Chapter 5.4
Functions

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Function Terminologies

- Let S and T be sets.
- A function (mapping) ***f* from *S* to *T***, $f : S \rightarrow T$, is a **subset of** $\boldsymbol{S \times T}$, where **each member** of S appears **exactly once** as the first component of an ordered pair.
- S is the **domain** and T the **codomain** of the function
  - Domain: Set of Starting Values
  - Codomain: Set from which associated values come
- If (s,t) belongs to the function, then
  - $s \in S$, $t \in T$
  - t is denoted by $f(s)$, i.e., $\boldsymbol{t = f(s)}$
  - t is the **image** of s under f
  - s is a **preimage** of t under f, and
  - f is said to map s to t

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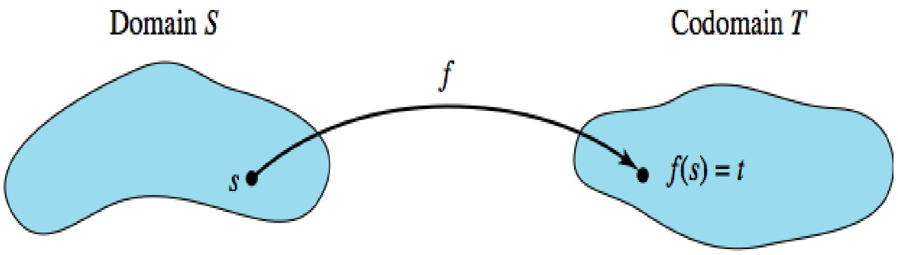
Function Representation



Function Examples

- Examples:
  - The **floor function** $\lfloor x \rfloor$ associates with each real number *x* the greatest integer less than or equal to *x*.
  - The **ceiling function** $\lceil x \rceil$ associates with each real number *x* the smallest integer greater than or equal to *x*.
  - Example: $\lfloor 2.8 \rfloor$ = ?, $\lceil 2.8 \rceil$ = ?, $\lfloor -4.1 \rfloor$ = ?, and $\lceil -4.1 \rceil$ = ?

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  - Example: $\lfloor 2.8 \rfloor = \mathbf{2}$, $\lceil 2.8 \rceil = \mathbf{3}$, $\lfloor -4.1 \rfloor = \mathbf{-5}$, and $\lceil -4.1 \rceil = \mathbf{-4}$
  - Both the floor function and the ceiling function are functions from $\mathbb{R}$ to $\mathbb{Z}$.
- Function from *S* to *T* is a subset of $S \times T$ with **certain restrictions on the ordered pairs it contains**.
  - **Each member of *S* must be used as a first component, exactly once**
  - By the definition of a function, a binary relation that is one-to-many (or many-to-many) cannot be a function

Function Example: Modulo Function

- Remember: Dividend = divisor * quotient + remainder
- For **any integer *x*** and any **positive integer *n*,** the **modulo function**, denoted by ***f* (*x*) = *x* mod *n*,** associates with *x* the remainder when *x* is divided by *n*
- One can write *x* as
  $x = qn + r$, $0 \le r < n$, where *q* is the quotient and *r* is the remainder, so the value of *x* mod *n* is *r*.
- Example:
  - 25 mod 2?

    $25 = 12 \cdot 2 + 1$, so 25 mod 2 = 1
  - -17 mod 5?

    $-17 = (-4) \cdot 5 + 3$, so -17 mod 5 = 3

Functions of more than one variable

- A function $f : S_1 \times S_2 \times \ldots \times S_n \to T$ that associates **each ordered *n*-tuple of elements** $(s_1, s_2, \ldots, s_n)$, $s_i \in S_i$ to unique **element of *T***
- Example
  - **f:Z ×N ×{1,2} →Z** is given by $f(x,y,z) = x^y+z$
  - **Then, f(-4,3,1) = ?**

    $(-4)^3+1 = -64+1 = -63$

Equal Functions

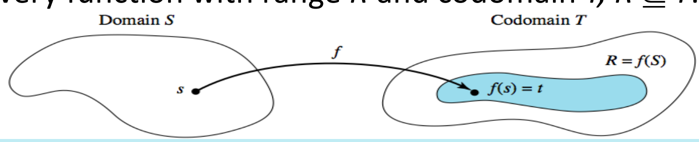
- g: **R** $\to$ **R,** where $g(x) = x^3$.
- f : **Z** $\to$ **R,** given by $f(x) = x^3$
- Are they same?
  - NO
  - f is not the same function as g
    - The domain has been changed, which **changes the set of ordered pairs**.

