Chapter 2.1
Theorems and Informal Proofs

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1

Proof Techniques

- **Proof methods :**
  - Inductive reasoning
  - Deductive reasoning
  - Proof by exhaustion
  - Direct proof
  - Proof by contraposition
  - Proof by contradiction
  - Serendipity
- **A few terms for proof:**
  - **Axioms**: Statements that are *assumed true.*
    - Example: Given two distinct points, there is exactly one line that contains them.
  - **Theorem**: A proposition that has been *proved to be true*.
    - Two special kinds of theorems: Lemma and Corollary.
    - **Lemma**: A theorem that is usually not too interesting in its own right but is **useful in proving another theorem**.
    - **Corollary**: A theorem that **follows quickly** from another theorem.

2

Inductive Reasoning: *Experience*

- Inductive Reasoning: Drawing a conclusion from a hypothesis based on **experience**.
- Hence the **more cases** you find where **Q follows from P**, the more confident you are about the conjecture $P \rightarrow Q$.
- Example:
  - Every time I've walked by that dog, it hasn't tried to bite me. So, the next time I walk by that dog it won't try to bite me.
- Usually, deductive reasoning is also applied to the same conjecture to ensure that it is indeed valid.

Deductive Reasoning: *Counter Example*

- Deductive reasoning looks for a **counter example** that disproves the conjecture, i.e., a case when P is true, but Q is false.
- Remember: **$(\exists x)A(x) \rightarrow (\forall x)A(x)$?**
- Example: Prove that "For every positive integer n, $n! \leq n^2$."
  - Start testing some cases say, $n = 1, 2, 3$ etc.
  - It might seem like it is true for some cases but how far do you test, say $n = 4$.
  - We get $n! = 24$ and $n^2 = 16$ which is a counter example for this theorem.
  - Hence, even **finding a single case** that doesn't satisfy the condition is enough to **disprove** the theorem.

Counter Example

- More examples of counter example:
  - **Prove: All animals living in the ocean are fish.**
    - Blue whale is a mammal (counter example)
  - **Prove: Every integer less than 10 is bigger than 5.**
    - 4 < 10, but 4 not less than 5 (counter example)

- Counter example is **not trivial** for all cases, so we have to use other proof methods.

Exhaustive Proof: *Try all Cases*

- If dealing with a finite domain in which the proof is to be shown to be valid, then using the exhaustive proof technique, **one can go over all the possible cases for each member of the finite domain**.
- Final result of this exercise: you prove or disprove the theorem, but you could be **definitely exhausted**.
- **Example**: For any positive integer less than or equal to 5 (say, n), the square of the integer is less than or equal to the sum of 10 and 5 times the integer(i.e., $n^2 \leq 10+5n$)

n	n^2	$10+5n$	$n^2 \leq 10+5n$
1	1	15	yes
2	4	20	yes
3	9	25	yes
4	16	30	yes
5	25	35	yes

Example: Exhaustive Proof

- If an integer between 1 and 20 is divisible by 6, then it is also divisible by 3.

TABLE 2.1

Number	Divisible by 6	Divisible by 3
1	no	
2	no	
3	no	
4	no	
5	no	
6	yes: 6 = 1×6	yes: 6 = 2×3
7	no	
8	no	
9	no	
10	no	
11	no	
12	yes: 12 = 2×6	yes: 12 = 4×3
13	no	
14	no	
15	no	
16	no	
17	no	
18	yes: 18 = 3×6	yes: 18 = 6×3
19	no	
20	no	

Direct Proof

- Used when **exhaustive proof doesn't work**.
- Using the **rules of propositional and predicate logic**, prove P $\rightarrow$ Q.
- Assume the hypothesis P and then try to prove Q. Hence, a formal proof would require a **proof sequence to go from P to Q**.
- Consider the conjecture

  x is an even integer $\wedge$ y is an even integer $\rightarrow$ the product *xy* is an even integer.

