Rutgers Math250 Intro to Linear Algebra **Homework - 4**
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Problem 1.7.11
Find the values(s) of h for which the vectors are linearly dependent. Justify each answer.

$$\begin{bmatrix}1\\-1\\4\end{bmatrix}, \begin{bmatrix}3\\-5\\7\end{bmatrix}, \begin{bmatrix}-1\\5\\h\end{bmatrix}$$

Solution: Assume that:

$$\begin{bmatrix}1\\-1\\4\end{bmatrix}x_1 + \begin{bmatrix}3\\-5\\7\end{bmatrix}x_2 + \begin{bmatrix}-1\\5\\h\end{bmatrix}x_3 = 0$$

We can get the augmented matrix:

$$\begin{bmatrix}1 & 3 & -1 & 0\\-1 & -5 & 5 & 0\\4 & 7 & h & 0\end{bmatrix} \xrightarrow[-\frac{1}{2}(R_2+R_1)]{R_3-4R_1} \begin{bmatrix}1 & 3 & -1 & 0\\0 & 1 & -2 & 0\\0 & -5 & h+4 & 0\end{bmatrix} \xrightarrow[R_1-3R_2]{R_3+5R_2} \begin{bmatrix}1 & 0 & 5 & 0\\0 & 1 & -2 & 0\\0 & 0 & h-6 & 0\end{bmatrix}$$

If the vectors are linearly dependent, $h-6=0$. Then values of h is $h=6$.

Problem 1.7.21

Mark each statement True or False. Justify each answer on the basis of a careful reading of the text.
a. The columns of a matrix A are linearly independent if the equation $Ax=0$ has the trivial solution.
b. If S is a linearly dependent set, then each vector is a linear combination of the other vectors in S.
c. The columns of any 4×5 matrix are linearly dependent.
d. If x and y are linearly independent, and if $\{x,y,z\}$ is linearly dependent, then z is in Span$\{x,y\}$.

Solution:
(a.): False. The correct condition is equation $Ax=0$ has only the trivial solution. In other words, the trivial solution must be the only solution.
(b.): False. If S is a linearly dependent set, it means that at least one vector in S can be written as a linear combination of the others, but it does not imply that each vector is a linear combination of the others.
(c.): True. A 4×5 matrix has 4 rows and 5 columns, meaning there are one more columns than rows. At least one column must be a linear combination of the others, making the columns linearly dependent.
(d.): True. If x and y are linearly independent, and if $\{x,y,z\}$ is linearly dependent, then z can be written as a linear combination of x and y.

Problem 1.8.5
With T defined by $T(x)=Ax$, find a vector x whose image under T is b, and determine whether x is unique.

$$A = \begin{bmatrix}1 & -5 & -7\\-3 & 7 & 5\end{bmatrix}, b = \begin{bmatrix}-2\\-2\end{bmatrix}$$

Solution: We set $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then we get $T(x) = Ax = b$

$$\begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 & -7 & -2 \\ -3 & 7 & 5 & -2 \end{bmatrix} \xrightarrow{-\frac{1}{8}(R_2+3R_1)} \begin{bmatrix} 1 & -5 & -7 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix} \xrightarrow{R_1+5R_2} \begin{bmatrix} 1 & 0 & 3 & 3 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

So, we have: $x_1 = 3 - 3x_3$, $x_2 = 1 - 2x_3$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 - 3x_3 \\ 1 - 2x_3 \\ x_3 \end{bmatrix}$$

where x_3 is a free parameter, it could be $x_3 = 1$. Then $x = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ is a vector whose image under T is b.

Problem 1.8.9
Find all x in R^4 that are mapped into the zero vector by the transformation $x \to Ax$ for the given matrix A.

$$A = \begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix}$$

Solution: We set $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, then we get:

$$\begin{bmatrix} 1 & -4 & 7 & -5 \\ 0 & 1 & -4 & 3 \\ 2 & -6 & 6 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 2 & -6 & 6 & -4 & 0 \end{bmatrix} \xrightarrow{R_3-2R_1} \begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 2 & -8 & 6 & 0 \end{bmatrix} \xrightarrow{R_3-2R_2} \begin{bmatrix} 1 & -4 & 7 & -5 & 0 \\ 0 & 1 & -4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Then we get equations:

$$x_1 - 4x_2 + 7x_3 - 5x_4 = 0$$

$$x_2 - 4x_3 + 3x_4 = 0$$

Which simplifies to: $x_1 = 9x_3 - 7x_4$, $x_2 = 4x_3 - 3x_4$
So the general solution is:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9x_3 - 7x_4 \\ 4x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 9 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Problem 1.8.17
Let T: $R^2 \to R^2$ be a linear transformation that maps $u = \begin{bmatrix}5\\2\end{bmatrix}$ into $\begin{bmatrix}2\\1\end{bmatrix}$ and maps $v = \begin{bmatrix}1\\3\end{bmatrix}$ into $\begin{bmatrix}-1\\3\end{bmatrix}$.
Use the fact that T is linear to find the images under T of $3u$, $2v$, and $3u + 2v$.

