Rutgers Math250 Intro to Linear Algebra **Homework - 3**
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Problem 1.4.11
Given A and b, write the augmented matrix for the linear system that corresponds to the matrix equation $AX = b$. Then solve the system and write the solution as vector.

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ -2 & -4 & -3 \end{bmatrix}, b = \begin{bmatrix} -2 \\ 2 \\ 9 \end{bmatrix}$$

Solution: From matrix A and vector b, we can get the augmented matrix for linear system that corresponds to the matrix equation $AX = b$:

$$\begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ -2 & -4 & -3 & 9 \end{bmatrix}$$

Now, solve it!

$$\begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ -2 & -4 & -3 & 9 \end{bmatrix} \xrightarrow{\frac{1}{5}(R_3+2R_1)} \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 5 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_2-5R_3} \begin{bmatrix} 1 & 2 & 4 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_1-2R_2-4R_3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

So the solution as a vector is: $x = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$

Problem 1.4.13
Let $u = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{bmatrix}$. Is u in the plane R^3 spanned by the columns of A?(See the figure.) Why or why not?

Solution: We get the augmented matrix from $Ax = b$:

$$\begin{bmatrix} 3 & -5 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{bmatrix}$$

Let's process it:

$$\begin{bmatrix} 3 & -5 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 3 & -5 & 0 \\ -2 & 6 & 4 \end{bmatrix} \xrightarrow[-\frac{1}{4}(R_2-3R_1)]{\frac{1}{4}(R_3+2R_1)} \begin{bmatrix} 1 & 1 & 4 \\ 0 & -2 & -3 \\ 0 & 2 & 3 \end{bmatrix} \xrightarrow[R_3+R_2]{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1-R_2} \begin{bmatrix} 1 & 0 & \frac{5}{2} \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

We can make the matrix into reduced echelon form, so there is a solution for this linear system. It means that the u in the plane R^3 spanned by the column of A.

Problem 1.4.21

Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$. Does $\{v_1, v_2, v_3\}$ span R^4? Why or why not?

Solution: We set $b = \{b_1, b_2, b_3, b_4\}$ in R^4, and we get the augmented matrix of $Ax = b$:

$$\begin{bmatrix} 1 & 0 & 1 & b_1 \\ 0 & -1 & 0 & b_2 \\ -1 & 0 & 0 & b_3 \\ 0 & 1 & -1 & b_4 \end{bmatrix} \xrightarrow[-R_2]{-R_3} \begin{bmatrix} 1 & 0 & 1 & b_1 \\ 0 & 1 & 0 & -b_2 \\ 1 & 0 & 0 & -b_3 \\ 0 & 1 & -1 & b_4 \end{bmatrix} \xrightarrow[R_1-R_3]{R_4-R_2} \begin{bmatrix} 0 & 0 & 1 & b_1+b_3 \\ 0 & 1 & 0 & -b_2 \\ 1 & 0 & 0 & -b_3 \\ 0 & 0 & -1 & b_4+b_2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -b_3 \\ 0 & 1 & 0 & -b_2 \\ 0 & 0 & 1 & b_1+b_3 \\ 0 & 0 & 0 & b_1+b_2+b_3+b_4 \end{bmatrix}$$

If and only if $b_1 + b_2 + b_3 + b_4 = 0$, $\{v_1, v_2, v_3\}$ span R^4.

Problem 1.5.5

Follow the method of Examples 1 and 2 to write the solution set of the given homogeneous system in parametric vector form.

$$x_1 + 3x_2 + x_3 = 0$$

$$-4x_1 - 9x_2 + 2x_3 = 0$$

$$-3x_2 - 6x_3 = 0$$

Solution: Let A be the matrix of coefficients of the system and row reduce the augmented matrix $[A, 0]$ to echelon form:

$$\begin{bmatrix} 1 & 3 & 1 & 0 \\ -4 & -9 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix} \xrightarrow[\frac{1}{3}(R_2+4R_1)]{-\frac{1}{3}R_3} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{R_3-R_2} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1-3R_2} \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since x_3 is a free variable, $Ax = 0$ has nontrivial solutions, and:

$$x_1 - 5x_3 = 0$$

$$x_2 + 2x_3 = 0$$

So with free x_3, $x_1 = 5x_3$, $x_2 = -2x_3$. As a vector, the general solution of $Ax = 0$ has the form:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} = x_3 v, \; where \; v = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

Problem 1.5.15

Follow the method of Example 3 to describe the solutions of the following system in parametric vector form. Also, give a geometric description of the solution set and compare it to that in Exercise 5.

$$x_1 + 3x_2 + x_3 = 1$$

$$-4x_1 - 9x_2 + 2x_3 = -1$$

$$-3x_2 - 6x_3 = -3$$

Solution: Let A be the matrix of coefficients of the system and row reduce the augmented matrix $[A, b]$ to echelon form:

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{bmatrix} \xrightarrow[\frac{1}{3}(R_2+4R_1)]{-\frac{1}{3}R_3} \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix} \xrightarrow[R_3-R_2]{R_1-3R_2} \begin{bmatrix} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since x_3 is free variable, and:

$$x_1 - 5x_3 = -2$$

$$x_2 + 2x_3 = 1$$

So with free x_3, $x_1 = 5x_3 - 2$, $x_2 = -2x_3 + 1$. As a vector, the general solution of $Ax = b$ has the form:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 - 2 \\ -2x_3 + 1 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} = x_3 v + p, \ where \ v = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \ p = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Problem 1.5.23
Mark each statement True or False. Justify each answer.
a. A homogeneous equation is always consistent.
b. The equation $Ax = 0$ gives an explicit description of its solution set.
c. The homogeneous equation $Ax = 0$ has the trivial solution if and only if the equation has at least one free variable.
d. The equation $x = p + tv$ describes a line through v parallel to p.
e. The solution set of $Ax = b$ is the set of all vectors of the form $w = p + v_h$, where v_h is any solution of the equation $Ax = 0$.

