Rutgers Math250 Intro to Linear Algebra **Homework - 5**
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Problem 2.1.1
Compute each matrix sum or product if it is defined. If an expression is undefined, explain why Let

$$A = \begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix}, B = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, D = \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix}, E = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

$$-2A, B - 2A, AC, CD$$

Solution:

$$-2A = -2\begin{bmatrix} 2 & 0 & -1 \\ 4 & -5 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 2 \\ -8 & 10 & -4 \end{bmatrix}$$

$$B - 2A = B + (-2A) = \begin{bmatrix} 7 & -5 & 1 \\ 1 & -4 & -3 \end{bmatrix} + \begin{bmatrix} -4 & 0 & 2 \\ -8 & 10 & -4 \end{bmatrix} = \begin{bmatrix} 3 & -5 & 3 \\ -7 & 6 & -7 \end{bmatrix}$$

$$CD = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 13 \\ -7 & -6 \end{bmatrix}$$

And the product of AC is not defined because the number of columns from A is not equal to the number of rows from C.

Problem 2.1.9
Let $A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}$. What value(s) of k, if any, will make $AB = BA$?

Solution:

$$AB = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix} = \begin{bmatrix} 23 & 5k - 10 \\ -9 & k + 15 \end{bmatrix}$$

$$BA = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 23 & 15 \\ -3k + 6 & k + 15 \end{bmatrix}$$

$$\begin{bmatrix} 23 & 5k - 10 \\ -9 & k + 15 \end{bmatrix} = \begin{bmatrix} 23 & 15 \\ -3k + 6 & k + 15 \end{bmatrix}$$

Equation $5k - 10 = 15$ and equation $-3k + 6 = -9$ have the same solution $k = 5$.
So if $k = 5$, will make $AB = BA$.

Problem 2.1.17
If $A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$ and $AB = \begin{bmatrix} -1 & 2 & -1 \\ 6 & -9 & 3 \end{bmatrix}$, determine the first and second columns of B.

Solution:

$$AB = \begin{bmatrix} Ab_1 | Ab_2 | Ab_3 \end{bmatrix}$$

$$Ab_1 = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} b_1 = \begin{bmatrix} -1 \\ 6 \end{bmatrix},\ b_1 = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$$

$$Ab_2 = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} b_2 = \begin{bmatrix} 2 \\ -9 \end{bmatrix}, \ b_2 = \begin{bmatrix} -8 \\ -5 \end{bmatrix}$$

So the first column of B is $\begin{bmatrix} 7 \\ 4 \end{bmatrix}$, the second column of B is $\begin{bmatrix} -8 \\ -5 \end{bmatrix}$

Problem 2.2.7
Let $A = \begin{bmatrix} 1 & 2 \\ 5 & 12 \end{bmatrix}, b_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}, b_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}, b_3 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, b_4 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$
a. Find A^{-1}, and use it to solve the four equations $Ax = b_1, Ax = b_2, Ax = b_3, Ax = b_4$
b. The four equations in part(a) can be solved by the same set of row operations, since the coefficient matrix is the same in each case. Solve the four equations in part(a) by row reducing the augmented matrix $\begin{bmatrix} A & b_1 & b_2 & b_3 & b_4 \end{bmatrix}$

Solution:
(a.)

$$\det(A) = 1 \times 12 - 2 \times 5 = 2 \neq 0$$

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 12 & -2 \\ -5 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{bmatrix}$$

$$Ax = b_1 \Rightarrow AA^{-1}x = A^{-1}b_1 \Rightarrow x = A^{-1}b_1 = \begin{bmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \end{bmatrix}$$

$$Ax = b_2 \Rightarrow AA^{-1}x = A^{-1}b_2 \Rightarrow x = A^{-1}b_2 = \begin{bmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ -5 \end{bmatrix} = \begin{bmatrix} 11 \\ -5 \end{bmatrix}$$

$$Ax = b_3 \Rightarrow AA^{-1}x = A^{-1}b_3 \Rightarrow x = A^{-1}b_3 = \begin{bmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$$

$$Ax = b_4 \Rightarrow AA^{-1}x = A^{-1}b_4 \Rightarrow x = A^{-1}b_4 = \begin{bmatrix} 6 & -1 \\ -\frac{5}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 13 \\ -5 \end{bmatrix}$$

(b.)

$$\begin{bmatrix} A & b_1 & b_2 & b_3 & b_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & 1 & 2 & 3 \\ 5 & 12 & 3 & -5 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -9 & 11 & 6 & 13 \\ 0 & 1 & 4 & -5 & -2 & -5 \end{bmatrix}$$

So, $b_1 = \begin{bmatrix} -9 \\ 4 \end{bmatrix}, b_2 = \begin{bmatrix} 11 \\ -5 \end{bmatrix}, b_3 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}, b_4 = \begin{bmatrix} 13 \\ -5 \end{bmatrix}$

Problem 2.2.9
Mark each statement True or False. Justify each answer.
a. In order for a matrix B to be the inverse of A, both equations $AB = I$ and $BA = I$ must be true.
b. If A and B are $n \times n$ and invertible, then $A^{-1}B^{-1}$ is the inverse of AB.
c. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ab - cd \neq 0$, then A is invertible.
d. If A is an invertible $n \times n$ matrix, then the equation $Ax = b$ is consistent for each b in R^n.
e. Each elementary matrix is invertible.

