Ejercicios del Capítulo 3

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```
#librerias
library(ggplot2)
```

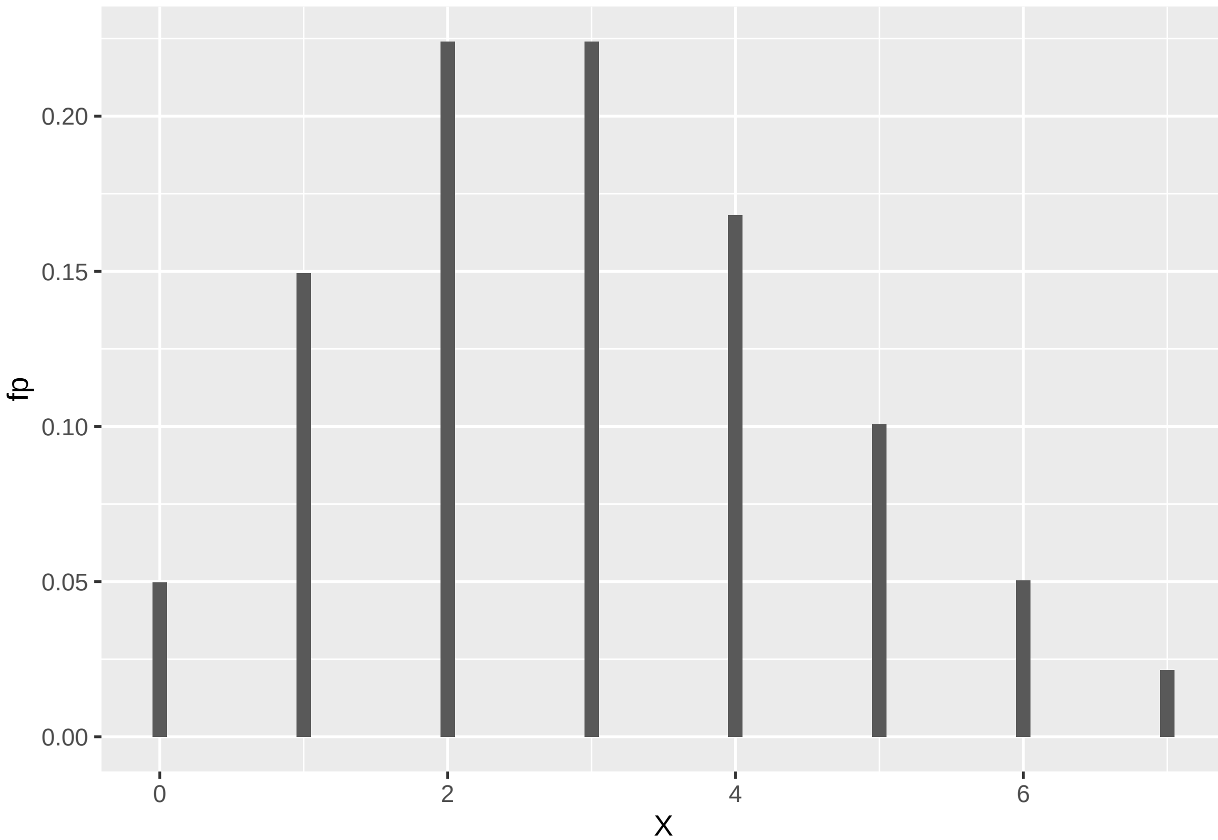
3.1

a)

```
X <- c(0,1,2,3,4,5,6,7)
fp <- c()
for (x in 0:7) {
  fp <- c(fp, exp(-3)*3^x / factorial(x))
}
df <- data.frame(X,fp)
```

b)

```
ggplot(data = df, mapping = aes(X,fp)) +
  geom_col(width = 0.1)
```



c)

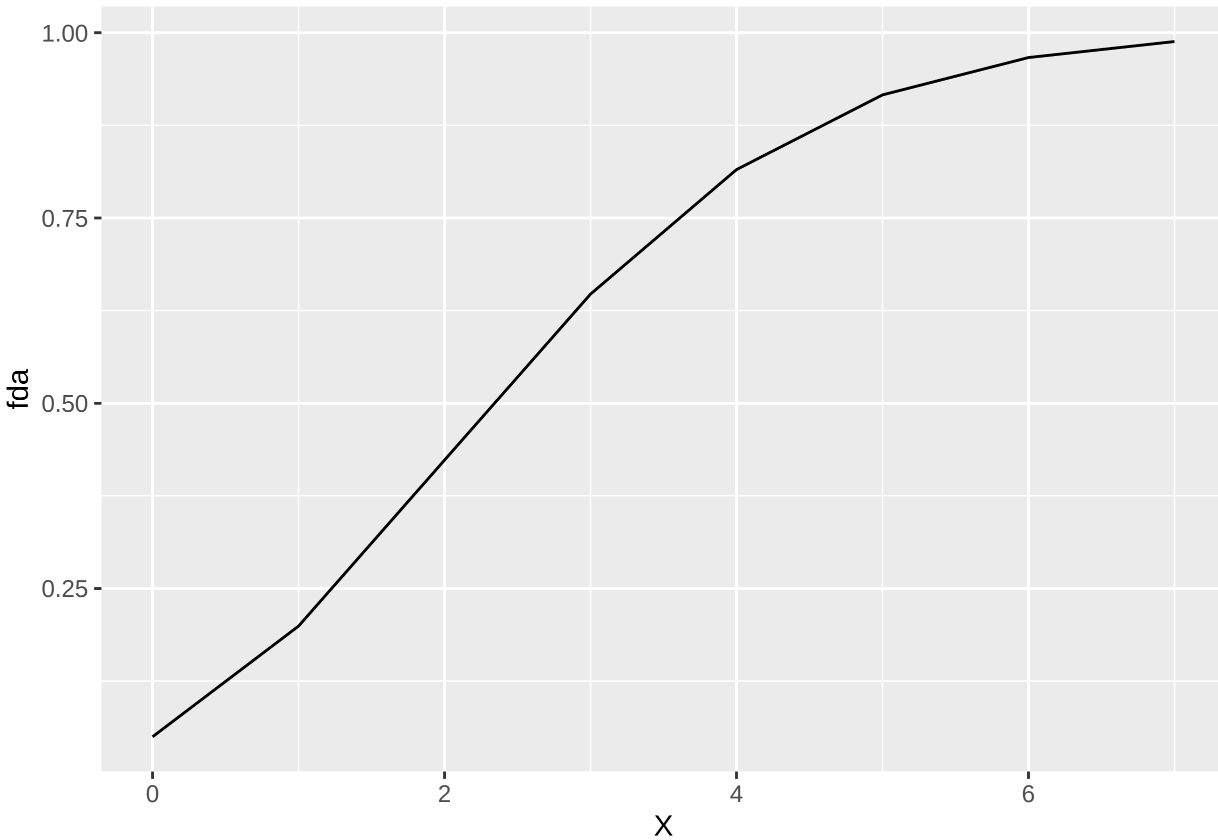
```
fda <- c()
sum <- 0
for (x in 0:7) {
  sum <- sum + (exp(-3)*3^x / factorial(x))
  fda <- c(fda,sum)
}
df <- data.frame(X,fp,fda)
print(fda)
```