Solution: For (a.): True. A homogeneous equation means that it can be written in the form $Ax = 0$, where A is an matrix and 0 is the zero vector in R^m. And such a system $Ax = 0$ always has at least one solution, $x = 0$. So it is always consistent.

For (b.): False. The equation $Ax = 0$ describes the solution set implicitly. It tells us that any vector x that satisfies this equation is part of the solution set, but it does not directly give an explicit form of the solutions.

For (c.): False. The homogeneous equation $Ax = 0$ always has the trivial solution $x = 0$, regardless of the number of free variables. The presence of free variables only affects the existence of non-trivial solutions, not the existence of the trivial solution.

For (d.): False. The equation $x = p + tv$ actually describes a line through the point p in the direction of the vector v. It means that as t varies, x traces a line that passes through p and is parallel to the direction v, not parallel to p.

For (e,): True. The solution set of a non-homogeneous equation $Ax = b$ can be written as $w = p+v_h$, where p is a particular solution of $Ax = b$ and v_h is any solution of the corresponding homogeneous equation $Ax = 0$. The set of solutions to $Ax = b$ is essentially a translation of the solution space of $Ax = 0$ by the particular solution p.

Problem 1.7.1
Determine if the vectors are linearly independent. Justify each answer.

$$\begin{bmatrix}5\\0\\0\end{bmatrix}, \begin{bmatrix}7\\2\\-6\end{bmatrix}, \begin{bmatrix}9\\4\\-8\end{bmatrix}$$

Solution: Assume that:

$$\begin{bmatrix}5\\0\\0\end{bmatrix} x_1 + \begin{bmatrix}7\\2\\-6\end{bmatrix} x_2 + \begin{bmatrix}9\\4\\-8\end{bmatrix} x_3 = 0$$

We can get the augmented matrix:

$$\begin{bmatrix}5 & 7 & 9 & 0\\0 & 2 & 4 & 0\\0 & -6 & -8 & 0\end{bmatrix} \xrightarrow{-\frac{1}{2}(2R_2+R_3)} \begin{bmatrix}5 & 7 & 9 & 0\\0 & 1 & 0 & 0\\0 & -6 & -8 & 0\end{bmatrix} \xrightarrow{-\frac{1}{8}(R_3+6R_2)} \begin{bmatrix}5 & 7 & 9 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix} \xrightarrow{\frac{1}{5}(R_1-7R_2-9R_3)} \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix}$$

The only solution for this linear system is $x_1 = x_2 = x_3 = 0$. So these vectors are linearly independent.

Problem 1.7.7
Determine if the columns of the matrix from a linearly independent set. Justify each answer.

$$\begin{bmatrix}0 & -8 & 5\\3 & -7 & 4\\-1 & 5 & -4\\1 & -3 & 2\end{bmatrix}$$

Solution: Assume that:

$$\begin{bmatrix}0 & -8 & 5\\3 & -7 & 4\\-1 & 5 & -4\\1 & -3 & 2\end{bmatrix} \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = 0$$

We can get the augmented matrix:

$$\begin{bmatrix}0 & -8 & 5 & 0\\3 & -7 & 4 & 0\\-1 & 5 & -4 & 0\\1 & -3 & 2 & 0\end{bmatrix} \rightarrow \begin{bmatrix}1 & -3 & 2 & 0\\0 & 2 & -2 & 0\\0 & 2 & -2 & 0\\0 & -8 & 5 & 0\end{bmatrix} \rightarrow \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

There is a solution of the linear system. So the columns of the matrix from a linearly independent set.

Problem 1.7.9
(a) for what values of h in v_3 in Span $\{v_1, v_2\}$, and (b) for what values of h is $\{v_1, v_2, v_3\}$ linearly dependent? Justify each answer.

$$v_1 = \begin{bmatrix}1\\-3\\2\end{bmatrix}, v_2 = \begin{bmatrix}-3\\9\\-6\end{bmatrix}, v_3 = \begin{bmatrix}5\\-7\\h\end{bmatrix}$$

Solution: For (a), we have augmented matrix:

$$\begin{bmatrix}1 & -3 & 5\\-3 & 9 & -7\\2 & -6 & h\end{bmatrix} \xrightarrow[R_2+3R_1]{R_3-2R_1} \begin{bmatrix}1 & -3 & 5\\0 & 0 & 8\\0 & 0 & h-10\end{bmatrix}$$

The R2 means this system has a built-in contradiction. There is no values of variables could satisfy the R2 equation because $0 = 8$ is never true. So this system is inconsistent, it has no solution. There is no values of h in v_3 in Span $\{v_1, v_2\}$.

For (b), we have augmented matrix:

$$\begin{bmatrix} 1 & -3 & 5 & 0 \\ -3 & 9 & -7 & 0 \\ 2 & -6 & h & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 5 & 0 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & h-10 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 0 & 0 \\ 0 & 0 & h-10 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Get equation from this linear system:

$$x_1 - 3x_2 = 0$$

We can see that if the system is linearly dependent has nothing to do with h, it means that for all real values of h, the system is linear dependent.