Chapter 1.3
Quantifiers, Predicates and Validity

Instructor: Abhishek Santra

Email: abhishek.santra@uta.edu

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Variable

- Definition
  - A symbol that stands for or represents **an individual** in a **collection or set**.
- Example
  - The variable x may stand for one of the days.
  - We may let x **= Monday** or x **= Tuesday**, etc.

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Incomplete and Complete Statement

- Definition (Incomplete Statement)
  - **A sentence containing a variable is called an incomplete statement**.
- Example
  - Example of an incomplete statement: "x has 30 days."
  - Here, x can be any month and substituting that, we will get a **complete statement**.
- An incomplete statement is about the **individuals in a definite domain or set**. When we *replace the variable by the name of an individual* in the set, we obtain a **statement about that individual**.

Quantifiers

- Quantifiers:
  - Quantifiers are phrases that refer to given quantities, such as "for some" or "for all" or "for every," indicating ***how many objects have a certain property.***
- Two kinds of quantifiers:
  - Universal and Existential
- **Universal Quantifier**: represented by ∀
  - The symbol is translated as and means "for all", "given any", "for each," or "for every," and is known as the universal quantifier.
- **Existential Quantifier**: represented by ∃
  - The symbol is translated as and means variously "for some," "there exists," "there is a," or "for at least one".

Predicates

- The verbal statement that describes the property of a variable.
- Usually represented by the letter P, the notation P(x) is used to represent some unspecified **property or predicate** that x may have
  - e.g. P(x) = x has 30 days.
  - P(April) = April has 30 days.
- Combining the quantifier and the predicate, we get a complete statement of the form $(\forall x)P(x)$ or $(\exists x)P(x)$.
- The collection of objects is called the **domain of interpretation**.
- **Truth value of expressions** formed using quantifiers and predicates
  - What is the truth value of **$(\forall x)P(x)$ where x is all the months** and **P(x) = x has less than 32 days**.
  - Undoubtedly, the above is true since no month has 32 days.

Truth value of the following expressions

- Truth of expression **$(\forall x)P(x)$**
  1. P(x) is the property that x is yellow, and the domain of interpretation is the collection of all flowers: **Not True**
  2. P(x) is the property that x is a plant, and the domain of interpretation is the collection of all flowers: **True**
  3. P(x) is the property that x is positive, and the domain of interpretation consists of integers: **Not True**
  - Can you find one interpretation in which both $(\forall x)P(x)$ is true and $(\exists x)P(x)$ is false? **Not possible**
  - Can you find one interpretation in which both $(\exists x)P(x)$ is true and $(\forall x)P(x)$ is false? **Case 1 as mentioned above**

- Predicates involving properties of single variables : **unary predicates**
- **Binary, ternary and *n*-ary** predicates are also possible.
  - $(\forall x)(\exists y)Q(x, y)$ is a binary predicate. This expression reads as "for every x there exists a y such that Q(x, y)., say x is a student of y"

Interpretation

- An interpretation for an expression (e.g. **($\forall x$)P(x)**), consists of the following:
  - A **collection of objects**, called the **domain of interpretation**, which must include at least one object. E.g., x is **all integers**
  - An **assignment of a property** of the objects in the domain to **each predicate** in the expression. E.g., **P(x) = x < 10**
  - An **assignment of a particular object** in the domain to **each constant symbol** in the expression. E.g., **P(7) = 7 < 10**
  - **How many interpretations are possible?**
- **Predicate wffs** can be built similar to *propositional wffs* using **logical connectives with predicates and quantifiers**.
- Examples of **predicate wffs**
  - $(\forall x)[P(x) \to Q(x)]$
  - $(\forall x)\ ((\exists\ y)[P(x, y) \lor Q(x, y)] \to R(x))$
  - $S(x, y) \land R(x, y)$

