Chapter 5.1
Relations

Instructor: Abhishek Santra
Email: abhishek.santra@uta.edu

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Binary Relations

- ***S* = {1, 2, 4}**
- *What is the Cartesian product of set S with itself?*
  $S \times S$ = {(1,1), (1,2), (1,4), (2,1), (2,2), (2,4), (4,1), (4,2), (4,4)}

- Certain **ordered pairs of objects** have **relationships**
- The notation $x \rho y$ implies that the ordered pair (x, y) satisfies the relationship ρ.

- **Find <u>the subset of $S \times S$</u>** satisfying the relation $x \rho y \leftrightarrow x = y/2$
  - {(1, 2), (2, 4)}

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Binary Relations

- DEFINITION: BINARY RELATION on a set S Given a set S, a **binary relation ρ** on a set S is a **subset of $S \times S$ (a set of ordered pairs of elements of S)**.
- A binary relation is always a subset with the property that:

$$\boldsymbol{x \rho y \leftrightarrow (x, y) \in \rho}$$

- What is the set where binary relation ρ on S is defined by $x \rho y \leftrightarrow x + y$ is odd where $S = \{1, 2\}$?
  - The set for ρ is $\{(1,2), (2,1)\}$.

Relations on Multiple Sets

- Given **two sets S and T**, a **binary relation from S to T is a subset of S × T**

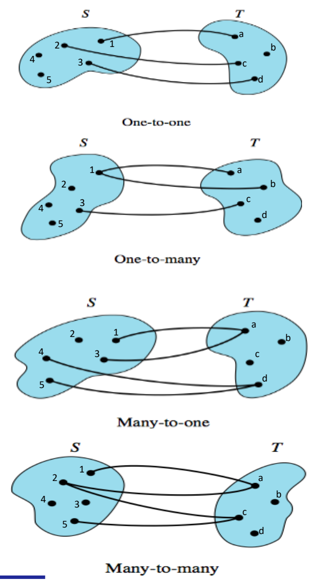
- S = {1, 2, 3} and T = {2, 4, 7}
  - What is the set that satisfies the relation $x \rho y \leftrightarrow x = y/2$
  - **{(1,2), (2,4)}**

- S = {2, 4, 6, 8} and T = {2, 3, 4, 6, 7}.
  - What is the set that satisfies the relation $x \rho y \leftrightarrow x = (y + 2)/2$
  - **{(2,2), (4,6)}**

- *How many elements of set S are paired with how many elements of Set T?*
- *How many times the 1st and 2nd component appear?*

Types of Relationships

- **One-to-one:** If each first component and each second component **only appear once in the relation.**
- **One-to-many:** If *some* first component is **paired with more than one second** component
- **Many-to-one:** If *some* second component is **paired with more than one first** component.
- **Many-to-many:** If *at least one* first component is **paired with more than one second** component and *at least one second* component is **paired with more than one first** component.



Relationships: Examples

- If S = {2, 5, 7, 9}, then identify the types of the following relationships:

  - {(2,5), (5,7), (7,2)}

    one-to-one *(1st and 2nd components appear only once)*

  - {(5,2), (7,5), (9,2)}

    many-to-one *(2nd component appears multiple times)*

  - {(7,9), (2,5), (9,9), (2,7)}

    many-to-many *(1st and 2nd components appear multiple times)*

Properties of Relationships

- Let ρ **be a binary relation on a set** S
  - ρ is reflexive means $(\forall x)\ (x \in S \rightarrow (x,x) \in \rho)$
  - ρ is symmetric means: $(\forall x)(\forall y)\ (x \in S \wedge y \in S \wedge (x,y) \in \rho \rightarrow (y,x) \in \rho)$
  - ρ is transitive means:
    $(\forall x)(\forall y)(\forall z)\ (x \in S \wedge y \in S \wedge z \in S \wedge (x,y) \in \rho \wedge (y,z) \in \rho \rightarrow (x,z) \in \rho)$
  - ρ is antisymmetric means:
    $(\forall x)(\forall y)\ (x \in S \wedge y \in S \wedge (x,y) \in \rho \wedge (y,x) \in \rho \rightarrow x = y)$

- Example: Consider the relation ρ of equality(=) on S.
  - For any $x \in S$, $x = x$, or $(x,x) \in \rho$ (reflexive)
  - For any $x,y \in S$, if $x = y$ then $y=x$, or $(x,y) \in \rho \rightarrow (y,x) \in \rho$ (symmetric)
  - For any $x,y,z \in S$, if $x=y$ and $y=z$, then $x=z$, or $[(x,y) \in \rho$ and $(y,z) \in \rho \rightarrow (x,z) \in \rho]$ (transitive)

Example

- Consider the **relation** $\leq$ on the set of non-negative integers **N**.
  - **Is it reflexive?**
    Yes, since for every nonnegative integer x, $x \leq x$.
  - **Is it symmetric?**
    No, since $x \leq y$ doesn't imply $y \leq x$
    - *Counter Example: (2,4) satisfies property, (4,2) does not*
    - If this was the case, then $x = y$. This property is called antisymmetric.
  - **Is it transitive?**
    Yes, since if $x \leq y$ and $y \leq z$, then $x \leq z$.