Solution:
(a.) True. A matrix B is called the inverse of A if and only if $AB = BA = I$, where I is the identity matrix. Both conditions are necessary because matrix multiplication is not generally commutative.
(b.) False. The inverse of a product of two invertible matrices A and B is given by $(AB)^{-1} = B^{-1}A^{-1}$. Thus, the inverse of AB is $B^{-1}A^{-1}$, not $A^{-1}B^{-1}$.
(c.) False. For a 2×2 matrix A, the determinant is $\det(A) = ad - bc$.

(d.) True. Since A is invertible, there exists an inverse matrix A^{-1} such that $x = A^{-1}b$. This solution exists and is unique for every b in R^n, so the equation $Ax = b$ is consistent for each b in R^n.
(e.) True. Each elementary matrix represents an elementary row operation, and since every elementary row operation is reversible, each elementary matrix is invertible.

Problem 2.2.13
Suppose $AB = AC$, where B and C are $n \times p$ matrices and A is invertible. Show that $B = C$. Is this true, in general, when A is not invertible?

Solution:
(a.) When A is invertible, we have:

$$AB = AC \Rightarrow AA^{-1}B = AA^{-1}C, AA^{-1} = I$$

So:

$$B = C$$

(b.) In general, when A is not invertible, the statement $AB = AC$ does not necessarily imply $B = C$.

Problem 2.2.21
Explain why the columns of an $n \times n$ matrix A are linearly independent when A is invertible.

Solution:
Proof:
When A is invertible, we have

$$Ax = 0 \Rightarrow AA^{-1}x = A^{-1} * 0,\ AA^{-1} = I$$

So:

$$x = 0$$

It means that $Ax = 0$ has only trivial solution $x = 0$, so matrix A are linearly independent when A is invertible.

Problem 2.3.11
Mark each statement True or False. Justify each answer.
a. If the equation $Ax = 0$ has only the trivial solution, then A is row equivalent to the $n \times n$ identity matrix.
b. If the columns of A span R^n, then the columns are linearly independent.
c. If A is an $n \times n$ matrix, then the equation $Ax = b$ has at least one solution for each b in R^n.
d. If the equation $Ax = 0$ has a nontrivial solution, then A has fewer than n pivot positions.
e. If A^T is not invertible, then A is not invertible.

Solution:
(a.) True. If $Ax = 0$ has only the trivial solution, then A is invertible. An invertible matrix is row equivalent to the identity matrix.
(b.) True. If the columns of A span R^n, then A has a pivot in every row. For an $n \times n$ matrix, this implies the columns are linearly independent.
(c.) False. An $n \times n$ matrix A may not be invertible. If A is singular, there exists at least one b for which $Ax = b$ has no solution.
(d.) True. A nontrivial solution to $Ax = 0$ means the columns of A are linearly dependent, so A has fewer

than n pivot positions.
(e.) True. If A^T is not invertible, then $\det(A^T) = 0$. Since $\det(A^T) = \det(A)$, it follows that A is also not invertible.

Problem 3.1.5
Compute the determinants using a cofactor expansion across the first row.

$$\begin{bmatrix} 2 & 3 & -3 \\ 4 & 0 & 3 \\ 6 & 1 & 5 \end{bmatrix}$$

Solution:

$$\det = 2det(\begin{bmatrix} 0 & 3 \\ 1 & 5 \end{bmatrix}) - 3det(\begin{bmatrix} 4 & 3 \\ 6 & 5 \end{bmatrix}) - 3det(\begin{bmatrix} 4 & 0 \\ 6 & 1 \end{bmatrix})$$

$$det = 2 \times (0-3) - 3 \times (20-18) - 3 \times (4-0) = -6 - 6 - 12 = -24$$

Problem 3.1.13
Compute the determinants by cofactor expansions. At each step, choose a row or column that involves the least amount of computation.

$$\begin{bmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{bmatrix}$$

Solution: We use a cofactor expansion across the second row:

$$det = -2\det(\begin{bmatrix} 4 & 0 & 3 & -5 \\ 7 & 3 & 4 & -8 \\ 5 & 0 & 2 & -3 \\ 0 & 0 & -1 & 2 \end{bmatrix}) = -2(\det(\begin{bmatrix} 4 & 0 & -5 \\ 7 & 3 & -8 \\ 5 & 0 & -3 \end{bmatrix}) + 2\det(\begin{bmatrix} 4 & 0 & 3 \\ 7 & 3 & 4 \\ 5 & 0 & 2 \end{bmatrix}))$$

$$det(\begin{bmatrix} 4 & 0 & -5 \\ 7 & 3 & -8 \\ 5 & 0 & -3 \end{bmatrix}) = 4det(\begin{bmatrix} 3 & -8 \\ 0 & -3 \end{bmatrix}) + 5det(\begin{bmatrix} 7 & 3 \\ 5 & 0 \end{bmatrix}) = -111$$

$$\det(\begin{bmatrix} 4 & 0 & 3 \\ 7 & 3 & 4 \\ 5 & 0 & 2 \end{bmatrix}) = 4\det(\begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}) - 3\det(\begin{bmatrix} 7 & 3 \\ 5 & 0 \end{bmatrix}) = 69$$

So:

$$det = -2 \times (-111 + 2 \times 69) = -54$$

Problem 3.1.21
Explore the effect of an elementary row operation on the determinant of a matrix. State the row operation and describe how it affects the determinant.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$$

Solution: The determinant of the $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$:

$$\det(A) = ad - bc$$

And multiply Row 2 with k, we get $B = \begin{bmatrix} a & b \\ kc & kd \end{bmatrix}$, and the determinant of the B:

$$\det(B) = kad - kbc = k(ad - bc) = kdet(A)$$

So in conclusion, multiplying a row of a matrix by a scalar k results in the determinant being multiplied by k. Therefore, the determinant of the new matrix is k times the determinant of the original matrix.