Direct Proof Example (Proof Sequence)

1. x is an even integer ∧ y is an even integer hyp
2. $(\forall x)[x$ is even integer $\rightarrow (\exists k)(k$ is an integer ∧ $x = 2k)]$ number fact (definition of even integer)
3. x is an even integer $\rightarrow (\exists k)(k$ is an integer ∧ $x = 2k)$ 2,ui
4. y is an even integer $\rightarrow (\exists k)(k$ is an integer ∧ $y = 2k)$ 2,ui
5. x is an even integer 1,sim
6. $(\exists k)(k$ is an integer ∧ $x = 2k)$ 3,5,mp
7. m is an integer ∧ $x = 2m$ 6, ei
8. y is an even integer 1,sim
9. $(\exists k)(k$ is an integer ∧ $y = 2k)$ 4,8,mp
10. n is an integer ∧ $y = 2n$ 9, ei

Direct Proof Example (Proof Sequence) contd.

11. $x = 2m$ 7,sim
12. $y = 2n$ 10, sim
13. $xy = (2m)(2n)$ 11, 12, substitution of equals
14. $xy = 2(2mn)$ 13, multiplication fact
15. m is an integer 7,sim
16. n is an integer 10, sim
17. $2mn$ is an integer 15, 16, number fact
18. $2mn$ is an integer ∧ $xy = 2(2mn)$ 17, 14, con
19. $(\exists k)(k$ is an integer ∧ $xy = 2k)$ 18,eg
20. $(\forall x)[(\exists k)(k$ is an integer ∧ $x = 2k) \rightarrow x$ is even integer] number fact (definition of even integer)
21. $(\exists k)(k$ an integer ∧ $xy = 2k) \rightarrow xy$ is even integer 20, ui
22. xy is an even integer 19, 21, mp

Direct Proof: Contraposition

- If you tried to prove but failed to produce a direct proof of your conjecture $P \to Q$
- You can use a **variant of direct proof, contraposition**
- **$Q' \to P'$ is the contrapositive of $P \to Q$**
- Example 1: Prove that "If the square of an integer is odd, then the integer must be odd."
  - P: n^2 is odd, Q: n is odd
  - Conjecture: $P \to Q$
  - **Try to prove, $Q' \to P'$**
  - Q' : n is even, P' : n^2 is even
  - Since n is even, $n^2 = n \times n$ is even
    ($n=2k$, $n^2=4k^2=2(2k^2)$)

Direct Proof: Contraposition

- Example 2: Prove that "If n+1 separate passwords are issued to n students, then some student gets ≥ 2 passwords."
  - The contrapositive is:
    - *If every student gets < 2 passwords, then n+1 separate passwords were NOT issued."*
  - Suppose every student has < 2 passwords
  - Then, every one of the n students has at most 1 password.
  - The total number of passwords issued is **at most *n*, not *n*+1**.

Indirect Proof: Proof by Contradiction

- Our aim is to prove $P \rightarrow Q$
- In a proof by contradiction, you **assume** $(P \rightarrow Q)$ is FALSE
  - Thus, $(P \rightarrow Q)'$ is TRUE, That is, $(P \wedge Q')$ is TRUE
  - That is, the *hypothesis is true and the negation of conclusion is true*
- Then, try to **deduce some contradiction from these assumptions**.

To prove $\mathbf{P \rightarrow Q}$, it is sufficient to prove $\mathbf{P \wedge Q' \rightarrow 0}$

Proof by Contradiction (Example)

- Example 1: Prove that "If a number added to itself gives itself, then the number is 0."
  - The hypothesis (P) is $x + x = x$ and the conclusion (Q) is $x = 0$. Hence, the hypotheses for the **proof by contradiction** are:
  - $x + x = x$ and $x \neq 0$ (P and Q')
  - Then $2x = x$ and $x \neq 0$,
  - Hence dividing both sides of P by x,
    the result is $2 = 1$,
    which is a **contradiction**.
  - Thus, $(x + x = x) \wedge (x \neq 0) \rightarrow 0$
  - Hence, $(x + x = x) \rightarrow (x = 0)$, which means we proved $P \rightarrow Q$.