```
## [1] 0.04978707 0.19914827 0.42319008 0.64723189 0.81526324 0.91608206 0.96649146
## [8] 0.98809550
```

d)

```
ggplot(data = df, mapping = aes(X,fda)) +
  geom_line(width = 2)
```

```
## Warning: Ignoring unknown parameters: width
```



3.2.

Por definición 3.4, sea

$$k + \frac{k}{2} + \frac{k}{3} + \frac{k}{4} = 1 \quad \Longrightarrow \quad k = \frac{12}{25}$$

entonces la función de probabilidad de una variable aleatoria discreta X estará dada por:

$$p(x) = \begin{cases} \frac{12}{25} & si \quad x = 1 \\ \frac{6}{25} & si \quad x = 2 \\ \frac{4}{25} & si \quad x = 3 \\ \frac{3}{25} & si \quad x = 4 \end{cases}$$

Así, la probabilidad de $P(1 \leq X \leq 3)$ será,

$$P(1 \leq X \leq 3) = 1 - P(X = 4) = 1 - \frac{3}{25} = \frac{22}{25} = 1.88$$

3.3.

a)

Según la definición 3.6 se tiene,

$$\int_{-x}^{x} kx^2 \, dx = \int_{-1}^{1} kx^2 \, dx = 1 \quad \Longrightarrow \quad k = \frac{3}{2}$$

```
integrate(function(x) 3/2*x^2, lower = -1, upper = 1)
```

```
## 1 with absolute error < 1.1e-14
```

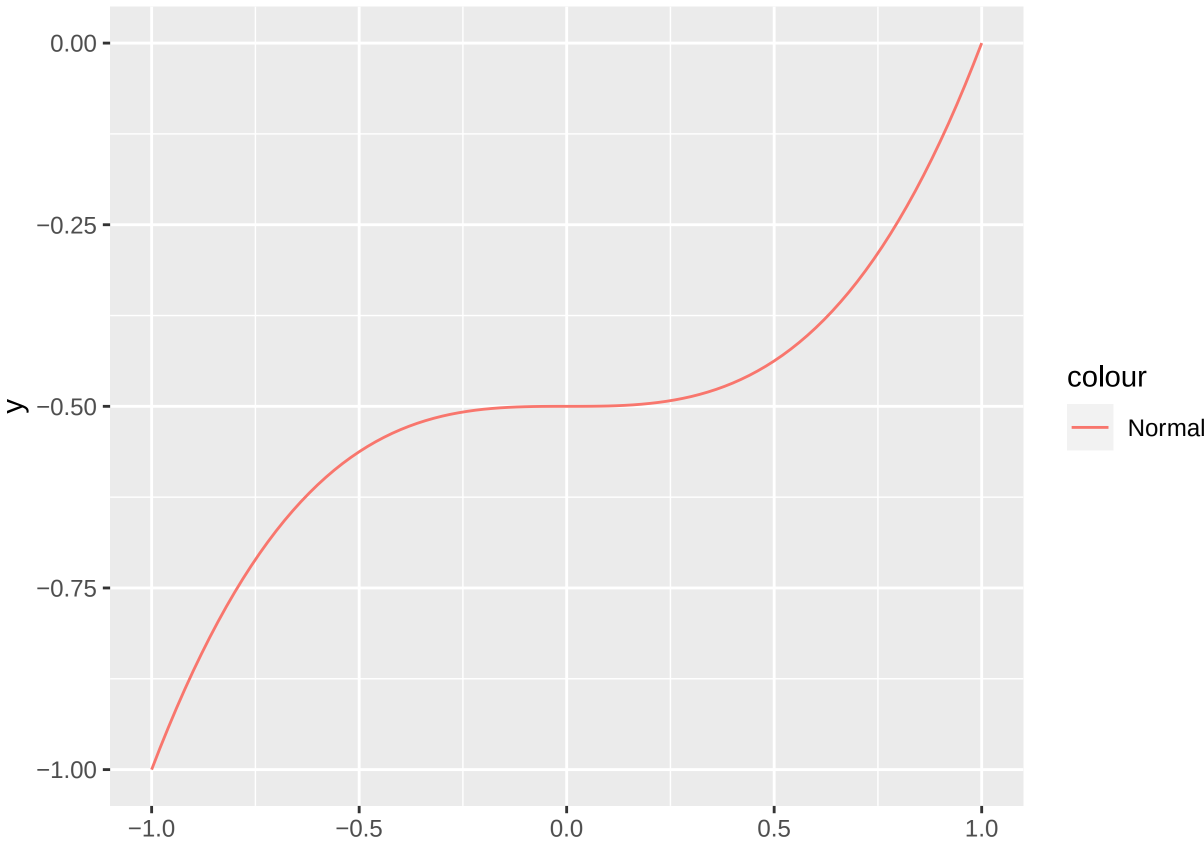
b)

$$P(X \leq x) = \int_{-\infty}^{x} f(t) \, dt = \int_{-\infty}^{-1} 0 \, dt + \int_{-1}^{x} \frac{3}{2} t^2 \, dt = \frac{3}{2} \cdot \frac{x^3+1}{3} = \frac{x^3+1}{2}$$

