UNIVERSITY OF TEXAS ARLINGTON

Chapter 8.1

Boolean Algebra

(Connecting Logic and Set Theory)

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Propositional Logic vs. Set Theory

- *Set of wffs*
- Operations
  - Conjunction (**∧**)
  - Disjunction (**∨**)
  - Negation (′)
- Properties
  - Commutative, Associative, Distributive, Identity, Complement
- Distinct Elements
  - **0/1**

- *Set of subsets of a set S*
- Operations
  - Intersection (**∩**)
  - Union (**∪**)
  - Complement (′)
- Properties
  - Commutative, Associative, Distributive, Identity, Complement
- Distinct Elements
  - **∅ (Empty) / S (Universe)**

Both are examples of **Boolean Algebra**

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Boolean Algebra (Formalism)

- It is a **Set B**, on which,
- Operations are defined using,
  - **Two Binary Operators: +** and ·
  - **One Unary Operator: ’**
- Has **two distinct elements:** 0 and 1
- and, a few **properties hold true**

Boolean Algebra (Properties)

In Boolean algebra, following hold for all x, y, z $\in$ B

- **Commutative Properties**

  $x + y = y + x$ $\qquad$ $x \cdot y = y \cdot x$
- **Associative Properties**

  $(x + y) + z = x + (y + z)$ $\qquad$ $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
- **Distributive Properties**

  $x + (y \cdot z) = (x + y) \cdot (x + z)$ $\qquad$ $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$
- **Identity Properties**

  $x + 0 = x$ $\qquad$ $x \cdot 1 = x$
- **Complement Properties**

  $x + x' = 1$ $\qquad$ $x \cdot x' = 0$

Proof for Boolean Algebra Properties

- x + x = x (**Idempotent Property**)
  - *A U A = A (Set Theory), A V A = A (Logic)*
- **Proof:**

x + x	= (x + x) · 1	*identity prop.*
	= (x + x) · (x + x’)	*complement prop.*
	= x + (x · x’)	*distributive prop.*
	= x + 0	*complement prop.*
	= x	*identity prop.*

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Proof for Boolean Algebra Properties

- x + 1 = 1 (**Universal Bound Property**)
  - *A U S = S (Set Theory), A V 1 = 1 (Logic)*
- **Proof:**

x + **1**	= (x + **x) + x’**	*complement prop.*
	= ((x + x) · 1) + x’	*identity prop.*
	= ((x + x) · (x + x’)) + x’	*complement prop.*
	= (x + (x · x’)) + x’	*distributive prop.*
	= (x + 0) + x’	*complement prop.*
	= x + x’	*identity prop.*
	= 1	*complement prop.*

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Other Properties

- $(x')' = x$ (**Double Negation**)
  - *$(A')' = A$ (Set Theory), $(A')' = A$ (Logic)*

- **De Morgan's Laws**

  $(x + y)' = x' \cdot y'$ $\qquad$ $(x \cdot y)' = x' + y'$

Home Exercise (Proofs)

- $\mathbf{x + (x \cdot y) = x}$

Proof:

$x + (x \cdot y)$	$= x \cdot 1 + x \cdot y$	*identity prop.*
	$= x \cdot (1+y)$	*distributive prop.*
	$= x \cdot (y+1)$	*commutative prop.*
	$= x \cdot 1$	*universal bound prop.*
	$= x$	*identity prop.*

Hints for Proving Boolean Algebra Equalities

TABLE 8.1
Hints for Proving Boolean Algebra Equalities
Usually the best approach is to start with the more complicated expression and try to show that it reduces to the simpler expression.
Think of adding some form of 0 (like $x \cdot x'$) or multiplying by some form of 1 (like $x + x'$).
Remember property 3a, the distributive property of addition over multiplication—it is easy to forget because it doesn't look like arithmetic.
Remember the idempotent property $x + x = x$ and its dual $x \cdot x = x$.
Remember the universal bound property $x + 1 = 1$ and its dual $x \cdot 0 = 0$.

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Discussion

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