UNIVERSITY OF TEXAS ARLINGTON

Chapter 1.4
Predicate Logic

Instructor: Abhishek Santra

Email: abhishek.santra@uta.edu

1

Predicate Logic

- Similar to propositional logic for solving arguments, build from **quantifiers, predicates and logical connectives**.
- **The meaning and the structure** of the quantifiers and predicates determines the interpretation and the validity of the arguments
- **Basic approach** to prove arguments:
  - Strip off quantifiers (*from left to right, one at a time*)
  - Manipulate the unquantified wffs
  - Reinsert the quantifiers, as necessary

Four new inference rules

Two rules to strip the quantifiers	Two rules to reinsert the quantifiers

3

Inference Rules

From	Can Derive	Name / Abbreviation	Restrictions on Use
$(\forall x)P(x)$	$P(t)$ where t is a variable or constant symbol	Universal Instantiation- ui	If t is a variable, it **<u>must not fall</u>** within the scope of a quantifier for t e.g.: $(\forall x)(\exists y)P(x,y)$ to $(\exists y)P(y,y)$ ✖ $(\forall x)(\exists y)P(x,y)$ to $(\exists y)P(x,y)$ ✓
$(\exists x)P(x)$	$P(a)$ where a is a constant symbol not previously used in a proof sequence	Existential Instantiation- ei	Must be the first rule used **that introduces *a***
$P(x)$ or $P(a)$	$(\exists x)P(x)$	Existential Generalization- eg	To go from $P(a)$ to $(\exists x)P(x)$, x must not appear in $P(a)$ e.g.: $P(a,y)$ to $(\exists y)P(y,y)$ ✖ $P(a,y)$ to $(\exists x)P(x,y)$ ✓

Note to remember: $P(x)$ could be $(\forall y)$ $(\forall z)$ $Q(x,y,z)$

Examples: Proofs using Predicate Logic (ui)

- Prove the following argument:
  - All students are humans. John is a student. Therefore, John is a human.
  - $P(x)$ is "x is a student"
  - a is a constant symbol (John)
  - $Q(x)$ is "x is a human"
- The argument is $(\forall x)[P(x) \rightarrow Q(x)] \land P(a) \rightarrow Q(a)$
- The proof sequence is as follows:

1.	$(\forall x)[P(x) \rightarrow Q(x)]$	hyp
2.	$P(a)$	hyp
3.	$P(a) \rightarrow Q(a)$	1, ui
4.	$Q(a)$	2, 3, mp

UI continued…

- (One more ui example) Prove the argument
  $(\forall x)[P(x) \rightarrow Q(x)] \wedge [Q(y)]' \rightarrow [P(y)]'$
- Proof sequence:
  1. $(\forall x)[P(x) \rightarrow Q(x)]$ hyp
  2. $[Q(y)]'$ hyp
  3. $P(y) \rightarrow Q(y)$ 1, ui
  4. $[P(y)]'$ 2, 3, mt

- **What is y called in the antecedent?**

Examples: Proofs using Predicate Logic (ei)

- The following would be **legitimate steps** in a proof sequence: $(\forall x)[P(x) \rightarrow Q(x)]$ ^ $(\exists y)[P(y)] \rightarrow Q(a)$
  1. $(\forall x)[P(x) \rightarrow Q(x)]$ hyp
  2. $(\exists y)[P(y)]$ hyp
  3. $P(a)$ 2, ei
  4. $P(a) \rightarrow Q(a)$ 1, ui
  5. $Q(a)$ 3,4 mp
- **What if we switch the order, 3 and 4?**

$(\exists x)P(x)$	$P(a)$ where a is a constant symbol not previously used in a proof sequence	Existential Instantiation- ei	Must be the first rule used **that introduces *a***

Examples: Proofs using Predicate Logic (eg)

