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Math 250, Linear Algebra (Section 12): Midterm 1
Fall 2024
"Cheese has holes. The more cheese, the more holes. The more holes, the less cheese. Therefore, more cheese implies less cheese." – Unknown

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Read the directions **very carefully** and then solve each problem.
Write out solutions with complete steps and in complete sentences when necessary.

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1. (10 points) State whether the following statements are **True** or **False**.
No justification is necessary.

(a) Given any matrix A, the homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable. True.

(b) If a set contains fewer vectors than there are entries in the vectors, then the set is linearly independent. True.

(c) If $T : \mathbf{R}^2 \to \mathbf{R}^2$ rotates vectors about the origin through an angle ϕ, then T is a linear transformation. True

(d) A system of 3 linear equations with 4 variables can have exactly one solution. False

(e) If $T : \mathbf{R}^3 \to \mathbf{R}$ is a linear transformation, then $T(x, y, z) = ax + by + cz$ for some constants $a, b,$ and c in $\mathbf{R}$. True

(b) $\left(\begin{array}{ccc|c} * & * & * \\ * & * & * \\ * & * & * \end{array}\right)$

(d) free $\left(\begin{array}{cccc|c} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ \cancel{*} & \cancel{*} & \cancel{*} & \cancel{*} & \end{array}\right)$

(e) $\underset{T}{(* \; * \; *)} \underset{R^3}{\begin{pmatrix} * \\ * \\ * \end{pmatrix}} = \underset{R}{(*)}$

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2. Consider the following system of linear equations:

$$\begin{cases} x_1 + 3x_2 - 5x_3 = 4 \\ x_1 + 4x_2 - 8x_3 = 7 \\ -3x_1 - 7x_2 + 9x_3 = -6. \end{cases}$$

(a) (10 points) Write down the augmented matrix representing this system and reduce it to its reduced row echelon form.
(b) (10 points) The system is consistent (you do not need to prove this). Write down its solution set in parametric vector form.

(a) the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{array}\right) \xrightarrow{R_2-R_1} \left(\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ -3 & -7 & 9 & -6 \end{array}\right) \xrightarrow{R_3+3R_1} \left(\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{array}\right)$$

$$\xrightarrow{\frac{1}{2}R_3} \left(\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 1 & -3 & 3 \end{array}\right) \xrightarrow{R_3-R_2} \left(\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1-3R_2} \left(\begin{array}{ccc|c} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

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(b) from (a), we know that:

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$$\left(\begin{array}{ccc|c} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right), \text{ So } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4x_3-5 \\ 3x_3+3 \\ x_3 \end{pmatrix} = \begin{cases} -4x_3-5 \\ 3x_3+3 \\ x_3 \text{ is free} \end{cases}$$

So that $x = \begin{cases} -4x_3-5 \\ 3x_3+3 \\ x_3 \text{ is free} \end{cases}$ is the solution set

And the set in parametric vector $x = x_3 \begin{pmatrix} -4 \\ 3 \\ 1 \end{pmatrix} + \begin{pmatrix} -5 \\ 3 \\ 0 \end{pmatrix}$

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3. (a) (10 points) Find the value(s) of h for which the following vectors are linearly dependent.

$$\begin{pmatrix}1\\-1\\3\end{pmatrix}, \begin{pmatrix}-5\\7\\8\end{pmatrix}, \begin{pmatrix}1\\1\\h\end{pmatrix}.$$

(b) (5 points) Determine by inspection whether the following vectors are linearly dependent. **Justify your answer.**

$$\begin{pmatrix}1\\4\\-7\\5\end{pmatrix}, \begin{pmatrix}-2\\5\\3\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\0\end{pmatrix}.$$

(c) (5 points) Determine by inspection whether the following vectors are linearly dependent. **Justify your answer.**

$$\begin{pmatrix}4\\4\end{pmatrix}, \begin{pmatrix}-1\\3\end{pmatrix}, \begin{pmatrix}2\\5\end{pmatrix}, \begin{pmatrix}8\\1\end{pmatrix}.$$

(a) if the vectors are linear dependent, their matrix A has $\overset{\text{many solutions}}{Ax=0}$, so:

$$\left(\begin{array}{ccc|c}1&-5&1&0\\-1&7&1&0\\3&8&h&0\end{array}\right) \xrightarrow[R_3-3R_1]{R_2+R_1} \left(\begin{array}{ccc|c}1&-5&1&0\\0&2&2&0\\0&23&h-3&0\end{array}\right) \xrightarrow{\frac{1}{2}R_2} \left(\begin{array}{ccc|c}1&-5&1&0\\0&1&1&0\\0&23&h-3&0\end{array}\right)$$

$$\xrightarrow{R_3-23R_1} \left(\begin{array}{ccc|c}1&-5&1&0\\0&1&1&0\\0&0&h-26&0\end{array}\right)$$

Because $Ax=0$ has many solutions, it must be free variables here.
So $h-26=0$, $h=26$

(b) We get the augmented matrix for $Ax=0$:

$$\left(\begin{array}{ccc|c}1&-2&0&0\\4&5&0&0\\-7&3&0&0\\5&0&0&0\end{array}\right) \xrightarrow{R_2-4R_1} \left(\begin{array}{ccc|c}1&-2&0&0\\0&13&0&0\\0&-11&0&0\\0&10&0&0\end{array}\right)$$

$$\xrightarrow{\cdots} \left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&1&0&0\\0&1&0&0\end{array}\right) \xrightarrow[R_4-R_2]{R_3-R_2} \left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{array}\right), \text{ we get } x=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{cases}0\\0\\x_3 \text{ is free}\end{cases}$$

There are many solutions because of free variable x_3, so they are linear dependent.

(c) can be viewed next page.

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(Additional space for Problem 3.)

(c) We get the augmented matrix of $Ax=0$:

$$\left(\begin{array}{cccc|c} 4 & -1 & 2 & 8 & 0 \\ 4 & 3 & 5 & 1 & 0 \end{array}\right) \xrightarrow{R_2-R_1} \left(\begin{array}{cccc|c} 4 & -1 & 2 & 8 & 0 \\ 0 & 4 & 3 & -7 & 0 \end{array}\right)$$

$$\xrightarrow[\frac{1}{4}R_2]{R_1+\frac{1}{4}R_2} \left(\begin{array}{cccc|c} 4 & 0 & \frac{11}{4} & \cancel{8}\,\frac{25}{4} & 0 \\ 0 & 1 & \frac{3}{4} & -\frac{7}{4} & 0 \end{array}\right) \xrightarrow{\frac{1}{4}R_1} \left(\begin{array}{cccc|c} 1 & 0 & \frac{11}{16} & \frac{25}{16} & 0 \\ 0 & 1 & \frac{3}{4} & -\frac{7}{4} & 0 \end{array}\right).$$

We get: $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\frac{11}{16}x_3 - \frac{25}{16}x_4 \\ -\frac{3}{4}x_3 + \frac{7}{4}x_4 \\ x_3 \\ x_4 \end{pmatrix} = \begin{cases} -\frac{11}{16}x_3 - \frac{25}{16}x_4 \\ -\frac{3}{4}x_3 + \frac{7}{4}x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \end{cases}$

There are 2 free variables, so there are many solutions.
The vectors are linear dependent.