Intro: Asymptotic Notation

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Data Structures and Algorithms
Algorithmic Toolbox

Learning Objectives

- Understand the basic idea behind asymptotic runtimes.
- Describe some of the advantages to using asymptotic runtimes.

Last Time

Computing Runtimes Hard

- Depends on fine details of program.
- Depends on details of computer.

Idea

All of these issues can multiply runtimes by (large) constant.

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All of these issues can multiply runtimes by (large) constant. So measure runtime in a way that ignores constant multiples.

Problem

Unfortunately, 1 second, 1 hour, 1 year only differ by constant multiples.

Solution

Consider asymptotic runtimes. How does runtime scale with input size.

Approximate Runtimes

	n	$n \log n$	n^2	2^n
$n = 20$	1 sec	1 sec	1 sec	1 sec
$n = 50$	1 sec	1 sec	1 sec	13 day
$n = 10^2$	1 sec	1 sec	1 sec	$4 \cdot 10^{13}$ year
$n = 10^6$	1 sec	1 sec	17 min	
$n = 10^9$	1 sec	30 sec	30 year	
max n	10^9	$10^{7.5}$	$10^{4.5}$	30

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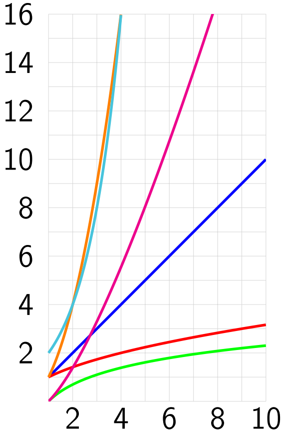

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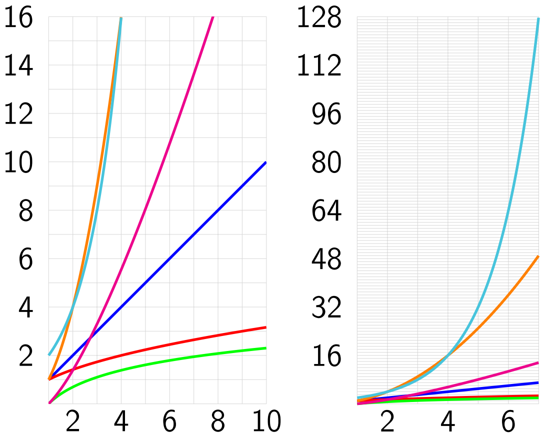

Intro: Big-O Notation

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Learning Objectives

- Understand the meaning of Big-O notation.
- Describe some of the advantages and disadvantages of using Big-O notation.

Big-O Notation

Definition

$f(n) = O(g(n))$ (f is Big-O of g) or $f \preceq g$ if there exist constants N and c so that for all $n \geq N$, $f(n) \leq c \cdot g(n)$.

Big-O Notation

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f is bounded above by some constant multiple of g.

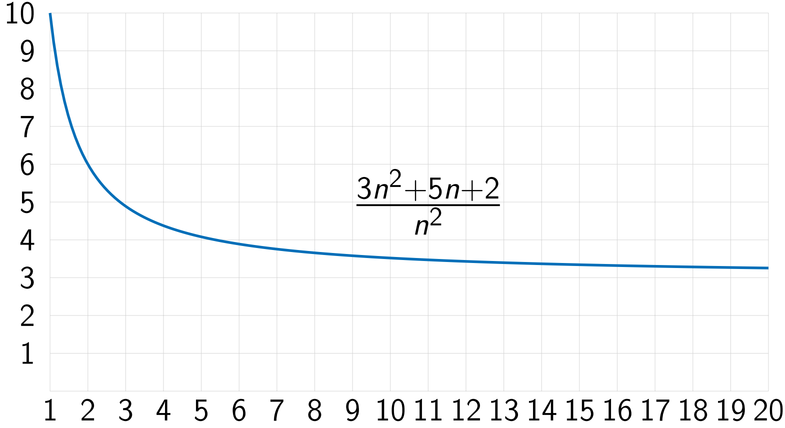
Big-O Notation

Example

$3n^2 + 5n + 2 = O(n^2)$ since if $n \geq 1$,
$3n^2 + 5n + 2 \leq 3n^2 + 5n^2 + 2n^2 = 10n^2$.

