UNIVERSITY OF TEXAS ARLINGTON

Chapter 4.1
Sets

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Set Theory Basics

- A set is a **collection of <u>distinct</u> objects** called elements.
- Traditionally, sets are **represented** by **capital letters**, and elements by **lower case letters**.
  - Set: A, B, C
  - Elements: *a,b,c*
- Symbol $\in$ means "belongs to" and is used to represent the fact that *an element belongs to a particular set*.
  - $a \in A$ means that element *a* belongs to set A.
  - $b \notin A$ implies that *b* is not an element of A.
- Braces {} are used to indicate a set.
  - A = {2, 4, 6, 8, 10}
  - $3 \notin A$ and $2 \in A$

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Set Theory Basics

- Ordering is not imposed on the set elements

- Two sets are equal if and only if they contain the same elements.
  - Example, A = { 1,2,3,4} and B = {4,3,2,1}
  - Hence, $A = B$ means
    - $(\forall x)[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$

- Finite and infinite set: described by number of elements in a set

Set Representation

- Two types of set representation
  - **List up** all the elements of a set
    - Infinite Sets cannot be listed

  - **Describe a property** that characterizes the set elements (for both finite and infinite sets)
    - E.g., ***S* = {*x* | *x* is a positive even integer}** or using predicate notation.
    - ***S* = {*x* | *P*(*x*)}** means $(\forall x)[(x \in S \rightarrow P(x)) \wedge (P(x) \rightarrow x \in S)]$ where *P* is the unary predicate. Hence, every element of *S* has the property *P* and everything that has a property *P* is an element of *S*.

Set Theory Examples

- Describe each of the following sets by **listing the elements**:
  - A = {*x* | *x* is a month with exactly thirty days}

    A = {April, June, September, November}
  - B = {*x* | *x* is an integer and $4 < x < 9$}

    B = {5, 6, 7, 8}
- **What is the predicate** for each of the following sets?
  - C = {1, 4, 9, 16}

    C = {*x* | *x* is one of the first four perfect squares}
  - D = {2, 3, 5, 7, 11, 13, 17, …}

    D = {*x* | *x* is a prime number}

Set Theory Notations

- Notations used for convenience of defining sets
  - N = set of all nonnegative integers (note that $0 \in N$)
  - Z = set of all integers
  - Q = set of all rational numbers
  - R = set of all real numbers
  - C = set of all complex numbers
- Using the above notations and predicate symbols, one can describe sets quite easily

Examples

- $A = \{x \mid (\exists y)[(y \in \{0,1,2\}) \text{ and } (x = y^2)]\}$
  - $A = \{0,1, 4\}$
- $B = \{x \mid x \in N \text{ and } (\exists y)(y \in N \text{ and } x \leq y)\}$
  - $B = N$
- $C = \{x \mid x \in N \text{ and } (\forall y)(y \in N \rightarrow x \leq y)\}$
  - $C = \{0\}$
- $D = \{x \mid x \in N \text{ and } (\forall y)(y \in \{2, 3, 4, 5\} \rightarrow x \geq y)\}$
  - $D = \{5, 6, 7, \ldots\}$
- $E = \{x \mid (\exists y)(\exists z)(y \in \{1,2\} \text{ and } z \in \{2,3\} \text{ and } x = z - y)\}$
  - $E = \{0, 1, 2\}$

The Empty Set

- A set that has no elements is called a null or empty set and is represented by $\varnothing$ or $\{\}$.
  - Example: $S = \{x \mid x < 3 \text{ and } x > 5\} = \{\} = \varnothing$
- Note that $\varnothing$ **is different from $\{\varnothing\}$**.
  - The latter is **a set with 1 element**, which is the empty set.

More Examples

- A = {Apple, Orange, Banana, …}
  - Set of Fruits
- B = {Ssn, MavID, Name, Age, EmailIDs}
  - Set of Student Attributes
- C = {123, 100112, John, 20, {john@uta.edu, j1@gmail.com}}
  - An example of a set with Student Attribute Values
- D = {-123, ABC, {∅}, {1,2,abcd}, iphone}
  - A random set

Open and Closed Interval

$$\{x \in R \mid -2 < x < 3\}$$

- Denotes the set containing all real numbers **between** -2 and 3. This is an open interval, meaning that the endpoints -2 and 3 are **not included**.
  - By all real numbers, we mean everything such as 1.05, -3/4, and every other real number within that interval.

$$\{x \in R \mid -2 \leq x \leq 3\}$$

- Similar set but on a closed interval
  - It includes all the numbers in the open interval described above, **plus the endpoints**.

