# Intro: Greatest Common Divisors I

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## Data Structures and Algorithms
## Algorithmic Toolbox

# Learning Objectives

- Define greatest common divisors.
- Compute greatest common divisors inefficiently.

# GCDs

- Put fraction $\frac{a}{b}$ in simplest form.
- Divide numerator and denominator by $d$, to get $\frac{a/d}{b/d}$.

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  - Want $d$ to be as large as possible.

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- Divide numerator and denominator by $d$, to get $\frac{a/d}{b/d}$.
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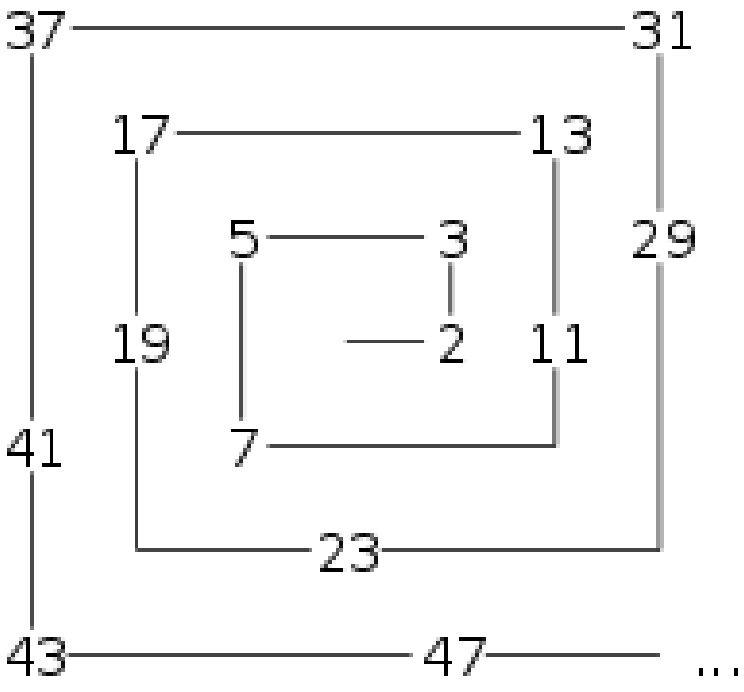
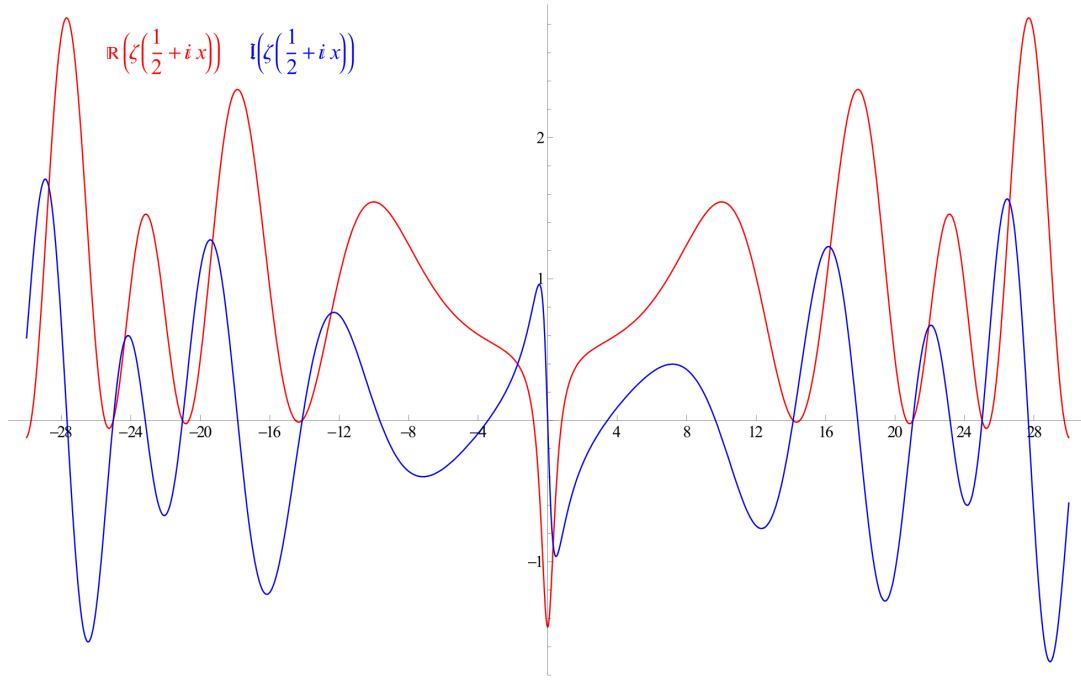
  - Want $d$ to be as large as possible.