```
funcdist <- function(x) (x^3+1)/2
```

$$P(X \leq x) = \begin{cases} \dfrac{x^3+1}{2} & si \quad x > 0 \\ \\ 0 & si \quad \text{Para cualquier otro valor} \end{cases}$$

```
ggplot() +
  xlim(-1, 1) +
  geom_function(
    aes(color = "Normal"),
    fun =~ (.x^3-1)/2
    )
```



c)

$$P(X \geq 1/2) = 1 - P(X \leq 1/2) = 1 - \frac{x^3+1}{2} = 1 - \frac{(1/2)^3+1}{2} = 1 - \frac{9/8}{2} = \frac{7}{16}$$

```
1-funcdist(1/2)
```

```
## [1] 0.4375
```

$$P(-1/2 \leq X \leq 1/2) = P(X \leq 1/2) - P(X \leq -1/2) = \frac{(1/2)^3}{2} - \frac{(-1/2)^3}{2} = \frac{1}{8}$$

```
# primera manera
funcdist(1/2) - funcdist(-1/2)
```

```
## [1] 0.125
```

```
# segunda manera
integrate(function(x) 3/2*x^2, lower = -1/2, upper = 1/2)
```

```
## 0.125 with absolute error < 1.4e-15
```

3.4.

a)

Por la definición 3.6 se tiene y sabiendo que $x \leq 0$ es cero.

$$\int_{-\infty}^{\infty} k \cdot e^{-x/5} \, dx = \int_{0}^{\infty} k \cdot e^{-x/5} \, dx$$

Luego igualando a uno,

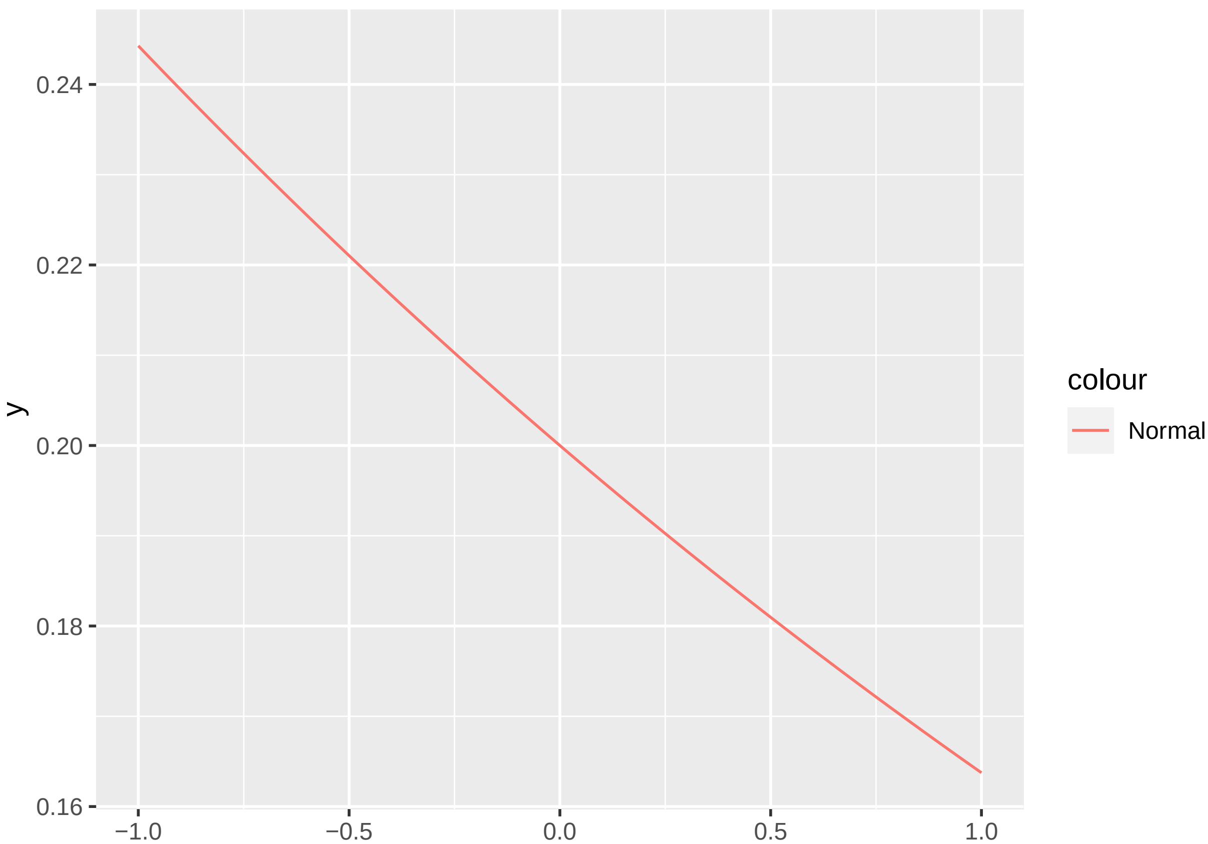
$$\int_0^{\infty} k \cdot e^{-x/5}\, dx = 1 \quad \implies \quad k = \frac{1}{5}$$

```
integrate(function(x) 1/5 * exp(-x/5), lower = 0, upper = Inf)
```

```
## 1 with absolute error < 2e-07
```

b)

```
ggplot() +
  xlim(-1, 1) +
  geom_function(
    aes(color = "Normal"),
    fun =~ 1/5 * exp(-.x/5)
    )
```



c)

$$P(X \leq 5) = \int_0^5 \frac{1}{5} e^{-x/5}\, dx = 1 - \frac{1}{e}$$

