vector is not limited to three-dimensional space; it can exist in any dimension, and it does not need to be visualized as an "arrow."
3. **False.** A subset H of V must satisfy three conditions to be a subspace: it must contain the zero vector, be closed under vector addition, and be closed under scalar multiplication. Having the zero vector alone is not sufficient.
4. **True.** By definition, a subspace of a vector space V is itself a vector space, with the inherited operations from V.

5. **False.** This statement is unrelated to the properties of vector spaces or linear transformations, so it does not apply to the context of linear algebra.

Problem 4.2.3 and 4.2.5
Find an explicit description of Nul A by listing vectors that span the null space.

3. $A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$

5. $A = \begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 1 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Solution:

3. For $A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$:

   To find Nul A, we need to solve $A\mathbf{x} = 0$. So we have: $\begin{bmatrix} 1 & 3 & 5 & 0 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7 & 6 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{bmatrix}$

   Now, express x_1 and x_2 in terms of the free variables x_3 and x_4:

   $$x_1 = 7x_3 - 6x_4, \quad x_2 = -4x_3 + 2x_4$$

   Thus, the general solution for $\mathbf{x}$ is: $\mathbf{x} = x_3 \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

   Therefore, $\text{Nul}\, A = \text{Span}\left\{ \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

5. For $A = \begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 1 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$:

   Similarly, we need to solve $A\mathbf{x} = 0$. So we have: $\begin{bmatrix} 1 & -2 & 0 & 4 & 0 & 0 \\ 0 & 1 & 1 & -9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

   From this, we see that $x_5 = 0$ and can express x_1, x_2 in terms of the free variables x_3 and x_4:

   $$x_1 = 2x_2 - 4x_4, \quad x_2 = -x_3 + 9x_4$$

   The general solution is then: $\mathbf{x} = x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 9 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

Therefore, $\text{Nul}\, A = \text{Span}\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 9 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Problem 4.2.7
Either use an appropriate theorem to show that the given set, W, is a vector space, or find a specific example to the contrary.

$$\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a+b+c=2 \right\}$$

Solution: To determine if W is a vector space, we can check if it includes the zero vector, which is a necessary condition for any vector space.

In $\mathbb{R}^3$, the zero vector is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. However, $0+0+0=0 \neq 2$. Therefore, the zero vector is not in W.

Since W does not contain the zero vector, it cannot be a vector space.

Problem 4.2.23
Let $A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Determine if $\mathbf{w}$ is in $\operatorname{Col} A$. Is $\mathbf{w}$ in $\operatorname{Nul} A$?

Solution:
To determine if $\mathbf{w}$ is in the column space of A, we need to determine if there exist scalars x and y such that $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. But we can see that vectors in A are linear dependent. So there are many solutions for $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{w}$ is in the column space of A.

To determine is $\mathbf{w}$ in $\operatorname{Nul} A$. We have:

$$Aw = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $\mathbf{w}$ is in $\operatorname{Nul} A$.

Problem 4.2.31
Define $T : \mathbb{P}_2 \to \mathbb{R}^2$ by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix}$. For instance, if

$$\mathbf{p}(t) = 3 + 5t + 7t^2, \text{ then } T(\mathbf{p}) = \begin{bmatrix} 3 \\ 15 \end{bmatrix}.$$

(a) Show that T is a linear transformation. *[Hint: For arbitrary polynomials* $\mathbf{p}, \mathbf{q}$ *in* $\mathbb{P}_2$*, compute* $T(\mathbf{p}+\mathbf{q})$ *and* $T(c\mathbf{p})$*.]*

(b) Find a polynomial $\mathbf{p}$ in $\mathbb{P}_2$ that spans the kernel of T, and describe the range of T.

Solution:

(a) Let $\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2$ and $\mathbf{q}(t) = b_0 + b_1 t + b_2 t^2$.
Then, $\mathbf{p}(0) + \mathbf{q}(0) = (a_0 + b_0)$ and $\mathbf{p}(1) + \mathbf{q}(1) = (a_0 + a_1 + a_2) + (b_0 + b_1 + b_2)$.
Thus,

$$T(\mathbf{p}+\mathbf{q}) = \begin{bmatrix} \mathbf{p}(0) + \mathbf{q}(0) \\ \mathbf{p}(1) + \mathbf{q}(1) \end{bmatrix} = T(\mathbf{p}) + T(\mathbf{q}).$$

Let $c\mathbf{p}(t) = c(a_0 + a_1 t + a_2 t^2)$, so $c\mathbf{p}(0) = ca_0$ and $c\mathbf{p}(1) = c(a_0 + a_1 + a_2)$.

Thus,

$$T(c\mathbf{p}) = \begin{bmatrix} c\mathbf{p}(0) \\ c\mathbf{p}(1) \end{bmatrix} = c \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} = cT(\mathbf{p}).$$

Since both additivity and scalar multiplication are satisfied, T is a linear transformation.

(b) Kernel of T: The kernel of T consists of all polynomials $\mathbf{p}(t)$ in $\mathbb{P}_2$ such that $T(\mathbf{p}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

For $T(\mathbf{p}) = 0$, we must have:

$$\mathbf{p}(0) = 0 \quad \text{and} \quad \mathbf{p}(1) = 0.$$

Let $\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2$. Then:

$$\mathbf{p}(0) = a_0 = 0,$$

$$\mathbf{p}(1) = a_0 + a_1 + a_2 = 0.$$

This implies $a_0 = 0$ and $a_1 + a_2 = 0$, so $a_1 = -a_2$. Therefore, any polynomial in the kernel has the form:

$$\mathbf{p}(t) = a_2(-t + t^2).$$

A basis for the kernel of T is $\{-t + t^2\}$.

