Chapter 4.4
Permutations and Combinations

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Permutations and Combinations

- **A = {1,2,3}; |A| = 3**
- What are the ways for ***arranging/ordering*** *the chosen 3 elements*?
  - (1,2,3),
  - (1,3,2),
  - (2,1,3),
  - (2,3,1),
  - (3,1,2),
  - (3,2,1)
- How many ways can you ***select/choosing*** 3 elements (number of 3 element combinations)
  - Only one: (1, 2, 3)

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Permutations

- **An <u>ordered arrangement</u> of objects**
  - Hence, a permutation of *n* <u>distinct elements</u> is an ordering of these *n* elements.

- ***Ordering*** of last four digits of a telephone number if digits are <u>allowed to repeat?</u>

  = 10*10*10*10 = 10000

- ***Ordering*** of four digits if <u>repetition is not allowed</u>

  = 10*(10-1)*(10-2)*(10-3) = 10*9*8*7 = 5040

  - Number of permutations/ orderings/ arrangements of 4 distinct digits chosen from 10 integers?

Permutations

- In general, the number of permutations/orderings of ***r* distinct objects chosen from *n* objects,** for $r \leq n$
- Denoted by $P(n,r)$ or ${}_nP_r$

$$P(n,r) = n*(n-1)*(n-2)*\ldots*(n-r+1) = \frac{n*(n-1)*(n-2)*\ldots*(n-r+1)*(n-r)!}{(n-r)!}$$

$$\Rightarrow P(n,r) = \frac{n!}{(n-r)!} \qquad \text{for } 0 \leq r \leq n$$

where $n! = n*(n-1)*(n-2)*\ldots*3*2*1$ and by definition $0! = 1$

- ***Ordering*** of four digits if <u>repetition is not allowed</u>
  - Hence, 10*9*8*7 = *P*(10,4) = 10! / (10-4)! = 10!/6! = 5040

Permutations: Some special cases

- $P(n,0) = n! / n! = 1$
  - This means that there is **only one ordered arrangement of 0 objects**, called the empty set.
- $P(n,1) = n!/ (n-1)! = n$
  - There are ***n* possible ordered arrangements of one object** (i.e. *n* ways of selecting one object from *n* objects).
- $P(n,n) = n!/(n-n)! = n!/0! = n!$
  - This means that **one can arrange *n* distinct objects in *n*! ways**, that is nothing but the multiplication principle.

Permutation Examples

- Ten athletes compete in an Olympic event. Gold, silver and bronze medals are awarded to the first three in the event, respectively. *How many ways can the awards be presented to the athletes?*
  - *How many possibilities of 1^{st}, 2^{nd} and 3^{rd} place exist?*
  - ***Ordering*** *of 3 athletes from a pool of 10 athletes*
  - Hence, $P(10,3) = 10!/7! = 10*9*8 = 720$

- How many ways can six people be seated on six chairs?
  - *How many possibilities for each chair?*
  - *Orderings of 6 people out of 6 people*
  - $P(6,6) = 6!/0! = 6*5*4*3*2*1 = 720$

Permutation Examples

- How many permutations of the letters **ABCDEF** contain the letters **DEF together** in any order?
  - **DEF** should be together. Thus, consider it as a single group
  - **Total groups** to be ordered = **1** (for DEF) + **3** (for A, B, C) = 4
  - Total Orderings of 4 groups = **4!**
  - **AND**, *DEF has 3 letters* which can be ***arranged in 3! Ways***
  - Total required orderings = **4!*3! = 144** (by ***multiplication principle***)

- The professor's dilemma: how to arrange four books on OS, seven on programming, and three on data structures on a shelf such that books on the same subject must be together?
  - *Books on the same subject must be together* = **3 groups** of books
    - Total Ordering for the 3 groups = **3!**
  - **AND**, Number of orderings ***within each group*** = 4! * 7! * 3!
  - Final number of orderings possible = ***(4!*7!*3!)*3!*** *= 24*5040*6*6 = 4,354,560*

Combinations

- When we are ***just interested in <u>selecting/choosing</u> r*** **objects** from ***n*** **distinct objects** *(and not their ordering)*, we talk of combinations denoted by

  $C(n,r)$ or ${}_nC_r$

- For each combination, there are <u>$r!$ ways of **ordering/arranging** those r chosen objects</u>
- Hence, from multiplication principle,

$$C(n,r) * r! = P(n,r) \Rightarrow \boxed{C(n,r) = \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!r!} \text{ for } 0 \le r \le n}$$

Combinations: Special Cases

- ***C*(*n*,0) = 1**
  - Only **one way to choose 0 objects from *n* objects**-chose the empty set
- ***C*(*n*,1) = n**
  - Obvious, since ***n* ways to choose one object from *n* objects**
- ***C*(*n*,*n*) = 1**
  - Only **one way to choose *n* objects from *n* objects**

Combinations: Examples

- In how many ways can **three athletes be declared winners** from a group of 10 athletes who compete in an Olympic event?
  - ***Choose*** 3 athletes from 10 athletes
    - *Ordering/arrangement* ***not a requirement***
  - *C*(10,3) = 10!/(7!*3!) = 120
    - *much less than to award three winners medals*

