Rutgers Math250 Intro to Linear Algebra **Homework - 6**
Name: Rui Wu RUID: 237009561
Email: rw761@scarletmail.rutgers.edu Term: Fall 2024
Instructor: Filip Dul Due Date: 25^{nd} October, 2024

Problem 3.2.5
Find the determinants by row reduction to echelon form.

$$\begin{bmatrix} 1 & 5 & -4 \\ -1 & -4 & 5 \\ -2 & -8 & 7 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 5 & -4 \\ -1 & -4 & 5 \\ -2 & -8 & 7 \end{bmatrix} \xrightarrow{R_2+R_1} \begin{bmatrix} 1 & 5 & -4 \\ 0 & 1 & 1 \\ -2 & -8 & 7 \end{bmatrix} \xrightarrow{R_3+2R_1} \begin{bmatrix} 1 & 5 & -4 \\ 0 & 1 & 1 \\ 0 & 2 & -1 \end{bmatrix} \xrightarrow{R_3-2R_2} \begin{bmatrix} 1 & 5 & -4 \\ 0 & 1 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

So $\det = 1 \times 1 \times (-3) = -3$

Problem 3.2.7
Find the determinants by row reduction to echelon form.

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & 3 & 0 & 2 \\ -2 & -5 & 7 & 4 \\ 3 & 5 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{bmatrix} \xrightarrow[R_3-3R_1]{R_2+2R_1} \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 1 & -1 & 2 & -3 \end{bmatrix} \xrightarrow{R_4-R_1} \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 0 & -4 & 2 & -5 \end{bmatrix} \xrightarrow{R_4+R_3} \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 1 & 7 & 8 \\ 0 & -4 & 2 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $\det = 0$

Problem 3.2.11
Combine the methods of row reduction and cofactor expansion to compute the determinants.

$$\begin{bmatrix} 3 & 4 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 3 \\ 6 & 8 & -4 & -1 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 3 & 4 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 3 \\ 6 & 8 & -4 & -1 \end{bmatrix} \xrightarrow[R_3+2R_1]{R_2-R_1} \begin{bmatrix} 3 & 4 & -3 & -1 \\ 0 & -4 & 4 & -2 \\ 0 & 8 & 10 & 1 \\ 6 & 8 & -4 & -1 \end{bmatrix} \xrightarrow[R_3-2R_2]{R_4-2R_1} \begin{bmatrix} 3 & 4 & -3 & -1 \\ 0 & -4 & 4 & -2 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 6 & -3 \end{bmatrix} \xrightarrow{R_4-3R_3} \begin{bmatrix} 3 & 4 & -3 & -1 \\ 0 & -4 & 4 & -2 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & -18 \end{bmatrix}$$

So: $\det = 3 \times (-4) \times 2 \times (-18) = 432$

Problem 3.2.15

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 7$$

Find the determinants:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 3g & 3h & 3i \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 7$$

And we can get $B = \begin{bmatrix} a & b & c \\ d & e & f \\ 3g & 3h & 3i \end{bmatrix}$ by multiplying R_3 by 3 in matrix A. So

$$\det(B) = 3\det(A) = 21$$

Problem 3.2.17

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 7$$

Find the determinants:

$$\begin{bmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 7$$

And we can get $B = \begin{bmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{bmatrix}$ by adding R_2 to R_1 in matrix A. So

$$\det(B) = \det(A) = 7$$

Problem 3.2.19

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 7$$

Find the determinants:

$$\begin{bmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{bmatrix}$$

Solution:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 7$$

And we can get $B = \begin{bmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{bmatrix}$ by multiplying R_2 by 2 in matrix A.

Then we get $C = \begin{bmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g & h & i \end{bmatrix}$ by adding R_1 to R_2 in matrix B. So

$$\det(C) = \det(B) = 2\det(A) = 14$$

Problem 3.2.25
Use determinants to decide if the set of vectors is linearly independent.

$$\begin{bmatrix} 7 \\ -4 \\ -6 \end{bmatrix}, \begin{bmatrix} -8 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 7 \\ 0 \\ -5 \end{bmatrix}$$

Solution: We can get the matrix from the set of vectors:

$$A = \begin{bmatrix} 7 & -8 & 7 \\ -4 & 5 & 0 \\ -6 & 7 & -5 \end{bmatrix}$$

$$\det(A) = 4 \times (40 - 49) + 5 \times (-35 + 42) = -1 \neq 0$$

So the set of vecotrs is linearly independent.

Problem 3.2.27
A and B are $n \times n$ matrices. Mark each statement True or False. Justify each answer.

a. A row replacement operation does not affect the determinant of a matrix.
b. The determinant of A is the product of the pivots in any echelon form U of A, multiplied by $(-1)^r$, where r is the number of row interchanges made during row reduction from A to U.
c. If the columns of A are linearly dependent, then $\det A = 0$.
d. $\det(A + B) = \det A + \det B$.

