Matroid Partitioning

Main result

Let $\mathcal{M}_1, \ldots, \mathcal{M}_k$ be matroids (otherwise known as "pregeometries," in the terminology of Crapo and Rota [1]), over the n-element set E. Edmonds [2] has given an efficient algorithm to determine whether or not the elements of E can be partitioned into k disjoint subsets, $E = E_1 \cup \cdots \cup E_k$, in such a way that E_j is independent in $\mathcal{M}_j$ for all j. The purpose of this note is to present his algorithm in a somewhat different manner, which indicates how he might have discovered it in the first place. We shall also extend the algorithm slightly so that bounds are placed on the number of elements in the subsets E_j.

Derivation of an Algorithm

Suppose we have painted certain elements of E, and let E_j be the set of elements that have color j. We assume that E_j is independent in $\mathcal{M}_j$. Let $E_0 = E \setminus (E_1 \cup \cdots \cup E_k)$ be the unpainted elements. If x is some element not of color j, we could paint it with that color if $x \cup E_j$ were independent in $\mathcal{M}_j$. (We write '$x \cup E_j$' as shorthand for '$\{x\} \cup E_j$'.) On the other hand, if $x \cup E_j$ is dependent, there is a unique circuit $P \subseteq x \cup E_j$, and we can paint x with color j if the color of any element y of $P \cap E_j$ is scraped off. Then perhaps we can paint y with some other color.

A sequence of such repaintings might be denoted by, say,

$$x \to y \to z \to 0_3$$

meaning "paint x with the current color of y, then repaint y with the current color of z, then repaint z with color 3." In general we may write

$$x \to y \iff x \cup E_j \setminus y \text{ is independent in } \mathcal{M}_j$$

when $y \in E_j$ and $x \notin E_j$; and

$$x \to 0_j \iff x \cup E_j \text{ is independent in } \mathcal{M}_j$$

where x is an element of $E \setminus E_j$ and 0_j is a special symbol distinct from the elements of E. (We may think of 0_j as a "standard" element of color j, whose color never needs to be washed off.) Notice that if $x \to 0_j$ then $x \to y$ for all $y \in E_j$.

Lemma. *In terms of the notation above, let*

$$x = x_0 \to x_1 \to \cdots \to x_m = 0_t,$$

where $x_i \not\to x_j$ for $j > i+1$. Then if x_i is painted the color of x_{i+1}, for $0 \le i < m$, the resulting elements of color j are independent in $\mathcal{M}_j$ for $1 \le j \le k$.

Proof. The result is trivial when $m = 1$. If $m > 1$, consider what happens after making just the mth step of the repainting: Let x_{m-1} have color s, and let

$$E'_j = \begin{cases} E_j \cup x_{m-1}, & \text{if } j = t; \\ E_j \setminus x_{m-1}, & \text{if } j = s; \\ E_j, & \text{otherwise.} \end{cases}$$

Derivation of Good Characterizations

Theorem 1. *Let $\mathcal{M}_1, \ldots, \mathcal{M}_k$ be matroids on a set E. It is possible to find disjoint subsets $E_1, \ldots, E_k$ of E, such that E_j is independent in $\mathcal{M}_j$ and $|E_j| = n_j$, if and only if*

$$|A| \le |E| - \sum_{j=1}^{k} \max(n_j - r_j(A), 0)$$

for all $A \subseteq E$, where r_j is the rank function in $\mathcal{M}_j$.

Theorem 2. *Let $\mathcal{M}_1, \ldots, \mathcal{M}_k$ be matroids on a set E. It is possible to find disjoint subsets $E_1, \ldots, E_k$ of E, such that E_j is independent in $\mathcal{M}_j$ and $|E_j| \le n'_j$ and $E = E_1 \cup \cdots \cup E_k$, if and only if*

$$|A| = \sum_{j=1}^{k} \min(r_j(A), n'_j)$$

for all $A \subseteq E$, where r_j is the rank function in $\mathcal{M}_j$.

Theorem 3. *Let $\mathcal{M}_1, \ldots, \mathcal{M}_k$ be matroids on a set E, and let (n_j, n'_j) be pairs of numbers with $n_j \le n'_j$ for $1 \le j \le k$. It is possible to find disjoint subsets $E_1, \ldots, E_k$ of E, such that E_j is independent in $\mathcal{M}_j$ and $n_j \le |E_j| \le n'_j$ and $E = E_1 \cup \ldots \cup E_k$, if and only if both the conditions of Theorems 1 and 2 hold for all $A \subseteq E$.*

The Algorithm

begin $E_0 := E$;
for $j := 1$ **until** k **do** $E_j := \emptyset$;
for $x \in E$ **do** *color*$(x) := 0$;
for $j := 1$ **until** k **do** *color*$(0_j) := j$;
for $j := 1$ **until** k **do for** $i := 1$ **until** n_j **do** *augment*(j);
while $E_0 \ne \emptyset$ **do** *augment*(0);
for $j := 1$ **until** k **do output** E_j;
exit: **end**.

procedure *augment* (**integer value** t);
begin for $x \in E$ **do** *succ*$(x) :=$ **none**;
$A := E$; $B :=$ **if** $t > 0$ **then** $\{0_t\}$ **else** $\{0_j \mid |E_j| < n'_j\}$;
comment Later *succ*(x) will be set to y if we find a shortest path $x \to y \to^* 0_j$ for some 0_j now in B. Also $A = \{x \mid succ(x) = \textbf{none}\}$;
while $B \ne \emptyset$ **do**
begin $C := \emptyset$;
for $y \in B$ **do for** $x \in A$ **do**
begin $j :=$ *color*(y);
if $x \cup E_j \setminus y$ is independent in $\mathcal{M}_j$ **then**
begin *succ*$(x) := y$; $A := A \setminus x$; $C := C \cup x$;
if *color*$(x) = 0$ **then go to** repaint;
end;
end;
$B := C$;
end;
output A;
output "This set A violates the condition of Theorem ";
output if $t > 0$ **then** "1" **else** "2"; **go to** exit;

References

[1] Henry H. Crapo and Gian-Carlo Rota, *On the Foundations of Combinatorial Theory: Combinatorial Geometries*, preliminary edition (Cambridge, Massachusetts: MIT Press, 1970).

[2] Jack Edmonds, "Matroid partition," in *Mathematics of the Decision Sciences*, part 1, edited by G. B. Dantzig and A. F. Veinott, Jr. (Providence, Rhode Island: American Mathematical Society, 1968), 335–345.

[3] Jack Edmonds, "Lehman's switching game and a theorem of Tutte and Nash-Williams," *Journal of Research of the National Bureau of Standards* **69B** (1965), 73–77.

[4] Jack Edmonds, "Submodular functions, matroids, and certain polyhedra," in *Combinatorial Structures and their Applications*, Proceedings of a conference held at the University of Calgary in 1969, edited by Richard Guy, Haim Hanani, Norbert Sauer, and Johanan Schönheim (New York: Gordon and Breach, 1970), 69–87.

[5] L. R. Ford, Jr., and D. R. Fulkerson, "Network flow and systems of representatives," *Canadian Journal of Mathematics* **10** (1958), 78–84.

[6] L. R. Ford, Jr., and D. R. Fulkerson, *Flows in Networks* (Princeton, New Jersey: Princeton University Press, 1962), §2.10.

[7] John E. Hopcroft and Richard M. Karp, "An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs," *SIAM Journal on Computing* **2** (1973), 225–231.