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Chapter 3.2
Recurrence Relations (Linear, First Order)

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Properties of recurrence relations

- Recurrence relation is an equation that defines a **sequence recursively**
  - Each term is defined as a **function of the preceding terms**
- A linear recurrence relation can be written as

$$S(n) = f_1(n)S(n-1) + f_2(n)S(n-2) + \ldots + f_k(n)S(n-k) + g(n)$$

where f's and g are or can be expressions involving n.

- **Example: for S(n) = 2*S(n-1), what is $f_1(n)$ = ?, $f_2(n)$ = ?, g(n) = ?**
  - **we have $f_1(n) = 2$, $g(n) = 0$**

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Linearity, Homogeneity, and Orders

- A linear relation is when the **earlier values** in the definition of S(n) as shown below have **power 1**.
  - The term "linear" means that each term of the sequence is defined as **a linear function of the preceding terms**.
  - Example: $F(n) = F(n\text{-}1) + F(n\text{-}2)$
- A nonlinear relation is the one that has **earlier values** in the definition as **powers other than 1**.
  - Example: $G(n) = 2nG(n\text{-}1) – 3G^2(n\text{-}2)$
  - Solutions are quite complex.

- Homogenous relation is a relation that has **g(n) = 0** for all n
  - Example: $S(n) = 2S(n\text{-}1)$, $F(n) = F(n\text{-}1) + F(n\text{-}2)$
- Non-homogenous relation:
  - Example: $a(n) = a(n\text{-}1) + 2n$

$$S(n) = f_1(n)S(n\text{-}1) + f_2(n)S(n\text{-}2) + \ldots + f_k(n)S(n\text{-}k) + g(n)$$

Linearity, Homogeneity, and Orders

- A recurrence relation is said to have **constant coefficients** if the *f's are all constants*.
  - Fibonacci relation is **homogenous, linear and constant coefficients**:
    - $F(n) = F(n\text{-}1) + F(n\text{-}2)$
  - Non-constant coefficients: $T(n) = 2nT(n\text{-}1) + 3n^2T(n\text{-}2)$
- **Order** of a relation is defined by the **number of previous terms in a relation for the n^{th} term**.
  - **First order:** $S(n) = 2S(n\text{-}1)$
    - n^{th} term depends only on term n-1
  - Second order: $F(n) = F(n\text{-}1) + F(n\text{-}2)$
    - n^{th} term depends only on term n-1 and n-2
  - Third Order: $T(n) = 3nT(n\text{-}2) + 2T(n\text{-}1) + T(n\text{-}3)$
    - n^{th} term depends only on term n-1, n-2 and n-3

$$S(n) = f_1(n)S(n\text{-}1) + f_2(n)S(n\text{-}2) + \ldots + f_k(n)S(n\text{-}k) + g(n)$$

Solving recurrence relations

- Solving a recurrence relation employs finding a **closed-form solution for the recurrence relation**.
- An equation such as $S(n) = 2^n$, where we can **substitute a value for n** and **get the output value back directly**, is called a **closed-form solution**. *No dependence on earlier values.*

- Two methods used to solve a recurrence relation:
  - **Expand, Guess, Verify**
    - **Repeatedly uses the recurrence relation to expand** the expression for the n^{th} term until the general pattern can be guessed.
    - Finally, the guess is **verified by mathematical induction.**
  - **Solution from a formula**
    - Known **solution formulas** can be derived for some types of recurrence relations.

Expand, Guess, and Verify

- **Find the closed form solution for the following recurrence relation:**

  $S(1) = 2$

  $S(n) = 2S(n\text{-}1)$ for $n \geq 2$
- Expansion: Using the recurrence relation *over again every time*
  - $S(n) = 2\boldsymbol{S(n\text{-}1)}$ *Expand using,* $S(n\text{-}1) = 2S(n\text{-}2)$

  $\Rightarrow S(n) = 2(2S(n\text{-}2)) = 2^2\boldsymbol{S(n\text{-}2)}$

  $\Rightarrow S(n) = 2^2(2S(n\text{-}3)) = 2^3S(n\text{-}3)$
- Looking at the developing pattern, we guess that after ***k* such expansions**, the equation has the form
  - $S(n) = 2^kS(\boldsymbol{n\text{-}k})$
- What is the last expansion? This should stop when $\boldsymbol{n\text{-}k = 1}$, hence $\boldsymbol{k = n\text{-}1}$,
  - As the base case provided is $S(1)$
- $S(n) = 2^{n\text{-}1}S(1) \Rightarrow S(n) = 2.2^{n\text{-}1} = 2^n$
- Do the verification step using mathematical induction

Verification step for Expand, Guess, and Verify

- Confirm derived closed-form solution by induction for all values of 'n' specified in recurrence relation
  - Statement to prove: **$S(n) = 2^n$ for $n \geq 1$**.
- For the **basis step**, $S(1) = 2^1$. This is true since $S(1)$ is provided in the problem.
- Assume that $S(k) = 2^k$, for any arbitrary positive integer k
- Then try to prove: $S(k+1) = 2^{k+1}$
  - $S(k+1) = 2S(k)$ (by using the recurrence relation definition)
    $= 2(2^k)$ (by using the above inductive hypothesis)

  $\Rightarrow S(k+1) = 2^{k+1}$
- This proves that S(n) is true for any positive integer n
- This proves that our closed-form solution is correct.

