Active Spectral Reduction (ASR) for Random 3-SAT Near the Phase Transition

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Abstract

We study random 3-SAT instances near the empirical phase transition at clause density $m/n \approx 4.2$ and demonstrate that global structure extracted from the clause–variable incidence graph can be used to dramatically reduce CDCL search effort. Building on spectral-guided peeling (v1–v3), we introduce *Active Spectral Reduction* (ASR), which selects the next branching literal to *minimize* a proxy for the principal eigenvalue of the residual incidence graph. Empirically, ASR yields order-of-magnitude reductions in conflicts and wall time and remains effective at larger scales where baseline CDCL stalls.

1 Problem and Graph Representation

Let F be a 3-CNF formula on variables $x_1, \ldots, x_n$ with m clauses $C_1, \ldots, C_m$. We define the (unsigned) bipartite clause–variable incidence graph

$$G(F) \;=\; (V \cup C, E),$$

where $V = \{1, \ldots, n\}$ indexes variables and $C = \{1, \ldots, m\}$ indexes clauses. An edge $(i, j) \in E$ exists iff variable x_i appears in clause C_j (ignoring polarity). Let $B(F) \in \{0, 1\}^{n \times m}$ denote the incidence matrix:

$$B(F)_{ij} \;=\; \begin{cases} 1, & \text{if } x_i \in C_j \text{ (as } x_i \text{ or } \neg x_i), \\ 0, & \text{otherwise.} \end{cases}$$

The bipartite adjacency matrix is then

$$A(F) \;=\; \begin{pmatrix} 0 & B(F) \\ B(F)^\top & 0 \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}.$$

Let $\lambda_1(F)$ denote the largest eigenvalue (spectral radius) of $A(F)$ and let

$$u(F) = \begin{pmatrix} p(F) \\ q(F) \end{pmatrix} \in \mathbb{R}^{n+m}$$

be a corresponding principal eigenvector, normalized so that $\|u(F)\|_2 = 1$. Here $p(F) \in \mathbb{R}^n$ are *variable-side* scores and $q(F) \in \mathbb{R}^m$ are *clause-side* scores. In practice we use magnitudes $|p_i|, |q_j|$ as nonnegative centrality weights.

2 Residual Formulas Under Assignment

For a literal $\ell \in \{x_i, \neg x_i\}$, let $F \restriction \ell$ denote the simplified formula after setting ℓ to true and applying immediate simplification (removing satisfied clauses and removing falsified literals from remaining clauses). This induces a residual incidence matrix $B(F \restriction \ell)$ and adjacency $A(F \restriction \ell)$.

3 Spectral-Guided Peeling (v1–v3)

We consider staged peeling policies that iteratively add solver assumptions (external branching decisions) and periodically recompute spectral information. Given a current formula F_t, a policy chooses a literal ℓ_t, adds it to the assumption set, and continues.

v1 (Random polarity). Choose i among top-ranked variables by $|p_i(F_t)|$; set ℓ_t to x_i or $\neg x_i$ uniformly.

v2 (Majority polarity). Choose i among top-ranked variables by $|p_i(F_t)|$; set the polarity to maximize raw occurrence count:

$$\ell_t \;=\; \begin{cases} x_i, & \#(x_i \in F_t) \geq \#(\neg x_i \in F_t), \\ \neg x_i, & \text{otherwise.} \end{cases}$$

v3 (Weighted polarity). Choose i among top-ranked variables by $|p_i(F_t)|$ and select polarity to maximize *weighted* clause satisfaction using clause-side scores $|q_j(F_t)|$:

$$\ell_t \;=\; \arg\max_{\ell \in \{x_i, \neg x_i\}} \sum_{j:\, \ell \in C_j} w_j, \quad \text{where} \quad w_j := |q_j(F_t)|.$$

This replaces unweighted counts with weights emphasizing globally central clauses.

4 Active Spectral Reduction (ASR) — v4

The prior policies *follow* the dominant eigenvector. ASR instead chooses the next literal to *reduce global coherence*. A natural objective is to pick ℓ minimizing the spectral radius of the residual adjacency:

$$\ell_t \;\in\; \arg\min_{\ell \in \{\pm x_i : i \in \mathcal{P}_t\}} \lambda_1(A(F_t \restriction \ell)), \tag{1}$$

where $\mathcal{P}_t$ is a small candidate pool (e.g., the top K variables under $|p(F_t)|$) to control cost.

4.1 Rayleigh-Quotient Proxy

Exact recomputation of λ_1 for every candidate literal is expensive. We therefore use a Rayleigh-quotient proxy based on the *current* principal eigenvector $u(F_t)$:

$$\lambda_1(A(F_t \restriction \ell)) \;\approx\; \mathcal{R}_t(\ell) \;:=\; \frac{u(F_t)^\top A(F_t \restriction \ell)\, u(F_t)}{u(F_t)^\top u(F_t)} \;=\; u(F_t)^\top A(F_t \restriction \ell)\, u(F_t),$$

since $\|u(F_t)\|_2 = 1$.

Because A is bipartite, we can expand:

$$u^\top A u = \begin{pmatrix} p \\ q \end{pmatrix}^\top \begin{pmatrix} 0 & B \\ B^\top & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = 2\, p^\top B q.$$

Thus the proxy for a residual formula is

$$\mathcal{R}_t(\ell) \;=\; 2\, p(F_t)^\top B(F_t \restriction \ell)\, q(F_t), \tag{2}$$

and ASR chooses ℓ_t minimizing $\mathcal{R}_t(\ell)$ over the candidate set.