Instant Exercises

- What is the **truth value** of each of the following wffs in the interpretation where the domain consists of the integers, $O(x)$ is "x is odd," $L(x)$ is "x<10," and $G(x)$ is "x>9"?
  - A. $(\exists x)\ O(x)$ **True**
  - B. $(\forall x)[L(x) \to O(x)\ ]$ **Not True**
  - C. $(\exists\ x)[L(x) \land G(x)]$ **Not True**
  - D. $(\forall x)[L(x) \lor G(x)]$ **True**
- $(\forall x)\ (\exists y)\ Q(x,y)$ vs. $(\exists y)\ (\forall x)\ Q(x, y)$?
  - Domain of interpretation is integers
  - $Q(x, y) : x < y$
  - $(\forall x)\ (\exists y)\ Q(x, y)$ → For any integers (x), there is a larger integer (y) **True, imagine 'y' inside the for loop over 'x'**
  - $(\exists y)\ (\forall x)\ Q(x, y)$ → There is a single integer (y), that is larger than any integer (x) **Not True, imagine 'y' being initialized before the for loop over 'x'**

Scope of a variable in an expression

- **Brackets** are used wisely to identify the **scope of the variable**.
  - $(\forall x)\ [\exists(y)[P(x, y) \vee Q(x, y)] \to R(x)]$
    - Scope of $(\exists y)$ is $P(x, y) \vee Q(x, y)$ while the scope of $(\forall x)$ is the entire expression.
  - $(\forall x)S(x) \vee (\exists y)R(y)$
    - Scope of x is $S(x)$ while the scope of y is $R(y)$.
  - $(\forall x)[P(x, y) \to (\exists y)\ Q(x, y)]$
    - Scope of variable y is not defined for $P(x, y)$ hence y is called a **free variable**. Such expressions might not have a truth value at all.
- What is the truth value of the expression
  - $\exists(x)[A(x) \wedge (\forall y)[B(x, y) \to C(y)]]$ in the interpretation, where
  - $A(x)$ is "$x > 0$" , $B(x, y)$ is "$x > y$" and $C(y)$ is "$y \le 0$" where the domain of x is positive integers, and the domain of y is all integers

True, $x = 1$ is a positive integer and any integer less than x is ≤ 0

Translation: Verbal statements to Symbolic form

- "Every person is nice" can be rephrased as "For any thing, if it is a person, then it is nice."
- So, if $P(x)$ is "x is a person" and $Q(x)$ be "x is nice," the statement can be symbolized as
  - $(\forall x)[P(x) \to Q(x)]$
  - Variations: "All persons are nice" or "Each person is nice"
  - *How about* $(\forall x)[P(x) \wedge Q(x)]$?
    - Domain: whole world, then not everything in the world is nice person.

Translation: Verbal statements to Symbolic form

- "There is a nice person" can be rewritten as "There exists something that is both a person and nice."
  - $(\exists x)[P(x) \wedge Q(x)]$
  - Variations: "Some persons are nice" or "There are nice persons."
  - *How about* $(\exists x)[P(x) \rightarrow Q(x)]$ *?*
    - If x is a person, then x is nice?
- So **almost always**, $\exists$ goes with $\wedge$ (conjunction) and $\forall$ goes with $\rightarrow$ (implication).

Translation

- To translate an English statement into wff, **use intermediate English statement**
- The word "only" can be tricky depending on its presence in the statement.
  - X loves **only** Y $\Leftrightarrow$ If X loves anything, then that thing is Y.
  - **Only** X loves Y $\Leftrightarrow$ If anything loves Y, then it is X.
  - X **only** loves Y $\Leftrightarrow$ If X does anything to Y, then it is love.
- Example for forming symbolic forms from predicate symbols
  - $J(x)$ is "x is John"; $M(x)$ is "x is Mary"; $L(x, y)$ is "x loves y"
  - John loves only Mary $\Leftrightarrow$

    For any thing, if it is John then, if it loves anything, that thing is Mary $\Leftrightarrow$

    $(\forall x)[J(x) \rightarrow (\forall y)(L(x, y) \rightarrow M(y))]$
  - Only John loves Mary $\Leftrightarrow$

    For any thing, if it is Mary then, if anything loves it, that thing is John $\Leftrightarrow$

    $(\forall x)\ [M(x) \rightarrow (\forall\ y)\ (L(y, x) \rightarrow J(y))]$
  - John only loves Mary $\Leftrightarrow$