```
integrate(function(x) 1/5 * exp(-x/5), lower = 0, upper = 5)
```

```
## 0.6321206 with absolute error < 7e-15
```

$$P(0 \leq X \leq 8) = \int_0^8 \frac{1}{5} e^{-x/5}\, dx = 1 - \frac{1}{e^{8/5}}$$

```r
integrate(function(x) 1/5 * exp(-x/5), lower = 0, upper = 8)
```

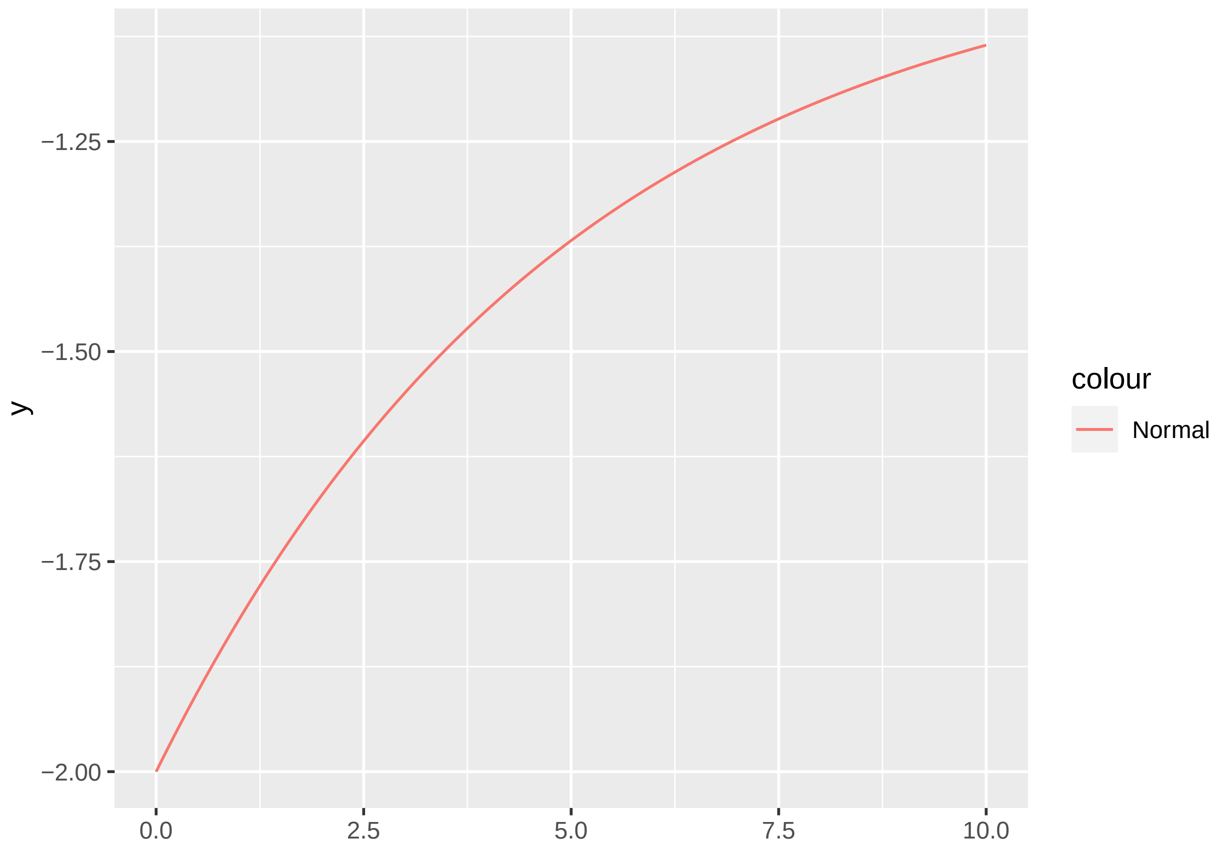
```
## 0.7981035 with absolute error < 8.9e-15
```

d)

La función de distribución acumulativa esta dado por

$$F(x) = \int_{\infty}^{x} f(t)\, dt = \int_{-\infty}^{0} 0\, dt + \frac{1}{5}\int_{0}^{x} e^{-t/5}\, dt = -e^{-x/5} - 1$$

```r
ggplot() +
  xlim(0, 10) +
  geom_function(
    aes(color = "Normal"),
    fun =~ -exp(-.x/5) - 1
    )
```



3.5

$$F(x) = 1 - exp(-x/100), \quad x > 0$$

a)

$$F^{'}(x) = 1 - e^{-x/100} = \frac{1}{100}e^{-x/100}$$

b)

$$1 - (1 - exp(-200/100)) = 0.13533$$