Growth Rate

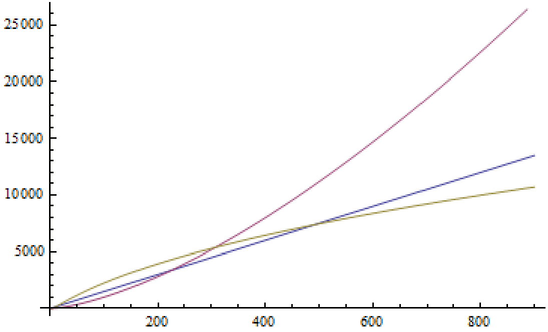
$3n^2 + 5n + 2$ has the same growth rate as n^2



Using Big-O

We will use Big-O notation to report algorithm runtimes. This has several advantages.

Clarifies Growth Rate



Cleans up Notation

- $O(n^2)$ vs. $3n^2 + 5n + 2$.
- $O(n)$ vs. $n + \log_2(n) + \sin(n)$.

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- $O(n)$ vs. $n + \log_2(n) + \sin(n)$.
- $O(n\log(n))$ vs. $4n\log_2(n) + 7$.
    - Note: $\log_2(n), \log_3(n), \log_x(n)$ differ by constant multiples, don't need to specify which.

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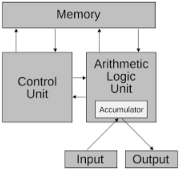
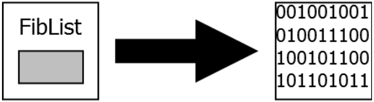
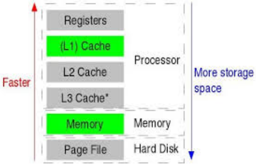
- $O(n \log(n))$ vs. $4n \log_2(n) + 7$.
  - Note: $\log_2(n), \log_3(n), \log_x(n)$ differ by constant multiples, don't need to specify which.
- Makes algebra easier.

Can Ignore Complicated Details

No longer need to worry about:







Warning

- Using Big-O loses important information about constant multiples.
- Big-O is *only* asymptotic.

Intro: Using Big-O

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Learning Objectives

- Manipulate expressions involving Big-O and other asymptotic notation.
- Compute algorithm runtimes in terms of Big-O.

Big-O Notation

Definition

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Common Rules

Multiplicative constants can be omitted:

$$7n^3 = O(n^3),\ \frac{n^2}{3} = O(n^2)$$

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Smaller terms can be omitted :

$$n^2 + n = O(n^2),\ 2^n + n^9 = O(2^n)$$

Recall Algorithm

```
Function FibList(n)
create an array F[0...n]
F[0] ← 0
F[1] ← 1
for i from 2 to n:
    F[i] ← F[i − 1] + F[i − 2]
return F[n]
```

Big-O in Practice

Operation	Runtime

Big-O in Practice

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create an array $F[0 \ldots n]$	$O(n)$

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$F[i] \leftarrow F[i-1] + F[i-2]$	$O(n)$
return $F[n]$	$O(1)$

Total:

$$O(n)+O(1)+O(1)+O(n)\cdot O(n)+O(1) = O(n^2).$$

Other Notation

Definition

For functions $f, g : \mathbb{N} \to \mathbb{R}^+$ we say that:

- $f(n) = \Omega(g(n))$ or $f \succeq g$ if for some c, $f(n) \geq c \cdot g(n)$ (f grows no slower than g).
- $f(n) = \Theta(g(n))$ or $f \asymp g$ if $f = O(g)$ and $f = \Omega(g)$ (f grows at the same rate as g).

Other Notation

Definition

For functions $f, g : \mathbb{N} \to \mathbb{R}^+$ we say that:

- $f(n) = o(g(n))$ or $f \prec g$ if $f(n)/g(n) \to 0$ as $n \to \infty$ (f grows slower than g).

Asymptotic Notation

- Lets us ignore messy details in analysis.
- Produces clean answers.
- Throws away a lot of practically useful information.