Range of T: For a general polynomial $\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2$, we have:

$$T(\mathbf{p}) = \begin{bmatrix} a_0 \\ a_0 + a_1 + a_2 \end{bmatrix}.$$

Let $y_1 = a_0$ and $y_2 = a_0 + a_1 + a_2$. Then any vector in the range of T has the form $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, where y_1 and y_2 are arbitrary real numbers. Therefore, the range of T is $\mathbb{R}^2$.

Problem 4.3.1 and 4.3.3

Determine which sets are bases for $\mathbb{R}^3$. Of the sets that are *not* bases, determine which ones are linearly independent and which ones span $\mathbb{R}^3$. Justify your answers.

1. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

3. $\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix}$

Solution:

1. The vectors in $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are linear independent and span $\mathbb{R}^3$, they are bases for $\mathbb{R}^3$.

3. Suppose $c_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} + c_3 \begin{bmatrix} -3 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

   This leads to the following system of equations:

$$c_1 + 3c_2 - 3c_3 = 0$$

$$2c_2 - 5c_3 = 0$$

$$-2c_1 - 4c_2 + c_3 = 0$$

   Solving this system, we find that the only solution is $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$, so the vectors are linearly independent and they span $\mathbb{R}^3$, they are bases for $\mathbb{R}^3$.

Problem 4.3.11
Find a basis for the set of vectors in $\mathbb{R}^3$ in the plane $x + 2y + z = 0$. *[Hint: Think of the equation as a "system" of homogeneous equations.]*

Solution: The plane $x + 2y + z = 0$ in $\mathbb{R}^3$ represents a subspace. To find a basis, we solve for x in terms of y and z: $x = -2y - z$.

Thus, any vector $\mathbf{v}$ on this plane can be written as: $\mathbf{v} = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Therefore, a basis for this plane is: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Problem 4.3.15
Find a basis for the space spanned by the given vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_5$.

$$\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -6 \\ 9 \end{bmatrix}$$

Solution: Construct the matrix with $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ as columns:

$$A = \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & -3 & 1 \\ -3 & 2 & 1 & -8 & -6 \\ 2 & -3 & 6 & 7 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 1 & 2 \\ 0 & 1 & -4 & 0 & -5 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

After row reducing, we find that the pivot columns correspond to $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_4$. Thus, a basis for the space spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_5$ is:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -8 \\ 7 \end{bmatrix} \right\}.$$

Problem 4.3.31
Reveal an important connection between linear independence and linear transformations and provide practice using the definition of linear dependence. Let V and W be vector spaces, let $T : V \to W$ be a linear transformation, and let $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ be a subset of V.

Show that if $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly dependent in V, then the set of images, $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)\}$, is linearly dependent in W. This fact shows that if a linear transformation maps a set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ onto a linearly independent set $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)\}$, then the original set is linearly independent, too (because it cannot be linearly dependent).

Solution: To prove the statement, let $T : V \to W$ be a linear transformation and suppose that $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is a linearly dependent set in V. By definition of linear dependence, there exist scalars $c_1, c_2, \dots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

Now, apply the linear transformation T to both sides of this equation:

$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p) = T(\mathbf{0}).$$

Since T is linear, we can distribute T over the sum and scalars:

$$c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + \cdots + c_pT(\mathbf{v}_p) = \mathbf{0}.$$

This equation shows that $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)\}$ is linearly dependent in W because there exist scalars $c_1, c_2, \dots, c_p$, not all zero, such that their linear combination is zero.

This fact also implies that if $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)\}$ is linearly independent in W, then $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ must be linearly independent in V. Otherwise, if $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ were linearly dependent, it would imply that $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_p)\}$ is linearly dependent as well, contradicting the assumption.

Problem 4.4.1
Find the vector $\mathbf{x}$ determined by the given coordinate vector $[\mathbf{x}]_B$ and the given basis B.

$$B = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}, \quad [\mathbf{x}]_B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Solution: To find the vector $\mathbf{x}$ from the coordinate vector $[\mathbf{x}]_B$ with respect to the basis B, we use the formula:

$$\mathbf{x} = 5 \begin{bmatrix} 3 \\ -5 \end{bmatrix} + 3 \begin{bmatrix} -4 \\ 6 \end{bmatrix}.$$

We get:

$$\mathbf{x} = \begin{bmatrix} 15 \\ -25 \end{bmatrix} + \begin{bmatrix} -12 \\ 18 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}.$$

Thus, $\mathbf{x} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$.

Problem 4.4.7
Find the coordinate vector $[\mathbf{x}]_B$ of $\mathbf{x}$ relative to the given basis $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$.

$$\mathbf{b}_1 = \begin{bmatrix} 1 \\ -1 \\ -3 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -3 \\ 4 \\ 9 \end{bmatrix}, \quad \mathbf{b}_3 = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 8 \\ -9 \\ 6 \end{bmatrix}$$

Solution: To find the coordinate vector $[\mathbf{x}]_B$ of $\mathbf{x}$ relative to the basis B, express $\mathbf{x}$ as a linear combination:

$$\mathbf{x} = c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3$$

This leads to the system:

$$\begin{cases} c_1 - 3c_2 + 2c_3 = 8, \\ -c_1 + 4c_2 - 2c_3 = -9, \\ -3c_1 + 9c_2 + 4c_3 = 6. \end{cases}$$

Solving, we get $c_1 = -1$, $c_2 = -1$, $c_3 = 3$. Thus,

$$[\mathbf{x}]_B = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}.$$