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Q2.1 Part 1

(i) $a = \Delta_h f(x) = [f(x+h) - f(x)]/h = ? + O(?)$

$$= \frac{\left(f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f''''(x) \ldots\right) - f(x)}{h}$$

$$= \frac{\left(hf'(x) + \frac{h^2}{2!}f''(x) \ldots\ldots \frac{h^n}{n!}f^{n}(x) + O(h^{n+1})\right)}{h}$$

$$= \frac{h\left[f'(x) + \frac{h}{2!}f''(x) \ldots\ldots \frac{h^{n-1}}{n!}f^{n}(x) + \frac{O(h^{n+1})}{h}\right]}{h}$$

$$= f'(x) + \frac{h}{2!}f''(x) + \ldots \frac{h^{n-1}}{n!}f^{n}(x) + O(h^{n+1}) \quad \text{— (1)}$$

(h was too small so we are ignoring it)

(ii) $b = \bar{\Delta}_h f(x) = \left[\frac{f(x) - f(x-h)}{h}\right]$

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$$= \frac{f(x) - \left[f(x) - hf'(x) + \frac{h^2}{2!} f''(x) - \frac{h^3}{3!} f'''(x) \ldots \frac{h^n}{n!} f^n(x) + O(h^{n+1}) \right.}{h}$$

$$= \frac{f(x) - f(x) + hf'(x) - \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) \ldots O(h^{n+1})}{h}$$

$$= \frac{\cancel{h}\left[f'(x) - \frac{h}{2!} f''(x) + \frac{h^2}{3!} f'''(x) \ldots \frac{h^{n-1}}{n!} f^n(x) \ldots \frac{O(h^{n+1})}{h} \right]}{\cancel{h}}$$

avoid this

$$= \begin{cases} f'(x) - \frac{h}{2!} f''(x) + \frac{h^2}{3!} f'''(x) - \ldots\ldots - \frac{h^{n-1}}{n!} f^n(x) + O(h^{n+1}), & n \text{ is even} \\ f'(x) - \frac{h}{2!} f''(x) + \frac{h^2}{3!} f'''(x) - \ldots \ldots + \frac{h^{n-1}}{n!} f^n(x) - O(h^{n+1}), & n \text{ is odd} \end{cases}$$

— (2)

(viii) $\frac{a+b}{2} = \frac{1}{2}\left(\frac{f(x+h) - f(x-h)}{h} \right)$

$$\frac{1}{2h}\left[\cancel{f(x)} + hf'(x) + \cancel{\frac{h^2}{2!}f''(x)} \ldots \cancel{O(h^{n+1})} - \cancel{f(x)} - hf'(x) + \cancel{\frac{h^2}{2!}f''(x)} \ldots \cancel{O(h^{n+1})}\right]$$

$$= \frac{1}{2h}\left[hf'(x) + \cancel{\frac{h^2}{2!}f''(x)} \ldots + hf'(x) - \cancel{h^2 f''(x)} + \ldots\right]$$

$$= \frac{1}{2h}\left[2hf'(x) + \frac{2h^3}{3!}f'''(x) + \frac{2h^5}{5!}f'''''(x) + \ldots\right]$$

$$= \frac{\cancel{2h}}{\cancel{2h}}\left[hf'(x) + \frac{h^3}{3!}f'''(x) + \frac{h^5}{5!}f'''''(x) + \ldots\right]$$

$$= hf'(x) + \frac{h^3}{3!}f'''(x) + \frac{h^5}{5!}f'''''(x) + \ldots + 0 \quad - (3)$$

(iv) $A = \Delta_{2h} f(x) = \frac{1}{2h}\left[f(x+2h) - f(x)\right]$

$$= \frac{1}{2h}\left[\cancel{f(x)} + 2hf'(x) + \frac{4h^2}{2!}f''(x) + \ldots + \frac{2^n h^n}{n!}f^n(x) - \cancel{f(x)}\right]$$

