Chapter 4.3

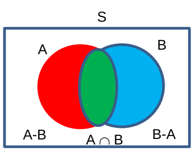
Principle of Inclusion and Exclusion, Pigeonhole Principle

Instructor: Abhishek Santra

Email: abhishek.santra@uta.edu

Principle of Inclusion & Exclusion

- If A and B are subsets of universal set S, then **(A-B)**, **(B-A)** and **(A ∩ B)** are disjoint sets.
  - **(A-B)** ∪ **(B-A)** ∪ **(A ∩ B)** is the same as **A ∪ B**.



- For three disjoint sets (addition principle)

  $|(A-B) \cup (B-A) \cup (A \cap B)| = |A-B| + |B-A| + |A \cap B|$
- We have solved for two finite sets A and B

  $|A-B| = |A| - |A \cap B|$ and $|B-A| = |B| - |A \cap B|$
- Hence, using this, we get

  $|(A-B) \cup (B-A) \cup (A \cap B)| = |A| - |A \cap B| + |B-A| + |A \cap B|$

  $= |A| - |A \cap B| + |B| - |A \cap B| + |A \cap B|$
- Hence, $|A \cup B| = |A| + |B| - |A \cap B|$

Principle of Inclusion & Exclusion

- The principle of inclusion and exclusion for two sets A and B.

  $$|A \cup B| = |A| + |B| - |A \cap B|$$

- The name comes from the fact that to calculate the elements in a union, we include the **individual elements of A** and **B** but subtract the elements common to A and B so that **we don't count them twice**.
- This principle can be generalized to *n* sets.

Example: Inclusion and Exclusion Principle

- *How many integers from 1 to 1000 are either multiples of 3 or multiples of 5?*
  - Let us assume that A = set of all integers from 1 to 1000 that are multiples of 3.
  - Let us assume that B = set of all integers from 1 to 1000 that are multiples of 5.
  - $A \cup B$ = The set of all integers from 1 to 1000 that are multiples of either 3 or 5.
  - $A \cap B$ = The set of all integers that are both multiples of 3 and 5, which also is the set of integers that are multiples of 15.
  - **We need to find out $|A \cup B|$**

Example: Inclusion and Exclusion Principle

- To obtain **$|A \cup B|$**, we need **$|A|$**, **$|B|$** and **$|A \cap B|$**.
  - From 1 to 1000, *every third integer is a multiple of 3*, each of this multiple can be represented as $3p$, for any integer p from 1 through 333
    - Hence **$|A| = 333$**.
  - Similarly for multiples of 5, *each multiple of 5 is of the form 5q for some integer q from 1 through 200*
    - Hence, we have **$|B| = 200$**.
  - To determine the *number of multiples of 15 from 1 through 1000, each multiple of 15 is of the form 15r for some integer r from 1 through 66*.
    - Hence, **$|A \cap B| = 66$**.
- From the principle, we have the number of integers either multiples of 3 or multiples of 5 from 1 to 1000 given by

$|A \cup B| = 333 + 200 - 66 = 467$

Example: Inclusion/exclusion principle **for 3 sets**

- *In a class of students undergoing a computer course the following were observed.*
  - Out of a total of 50 students: 30 know C, 18 know Python, 26 know C#, 9 know both C and Python, 16 know both C and C#, 8 know both Python and C#, **47** know *at least one of the three languages*.

- From this we have to determine,
  a) *How many students know none of these languages?*
  b) *How many students know all three languages?*

Example: Inclusion/exclusion principle for 3 sets

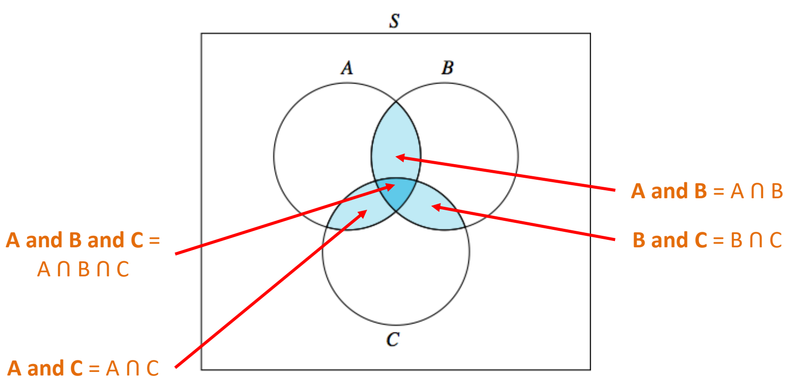
➢ *How many students know none of these languages?*