## Definition

For integers, $a$ and $b$, their greatest common divisor or $\gcd(a, b)$ is the largest integer $d$ so that $d$ divides both $a$ and $b$.

# Number Theory



# Cryptography

# Computation

## Compute GCD

Input: Integers $a, b \geq 0$.

Output: $\gcd(a, b)$.

# Computation

## Compute GCD

Input: Integers $a, b \geq 0$.

Output: $\gcd(a, b)$.

Run on large numbers like

$$\gcd(3918848, 1653264).$$

# Naive Algorithm

**Function NaiveGCD($a, b$)**

$best \leftarrow 0$
for $d$ from 1 to $a + b$:
    if $d|a$ and $d|b$:
        $best \leftarrow d$
return $best$

# Naive Algorithm

**Function NaiveGCD($a, b$)**

```
best ← 0
for d from 1 to a + b:
    if d|a and d|b:
        best ← d
return best
```

- Runtime approximately $a + b$.
- Very slow for 20 digit numbers.

# Intro: Greatest Common Divisors II

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## Data Structures and Algorithms
## Algorithmic Toolbox

# Learning Objectives

- Implement the Euclidean Algorithm.
- Approximate the runtime.

# GCDs

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# GCDs

## Definition

For integers, $a$ and $b$, their greatest common divisor or $\gcd(a, b)$ is the largest integer $d$ so that $d$ divides both $a$ and $b$.

## Compute GCD

Input: Integers $a, b \geq 0$.

Output: $\gcd(a, b)$.

# Key Lemma

## Lemma

Let $a'$ be the remainder when $a$ is divided by $b$, then

$$\gcd(a, b) = \gcd(a', b) = \gcd(b, a').$$

# Proof

## Proof (sketch)

- $a = a' + bq$ for some $q$
- $d$ divides $a$ and $b$ if and only if it divides $a'$ and $b$

# Euclidean Algorithm

**Function EuclidGCD($a, b$)**

```
if b = 0:
    return a
a' ← the remainder when a is
     divided by b
return EuclidGCD(b, a')
```

# Euclidean Algorithm

**Function EuclidGCD($a, b$)**

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if b = 0:
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a' ← the remainder when a is
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Produces correct result by Lemma.

# Example

$$\gcd(3918848, 1653264)$$

# Example

$$\begin{aligned} & \gcd(3918848, 1653264) \\ = & \gcd(1653264, 612320) \end{aligned}$$

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$$\begin{aligned} & \gcd(3918848, 1653264) \\ = & \gcd(1653264, 612320) \\ = & \gcd(612320, 428624) \end{aligned}$$

# Example

$$
\begin{aligned}
&\gcd(3918848, 1653264) \\
=\ &\gcd(1653264, 612320) \\
=\ &\gcd(612320, 428624) \\
=\ &\gcd(428624, 183696)
\end{aligned}
$$

# Example

$$
\begin{aligned}
&\gcd(3918848, 1653264) \\
=\ &\gcd(1653264, 612320) \\
=\ &\gcd(612320, 428624) \\
=\ &\gcd(428624, 183696) \\
=\ &\gcd(183696, 61232)
\end{aligned}
$$

# Example

$$
\begin{aligned}
&\gcd(3918848, 1653264) \\
=\ &\gcd(1653264, 612320) \\
=\ &\gcd(612320, 428624) \\
=\ &\gcd(428624, 183696) \\
=\ &\gcd(183696, 61232) \\
=\ &\gcd(61232, 0)
\end{aligned}
$$

# Example

$$
\begin{aligned}
&\gcd(3918848, 1653264)\\
=&\gcd(1653264, 612320)\\
=&\gcd(612320, 428624)\\
=&\gcd(428624, 183696)\\
=&\gcd(183696, 61232)\\
=&\gcd(61232, 0)\\
=&61232.
\end{aligned}
$$

# Runtime

- Each step reduces the size of numbers by about a factor of 2.
- Takes about $\log(ab)$ steps.

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- Each step reduces the size of numbers by about a factor of 2.
- Takes about $\log(ab)$ steps.
- GCDs of 100 digit numbers takes about 600 steps.
- Each step a single division.

# Summary

- Naive algorithm is too slow.
- The correct algorithm is much better.
- Finding the correct algorithm requires knowing something interesting about the problem.