```
1-(1-exp(-200/100))
```

```
## [1] 0.1353353
```

```
integrate(function(x) 1/100 * exp(-x/100), lower = 0, upper = 200)
```

```
## 0.8646647 with absolute error < 9.6e-15
```

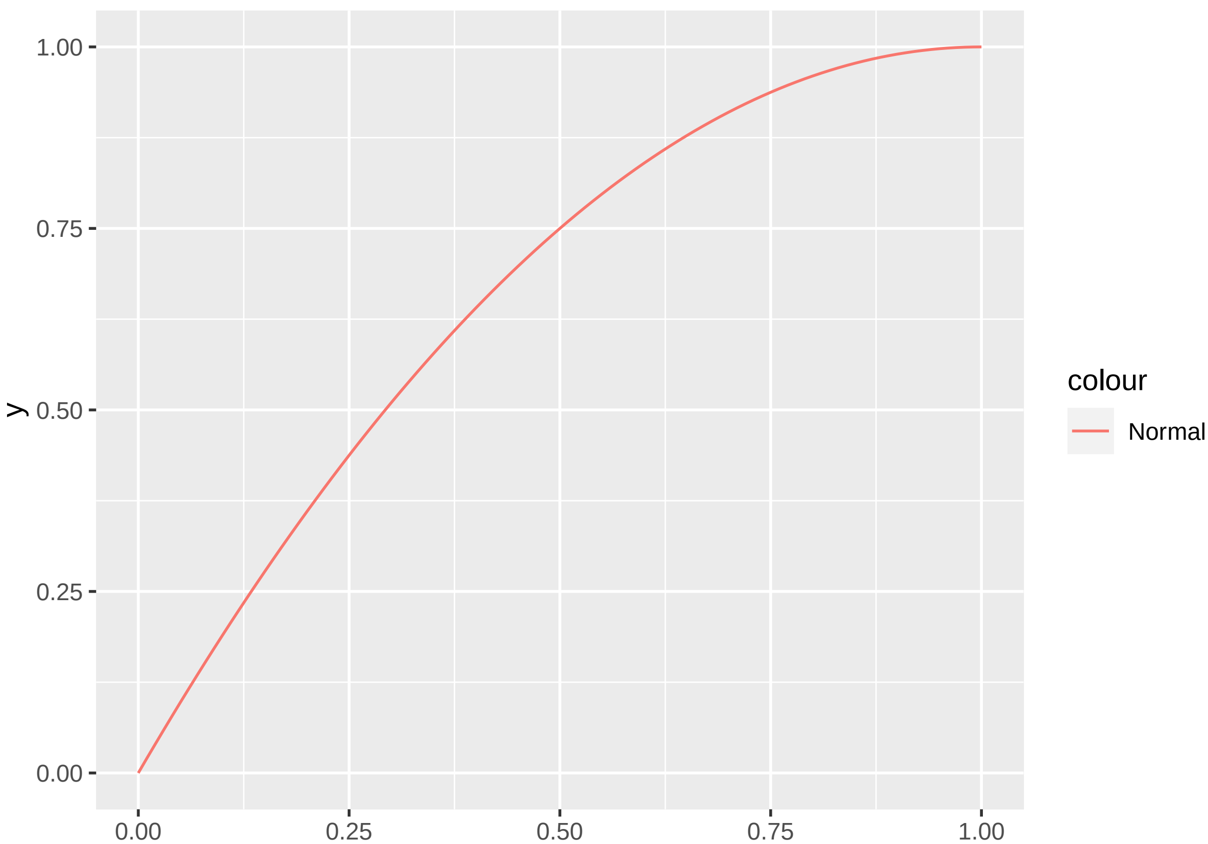
3.6

Función acumulada de v.a. es

$$F(X) = \begin{cases} 0 & x < 0 \\ 2x - x^2 & 0 < x < 1 \\ 1 & x > 1 \end{cases}$$

a)

```
# gráfico de F(x)
ggplot() +
   xlim(0, 1) +
   geom_function(
     aes(color = "Normal"),
     fun =~ 2*.x - .x^2,
     )
```



b)

$P(X < 1/2) = 2 \cdot 1/2 - (1/2)^2 = 3/4 = 0.75$

```
func <- function(x) 2*x - x^2
func(0.5)
```

```
## [1] 0.75
```

$1 - P(X > 3/4) = 1 - (2 \cdot 3/4 - (3/4)^2) = 1 - (-3/4)1 - (15/16) = 1/16 = 0.0625$

```
1-func(3/4)
```

```
## [1] 0.0625
```

c)

$F^{'}(x) = 2x - 2^2 = 2 - x$

3.7

```
x <- c(0,1,2,3,4,5,6,7,8)
px <- c(0.05,0.1,0.1,0.1,0.2,0.25,0.1,0.05,0.05)
```

$E(X) = \sum_{x=1}^{n} x \cdot p(x) = 0 \cdot 0.05 + 1 \cdot 0.1 + \ldots + 8 \cdot 0.05 = 4$

```
sum <- 0
for (i in 1:length(x)){
 sum <- sum + x[i]*px[i]
}
sum
```

```
## [1] 4
```

$E(X^2) = \sum_{x=1}^{n} x^2 p(x) = 0^2 \cdot 0.05 + 1^2 \cdot 0.1 + \ldots + 8^2 \cdot 0.05 = 20.1$

```
sum2 <- 0
for (i in 1:length(x)){
 sum2 <- sum2 + x[i]^2 * px[i]
}
sum2
```

```
## [1] 20.1
```

$Var(X) = E(X^2) - E^2(X) = 20.1 - 4^2 = 4.1$

```
sum2 - sum^2
```

```
## [1] 4.1
```

3.8

Calculamos el valor para un ganancia nula

$$E[X] = 0 = \frac{C \cdot 995}{1000} - \frac{50000 \cdot 5}{100} \implies 995C = 250000 \implies C = 251.26$$

3.9

Función de densidad de una v.a. X está dada por:

$$f(x) = \begin{cases} 2(1-x) & 0 < x < 1 \\ 0 & \text{para cualquier otro caso} \end{cases}$$

a)

$$E(X) = \int_0^1 x \cdot 2(1-x)\ dx = \int_0^1 2x - 2x^2\ dx = \frac{1}{3}$$

```
func <- function(x) x*2*(1-x)
integrate(func, lower = 0, upper = 1)
```

```
## 0.3333333 with absolute error < 3.7e-15
```

b)

$$E(X^2) = \int_0^1 x^2 \cdot 2(1-x)\ dx = \int_0^1 2x^2 - 2x^3\ dx = \frac{1}{6}$$

```
func <- function(x) x^2*2*(1-x)
integrate(func, lower = 0, upper = 1)
```

```
## 0.1666667 with absolute error < 1.9e-15
```

$$Var(X) = E(X^2) - E(X)^2 = \frac{1}{6} - \left(\frac{1}{3}\right)^2 = \frac{1}{18}$$

```
# varianza
1/6 - (1/3)^2
```

```
## [1] 0.05555556
```

3.10

Función de densidad de una v.a. de X está dada por:

$$f(x) = \begin{cases} 1/10 & 0 < x < 10 \\ 0 & \text{para cualquier otro valor} \end{cases}$$

a)

$$E(X) = \int_0^{10} \frac{1}{10}x \cdot\ dx = \frac{1}{10}\int_0^{10} x\ dx = 5$$

```
func <- function(x) 1/10 * x
integrate(func, lower = 0, upper = 10)
```

```
## 5 with absolute error < 5.6e-14
```

b)

$$E(X) = \int_0^{10} \frac{1}{10}x^2 \cdot\ dx = \frac{1}{10}\int_0^{10} x^2\ dx = \frac{100}{3}$$