Instant Exercises

- Test each binary relation on the given set S for reflexivity, symmetry, antisymmetry, and transitivity
  - S = {1,2,3}; $\rho = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$
    - Reflexive? Yes, symmetric? Yes, transitive? Yes, antisymmetric? No
  - S = {0,1}; x ρ y $\leftrightarrow$ x = y^2
    - $\rho = \{(0,0), (1,1)\}$
    - Reflexive? Yes, symmetric? Yes, transitive? Yes, antisymmetric? Yes
  - S = set of all lines in the plane; x ρ y $\leftrightarrow$ x is parallel to y or x coincides with y
    - Reflexive? Yes, symmetric? Yes, transitive? Yes, antisymmetric? No
  - S = N; x ρ y $\leftrightarrow$ x = y^2
    - Reflexive? No, symmetric? No, transitive? No, antisymmetric? Yes

Closures of Relations

- A binary relation ρ^* on set *S* is the **closure of a relation** ρ on *S* with respect to property *P* if:
  1. ρ^* has the property *P*
  2. $\rho \subseteq \rho^*$
  3. ρ^* is a **smallest subset** of any other relation on S that includes ρ and has the property *P*
- **Example**: Let *S* = {1, 2, 3} and ρ = {(1,1), (1,2), (1,3), (3,1), (2,3)}.
  - This is **not reflexive, not transitive and not symmetric**.
  - **Closure of ρ with respect to reflexivity** is {(1,1),(1,2),(1,3), (3,1), (2,3), (2,2), (3,3)} and it contains ρ
  - **Closure of ρ with respect to symmetry is** {(1,1), (1,2), (1,3), (3,1), (2,3), (2,1), (3,2)}
  - **Closure of ρ with respect to transitivity** is {(1,1), $\boxed{\textbf{(1,2)}}$, (1,3), (3,1), (2,3), (3,2), (3,3), $\boxed{\textbf{(2,1)}}$, **(2,2)**}

Exercise: Closures of Relations

- Find the **reflexive, symmetric and transitive closure** of the relation {(a,a), (b,b), (c,c), (a,c), (a,d), (b,d), (c,a), (d,a)} on the set S = {a, b, c, d}
- Reflexive Closure
  {(a,a),(b,b),(c,c),(a,c),(a,d),(b,d),(c,a),(d,a), (d,d)}
- Symmetric Closure
  {(a,a),(b,b),(c,c),(a,c),(a,d),(b,d),(c,a),(d,a), (d,b)}
- Transitive Closure
  {(a,a),(b,b),(c,c), (a,c) ,(a,d),(b,d),(c,a),(d,a), (b,a) , (c,d), (d,d), (d,c), **(b,c)**}

Partial Ordering

- A **binary relation on a set *S*** that is **reflexive, antisymmetric, and transitive** is called a **partial ordering on *S*.**
- If ρ is a partial ordering on *S*, then the ordered pair (*S*,ρ) is called a **partially ordered set** (also known as a **poset**).
- Examples:
  - On **N**, $x \rho y \leftrightarrow x \leq y$.
  - On {0,1}, $x \rho y \leftrightarrow x = y^2 \Rightarrow \rho = \{(0,0), (1,1)\}$.
- Denote an *arbitrary, partially ordered set* by **(*S*, $\leq$)** with ***(x,y)*** pairs
  - If $x \leq y$, then either x=y or x≠y.
  - If $x \leq y$, but $x \neq y$, we write < and say that
    - x is a **predecessor** of y or y is a **successor** of x
  - If x<y and **there is no *z*** with x<z<y, then x is an **immediate predecessor** of y
    - *That is, (x, z) and (z, y) are NOT part of the relation.*

Example

- Consider the relation "**x divides y**" on {1,2,3,6,12,18}
  - Write the ordered pairs (x,y) of this relation

(1,2), (1,3), **(1,6)**, (1,12), (1,18), **(2,6)**, (2,12), (2,18), **(3,6)**, (3,12), (3,18), (6,12), (6,18), (1,1), (2,2), (3,3), (6,6), (12,12), (18,18)

  - Write all the predecessors of 6

1, 2, 3

  - Write all the immediate predecessors of 6

2, 3 Why?