- Number of students who *do not know any of three languages* = (Number of students in class) **-** (Number of students who know *at least one language*)
- We know that ***47 students know at least one of the three languages*** in the class of 50.
- Hence, the students who know none of these languages = **50 – 47 = 3**

Example: Inclusion/exclusion principle for 3 sets

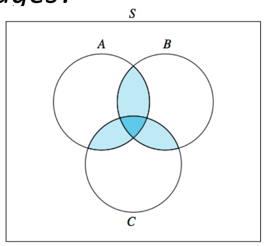
➢ Based on the requirement, we have three sets

- **A** = Set of students who know *C in class*
- **B** = Set of students who know *Python in the class*
- **C** = Set of students who know *C# in class.*



Example: Inclusion/exclusion principle for 3 sets

- $|A \cup B \cup C| = |A \cup (B \cup C)| = |A| + |B \cup C| - |A \cap (B \cup C)|$
  $= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)|$
  $= |A| + |B| + |C| - |B \cap C| - (|A \cap B| + |A \cap C| - |A \cap B \cap C|)$
  $= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$
- *How many students know all three languages?*
  - Find $|A \cap B \cap C|$?



- Given in the problem are the following:
  $|A \cap B| = 9$ (both C & Python)
  $|A \cap C| = 16$ (both C & C#)
  $|B \cap C| = 8$ (both Python & C#)

$|A|=30$, $|B|=18$, $|C|=26$, $|A \cup B \cup C| = 47$

- Hence, using the above formula, we have
  $47 = 30 + 18 + 26 - 8 - 9 - 16 + |A \cap B \cap C|$
  Hence, $\mathbf{|A \cap B \cap C| = 6}$

Pigeonhole Principle

- If more than *k* items are placed into *k* bins, then at least one bin has more than one item.
  - 10 Pigeons in 9 pigeonholes!
- *How many times must a single die be rolled **in order to guarantee** getting the same value twice?*
  - 6 sides. Thus, at least 7 rolls required
- *How many people must be in a room **to guarantee that two people** have the last name begin with the same initial?*
  - 26 alphabets. Thus, at least 27 people

(From 2-1) Direct Proof: Contraposition

- Example 2: Prove that "If n+1 separate passwords are issued to n students, then some student gets ≥ 2 passwords."
  - The contrapositive is:
    - *If every student gets < 2 passwords, then n+1 separate passwords were NOT issued."*
  - Suppose every student has < 2 passwords
  - Then, every one of the n students has at most 1 password.
  - The total number of passwords issued is **at most n, not n+1**.

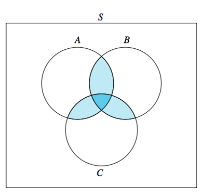
Home Exercise (Solution)

- *A group of students plan to order pizza. If 13 will eat sausage topping, 10 will eat pepperoni, 12 will eat extra cheese, 4 will eat both sausage and pepperoni, 5 will eat both pepperoni and extra cheese, 7 will eat both sausage and extra cheese, and 3 will eat all three toppings,* ***how many students are in the group?***
- Let, A = Set of students who will eat sausage
  B = Set of students who will eat pepperoni
  C = Set of students who will eat extra cheese

Then,

$|A| = 13$, $|B| = 10$, $|C| = 12$, $|A \cap B| = 4$, $|B \cap C| = 5$, $|A \cap C| = 7$, and $|A \cap B \cap C| = 3$

$\mathbf{|A \cup B \cup C| = ?}$



- Hence, using the formula, we have

$|A \cup B \cup C| = 13 + 10 + 12 - 4 - 5 - 7 + 3$

Hence, $\mathbf{|A \cup B \cup C| = 22}$

Discussion