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$$= \frac{\cancel{2h}}{\cancel{2h}}\left[f'(x) + \frac{2h}{2!}f''(x) + \cdots + \frac{2^{n-1}h^{n-1}}{n!}f^{n}(x) + O((2h)^{n+1})\right]$$

$$= f'(x) + \frac{2h}{2!}f''(x) + \cdots + \frac{2^{n-1}h^{n-1}}{n!}f^{n}(x) + O((2h)^{n+1}) \quad \text{—} (4)$$

(ii) $B = \frac{1}{2h}\left[f(x) - f(x-2h)\right]$

$$B = \frac{1}{2h}\left[\cancel{f(x)} - \left(\cancel{f(x)} - 2hf'(x) + \frac{4h^2}{2!}f''(x) \cdots + O((-2h)^{n+1})\right)\right]$$

$$= \frac{1}{2h}\left[2hf'(x) - \frac{4h^2}{2!}f''(x) + \cdots - O((-2h)^{n+1})\right]$$

$$= \frac{\cancel{2h}}{\cancel{2h}}\left[f'(x) - \frac{2h}{2!}f''(x) + \cdots - O((-2h)^{n+1})\right]$$

$$= f'(x) - \frac{2h}{2!}f''(x) + \cdots - O((-2h)^{n+1})$$

— (5)

Q2.2 Part-2

T.P. $\frac{2a}{3} + \frac{2b}{3} - \left(\frac{A}{6} + \frac{B}{6}\right) = \frac{df(x)}{dx} + O(h^3)$

using equation ①, ②, ④, ⑤

=

$= \frac{2}{3}\left(f'(x) + \cancel{\frac{h f''(x)}{2!}} + \frac{h^2 f'''(x)}{3!} + \cancel{\frac{h^3 f''''(x)}{4!} + O(h^5)}\right) + \frac{2}{3}\left(f'(x) - \cancel{\frac{h}{2!} f''(x)} + \frac{h^2}{3!} f'''(x) - \cancel{\frac{h^3}{4!} f(x) - O(h^5)}\right)$

$-\frac{1}{6}\left(\left(f'(x) + \cancel{\frac{2h}{2!} f''(x)} + \frac{2^2 h^2}{3!} f'''(x) + \cancel{\frac{2^3 h^3}{4!} f^4(x) + O(2^5 h^5)}\right) + \left(f'(x) - \cancel{\frac{2h}{2!} f''(x)} + \frac{2^2 h^2}{3!} f'''(x) - \cancel{\frac{2^3 h^3}{4!} f^4(x) - O(2h)^{4}}\right)\right)$

$= \frac{2}{3}\left(2f'(x) + \frac{2h^2}{3!} f'''(x)\right) - \frac{1}{6}\left(2f'(x) + \frac{2^3 h^2}{3!} f(x)\right)$

- multiplication by 2
divide

$= \frac{4}{6}\left(2f'(x) + \frac{2h^2}{3!} f'''(x)\right) - \frac{1}{6}\left(2f'(x) + \frac{2^3 h^2}{3!} f(x)\right)$

$= \frac{1}{6}\left(8f'(x) + \frac{8h^2}{3!} f'''(x) - 2f'(x) - \frac{2^3 h^2}{3!} f(x)\right)$

$= \frac{1}{6}\left(6f'(x) + \frac{\cancel{8} h^2 f''''(x)}{3!}\right) = \boxed{f'(x) + \frac{h^2 f'''(x)}{3!}}$

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LHS

$$X = \frac{2a}{3} + \frac{2b}{3} - \frac{1}{6}\left(A + B\right) = \frac{d f(x)}{dx} + O(h^5)$$

$$\cancel{X} = LHS = \frac{2}{3}\left(\frac{f(x+h) - f(x)}{h} + \frac{f(x) - f(x-h)}{h}\right)$$