Equal Functions

- Two functions are **equal** if they have the **same domain**, the **same codomain**, and the s**ame association of values** of the codomain with values of the domain.
- To show that two functions with the same domain and the same codomain are equal, **one must show that the associations are the same.**
- This can be done by showing that
  - given an arbitrary element of the domain,
  - both functions produce the same associated value for that element; that is, they map it to the same place.

Properties of Function: Onto Functions

- **Range**: **Set** of <u>**images**</u> in $f: S \to T$
- In every function with range R and codomain T, $R \subseteq T$.



- A function $f: S \to T$ is an **onto,** or **surjective,** function **if the range of f equals the codomain of f.**
- **To prove** that a given function is onto,
  - Show that $T \subseteq R$; then it will be true that $R = T$.
    - Show that an **arbitrary member of the codomain is a member of the range**
  - **State a counter example to say not onto.**
- Is g: $\mathbf{R} \to \mathbf{R}$ where $g(x) = x^3$ an onto function?
  - For any y in R, is it a cube value? Yes
- Is g: $\mathbf{N} \to \mathbf{N}$ where $g(x) = x^3$ an onto function?
  - NO. '2' belongs to Codomain, but does not belong to Range

Properties of Function: One-to-One Functions

- A function $f: S \to T$ is **one-to-one,** or **injective,** if no member of T is the image under f of two distinct elements of S.
- Idea same as for binary relations in general, except that ***every element of S*** must appear as a first component in an ordered pair.
- To prove that a function is one-to-one, we **assume** that there are elements s_1 and s_2 of S such that $f(s_1) = f(s_2)$ and then show that $\boldsymbol{s_1 = s_2}$
- Is function g: $\mathbf{R} \to \mathbf{R}$ defined by $g(x) = x^3$ one-to-one?
  - Assume, a and b are real numbers with $g(a) = g(b)$, thus $a^3 = b^3$
  - This is only possible when $a = b$
  - *Thus, it is one-to-one*

Example: One-to-one function

- The function g: $\mathbf{R} \to \mathbf{R}$ defined by $g(x) = x^2$
  - *For any 2 real numbers a, b where $a^2 = b^2$, does it mean $a = b$?*
  - *Counter Example*

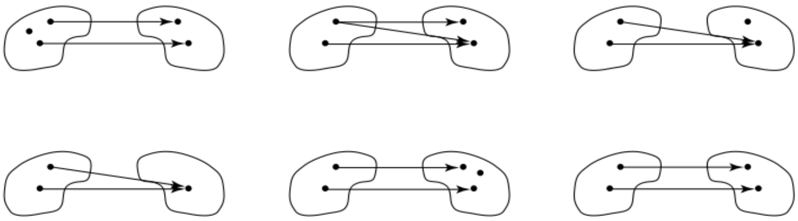
    g(2) = g(-2) = 4

    But, 2 is not equal to -2
  - **Not one-to-one**
- The function h: $\mathbf{N} \to \mathbf{N}$ defined by $h(x) = x^2$
  - **Is one-to-one**
  - If a and b are nonnegative integers with $h(a)=h(b)$, then $a^2=b^2$
  - Because a and b are both nonnegative, $a = b$

Properties of Function: Bijections

- A function $f{:}S \to T$ is **bijective** (a **bijection**) if it is **both one-to-one and onto**.
- The function g: **R** $\to$ **R** given by $g(x) = x^3$ is a bijection.

Are these functions? If yes, one-to-one / onto?



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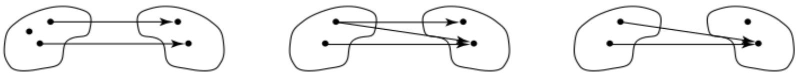
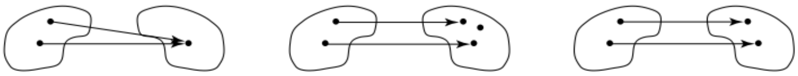


