Rutgers Math250 Intro to Linear Algebra **Homework - 2**

Name: Rui Wu RUID: 237009561
Email: rw761@scarletmail.rutgers.edu Term: Fall 2024
Instructor: Filip Dul Due Date: 20^{nd} September, 2024

Problem 1.2.11
Find the general solutions of the system whose augmented matrix is given in Exercise:

$$\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix} \xrightarrow{R_2+3R_1} \begin{bmatrix} 3 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ -6 & 8 & -4 & 0 \end{bmatrix} \xrightarrow{R_3+2R_1} \begin{bmatrix} 3 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So we can get equation: $3x_1 - 4x_2 + 2x_3 = 0$
The general solution of this system is:

$$\begin{cases} x_1 = \frac{4}{3}x_2 - \frac{2}{3}x_3 \\ x_2 = \frac{3}{4}x_1 + \frac{1}{2}x_3 \\ x_3 \ is \ free \end{cases}$$

Problem 1.2.19
This exercise use the notation of Example 1 for matrices in echelon form. Suppose each matrix represents the augmented matrix for a system of linear equations. In each case, determine if the system is consistent. If the system is consistent, determine if the solution is unique.

$$a. \begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & 0 \end{bmatrix} \quad b. \begin{bmatrix} 0 & \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & \blacksquare \end{bmatrix}$$

Solution: From Example 1, we can know that the black square represents a non-zero number, and the star represents a number which could be zero.
For a.: from R_3, we can know that $x_3 = 0$, so:

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \blacksquare & * & 0 & * \\ 0 & \blacksquare & 0 & * \\ 0 & 0 & \blacksquare & 0 \end{bmatrix} \rightarrow \begin{bmatrix} \blacksquare & 0 & 0 & * \\ 0 & \blacksquare & 0 & * \\ 0 & 0 & \blacksquare & 0 \end{bmatrix}$$

The system of a. is consistent, and the solution is unique.

For b.: from provided augmented matrix of b. we can notice one thing:
the R_3 means this system has a built-in contradiction. There is no values of variables could satisfy the R_3 equation because $0 = \blacksquare$ is never true. So this system is inconsistent, it has no solution.

Problem 1.2.23
Choose h and k such that the system has (a) no solution, (b) a unique solution, and (c) many solutions. Give separate answers for each part.

$$x_1 + hx_2 = 2$$

$$4x_1 + 8x_2 = k$$

Solution: We can get the augmented matrix from the equations:

$$\begin{bmatrix} 1 & h & 2 \\ 4 & 8 & k \end{bmatrix} \xrightarrow{R_2 - 4R_1} \begin{bmatrix} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{bmatrix}$$

So for (a), we have: If there is no solution for the system, the equations $8 - 4h = 0, k - 8 \neq 0$ should be satisfied.
So I choose $h = 2$ and $k = 1$ for (a).

For (b), we have:
If equations from R_1 and R_2 are satisfied, and the equations are not the same, there is a unique solution for the system.
So I choose $h = 3$ and $k = 6$ for (b).

For (c), we have:
If equations from R_1 and R_2 are the same, there are many solutions for the system.
So I choose $h = 2$ and $k = 8$ for (c).

Problem 1.3.5
Write a system of equations that is equivalent to the given vector equation.

$$x_1 \begin{bmatrix} 6 \\ -1 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ -5 \end{bmatrix}$$

Solution: A system of equations that is equivalent to the given vector equation:

$$\begin{cases} 6x_1 - 3x_2 = 1 \\ -x_1 + 4x_2 = -7 \\ 5x_1 = -5 \end{cases}$$

Problem 1.3.7
Use the accompanying figure(on textbook) to write vectors a,b,c and d as a linear combinations of u and v. Is every vector in R^2 a linear combination of u and v?

Solution: From the accompanying, we can easily know:

$$\begin{cases} a = u - 2v \\ b = 2u - 2v \\ d = 3u - 4v \end{cases}$$

And c is not a linear combination of u and v.