    For any thing, if it is John then, for any other thing, if that thing is Mary, then John loves it $\Leftrightarrow$

    $(\forall x)[J(x) \rightarrow (\forall\ y)(M(y) \rightarrow L(x, y))]$

Tips for translation to predicate wff

- Textbook p45
- Look for the key words that signify the **type of quantifier**
  - For all, for every, for any, for each : **universal quantifier**
  - For some, there exists: **existential quantifier**
- English sometimes uses **“understood”** universal quantifiers
  - “Dogs chase rabbits” is understood to mean, “All dogs chase all rabbits.”
- If you use a universal quantifier, the connective that goes with is almost always “implication”
- If you use an existential quantifier, the connective that goes with is almost always “conjunction”
- Whatever comes after the word “**only**” is the conclusion of an implication.
- You are most apt to arrive at a correct translation if you follow the **order of the English words**

Class exercise

- $S(x)$: x is a student; $I(x)$: x is intelligent; $M(x)$: x likes music
- Write wffs that express the following statements:
  - **All students are intelligent.**
  - For anything, if it is a student, then it is intelligent ⇔
  - $(\forall x)[S(x) \rightarrow I(x)]$

  - **Some intelligent students like music.**
  - There is something that is intelligent, and it is a student, and it likes music ⇔
  - $(\exists x)[I(x) \wedge S(x) \wedge M(x)]$

  - **Everyone who likes music is a stupid student.**
  - For anything, if that thing likes music, then it is a student and it is not intelligent ⇔
  - $(\forall x)(M(x) \rightarrow S(x) \wedge [I(x)]')$

  - **Only intelligent students like music.**
  - For any thing, if it likes music, then it is a student and it is intelligent ⇔
  - $(\forall x)(M(x) \rightarrow S(x) \wedge I(x))$

Negation of statements

- A(x): x is fun, $(\forall x)$A(x): Everything is fun
- Negation will be "it is false that everything is fun," i.e., "*something* is *not fun*."
- *Opposite: Every -> Some*
- In symbolic form, $[(\forall x)A(x)]' \Leftrightarrow (\exists x)[A(x)]'$

- Similarly, negation of "Something is fun" is "Nothing is fun" or "Everything is not fun/boring."
- Hence, $[(\exists x)A(x)]' \Leftrightarrow (\forall x)[A(x)]'$

Class Exercise

- What is the negation of "Everybody loves somebody sometime."
  - Everybody hates somebody sometime
  - Somebody loves everybody all the time
  - Everybody hates everybody all the time
  - Somebody hates everybody all the time √

- What is the negation of the following statements?
  - **Some pictures are old or faded.**
  - Every picture is neither old nor faded.
  - **All people are tall and thin.**
  - Someone is short or fat.
  - **Some students eat only pizza.** *(Recall, the John and Mary example)*

  - **Only students get t-shirts.**

Negation Approach

- What is the negation of the following statements?
  - **Some students eat only pizza.**
    - *Recall, John loves only Mary example:* $(\forall x)[J(x) \rightarrow (\forall y)(L(x, y) \rightarrow M(y))]$
    - Let S(x): x is student, E(x,y): x eats y, P(x): x is pizza
    - However, "John" is "some students". Thus, change the quantifier
    - Final Expression: $(\exists x)[S(x) \rightarrow (\forall y)(E(x, y) \rightarrow P(y))]$
    - Take Negation: $((\exists x)[S(x) \rightarrow (\forall y)(E(x, y) \rightarrow P(y))])'$

1.	$(\forall x)[S(x) \rightarrow (\forall y)(E(x, y) \rightarrow P(y))]'$	
2.	$(\forall x)[S(x)' \vee (\forall y)(E(x, y)' \vee P(y))]'$	1, Imp
3.	$(\forall x)[S(x) \wedge ((\forall y)(E(x, y)' \vee P(y)))']$	2, DM
4.	$(\forall x)[S(x) \wedge (\exists y)(E(x, y)' \vee P(y))']$	3, Negation
5.	$\mathbf{(\forall x)[S(x) \wedge (\exists y)(E(x, y) \wedge P(y)')]}$	4, DM