Proof by Contradiction (Example)

- **Example 2: Prove "For all real numbers x and y, if $x + y \geq 2$, then either $x \geq 1$ or $y \geq 1$."**
  - P: $x + y \geq 2$ Q: $x \geq 1$ or $y \geq 1$ and try to show **P ∧ Q′ → 0**
  - Proof: Say the conclusion (Q) is false, i.e. $x < 1$ and $y < 1$. (Q′ is true)
  - Adding the two conditions, the result is $x + y < 2$.
  - At this point, we also have P = $x + y \geq 2$
  - Hence, P ∧ Q′ which is a contradiction
  - Assumption is incorrect. Hence, proved by contradiction
- **Example 3: The sum of even integers is even.**
  - Proof: Let $x = 2m$, $y = 2n$ for integers m and n and assume that $x + y$ is odd.
  - Then $x + y = 2m + 2n = 2k + 1$ for some integer k.
  - Hence, $2*(m + n - k) = 1$, where $m + n - k$ is some integer.
  - *This is a contradiction since 1 is not even.*
  - Assumption is incorrect. Hence, proved by contradiction

Class Exercise

- Prove that √5 is not a rational number.
- What kind of proof method and How?
- Definition of rational number: a number that can be represented as a form of b/a (a,b, **integers, a≠0**, and **a and b have no common factors other than ±1**)

1. Assume that √5 is rational number, then $\sqrt{5} = b/a$, $5 = b^2/a^2$,
2. $5a^2 = b^2$ ------- (1)
3. b^2 is a multiple of 5, 5 is a prime number, therefore b is a multiple of 5
4. If we let $b=5k$ (k is an integer) in (1), then, $5a^2=25k^2$ $a^2=5k^2$
5. a^2 is a multiple of 5, 5 is a prime number, therefore a is a multiple of 5
6. Now, **5 is factor of both *a* and *b*.**
7. This is a contradiction that a and b have no common factors **other than ±1**
8. Therefore √5 is not rational.

Serendipity (Just for Fun)

- **Serendipity:** Fortuitous happening or something **by chance or good luck**.
- **Not a formal proof technique.**
- Interesting proofs provided by this method although other methods can be used as well.

- **Example:** 2048 players in a tennis tournament. One winner in the end. Each match is between two players with exactly one winner and the loser gets eliminated.

Prove the total number of matches played in the tournament are 2047.

Solution: Only one champion at the end of the tournament, hence 2047 losers at the end, hence 2047 matches should have been played to have 2047 losers.

Serendipity (Just for Fun)

Röntgen's discovery occurred accidentally in his Wurzburg, Germany, lab, where he was testing whether cathode rays could pass through glass when he noticed **a glow coming from a nearby chemically coated screen**. He dubbed the rays that caused this glow X-rays because of their unknown nature.

Summarizing Proof Techniques

Proof Technique	Approach to prove P → Q	Remarks
Exhaustive Proof	Demonstrate P → Q for all cases	May only be used for finite number of cases, when domain is small
Direct Proof	Assume P is true, deduce Q	The standard approach- usually the thing to try
Proof by Contraposition	Assume Q′ is true, derive P′	Use this when Q′ as a hypothesis seems to give more ammunition than P would
Proof by Contradiction	Assume P Λ Q′ is true, deduce a contradiction	Use this when Q says something is not true
Serendipity	Not really a proof technique	Fun to know

Further Study – Home Practice

1. Product of any 2 consecutive integers is even.
2. The sum of 3 consecutive integers is even.
3. Product of 3 consecutive integers is even.
4. The square of an odd integer equals $8k+1$ for some integer k.
5. The sum of two rational numbers is rational.
6. For some positive integer x, $x + 1/x \geq 2$.

Discussion