Problem 1.3.11

Determine if b is a linear combination of a_1,a_2 and a_3.

$$a_1 = \begin{bmatrix} 1 \\ -2 \\ -0 \end{bmatrix}, a_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, a_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}, b = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix},$$

Solution: We can get the augmented matrix:

$$\begin{bmatrix} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{bmatrix} \xrightarrow{R_2+2R_1} \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 2 & 8 & 6 \end{bmatrix} \xrightarrow{R_3-2R_2} \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From the echelon form of matrix, we can know that the linear system has many solutions.
So b is a linear combination of a_1,a_2 and a_3.

Problem 1.3.13
Determine if b is a linear combination of the vectors formed from the columns the matrix A.

$$A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix}, b = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$$

Solution: We can get the augmented matrix:

$$\begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ -2 & 8 & -4 & -3 \end{bmatrix} \xrightarrow{R_3+2R_1} \begin{bmatrix} 1 & -4 & 2 & 3 \\ 0 & 3 & 5 & -7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

From matrix, we can notice one thing:
The R_3 means this system has a built-in contradiction. There is no values of variables could satisfy the R_3 equation because $0 = 3$ is never true. So this system is inconsistent, it has no solution.
It means that b is not a linear combination of the vectors formed from the columns the matrix A.

Problem 1.3.17
Let $a_1 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$, $a_2 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$, and $b = \begin{bmatrix} 4 \\ 1 \\ h \end{bmatrix}$. For what value(s) of h is b in the plane spanned by a_1 and a_2?

Solution: We can get the augmented matrix:

$$\begin{bmatrix} 1 & -2 & 4 \\ 4 & -3 & 1 \\ -2 & 7 & h \end{bmatrix} \xrightarrow{\frac{1}{5}(R_2-4R_1)} \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ -2 & 7 & h \end{bmatrix} \xrightarrow{R_1+2R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ -2 & 7 & h \end{bmatrix} \xrightarrow{R_3+2R_1-7R_2} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & h+17 \end{bmatrix}$$

b is spanned by a_1 and a_2, it means that the linear system has solution, so we have: $h + 17 = 0$, $h = -17$.

Problem 1.4.1 and 1.4.3
Compute the products using (a) the definition, as in Example 1, and (b) the row-vector rule for computing Ax. If a product is undefined, explain why.

$$1. \begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 7 \end{bmatrix} \quad 3. \begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

Solution:
For 1.: The A is in $R^{3\times 2}$, but x is in R^3. So the product is not defined.

For 3.: The A is in $R^{3\times 2}$, and x is in R^2. They can be produced:

$$\begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = 2 \begin{bmatrix} 6 \\ -4 \\ 7 \end{bmatrix} - 3 \begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ -4 \end{bmatrix}$$

Problem 1.4.5 and 1.4.7
Use the definition of Ax to write the matrix equation as a vector equation, or vice versa.

$$5. \begin{bmatrix} 5 & 1 & -8 & 4 \\ -2 & -7 & 3 & -5 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \end{bmatrix} \quad 7. x_1 \begin{bmatrix} 4 \\ -1 \\ 7 \\ -4 \end{bmatrix} + x_2 \begin{bmatrix} -5 \\ 3 \\ -5 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ -8 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$$

Solution:
For 5.:

$$5 \begin{bmatrix} 5 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ -7 \end{bmatrix} + 3 \begin{bmatrix} -8 \\ 3 \end{bmatrix} - 2 \begin{bmatrix} 4 \\ -5 \end{bmatrix} = \begin{bmatrix} -8 \\ 16 \end{bmatrix}$$

For 7.:

$$\begin{bmatrix} 4 & -5 & 7 \\ -1 & 3 & -8 \\ 7 & -5 & 0 \\ -4 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 0 \\ -7 \end{bmatrix}$$