```r
func <- function(x) 1/10 * x^2
integrate(func, lower = 0, upper = 10)
```

```
## 33.33333 with absolute error < 3.7e-13
```

$$Var(X) = E(X^2) - E(X)^2 = \frac{100}{3} - 5^2 = \frac{25}{3} = 8.33333$$

```r
# varianza
100/3 - 5^2
```

```
## [1] 8.333333
```

c)

α_3 también llamado coeficiente de asimetría estará dado por

$$\alpha_3 = \frac{\mu_3}{(\mu_2)^{3/2}}$$

de donde μ_3 será:

$$\mu_3 = E(X-5)^3 = \int_0^{10} \frac{1}{10} \cdot (x-5)^3 \, dx = \frac{1}{10} \cdot \int_0^{10} (x^3 - 15x^2 + 75x - 125) \, dx = 0$$

para μ_2 tenemos

$$Var(X) = E(X-5)^2 = \int_0^{10} \frac{1}{10} \cdot (x-5)^2 \, dx = \frac{1}{10} \int_0^{10} (x^2 - 10x + 25) \, dx = \frac{25}{3}$$

Entonces

$$\alpha_3 = \frac{\mu_3}{(\mu_2)^{3/2}} = \frac{0}{\left(\frac{25}{3}\right)^{3/2}} = 0$$

```r
func <- function(x) 1/10 * (x-5)^3
func2 <- function(x) 1/10 * (x-5)^2
integrate(func, lower = 0, upper = 10)
```

```
## 1.204593e-15 with absolute error < 3.5e-13
```

```r
integrate(func2, lower = 0, upper = 10)
```

```
## 8.333333 with absolute error < 9.3e-14
```

Donde nos menciona que se tiene un coeficiente de asimetría simétrica.

d)

Para α_4 como medida relativa de la curtosis tenemos

$$\alpha_4 = \frac{\mu_4}{\mu_2^2}$$

de donde tenemos μ_4 de la siguiente manera

$$\mu_4 = E(X-5)^4 = \int_0^{10} \frac{1}{10} \cdot (x-5)^4 \, dx = 125$$

```r
func4 <- function(x) 1/10 * (x-5)^4
integrate(func4, lower = 0, upper = 10)
```

```
## 125 with absolute error < 1.4e-12
```

para luego:

$$\alpha_4 = \frac{\mu_4}{\mu_2^2} = \frac{125}{\frac{25^2}{3^3}} = 5.4$$

Donde nos dice que la distribución presenta un pico relativamente alto

3.11

La función de densidad viene dada por

$$f(x) = \begin{cases} \frac{1}{4}e^{-x/4} & x > 0 \\ 0 & \text{para cualquier otro valor} \end{cases}$$

a)

$$E(X) = \int_0^\infty x \cdot \frac{1}{4}e^{-x/4}\,dx = \frac{1}{4}\int_0^\infty x \cdot e^{-x/4}\,dx = 4$$

```r
func <- function(x) 1/4*x*exp(-x/4)
integrate(func, lower = 0, upper = Inf)
```

```
## 4 with absolute error < 1.2e-05
```

b)

$$Var(X) = E(X-4)^2 = \int_0^\infty \frac{1}{4}e^{-x/4} \cdot (x-4)^2\,dx = \frac{1}{4}\int_0^\infty e^{-x/4} \cdot (x-4)^2\,dx = 16$$

```r
func <- function(x) 1/4*(x-4)^2*exp(-x/4)
integrate(func, lower = 0, upper = Inf)
```

```
## 16 with absolute error < 0.00051
```

c)

$$\mu_3 = E(X-4)^3 = \frac{1}{4}\int_0^\infty (x-4)^3 \cdot e^{-x/4}\,dx = 128$$

```r
func <- function(x) 1/4*(x-4)^3*exp(-x/4)
integrate(func, lower = 0, upper = Inf)
```

```
## 128 with absolute error < 0.00029
```

$$\alpha_3 = \frac{\mu_3}{\mu_2^{3/2}} = \frac{128}{16^{3/2}} = 2$$

```r
128/(16^(3/2))
```

```
## [1] 2
```

De donde podemos mencionar que se tiene una asimetría positiva.

d)

$$\mu_4 = E(X-4)^4 = \frac{1}{4}\int_0^{\infty} (x-4)^4 \cdot e^{-x/4}\, dx = 2304$$

```
func <- function(x) 1/4*(x-4)^4*exp(-x/4)
integrate(func, lower = 0, upper = Inf)
```

```
## 2304 with absolute error < 0.0025
```

así,

$$\alpha_4 = \frac{\mu_4}{\mu_2^2} = \frac{2304}{16^2} = 9$$

```
2304/16^2
```

```
## [1] 9
```

e)

Del ejercico 3.10

$$V_X = \frac{E(X)}{Var(X)} = \frac{5}{8.33333}$$

```
5/8.33333
```

```
## [1] 0.6000002
```

Del ejercio 3.11

$$V_Y = \frac{E(X)}{Var(X)} = \frac{4}{16}$$

```
4/16
```

```
## [1] 0.25
```

De donde V_Y muestra mayor dispersión relativa con respecto a la media que la distribución correspondiente a X

3.12

Sea

$$E(X) = 62.5, \qquad y \qquad \sqrt{Var(X)} = 10 \;\Rightarrow\; Var(X) = 100$$

y

$$E(aX+b) = 70 \qquad y \qquad Var(aX+b) = 80$$

entonces

$$a^2 Var(X) = 80 \implies a = 2\sqrt{\frac{1}{5}}$$

y

$$aE(X) + b = 70 \implies b = 14.098$$

Entonces la respuesta estará dada por

$$aX + b = 2\sqrt{\frac{1}{5}}X + \left(70 - 125\sqrt{\frac{1}{5}}\right)$$