Combinations: Examples

- How many ways can we select a committee of **three from 10**?
  - Choose 3 people from 10 people
    - *Again, ordering does not matter here*
  - C(10,3) = 120

- How many ways can a committee of **two women** and **three men** be selected from a group of **five different women** and **six different men**?
  - Selecting two out of five women: C(5,2) ways = 10, AND
  - Selecting three out of six men: C(6,3) ways = 20.
  - Total number of ways for selecting the committee = 10*20 = 200
    - *Multiplication principle: Sequence of events is required.*

Combinations: Examples

- How many **five-card poker hands** can be dealt from a standard **52-card deck**?
  - Choosing 5 cards from 52 cards
  - **C(52,5) = 2,598,960**

- How many **5-card** poker hands contain **all cards from the same suit**?
  - Choosing 5 cards from a given suit = C(13,5)
  - Number of suits = 4
  - Total 5-card hands where all cards from same suit
    = 4*C(13,5) = 5148

Combinations: Examples

- How many poker hands contain **three cards of one denomination** and **two cards of another denomination**?
  - **Sequence of events**
    - Select first denomination: **13 possibilities**
    - Select 3 cards from this selected denomination from the 4 suits: **C(4,3)**
    - Select the *second* denomination: **12 possibilities**
    - Select two cards from this second selected denomination: **C(4,2)**
  - Total = **13*C(4,3)*12*C(4,2) = 3744**

Always read carefully

- How many ways can a **committee of two** be chosen from **four men** and **three women** and it ***must include at least one man***.
  - Total possible committees with 2 people = C(7,2)
  - Possibilities: MM, MW, WW (not allowed)
  - Total possibilities with both women = C(3,2)
  - **Required committee possibilities = C(7,2) – C(3,2)**

Eliminating Duplicates

- How many **distinct permutations** can be made from the **characters** in the word **FLORIDA**?
  - Total Characters = 7
  - Total Orderings of 7 characters = 7!

- How many **distinct permutations** can be made from the characters in the word **MISSISSIPPI**?
  - 11! – Is this correct?
    - **MISSISSIPPI is same as MISSISSIPPI:** Duplicates/Look-Alikes
    - **4 S** present: Thus, ***each** arrangement has 4! look-alikes*
    - Need to be **considered once**. Thus, 11!/4!
  - Actual Distinct Permutations = $\frac{11!}{4!4!2!}$
    - 4 **S**s, 4 **P**s and 2 **I**s

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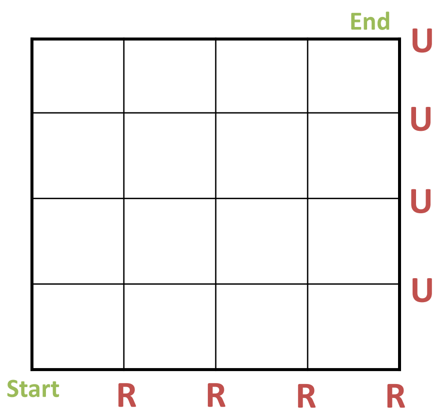
Combinations: Examples

- How many unique routes are there from the **lower-left corner of a *4* by *4* square grid** to the **upper-right corner** if we are ***restricted to traveling only to the right (R) or upward (U)***?



- One possible route
  - RRRRUUUU
- All Routes will be made up of 4 Rs and 4 Us
- For each route, there are 4! duplicate orderings for R (same with U)
- *Problem: Number of unique arrangements of 4 **R** and 4 **U***
- Final = 8!/(4!*4!)

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Permutations with Repetitions

- **Example:** We have 26 alphabets. We'd like to make 3-letter codes with *repetition allowed in every position*.
  - First position options: 26
  - Second position options: 26
  - Third position options: 26
  - **Total 3-letter codes = 26^3**

- **General Concept**: If out of n distinct objects, we are to find the *number of orderings/permutations* of r objects with *repetition allowed*, then it is n^r

Summary of Counting Techniques

TABLE 4.2

You Want to Count the Number of ...	Technique to Try
Subsets of an n-element set	Use formula 2^n.
Outcomes of successive events	Multiply the number of outcomes for each event.
Outcomes of disjoint events	Add the number of outcomes for each event.
Elements in overlapping sections of related sets	Use principle of inclusion and exclusion formula.
Ordered arrangements of r out of n distinct objects	Use $P(n, r)$ formula.
Ways to select r out of n distinct objects	Use $C(n, r)$ formula.

Class Exercises

- How many permutations of the characters in the word COMPUTER are there? How many of these end in a vowel? **8!, 3·7!**
- How many distinct permutations of the characters in ERROR are there? **5!/3!**
- A set of four coins is selected from a box containing five dimes and seven quarters.
  - Find the number of sets which has two dimes and two quarters.
    C(5,2)*C(7,2) = 10*21 = 210
  - Find the number of sets composed of all dimes or all quarters.
    C(5,4) + C(7,4) = 5 + 35 = 40
- In how many ways can you seat 11 men and 8 women in a row if no two women are to sit together?
  - Number of slots where women can sit = 12 (only between, before and after men)
  - Choose 8 slots out of 12 slots for women to sit AND then arrange them = C(12,8) * 8! AND
  - Arrange the Men too = **C(12,8) * 8! * 11!**

Discussion