Chapter 4.5
Binomial Theorem

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1

Coefficients of Expansion

- What are the expansions of following expressions?

$(a+b)^0 = \quad 1$
$(a+b)^1 = \quad a+b$
$(a+b)^2 = \quad a^2+2ab+b^2$
$(a+b)^3 = \quad a^3+3a^2b+3ab^2+b^3$
$(a+b)^4 = \quad a^4+4a^3b+6a^2b^2+4ab^3+b^4$
$(a+b)^5 = \quad a^5+5a^4b^1+10a^3b^2+10a^2b^3+5a^1b^4+b^5$

$(a+b)^n = ?$

2

Pascal's Triangle

Coefficients	Power
1	0
1 1	1
1 2 1	2
1 3 3 1	3
1 4 6 4 1	4
1 5 10 10 5 1	5

Pascal's Triangle

- Expansions
  $(a + b)^0 = 1$
  $(a + b)^1 = a + b$
  $(a + b)^2 = a^2 + 2ab + b^2$
  $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
  $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
  $(a + b)^5 = a^5 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + b^5$

Coefficients	Power
1	0
1 1	1
1 2 1	2
1 3 3 1	3
1 4 6 4 1	4
1 5 10 10 5 1	5

- Row ***n*** of the triangle ($n \geq 0$) consists of all the values **$C(n, r)$ for $0 \leq r \leq n$**. Thus, the **Pascal's Triangle** looks like this:

	Row/Power (n)
$C(\mathbf{0},0)$	**0**
$C(\mathbf{1},0)$ $C(1,1)$	1
$C(\mathbf{2},0)$ $C(\mathbf{2},1)$ $C(\mathbf{2},2)$	2
$C(\mathbf{3},0)$ $C(\mathbf{3},1)$ $C(\mathbf{3},2)$ $C(\mathbf{3},3)$	3
$C(\mathbf{4},0)$ $C(\mathbf{4},1)$ $C(\mathbf{4},2)$ $C(\mathbf{4},3)$ $C(\mathbf{4},4)$	4
$C(\mathbf{5},0)$ $C(\mathbf{5},1)$ $C(\mathbf{5},2)$ $C(\mathbf{5},3)$ $C(\mathbf{5},4)$ $C(\mathbf{5},5)$	5
$C(n,0)$ $C(n,1)$ …………… $C(n,n-1)$ $C(n,n)$	n

Binomial Theorem

- The binomial theorem provides us with a formula for the expansion of $(a + b)^n$.
- It states that for **every nonnegative integer** n:

$$(a+b)^n = C(n,0)a^n b^0 + C(n,1)a^{n-1}b^1 + C(n,2)a^{n-2}b^2 + \dots + C(n,k)a^{n-k}b^k + \dots + C(n,n-1)a^1 b^{n-1} + C(n,n)a^0 b^n = \sum_{k=0}^{n} C(n,k)a^{n-k}b^k$$

- The terms $C(n,k)$ in the above series is called the **binomial coefficient**
- What is the kth term? How many terms are there?
  - **$C(n, k-1)\ a^{(n-(k-1))}\ b^{(k-1)}$**
  - There are (n+1) terms
- Binomial theorem can be proved using mathematical induction.

Exercises

- Expand $(x + 1)^5$ using the binomial theorem

  $C(5,0)x^5 + C(5,1)x^4 + C(5,2)x^3 + C(5,3)x^2 + C(5,4)x^1 + C(5,5)x^0$ $\quad \sum_{k=0}^{n} C(n,k)a^{n-k}b^k$

- Expand $(x - 3)^4$ using the binomial theorem

  $C(4,0)x^4(-3)^0 + C(4,1)x^3(-3)^1 + C(4,2)x^2(-3)^2 + C(4,3)x^1(-3)^3 + C(4,4)x^0(-3)^4$

- What is the **fifth** term of $(3a + 2b)^7$? How many terms are there?

  $C(7,4)(3a)^{7-4}(2b)^4 = 15120\ a^3 b^4$

- What is the coefficient of $x^5y^2z^2$ in the expansion of $(x + y + 2z)^9$? (*hint: group (x+y)*)

  $C(9,2)(x+y)^{9-2}(2z)^2 = C(9,2)[C(7,2)(x)^5(y)^2](2z)^2 = 3024x^5y^2z^2$

  Therefore, the answer: 3024

- Use binomial theorem to prove that: (*Hint: Think, what can be "a" and "b"*)

  $C(n,0) - C(n,1) + C(n,2) - \dots + (-1)^n C(n,n) = 0$

  Try to find $(a+b)^n$ $\quad C(n,0) - C(n,1) + C(n,2) - \dots + (-1)^n C(n,n) = (1+(-1))^n = 0^n = 0$

Summary: Module 4

- **Set Theory Basics (4.1)**
  - Curly Brace: {}, No Ordering, No Duplicates
  - **Notations:** N, Z, …
  - **Terminologies:** Set, Subset, Power Set, Super Set, Proper Subset, Proper Superset, Cardinality (|A|), Empty Set, Universal Set
  - **Set Operations:** Union, Intersection, Difference, Cartesian Product, Complement
- **Counting, Permutations and Combinations (4.2, 4.3, 4.4)**
  - Addition, Multiplication, Inclusion and Exclusion Principles(for 2 and 3 sets)
  - Pigeon Hole Principle
  - **Permutations** P(n ,r)**:** Orderings / Arrangements
  - **Combinations** C(n, r)**:** Selections / Choices
- **Binomial Theorem (4.5)**
  - Expansion, kth term, number of terms, coefficients
- **The concept of Boolean Algebra (8.1)**

Discussion