Solution: Using the linearity of T, we know that:

$$T(3u) = 3T(u)$$

Since $T(u) = \begin{bmatrix}2\\1\end{bmatrix}$, we get:

$$T(3u) = \begin{bmatrix}6\\3\end{bmatrix}$$

And the same way, we can get:

$$T(2v) = 2T(v) = \begin{bmatrix}-2\\6\end{bmatrix}$$

So:

$$T(3u + 2v) = T(3u) + T(2v) = \begin{bmatrix}6\\3\end{bmatrix} + \begin{bmatrix}-2\\6\end{bmatrix} = \begin{bmatrix}4\\9\end{bmatrix}$$

Problem 1.8.21
Mark each statement True or False. Justify each answer.
a. A linear transformation is a special type of function.
b. If A is a 3×5 matrix and T is a transformation defined by $T(x) = Ax$, then the domain of T is R^3.
c. If A is an $m \times n$ matrix, then the range of the transformation $x \to Ax$ is R^m.
d. Every linear transformation is a matrix transformation.
e. A transformation T is linear if and only if $T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2)$ for all v_1 and v_2 in the domain of T and for all scalars c_1 and c_2.

Solution:
(a.): True. A linear transformation is indeed a function that satisfies the properties of linearity, making it a special type of function.
(b.): False. If A is a 3×5 matrix, it has 5 columns, meaning the domain of $T(x) = Ax$ is R^5, not R^3.
(c.): True. The range of the transformation $x \to Ax$, where A is an $m \times n$ matrix, lies in R^m, meaning it is mapped into R^m.
(d.): False. Not every linear transformation is a matrix transformation. Some linear transformations, particularly in infinite-dimensional spaces, cannot be represented by a matrix.
(e.): True. This is the definition of linearity: a transformation T is linear if and only if $T(c_1v_1 + c_2v_2) = c_1T(v_1) + c_2T(v_2)$ for all vectors v_1, v_2 and scalars c_1, c_2

Problem 1.9.1
Assume that T is a linear transformation. Find the standard matrix of T.
$T : R^2 \to R^4, T(e_1) = (3, 1, 3, 1)$ and $T(e_2) = (-5, 2, 0, 0)$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

Solution: We are given that:

$$T(e_1) = \begin{bmatrix}3\\1\\3\\1\end{bmatrix}, T(e_2) = \begin{bmatrix}-5\\2\\0\\0\end{bmatrix}$$

The standard matrix A is formed by placing the images of e_1 and e_2 as the columns of the matrix. Therefore, we have:

$$A = \begin{bmatrix} 3 & -5 \\ 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}$$

So, the standard matrix of T is $\begin{bmatrix} 3 & -5 \\ 1 & 2 \\ 3 & 0 \\ 1 & 0 \end{bmatrix}$

Problem 1.9.19
Show that T is a linear transformation finding a matrix that implements the mapping. Note that $x_1, x_2, ...$ are not vectors but are entries in vectors.

$$T(x_1, x_2, x_3) = (x_1 - 5x_2 + 4x_3,\ x_2 - 6x_3)$$

Solution: From:

$$T(x_1, x_2, x_3) = Ax = (x_1 - 5x_2 + 4x_3,\ x_2 - 6x_3)$$

We can get:

$$T(x_1, x_2, x_3) = \begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

So the matrix that implements the mapping is:

$$A = \begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & -6 \end{bmatrix}$$

Problem 1.9.21
Let $T : R^2 \to R^2$ be a linear transformation such that $T(x_1, x_2) = (x_1 + x_2,\ 4x_1 + 5x_2)$. Find x such that $T(x) = (3, 8)$.

Solution: From:

$$T(x_1, x_2) = (x_1 + x_2, 4x_1 + 5x_2) = (3, 8)$$

We can get:

$$T(x_1, x_2) = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 3 \\ 4 & 5 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -4 \end{bmatrix}$$

So, the $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ -4 \end{bmatrix}$ fits $T(x) = (3, 8)$

Problem 1.9.23
Mark each statement True or False. Justify each answer.
a. A linear transformation $T : R^n \to R^m$ is completely determined by its effect on the columns of the $n \times n$ identity matrix.
b. If $T : R^2 \to R^2$ rotates vectors about the origin through an angle ϕ, then T is a linear transformation.
c. When two linear transformations are performed one after another, the combined effect may not

always be a linear transformation.
d. A mapping $T : R^n \to R^m$ is onto R^m if every vector x in R^n maps onto some vector in R^m.
e. If A is a 3×2 matrix, then the transformation $x \to Ax$ cannot be one-to-one.

Solution:
(a.): True. Knowing how T acts on the standard basis vectors (the columns of the identity matrix) allows us to determine $T(x)$ for any vector $x \in R^n$.
(b.): True. Rotations about the origin preserve vector addition and scalar multiplication, satisfying the conditions of linearity, which makes T a linear transformation.
(c.): False. The composition of two linear transformations is always a linear transformation, as the properties of linearity are preserved. Therefore, the combined effect is still linear.
(d.): False. A mapping $T : R^n \to R^m$ is onto ^{m} if every vector in R^m has a preimage in R^n, not just that every vector in R^n maps to some vector in R^m.
(e.): True. A 3×2 matrix has more rows than columns, meaning there are more equations than unknowns. This implies that the transformation cannot be injective (one-to-one), since there will be multiple vectors in R^2 mapping to the same vector in R^3.