Class Exercise

Find the closed form solution for the following recurrence relation:

- $T(1) = 1$
- $T(n) = T(n-1) + 3$ for $n \geq 2$

Solution:

[EXPAND Step]

$T(n) = \boldsymbol{T(n-1)} + 3$

$= [T(n-2)+3] + 3 = \boldsymbol{T(n-2)} + 3 \times \mathbf{2}$

$= [T(n-3)+3] + 3 \times 2 = \boldsymbol{T(n-3)} + 3 \times \mathbf{3}$

[GUESS] In general after 'k' expansions, we guess that $\boldsymbol{T(n) = T(n-k) + 3k}$

Last Expansion: When $n-k = 1$, i.e., $k = n-1$

$T(n) = T(1) + (n-1) \times 3 = 1 + (n-1) \times 3 = 3n-2$

[VERIFY Step] Prove by induction that, T(n) = 3n-2, for n >= 1

Base Step: $T(1) = 3(1)-2 = 1$, true as given already

Inductive Step: Assume $T(k) = 3k-2$ is true for any arbitrary positive integer k

To show: $T(k+1) = 3(k+1) -2 = 3k+1$

$T(k+1) = T(k) + 3$ *(from the given recurrence relation)*

$T(k+1) = 3k-2+3 = 3k+1$ *(by inductive hypothesis)*

Hence, $T(k+1) = 3k+1$, Thus verified. Thus, T(n) is true for any positive integer n

Solution from a formula

- Solution formula for **linear first order constant coefficient relation**

  $S(n) = f_1(n)S(n\text{-}1) + f_2(n)S(n\text{-}2) + \ldots + f_k(n)S(n\text{-}k) + g(n)$

  $S(n) = cS(n\text{-}1) + g(n)$ -------------------- (1)

  General form of **linear first order recurrence relation with constant coefficient**

  $S(n) = c[cS(n\text{-}2)+g(n\text{-}1)] + g(n) = c[c[cS(n\text{-}3)+g(n\text{-}2)] + g(n\text{-}1)] + g(n)$.

  After k expansions, $S(n) = c^kS(n\text{-}k) + c^{k\text{-}1}g(n\text{-}(k\text{-}1)) + \ldots + cg(n\text{-}1) + g(n)$

  The lowest value of n-k is 1 (n-k=1, so k=n-1)

  Hence, $S(n) = c^{n\text{-}1}S(1) + c^{n\text{-}2}g(2) + c^{n\text{-}3}g(3) + \ldots + g(n)$

$$\boxed{S(n) = c^{n-1}S(1) + \sum_{i=2}^{n} c^{n-i}g(i)}$$

For $S(n) = 2S(n\text{-}1)$, $c = 2$ and $g(n) = 0$ (Using (1))

Hence, $S(n) = 2^{n\text{-}1}S(1) + 0 = 2.2^{n\text{-}1} = 2^n$ since $S(1) = 2$

How to solve a recurrence relation?

To solve recurrence relations of the form S(n) = cS(n-1)+g(n) subject to basis S(1)	
Method	Steps
Expand, guess, and verify	1. **Repeatedly use** the recurrence relation until you can **guess a pattern** 2. Decide what that pattern will be when n-k=1 3. **Verify** the resulting formula by induction
Solution formula	1. Match your recurrence relation to the form **S(n)=cS(n-1) + g(n) to find c and g(n)** 2. Use c, g(n), and S(1) in the formula $\boxed{S(n) = c^{n-1}S(1) + \sum_{i=2}^{n} c^{n-i}g(i)}$ 3. **Evaluate** the resulting summation to get the final expression

Class Exercise

- Find a closed-form solution to the recurrence relation
  - $S(n) = 2S(n\text{-}1) + 3$ for $n \geq 2$ and given $S(1) = 4$
  - **Here, c=2 and g(*n*) = 3, Thus,**

$$\begin{aligned} \boldsymbol{S(n)} &= \boldsymbol{2^{n-1}S(1) + \textstyle\sum_{i=2}^{n} 2^{n-i}3} \\ &= 2^{n-1}4 + 3(2^{n-2} + 2^{n-3} + \cdots + 2^2 + 2^1 + 2^0) \\ &= 2^{n-1}\,2^2 + 3(2^{n-2} + 2^{n-3} + \cdots + 2^2 + 2^1 + 2^0) \\ &= \boldsymbol{2^{n+1} + 3(2^{n-1} - 1)} \end{aligned}$$

*(using sum of terms in a **geometric progression**)*

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

- Find a closed-form solution to the recurrence relation
  - $T(n) = T(n\text{-}1) + (n\text{+}1)$ for $n \geq 2$ and given $T(1) = 2$
  - **Here, c = 1 and g(*n*) = *n*+1, Thus**

$$\begin{aligned} \boldsymbol{T(n)} &= \boldsymbol{1^{n-1}}T(1) + \textstyle\sum_{i=2}^{n} \boldsymbol{1^{n-i}g(i)} \\ &= \boldsymbol{1^{n-1}}2 + [\boldsymbol{g(2)} + \boldsymbol{g(3)} + \cdots + \boldsymbol{g(n)}] \\ &= 2 + [\boldsymbol{3} + \boldsymbol{4} + \cdots + (\boldsymbol{n} + \boldsymbol{1})] \\ &= \tfrac{n}{2}(2 * 2 + (n-1) * 1) = \tfrac{n}{2}\,(n+3) \end{aligned}$$

*(using sum of terms in an **arithmetic progression**)*

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

- Solutions for these exercises is in the text (pg. 184-186, Example 17 & 18)

$$\boxed{S(n) = c^{n-1}S(1) + \sum_{i=2}^{n} c^{n-i}g(i)}$$

Discussion