```
EX <- function(x) 2*(1/5)^(1/2)*x + 70-125*(1/5)^(1/2)
EX(62.5)
```

```
## [1] 70
```

```
VarX <- function(varx) (2*(1/5)^(1/2))^2 * varx
VarX(100)
```

```
## [1] 80
```

3.13

a)

Sea $E(X) = \mu$ y $Var(X) = \sigma^2$ entonces,

$$E(X-c)^2 = E(X^2 - 2cX + c^2) = E(X^2) - 2cE(X) + c^2 = E(X^2) - E^2(X) + E(X)\cdot E(X) - 2cE(X) + c^2 =$$

$$Var(X) + (E(X) - c)^2 = \sigma^2 + (\mu - c)^2$$

b)

Cuando $c = \mu$

3.14

$$E(Y) = E\left(\frac{X-4}{4}\right) = \int_0^\infty \left(\frac{x-4}{4}\right)\cdot\frac{1}{4}\cdot e^{-x/4}\,dx = \frac{1}{16}\int_0^\infty (x-4)\cdot e^{-x/4}\,dx = 0$$

```
fx = function(x) ((x-4)/4) * 1/4 * exp( -x/4 )
integrate(fx,lower = 0, upper = Inf)
```

```
## -5.632717e-13 with absolute error < 3e-06
```

$$Var(Y) = Var\left(\frac{X-4}{4}\right)^2 = \int_0^\infty \left(\frac{x-4}{4}\right)^2\cdot\frac{1}{4}\cdot e^{-x/4}\,dx = \frac{1}{64}\int_0^\infty (x-4)^2\cdot e^{-x/4}\,dx = 1$$

```
fx = function(x) 1/64 * (x-4)^2 * exp(-x/4)
integrate(fx,lower = 0, upper = Inf)
```

```
## 1 with absolute error < 3.2e-05
```

3.15

$$E\left|X - \frac{1}{3}\right| = \int_0^1 \left|x - \frac{1}{3}\right|\cdot 2(1-x)\ \,dx = \int_0^{1/3}\left(\frac{1}{3} - x\right) * 2(1-x)\,dx + \int_{1/3}^1 \left(x - \frac{1}{3}\right) * 2(1-x)\,dx = 0.1975309$$

```
f <- function(x) abs(x-1/3)*2*(1-x)
integrate(f,lower = 0,upper = 1)
```

```
## 0.1975309 with absolute error < 5e-06
```

La desviación estandar del ejercicio 3.9 viene dada por $\sqrt{\dfrac{1}{18}} = 0.2357$

Luego comparando con la desviación media vemos que se tiene poca diferencia.

3.16

$$E|X-\mu| = \int_0^{10} |x-5|\cdot\frac{1}{10}\,dx = \frac{1}{10}\left[\int_0^5 (5-x)\,dx + \int_5^{10}(x-5)\,dx\right] = \frac{1}{10}\left(5x\Big|_0^5 - \frac{x^2}{2}\Big|_0^5 + \frac{x^2}{2}\Big|_5^{10} - 5x\Big|_5^{10}\right) =$$

$$\frac{1}{10}\left(25 - \frac{25}{2} + \frac{75}{2} - 25\right) = \frac{5}{2}$$

```
f <- function(x) abs(x-5)*1/10
integrate(f,lower = 0,upper = 10)
```

```
## 2.5 with absolute error < 2.8e-14
```

```
# desviación típica del ejercicio 10
(25/3)^(1/2)
```

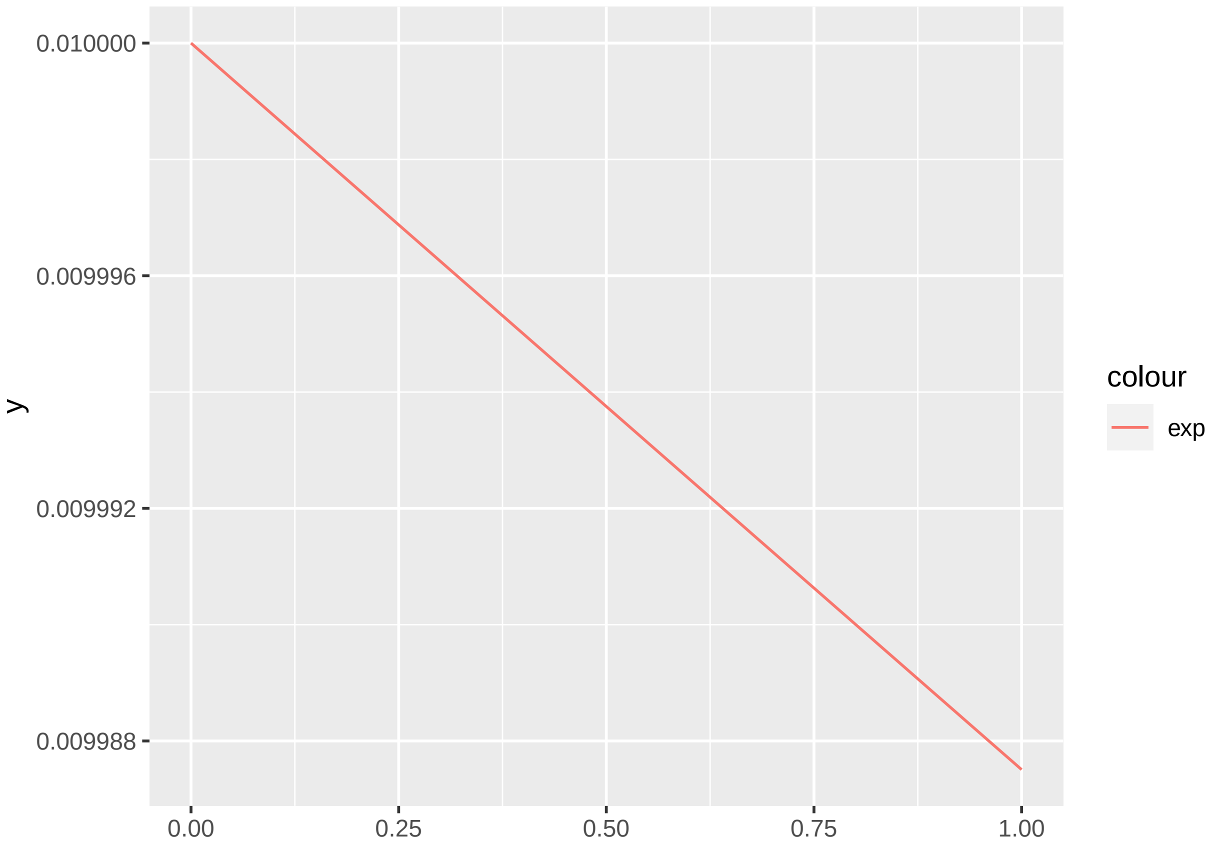
```
## [1] 2.886751
```

De lo que concluimos que entre la desviación media de X y la desviación estándar de ejercicio 10 se tiene una diferencia de 0.33.

3.17

$$$$