  - ***Every student eats something that is not pizza.***

Negation Approach

- What is the negation of the following statements?
  - **Only students get t-shirts.**
    - Let S(x): x is student, T(x): x gets t-shirt
    - Final Expression: $(\forall x)[T(x) \rightarrow S(x)]$
    - Take Negation: $((\forall x)[T(x) \rightarrow S(x)])'$

1.	$(\exists x)\ [T(x) \rightarrow S(x)]'$	
2.	$(\exists x)[T(x)' \vee S(x)]'$	1, Imp
3.	$\mathbf{(\exists x)[T(x) \wedge S(x)']}$	**2, DM**

  - ***There is a non-student who gets a t-shirt.***

Validity of Predicate wff

- Analogous to a **tautology of propositional logic.**
- Truth of a predicate wff *depends on the interpretation*.
- **A predicate wff is valid if it is true in all possible interpretations just like a propositional wff is true if it is true for all rows of the truth table.**
- A valid predicate wff is intrinsically true.

	Propositional Wffs	Predicate Wffs
Truth values	True or false – depends on the truth value of statement letters	True, false or neither (if the wff has a free variable)
Intrinsic truth	Tautology – true for all truth values of its statements	Valid wff – true for all interpretations
Methodology	Truth table to determine if it is a tautology	**No algorithm to determine validity** *(Use Reasoning; Can you find an interpretation where the expression is false?)*

Validity Examples

- $(\forall x)P(x) \rightarrow (\exists x)P(x)$
  - This is **valid** because if every object of the domain has a certain property, then there exists an object of the domain that has the same property.
- $(\forall x)P(x) \rightarrow P(a)$
  - **Valid** – quite obvious since a is a member of the domain of x.
- $(\exists x)P(x) \rightarrow (\forall x)P(x)$
  - **Not valid** since the property cannot be valid for **all objects** in the domain if it is valid for **some objects** of that domain. Can use a **mathematical context** to check as well.
  - *Say $P(x)$ = "x is even," then there exists an integer that is even but not every integer is even (domain: integer)*
- $(\forall x)[P(x) \vee Q(x)] \rightarrow (\forall x)P(x) \vee (\forall x)Q(x)$
  - **Invalid**, can show by mathematical context by taking $P(x) = x$ is even and $Q(x) = x$ is odd (domain: integer). **What is the scope of 'x' in each case?**
  - In that case, the hypothesis is true but the conclusion is false because it is not the case that every integer is even or that every integer is odd.

Class + Home Exercise - 1

- What is the truth of the following wffs where the domain consists of **integers**:
  1. $(\forall x)[L(x) \to O(x)]$,
     where $O(x)$ is "x is odd" and $L(x)$ is "$x < 10$"?
     False *since not all integers < 10 are odd, e.g., 4, 6*
  2. $(\exists y)(\forall x)(x + y = 0)$?
     False*, no one y works for all x's, imagine y being initialized before the "for loop" over x*
  3. $(\exists y)(\exists x)(x^2 = y)$, where x, y > 1?
     True*, e.g., y = 4 and x = 2*
  4. $(\forall x)[x < 0 \to (\exists y)(y > 0 \wedge x + y = 0)]$?
     True*, pick y = -x*

Class + Home Exercise - 2

- Using predicate symbols and appropriate quantifiers, write the symbolic form of the following English statement:
  - $D(x)$ is "x is a day" M is "Monday" T is "Tuesday"
  - $S(x)$ is "x is sunny" $R(x)$ is "x is rainy"

1. Some days are sunny and rainy.
   $(\exists x)(D(x) \wedge S(x) \wedge R(x))$
2. It is always a sunny day only if it is a rainy day.
   $(\forall x)[D(x) \wedge S(x) \to D(x) \wedge R(x)]$
3. It rained both Monday and Tuesday.
   $R(M) \wedge R(T)$
4. Every day that is rainy is not sunny.
   $(\forall x)[D(x) \wedge R(x) \to (S(x))']$

Discussion