- Prove the argument $(\forall x)P(x) \rightarrow (\exists x)P(x)$
- Proof sequence:
  1. $(\forall x)P(x)$ — hyp
  2. $P(a)$ — 1, ui
  3. $(\exists x)P(x)$ — 2, eg

Inference Rules (ug)

From	Can Derive	Name / Abbreviation	Restrictions on Use
$P(x)$	$(\forall x)P(x)$	Universal Generalization-ug	$P(x)$ has not been deduced from any hypotheses in which x is a free variable nor has $P(x)$ been deduced by ei from any wff in which x is a free variable

Note to remember: $P(x)$ could be $(\forall y)\ (\forall z)\ Q(x,y,z)$

Examples: Proofs using Predicate Logic (ug)

- Prove the argument $(\forall x)[P(x) \to Q(x)] \wedge (\forall x)P(x) \to (\forall x)Q(x)$
- Proof sequence:

1. $(\forall x)[P(x) \to Q(x)]$ hyp
2. $(\forall x)P(x)$ hyp
3. $P(x) \to Q(x)$ 1, ui
4. $P(x)$ 2, ui

   no restriction on ui about reusing a name

5. $Q(x)$ 3, 4, mp
6. $(\forall x)Q(x)$ 5, ug

- Note: step 6 is legitimate since x **is not a free variable in any hypotheses nor was ei used before**

Restrictions on ug (Why do we need?)

- **Incorrect ug 1**

1. $P(x)$ hyp
2. $(\forall x)P(x)$ 1, **incorrect ug**; x was free variable in the hypothesis

Example: x is set of flowers, P(x): x is yellow. If P(x) is true, does that mean all flowers are yellow?

- **Incorrect ug 2**

1. $(\forall x)(\exists y)Q(x,y)$ hyp
2. $(\exists y)Q(x,y)$ 1, ui
3. $Q(x,a)$ 2, ei
4. $(\forall x)Q(x,a)$ 3, **incorrect ug**; $Q(x,a)$ was deduced by ei from the wff in step2, in which x is free variable

Example: x is set of integers, Q(x,y): x + y = 0. If, for every x, there exists some y such that x + y = 0 is true, then does it mean that adding the a fixed integer 'a' to every x will give x + a = 0?

Examples: Proofs using Predicate Logic

- Prove the argument

$$(\forall x)[P(x) \land Q(x)] \to (\forall x)P(x) \land (\forall x)Q(x)$$

- Proof sequence:

1.	$(\forall x)[P(x) \land Q(x)]$	hyp
2.	$P(x) \land Q(x)$	1, ui
3.	$P(x)$	2, sim
4.	$Q(x)$	2, sim
5.	$(\forall x)P(x)$	3, ug
6.	$(\forall x)Q(x)$	4, ug
7.	$(\forall x)P(x) \land (\forall x)Q(x)$	5, 6, con

Examples: Proofs using Predicate Logic

- Prove the argument

$$(\forall y)[P(x) \to Q(x,y)] \to [P(x) \to (\forall y)Q(x,y)]$$

- Using the deduction method, we can derive

$$(\forall y)[P(x) \to Q(x,y)] \land P(x) \to (\forall y)Q(x,y)$$

- Proof sequence:

1.	$(\forall y)[P(x) \to Q(x,y)]$	hyp
2.	$P(x)$	hyp
3.	$P(x) \to Q(x,y)$	1, ui
4.	$Q(x,y)$	2, 3, mp
5.	$(\forall y)Q(x,y)$	4, ug

Temporary hypotheses

- A temporary hypothesis can be inserted into a proof sequence. If T is inserted as a temporary hypothesis and eventually W is deduced from T and other hypotheses, then the wff T → W has been deduced from other hypotheses and can be reinserted into the proof sequence
- Prove the argument

$$[P(x) \to (\forall y)Q(x,y)] \to (\forall y)[P(x) \to Q(x,y)]$$

- Proof sequence:

1.	$P(x) \to (\forall y)Q(x,y)$	hyp
2.	$P(x)$	temporary hypothesis (T)
3.	$(\forall y)Q(x,y)$	1, 2, mp
4.	$Q(x,y)$	3, ui (W)
5.	$P(x) \to Q(x,y)$	temp. hyp discharged (T→ W)
6.	$(\forall y)[P(x) \to Q(x,y)]$	5, ug

Proving Verbal Arguments

- Every crocodile is bigger than every alligator. Sam is a crocodile. But there is a snake, and Sam isn't bigger than that snake. Therefore, something is not an alligator.
  - Use C(x): x is a crocodile; A(x): x is an alligator, B(x,y): x is bigger than y, s is a constant (Sam), S(x): x is a Snake
- Hence prove argument

$$(\forall x)(\forall y)[C(x) \wedge A(y) \to B(x,y)] \wedge C(s) \wedge (\exists x)(S(x) \wedge [B(s,x)]') \to (\exists x)[A(x)]'$$

1.	$(\forall x)(\forall y)[C(x) \wedge A(y) \to B(x,y)]$	hyp
2.	$C(s)$	hyp
3.	$(\exists x)(S(x) \wedge [B(s,x)]')$	hyp
4.	$(\forall y)[C(s) \wedge A(y) \to B(s,y)]$	1, ui
5.	$S(a) \wedge [B(s,a)]'$	3, ei
6.	$C(s) \wedge A(a) \to B(s,a)$	4, ui
7.	$[B(s,a)]'$	5, sim
8.	$[C(s) \wedge A(a)]'$	6, 7, mt
9.	$[C(s)]' \vee [A(a)]'$	8, De Morgan
10.	$[C(s)] \to [A(a)]'$	9, imp
11.	$[A(a)]'$	2, 10, mp
12.	$(\exists x)[A(x)]'$	11, eg

Class + Home Exercise - 1

- Prove the argument
  $(\forall x)[(B(x) \vee C(x)) \rightarrow A(x)] \rightarrow (\forall x)[B(x) \rightarrow A(x)]$

Proof sequence:

1.	$(\forall x)[(B(x) \vee C(x)) \rightarrow A(x)]$	hyp
2.	$(B(x) \vee C(x)) \rightarrow A(x)$	1, ui
3.	$B(x)$	temp. hyp
4.	$B(x) \vee C(x)$	3, add
5.	$A(x)$	2, 4, mp
6.	$B(x) \rightarrow A(x)$	temp. hyp discharged
7.	$(\forall x)[B(x) \rightarrow A(x)]$	6, ug

17

Class + Home Exercise - 2

- Every ambassador speaks only to diplomats. Some ambassadors speak to someone. Therefore, there is a diplomat.
- Use $A(x)$: x is an ambassador; $S(x,y)$: x speaks to y; $D(x)$: x is a diplomat

Prove the argument

$(\forall x)\,[A(x) \rightarrow (\forall y)(S(x,y) \rightarrow D(y))] \wedge (\exists x)(A(x) \wedge (\exists y)S(x,y)) \rightarrow (\exists x)D(x)$

Proof Sequence:

1. $(\forall x)\,[A(x) \rightarrow (\forall y)(S(x,y) \rightarrow D(y))]$	hyp
2. $(\exists x)(A(x) \wedge (\exists y)S(x,y))$	hyp
3. $A(a) \wedge (\exists y)S(a,y)$	2, ei
4. $A(a) \wedge S(a,b)$	3, ei
5. $A(a) \rightarrow (\forall y)(S(a,y)) \rightarrow D(y))$	1, ui
6. $A(a) \rightarrow (S(a,b) \rightarrow D(b))$	5, ui
7. A(a)	4, sim
8. $S(a,b) \rightarrow D(b)$	6,7, mp
9. $S(a,b)$	4, sim
10. $D(b)$	8,9, mp
11. $(\exists x)D(x)$	10, eg

18

Discussion