```
ggplot() +
   xlim(0, 1) +
   geom_function(
     aes(color = "exp"),
     fun =~ 1/100*exp(-.x/800)
     )
```



a)

La media es:

$$E(X) = \int_0^\infty x \cdot \frac{1}{800} \cdot e^{-x/800}\, dx = \frac{1}{800} \int_0^\infty x \cdot e^{-x/800} = 800$$

```
integrate(function(x) 1/800*x*exp(-x/800), lower = 0, upper = Inf)
```

```
## 800 with absolute error < 0.034
```

La mediana es:

$$F(x_{0.5}) = P(X \leq x_{0.5}) = \frac{1}{800} \int_0^{x_{0.5}} e^{(-x/800)}\, dx = 0.5 \implies -e^{-x_{0.5}/800} + 1 = 0.5 \implies x_{0.5} = 554.5177$$

```
-800*log(0.5)
```

```
## [1] 554.5177
```

b)

Recorrido intercuartil

$$F(x_{0.25}) = P(X \leq x_{0.25}) = \frac{1}{800} \int_0^{x_{0.25}} e^{(-x/800)}\, dx = 0.25 \implies -e^{-x_{0.25}/800} + 1 = 0.25 \implies x_{0.25}$$

$$\implies x_{0.25} = -800 \cdot ln(0.25) = 1109.03548$$

$$F(x_{0.75}) = -800 \cdot ln(0.75) = 230.14565$$

por lo que

$$x_{0.75} - x_{0.25} = |230.14565 - 1109.03548| = 878.8898$$

c)

Recorrido interdecil

$$F(x_{0.1}) = P(X \leq x_{0.1}) = \frac{1}{800}\int_0^{x_{0.1}} e^{(-x/800)}\, dx = 0.1 \implies -e^{-x_{0.1}/800} + 1 = 0.1$$
$$\implies x_{0.1} \implies x_{0.1} = -800 \cdot ln(0.1) = 1842.068$$

$$F(x_{0.9}) = -800 \cdot ln(0.9) = 84.2884$$

$$x_{0.9} - x_{0.1} = |84.2884 - 1842.068| = 1757.78$$

d)

$$P(X \geq 800) = 1 - P(X \leq 800) = 1 - \frac{1}{800}\int_0^{800} e^{-x/800}\, dx = 1 - 0.3678794 = 0.3679$$

```
integrate(function(x) 1/800*exp(-x/800), lower = 0, upper = 800)
```

```
## 0.6321206 with absolute error < 7e-15
```

```
1 - 0.6321206
```

```
## [1] 0.3678794
```

3.18

$$\begin{aligned}
\left.\frac{d^r m_{X-\mu}(t)}{dt^r}\right|_{t=\mu} &= \left.\frac{d^r}{dt^r}E[e^{t(X-\mu)}]\right|_{t=0} \\
&= E\left\{\frac{d^r}{dt^r}\left[e^{t(X-\mu)}\right]\right\} \\
&= \left.E\left[(X-\mu)^r e^{t(X-\mu)}\right]\right|_{t=0} \\
&= E[X-\mu]^r \\
&= u_r
\end{aligned}$$

3.19

a)

$$m_X(t) = E\left[e^{tX}\right] = \int_0^{\infty} e^{tx} \cdot \frac{1}{16} \cdot x \cdot e^{-\frac{x}{4}}\, dx = \frac{1}{16}\int_0^{\infty} x \cdot e^{\frac{x(4t-1)}{4}}\, dx = (1-4t)^{-2}$$

b)

$$\left.\frac{dm_x(t)}{dt}\right|_{t=0} = \frac{d}{dt}\cdot(1-4t)^{-2}\bigg|_{t=0} = \left.\frac{8}{(1-4x)^3}\right|_{t=0} = 8 = E(X)$$

$$\left.\frac{d^2m_X(t)}{dt^2}\right|_{t=0} = \frac{d^2}{dt^2}(1-4x)^{-2} = \frac{d}{dt}\left.\frac{8}{(1-4x)^2}\right|_{t=0} = \left.\frac{96}{(-4x+1)^4}\right|_{t=0} = 96 = E(X^2)$$

$$Var(X) = E(X^2) - E^2(X) = 96 - 8^2 = 32$$

3.20

$$m_X(t) = \frac{1}{4}\int_0^{\infty} e^{tx}\cdot e^{-\frac{x}{4}}\ dx$$

3.21

$E(c) = \sum_x c\cdot p(x) = c\sum_x p(x) = c$

$E(cX+b) = \sum_x (cx+b)p(x) = c\sum_x x\cdot p(x) + b\sum_x p(x) = cE(x) + b$

$E[g(X)+h(X)] = \sum_x [g(x)+h(x)]p(x) + \sum_x g(x)p(x) + \sum_x h(x)p(x) = E[g(X)] + E[h(X)]$

3.22

$$Var(X) = (X-\mu)^2 = \sum_x (x-\mu)^2 p(x) = \sum_x x^2 p(x) - 2\mu\sum_x xp(x) + \mu^2\sum_x p(x) =$$
$$E(X^2) - 2E^2(X) + E^2(X) = E(X^2) - E^2(X)$$