1 < 2 < 6, (1,2): 1 predecessor of 2 and, then (2,6): 2 predecessor of 6

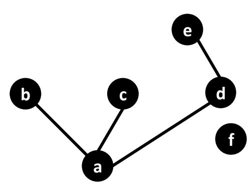
So, 1 is <u>not an immediate predecessor</u> of 6

Hasse Diagram

- *A diagram used to **visually depict a partially ordered set (S, ≤)** if S is finite.*
  - Each of the elements of *S* is represented by a dot, called a **node,** or **vertex,** of the diagram.
  - If ***x* is an immediate predecessor of *y***, then the node for *y* is **placed above** the node for *x* and the two nodes are **connected** by a straight-line segment.
  - **<u>Example:</u>**

Given the partial ordering on a set *S* = {*a, b, c, d, e, f*} as, {(*a,a*), (*b,b*), (*c,c*), (*d,d*), (*e,e*), (*f, f*), (*a, b*), (*a,c*), (*a,d*), (*a,e*), (*d,e*)},

Immediate predecessors: b: a, c: a, d: a, e: d; the Hasse diagram is:



Equivalence Relation

- A **binary relation on a set *S*** that is **reflexive, symmetric, and transitive** is called an **equivalence relation on *S***

- Examples:
  - On **N,** $x \rho y \leftrightarrow x + y$ is even.
  - On {1, 2, 3}, $\rho = \{(1,1), (2,2), (3,3), (1,2), (2,1)\}$.

Equivalence Class

- For **equivalence relation** ρ on set *S* and $x \in S$, the **equivalence class** of *x* (denoted by [*x*]) is the set of all members of *S* to which *x* is related.

$$[x] = \{y \mid y \in S \wedge x \rho y\}$$

- **Example**, for $\rho = \{(a,a), (b,b), (c,c), (a,c), (c,a)\}$ on S = {a,b,c}

$$[a] = \{a, c\} = [c]$$
$$[b] = \{b\}$$

Exercises

- Which of the following ordered pairs belong to the binary relation ρ on N?
  - $x \rho y \leftrightarrow x + y < 7$; (1,3), (2,5), (3,3), (4,4)
  - $x \rho y \leftrightarrow 2x + 3y = 10$; (5,0), (2,2), (3,1), (1,3)

- Identify each relation on **N** as one-to-one, one-to-many, many-to-one or many-to-many:
  - $\rho = \{(12,5), (8,4), (6,3), (7,12)\}$ **One-to-one**
  - $\rho = \{(2,7), (8,4), (2,5), (7,6), (10,1)\}$ **One-to-many**
  - $\rho = \{(1,2), (1,4), (1,6), (2,3), (4,3)\}$ **Many-to-many**

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Exercises

- Test if reflexive? symmetric? Transitive? Antisymmetric?
  - S = N; $x \rho y \leftrightarrow x + y$ is even
    - Reflexive? Yes, symmetric? Yes, transitive? Yes, antisymmetric? No
  - S = {x|x is a person living in Dallas}; $x \rho y \leftrightarrow$ x is older than y
    - Reflexive? No, symmetric? No, transitive? Yes, antisymmetric? Yes (hypothesis of it is false, which makes whole statement TRUE)
  - S = {x|x is a student in the class}; $x \rho y \leftrightarrow$ x sits in the same row as y
    - Reflexive? Yes, symmetric? Yes, transitive? Yes, antisymmetric? No
  - S = Z^+ (positive integers); $x \rho y \leftrightarrow$ x divides y
    - Reflexive? Yes, symmetric? No, transitive? Yes, antisymmetric? Yes

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Exercises

- S = {0, 1, 2, 4, 6}. Test the following binary relations on S for reflexivity, symmetry, antisymmetry, and transitivity. Find the closures for each category for all of them:
  - $\rho = \{(0,0), (1,1), (2,2), (4,4), (6,6), (0,1), (1,2), (2,4), (4,6)\}$

    Reflexive, Antisymmetric; Symmetric Closure: add (1,0),(2,1),(4,2),(6,4),
    Transitive Closure: add (1,4),(2,6),(1,6),(0,2),(0,4),(0,6)
  - $\rho = \{(0,0), (1,1), (2,2), (4,4), (6,6), (4,6), (6,4)\}$

    Reflexive, symmetric, transitive
  - $\rho = \{(0,1), (1,0), (2,4), (4,2), (4,6), (6,4)\}$

    Symmetric; Reflexive Closure: add (0,0),(1,1),(2,2),(4,4),(6,6);
    Transitive Closure: add (0,0),(1,1),(2,2),(2,6),(4,4),(6,2),(6,6)
- For the relation {(1,1), (2,2), (1,2), (2,1), (1,3), (3,1), (3,2), (2,3), (3,3), (4,4), (5,5), (4,5), (5,4)}
  - What is [3] and [4]? **[3] = {1,2,3}** **[4] = {4,5}**

Summary

- Binary Relation
  - On S
  - From S to T
- Types of Relationships
- Properties of Relationships
- Closure of Relations
- Partial Ordering and Hasse Diagram
- Equivalence Relation and Equivalence Class

Discussion