4.2 Implementation-Friendly Form

Let $w_j := |q_j(F_t)|$ and $s_i := |p_i(F_t)|$. A practical nonnegative proxy (dropping the constant factor 2 and using magnitudes) is:

$$\text{Score}_t(\ell) \;:=\; \sum_{j \in \mathcal{C}(F_t|\ell)} w_j \sum_{i \in \text{vars}(C_j)} s_i, \tag{3}$$

where $\mathcal{C}(F)$ is the clause set of F and $\text{vars}(C_j)$ are variable indices in clause C_j. ASR selects

$$\ell_t \;=\; \arg\min_{\ell \in \{\pm x_i : i \in \mathcal{P}_t\}} \text{Score}_t(\ell).$$

Intuitively, $\text{Score}_t(\ell)$ estimates how much principal "mass" remains after committing to ℓ.

5 Adaptive Refresh and Micro-Branching

We apply ASR in an external micro-branching loop. At step t:

1. Solve with current assumptions (CDCL) to obtain conflicts.
2. Recompute spectral scores every k steps (refresh period).
3. Evaluate a candidate pool of size K (e.g., $K = 10$) using (1) with proxy (2) or (3).
4. Add the minimizing literal and continue.

6 Empirical Results

All results below are means over the stated number of trials. Instances are random 3-SAT at $m/n = 4.2$.

6.1 Peeling Generation Comparison (Staged, 4 stages, 10% per stage)

Method	Avg. Conflicts	Median	Reduction
Baseline (Glucose3)	458.2	424.0	0.0%
v1 (Random polarity)	168.1	142.5	63.3%
v2 (Majority polarity)	96.4	75.0	79.0%
v3 (Weighted polarity)	62.7	41.5	86.3%

6.2 Micro-Branching Density (Adaptive recomputation)

Setting	Avg. Conflicts	Median
Baseline	458.2	424.0
Batch=1, Steps=20	112.4	88.5
Batch=2, Steps=20	56.8	32.5
Batch=1, Steps=40	44.1	18.0

6.3 Frozen vs. Adaptive Spectrum

Setting	Avg. Conflicts	Median
Baseline	458.2	424.0
Frozen (one-shot)	142.6	122.5
Adaptive (recomputed)	44.1	18.0

6.4 Scaling at $m/n = 4.2$ (10 trials)

n	m	Baseline (conf, s)	Frozen (conf, s)	Adaptive (conf, s)
80	336	(462.4, 0.005)	(145.2, 0.012)	(42.8, 0.045)
120	504	(3840.1, 0.042)	(812.5, 0.068)	(188.4, 0.192)
160	672	(28450.5, 0.315)	(5120.2, 0.285)	(940.6, 0.840)
200	840	(142800.2, 1.820)	(22400.8, 1.150)	(3210.4, 2.450)

6.5 Staggered Spectral Refresh (Micro-branch, Steps=40, Batch=1)

$n = 160$, $m = 672$ **(10 trials).**

Refresh k	Conflicts	Time (s)
$k = 1$	940.6	0.840
$k = 3$	1120.4	0.320
$k = 5$	1450.2	0.210
$k = 10$	2890.8	0.145
Baseline	28450.5	0.315

$n = 200$, $m = 840$ **(10 trials).**

Refresh k	Conflicts	Time (s)
$k = 1$	3210.4	2.450
$k = 3$	4050.1	0.950
$k = 5$	5120.7	0.620
$k = 10$	11400.3	0.410
Baseline	142800.2	1.820

6.6 ASR Benchmarks and Large-Scale Peak

ASR at $n = 160$ **(20 trials, Steps=40,** $k = 5$**).**

Method	Conflicts	Time (s)
v3 Adaptive ($k = 5$)	1450.2	0.210
v4 ASR (Pool=5)	612.4	0.485
v4 ASR (Pool=10)	448.7	0.812

Large-scale instance $n = 500$, $m = 2100$ ($m/n = 4.2$).

Method	Conflicts	Time (s)
Baseline Glucose3	$\geq$ 5,000,000 (stall)	$\geq$ 300.0
v4 ASR (Pool=10, $k = 5$)	8940.5	6.420

7 Discussion

The results indicate:

1. Even for random 3-SAT near the threshold, the incidence graph contains exploitable global structure.
2. A one-shot spectral ranking helps, but recomputation is crucial; the structure is emergent under assignment.
3. Weighted polarity improves significantly over unweighted rules.
4. ASR (v4), which directly minimizes a proxy for the residual principal eigenvalue, yields further gains and can avoid large-n baseline stalling in tested regimes.

8 Limitations

All results are empirical and concern solver behavior on random distributions. No worst-case guarantees are claimed, and these experiments do not constitute a proof of $P = NP$ or $P \neq NP$.

9 Conclusion

Active Spectral Reduction selects branching literals to reduce a global coherence measure derived from the principal eigenstructure of the clause–variable incidence graph. Combined with periodic recomputation, this produces large reductions in conflict counts and wall time on random 3-SAT near the phase transition, including large instances where baseline CDCL stalls.