Relationship between Sets

- Say **S** is the **set of all people**, **M** is the **set of all male humans**, and **C** is the **set of all computer science students**.
  - All elements of M and C are also elements of S
    - M and C are both **subsets** of S
  - Some elements of M that are not in C (specifically, all males who are not studying computer science)
    - **M is *not a subset* of C**

Relationship between Sets

- *For sets S and M, M is a subset of S if, and only if, every element in M is also an element of S.*
  - Symbolically: $\mathbf{M} \subseteq \mathbf{S} \Leftrightarrow (\forall x)$, if $x \in$ M, then $x \in$ S.
  - E.g., S = {1, 2, 3, 4}, M = {1, 2, 3}
- *If $M \subseteq S$ and $M \neq S$, then there is at least one element of S that is not an element of M, then M is a proper subset of S.*
  - Symbolically, denoted by $\mathbf{M} \subset \mathbf{S}$

Relationship between Sets

E.g., S = {1, 2, 3, 4}, M = {1, 2, 3}

- A superset is the opposite of subset. If M is a subset of S, then S is a superset of M.
  - Symbolically, denoted $S \supseteq M$.
- Likewise, if M is a proper subset of S, then S is a proper superset of M.
  - Symbolically, denoted $S \supset M$.
- Cardinality of a set is simply the number of elements within the set.
  - Cardinality of S is denoted by |S|, Example: **|S| = 4, |M| = 3**
- By the above definition of subset, it is clear that set M must have fewer members than S, which yields the following symbolic representation:

$$S \supset M \Rightarrow |M| < |S|$$

Example

- For the following sets, prove $A \subset B$.
  - $A = \{ x \mid x \in R \text{ such that } x^2 - 4x + 3 = 0\}$
    $A = \{1, 3\}$
  - $B = \{ x \mid x \in N \text{ and } 1 \leq x \leq 4\}$
    $B = \{1, 2, 3, 4\}$

All elements of A exist in B, hence $A \subset B$.

Set of Sets

- From every set, **many subsets** can be generated.
- A set whose elements are **all such subsets** is called the power set.
- For a set S, $\wp(S)$ is termed as the power set.
  - For a set **S** = {1, 2, 3}
  - $\wp$(**S**) = {
    - ∅, — *Subset with 0 elements*
    - {1}, {2}, {3}, — *Subsets with 1 element*
    - {1,2}, {1,3}, {2,3}, — *Subsets with 2 elements*
    - {1,2,3} — *Subset with 3 elements*

    }
  - ∅ **(Empty Set)** is a subset of every set
- For a set with n elements, the power set has **2^n elements**.

Class Exercise

- $A = \{x \mid x \in N \text{ and } x \geq 5\} \Rightarrow A = \{5, 6, 7, 8, 9, \ldots\}$
- $B = \{10, 12, 16, 20\}$
- $C = \{x \mid (\exists y)(y \in N \text{ and } x = 2y)\} \Rightarrow C = \{0, 2, 4, 6, 8, 10, \ldots\}$

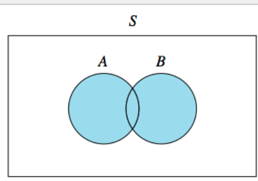
$B \subseteq C$	TRUE	$B \subset A$	TRUE
$A \subseteq C$	FALSE	$26 \in C$	TRUE
$\{11, 12, 13\} \subseteq A$	TRUE	$\{11, 12, 13\} \subset C$	FALSE
$\{12\} \in B$	FALSE. $12 \in B$	$\{12\} \subseteq B$	TRUE
$\{x \mid x \in N \text{ and } x < 20\} \not\subset B$	TRUE	$5 \subseteq A$	FALSE, $\{5\} \subseteq A$
$\{\varnothing\} \subseteq B$	FALSE, $\varnothing \subseteq B$	$\varnothing \in A$	FALSE, $\varnothing \subseteq A$

Class Exercise

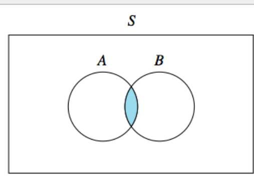
- $A = \{ 1, 2, \varnothing, \{1,2\}, \text{iphone} \}$
  - Q1: $|A| = 6$? **False**
  - Q2: $\varnothing \in A$? **True**
  - Q3: $\varnothing \subset A$? **True**
  - Q4: $\{1,2\} \in A$? **True**
  - Q5: $\{1,2\} \subset A$? **True**

Operation on Sets

- New sets can be formed in a variety of ways, and can be described using both set builder notation and Venn diagrams.
- Let $A, B \in \wp(S)$.
- The **union** of set A and B, denoted by $A \cup B$ is the set that contains all elements in either set A or set B, i.e. $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
- The **intersection** of set A and B, denoted by $A \cap B$ contains all elements that are common to both sets i.e. $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$



$A \cup B$

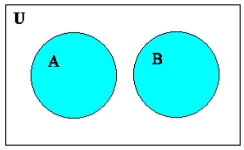


$A \cap B$

- If A = { 1,3,5,7,9} and B = {3,7,9,10,15} ;
  $A \cup B = \{1, 3, 5, 7, 9, 10, 15\}$,
  $A \cap B = \{3, 7, 9\}$.

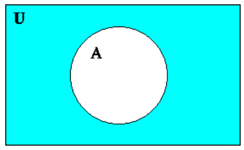
Disjoint, Universal and Difference Sets

- Given set A and set B, if $A \cap B = \varnothing$, then A and B are disjoint sets. In other words, there are **no elements in A that are also in B**.