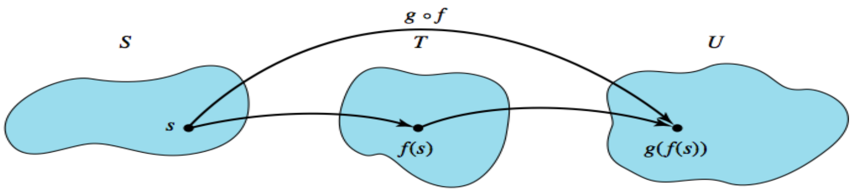
Not a function | Not a function | Function, not one-to-one, not onto



Function, not one-to-one, onto | Function, one-to-one, not onto | Function, one-to-one, onto

Composition of Functions

- Let $f: S \to T$ and $g: T \to U$. Then the **composition function,** $g \circ f$, is a function from S to U defined by $(g \circ f)(s) = g(f(s))$
- The function $g \circ f$ is applied right to left; function f is applied first and then function g.



- Function composition preserves the properties of being onto and being one-to-one.
- ***THEOREM:*** *The composition of two bijections is a bijection.*

Composition of Functions: Examples

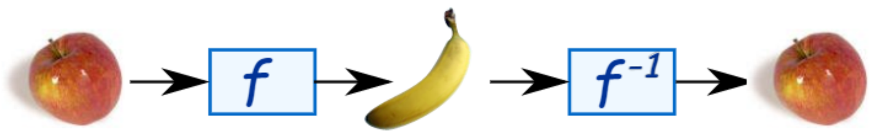
- Let f:R→R be defined by $f(x)=x^2$
- Let g:R→R be defined by g(x)= -x
- Let h:R→R be defined by h(x)= 2x
  - **What is the value of (g ∘ f)(4)?**

    (g ∘ f)(4) = g(**f(4)**)
    = g(**16**)
    = -16
  - **What is the value of (f ∘ g)(4)?**

    f(**g(4)**) = f(**-4**) = 16
  - **What is the value of (h ∘ f ∘ g)(4)?**

    h(f(**g(4)**)) = h(**f(-4)**) = h(**16**) = 32
- Solve practice question 31

Inverse Functions

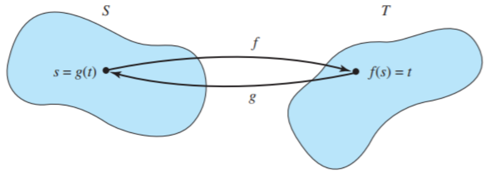


When the function **f** turns the apple into a banana,

Then the **inverse** function f^{-1} turns the banana back to the apple

Inverse Functions

- Let $f: S \rightarrow T$ be a **bijection**
  - Because ***f* is onto**, every $t \in T$ has a preimage in S
  - Because ***f* is one-to-one**, that preimage is unique.
- The function that maps each element of a set S to itself, that is, that leaves each element of S unchanged, is called the **identity function** on S and denoted by i_S.
- **DEFINITION:** Let f be a function, $f: S \rightarrow T$. If there exists a function $\boldsymbol{g: T \rightarrow S}$ such that $\boldsymbol{g \circ f = i_S}$ and $\boldsymbol{f \circ g = i_T}$, then g is called the **inverse function** of f, denoted by f^{-1}.



Inverse Function: Example

- **THEOREM**
  Let $f: S \rightarrow T$. Then f is a bijection **if and only if** f^{-1} exists.

- f: R → R given $f(x) = 3x + 4$ is a bijection
- Find f^{-1}

Solution:

- For any $x \in R$, $f(x) = 3x + 4$. **Let $f(x) = y$. To find, $x = f^{-1}(y)$**
- Thus, $y = 3x + 4$
- **For inverse, we need to represent x in terms of y**
- Thus, $y = 3x + 4 \Rightarrow$ **$x = (y - 4)/3 = f^{-1}(y)$**
- Thus, $f^{-1}(y)$: R → R given by $f^{-1}(y) = (y - 4)/3$

- **Verify:** For any element $s \in R$
  - $(f^{-1} \circ f)(s) = f^{-1}(f(s)) = f^{-1}(3s+4) = ((3s+4) - 4) / 3 = s$
  - $(f \circ f^{-1})(s) = f(f^{-1}(s)) = f((s-4)/3) = (3((s-4)/3) + 4) = s$

Summary

- What is a function?
  - Terminologies: Domain, CoDomain, Image, PreImage, Range, …
  - Examples: Modulo, Floor, Ceil, …
- Equal Function
- Properties
  - One-to-one, Onto, Bijective
- Composition of function
- Inverse Function

Discussion