Solution:
(a) True. Row replacements do not change the determinant, only row interchanges do.
(b) True. The determinant is the product of the pivots, adjusted by $(-1)^r$ to account for row swaps.
(c) True. If the columns are dependent, the matrix is singular, so $\det A = 0$.
(d) False. Determinants do not distribute over addition: $\det(A + B) \neq \det A + \det B$.

Problem 3.2.33
Let A and B be square matrices. Show that even though AB and BA may not be equal, it is always true that $\det(AB) = \det(BA)$.

Solution: Let:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \ B = \begin{bmatrix} 5 & 6 \\ 0 & 7 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 5 & 20 \\ 15 & 46 \end{bmatrix}, \ BA = \begin{bmatrix} 5 & 6 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 23 & 34 \\ 21 & 28 \end{bmatrix}$$

We can see that: $AB \neq BA$, but:

$$\det(AB) = -70, \ \det(BA) = -70$$

$$\det(AB) = \det(BA)$$

Problem 4.1.1
Let V be the first quadrant in the xy-plane; that is, let

$$V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x \geq 0, y \geq 0 \right\}$$

a. If $\mathbf{u}$ and $\mathbf{v}$ are in V, is $\mathbf{u} + \mathbf{v}$ in V? Why?

b. Find a specific vector $\mathbf{u}$ in V and a specific scalar c such that $c\mathbf{u}$ is not in V.(This is enough to show that V is not a vector space).

Solution:

a. Yes, $\mathbf{u} + \mathbf{v}$ is in V. Let $\mathbf{u} = \begin{pmatrix} u_1 & u_2 \end{pmatrix}$ and $\mathbf{v} = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$ be in V, meaning $u_1 \geq 0$, $u_2 \geq 0$, $v_1 \geq 0$, and $v_2 \geq 0$. Then $\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 + v_1 & u_2 + v_2 \end{pmatrix}$, and since $u_1 + v_1 \geq 0$ and $u_2 + v_2 \geq 0$, it follows that $\mathbf{u} + \mathbf{v}$ is also in V. Therefore, if both $\mathbf{u}$ and $\mathbf{v}$ are in V, their sum is also in V.

b. Consider the specific vector $\mathbf{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which is in V because $1 \geq 0$ and $1 \geq 0$. Now, choose a scalar $c = -1$. Then $c\mathbf{u} = -1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$. Since $-1 < 0$, $c\mathbf{u}$ does not satisfy the condition that both $x \geq 0$ and $y \geq 0$. Therefore, $c\mathbf{u}$ is not in V. This shows that V is not closed under scalar multiplication, so V is not a vector space.

Problem 4.1.5
Determine if the given set is a subspace of P_n for an appropriate value of n. Justify your answers. All polynomials of the form $p(t) = at^2$, where a is in $\mathbb{R}$.

Solution:
(1) When $a = 0$, $p(t) = 0$. So the zero vector is in $p(t) = at^2$.
(2) Let $p_1(t) = a_1t^2$, $p_2(t) = a_2t^2$. $p_1(t) + p_2(t) = (a_1 + a_2)t^2$ is in $p(t) = at^2$.
(3) $cp(t) = c(at^2) = (ca)t^2$, where c is a non-zero scalar.
For above reasons, the given set is a subspace of P_n for an appropriate value of n.

Problem 4.1.7

Determine if the given set is a subspace of P_n for an appropriate value of n. Justify your answers.
All polynomials of degree at most 3, with integers as coefficients.

Solution:
Let $p(t) = a_0 + a_1t + a_2t^2 + a_3t^3$
(1) When $a_0, a_1, a_2, a_3 = 0$, $p(t) = 0$. So the zero vector is in $p(t) = a_0 + a_1t + a_2t^2 + a_3t^3$.
(2) Let $p_1(t) = k_1p(t)$, $p_2(t) = k_2p(t)$, where k is a non-zero scalar. $p_1(t) + p_2(t) = (k_1 + k_2)p(t) = (k_1 + k_2)a_0 + (k_1 + k_2)a_1t + (k_1 + k_2)a_2t^2 + (k_1 + k_2)a_3t^3$ is in $p(t) = a_0 + a_1t + a_2t^2 + a_3t^3$.
(3) $cp(t) = c(a_0 + a_1t + a_2t^2 + a_3t^3) = ca_0 + ca_1t + ca_2t^2 + ca_3t^3$ is in $p(t) = a_0 + a_1t + a_2t^2 + a_3t^3$, where c is a non-zero scalar.
For above reasons, the given set is a subspace of P_n for an appropriate value of n.