- For a set $A \in \wp(S)$, the complement of set A, denoted as ~A or A′, is the **set of all elements that are not in A**.

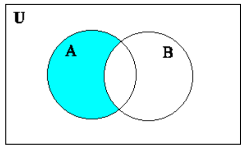
  $A' = \{x \mid x \in S \text{ and } x \notin A\}$



- The difference of A-B is the **set of elements in A that are not in B**. This is also known as the complement of B relative to A.

  $A - B = \{x \mid x \in A \text{ and } x \notin B\}$

  Note $A - B = A \cap B' \neq B - A$



Class Exercises

- Let A = {1, 2, 3, 5, 10}
  B = {2, 4, 7, 8, 9}
  C = {5, 8, 10}
  be subsets of S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. Find
- $A \cup B$ {1, 2, 3, 4, 5, 7, 8, 9, 10}
- $A - C$ {1, 2, 3}
- $B' \cap (A \cup C)$ {1, 3, 5, 10}
- $A \cap B \cap C$ $\varnothing$ or {}
- $(A \cup B) \cap C'$ {1, 2, 3, 4, 7, 9}

Ordered Pairs

- An ordered pair of elements is written as (x,y) and is different from (y,x).
  - Two ordered pairs (a,b) and (c,d) are equal if and only if a = c and b = d.
  - If S = {2,3},
    the ordered pairs of this set are (2,2), (2,3), (3,2), (3,3).
- ***Example:***
  - Coordinates of points on a graph are ordered pairs, where the first value must be the *x* coordinate, and the second value must be the y coordinate.

Cartesian Product

- If A and B are subsets of S, then the cartesian product (cross product) of A and B denoted symbolically by $\mathbf{A \times B}$ is defined by

$$\mathbf{A \times B = \{(\mathit{x,y}) \mid \mathit{x} \in A \text{ and } \mathit{y} \in B\}}$$

- Example: A = {a, b, c} and B = {1, 2, 3}

$\boldsymbol{A \times B} = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (c, 2), (c, 3)\}$

- **Is A × B = B × A?**

$\boldsymbol{B \times A} = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c)\}$

- Cross-product of a set with itself is represented as $\mathbf{A \times A}$ or $\mathbf{A^2}$

Class Exercise

- Let A={1,2} and B={3,4}

  - Find A x B
    {(1,3),(1,4),(2,3),(2,4)}
  - Find B x A
    {(3,1),(3,2),(4,1),(4,2)}
  - Find A^2
    {(1,1),(1,2),(2,1),(2,2)}
  - Find A^3
    {(1,1,1),(1,1,2),(1,2,1),(1,2,2),(2,1,1),(2,1,2),(2,2,1),(2,2,2)}

Basic Set Identities

- Given sets A, B, and C, and a universal set S and a null/empty set $\varnothing$, the following properties hold:

- **Commutative property (cp)**
  $A \cup B = B \cup A$ $A \cap B = B \cap A$

- **Associative property (ap)**
  $A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$

- **Distributive properties (dp)**
  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

- **Identity properties (ip)**
  $\varnothing \cup A = A \cup \varnothing = A$ $S \cap A = A \cap S = A$

- **Complement properties (comp)**
  $A \cup A' = S$ $A \cap A' = \varnothing$

Exercise

- Use the set identities to prove
  $[A \cup (B \cap C)] \cap ([A' \cup (B \cap C)] \cap (B \cap C)') = \varnothing$

- **Proof:**

$[A \cup (B \cap C)] \cap ([A' \cup (B \cap C)] \cap (B \cap C)') =$

$([A \cup (B \cap C)] \cap [A' \cup (B \cap C)]) \cap (B \cap C)'$	*using ap*
$([(B \cap C) \cup A] \cap [(B \cap C) \cup A']) \cap (B \cap C)'$	*using cp twice*
$[(B \cap C) \cup (A \cap A')] \cap (B \cap C)'$	*using dp*
$[(B \cap C) \cup \varnothing] \cap (B \cap C)'$	*using comp*
$(B \cap C) \cap (B \cap C)'$	*using ip*
$\varnothing$	*using comp*

Review

- **A = {1,2,3,4,5, ∅}**
- **B = {2,4}**
- **C = {4,2}**

$B \subset A$	TRUE	$B \subseteq A$	TRUE
$B \subset C$	False	$B \subseteq C$	TRUE
$\varnothing \in A$	TRUE	$\varnothing \in B$	False
$\varnothing \subset A$	TRUE	$\varnothing \subset B$	TRUE
$\varnothing \in \wp(A)$	TRUE	$\varnothing \in \wp(B)$	TRUE
$\varnothing \subset \wp(A)$	TRUE	$\varnothing \subset \wp(B)$	TRUE
$\{\varnothing\} \subseteq A$	TRUE	$\{\varnothing\} \subseteq B$	False

Discussion