UNIVERSITY OF TEXAS ARLINGTON

INFORMATION TECHNOLOGY LABORATORY @ UTA-CSE

Chapter 2.2
Induction

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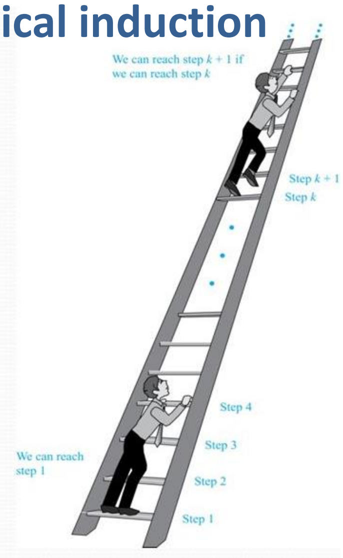
Principles of mathematical induction

- **First Principle:**
  1. P(1) is true, and,
  2. $(\forall k)P(k)$ true $\rightarrow$ $P(k+1)$ is true

  $\rightarrow$ $P(n)$ is true for all positive integers n
- P(1) is true: **Base Case**
  - "1" has the property P
  - **Get to the first step of a ladder**
- For <u>any positive integer k</u>, $P(k) \rightarrow P(k+1)$
  - *If any number 'k' has the property P, so does the next number*
  - ***Once you get to a step, you can climb to the next step of the ladder***



Can we reach every step?

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Principles of mathematical induction

- An example of an implication
  - 2 hypotheses and 1 conclusion
- **Basis (Basis step)**
  - Establishing the truth of P(1)
- **Inductive step**
  - Establishing the truth of **P(k) $\rightarrow$ P(k+1), for any 'k'**
  - For any 'k', when we assume P(k) to be true to prove the inductive step, P(k) is called **inductive assumption (inductive hypothesis)**.
  - Finally, using universal generalization,
  - **($\forall k$)P(k) true $\rightarrow$ P(k+1)**
- Thus, **P(n) is true for all positive integers n**

Example 1

- Prove that $1+3+5+\ldots+(2n-1) = n^2$ for any $n \geq 1$ (n, integer)

When n=1: (LHS) 1 = 1^2 (RHS)

When n=2: (LHS) **1** + 3 = 2^2 (RHS)

When n=3: (LHS) **1 + 3** + 5 = 3^2 (RHS)

When n=4: (LHS) **1 + 3 + 5** + 7 = 4^2 (RHS)

….. Until when???

When n=4, I would like to **use the result** from when n = 3, $3^2+7 = 16 = 4^2$

When n=3, result from when n=2, $2^2+5 = 9 = 3^2$

When n=2, result from when n=1, $1^2+3 = 4 = 2^2$

Now… the idea is …

P(1) is true **($\forall k$)P(k) true $\rightarrow$ P(k+1) true** **P(n) is true for all positive integers n**	**1. Prove the base case** **2. Assume P(k)** **3. Prove P(k+1)**

Example 1 (contd.)

- Prove that $1+3+5+\ldots+(2n-1) = n^2$ for any $n \geq 1$ (n, integer)
- What is P(n)?
- Basis step; P(1) is the equation $1 = 1^2$, which is true.
- Inductive hypothesis; $P(k)$: $1+3+5\ldots+(2k-1) = k^2$ (assumed true for any arbitrary positive integer 'k')
- To Prove $P(k+1)$: $1+3+5\ldots+[2(k+1)-1] = (k+1)^2$
- Again, rewriting the sum on the left side of P($k+1$) reveals how the inductive assumption can be used:

$$\begin{aligned} 1+3+5\ldots+[2(k+1)-1] &= 1+3+5\ldots+(2k-1) + [2(k+1)-1] \\ &= k^2 + [2(k+1)-1] \\ &= k^2 + 2k + 1 \\ &= (k+1)^2 \end{aligned}$$

- Therefore,

$$1+3+5\ldots+[2(k+1)-1] = (k+1)^2$$

and verifies P($k+1$)
- This proves that P(n) is true for any positive integer n

Example 2

- Prove that $1+2+2^2\ldots+2^n = 2^{n+1}-1$ for any $n \geq 1$.
- What is P(n)?
- Basis; $P(1)$ is the equation $1+2 = 2^{1+1}-1$ or $3 = 2^2-1$, which is true.
- Inductive Hypothesis: We assume $P(k)$: $1+2+2^2\ldots+2^k = 2^{k+1}-1$ to be true for any arbitrary positive integer 'k'
- To prove: $P(k+1)$: $1+2+2^2\ldots+2^{k+1} = 2^{k+1+1}-1$
- Again, rewriting the sum on the left side of $P(k+1)$ reveals how the inductive assumption can be used:

$$\begin{aligned} 1+2+2^2\ldots+2^{k+1} &= 1+2+2^2\ldots+2^k + 2^{k+1} \\ &= 2^{k+1}-1 + 2^{k+1} \text{ (by inductive hypothesis)} \\ &= 2(2^{k+1})-1 \\ &= 2^{k+1+1}-1 \end{aligned}$$

- Therefore,

$$1+2+2^2\ldots+2^{k+1} = 2^{k+1+1}-1$$

and verifies $P(k+1)$
- This proves that P(n) is true for any positive integer n

Example 3

- Prove that for any positive integer n, the number 2^{2n}-1 is divisible by 3.
- **Basis Step:** To show P(1) is true.
  - $2^{2(1)}-1 = 4-1 = 3$ is divisible by 3. Clearly this is true.
- Inductive Hypothesis: Assume P(k) is true
  - We assume that *for any positive integer 'k'*, **2^{2k}-1 is divisible by 3**, which means that $2^{2k}-1 = 3m$ for some integer m, or $2^{2k} = 3m+1$.
- To Prove P(k+1) is true: We want to show that **$2^{2(k+1)}$ -1 is divisible by 3**.

$$2^{2(k+1)} -1 = 2^{2k+2} -1$$
$$= 2^2 \bullet 2^{2k}-1$$
$$= 2^2(3m+1) -1 \text{ (by the inductive hypothesis)}$$
$$= 12m+4 -1$$
$$= 12m+3$$
$$= 3(4m+1) \text{ where } 4m+1 \text{ is an integer}$$

- Thus $2^{2(k+1)}$ -1 is divisible by 3. This verifies P(k+1)
- This proves that P(n) is true for any positive integer n

Example 4 (Number of Unique Handshakes)

- In a group of 'n' friends (n >= 1), each person is to shake hands with every other person.
- **What is P(n)?**
  - What is the number of unique handshakes for the 1st person? 2nd person? … nth/last person?
  - Can you find the corresponding *summation expression*?
- **Can you prove that the summation is equal to n(n-1)/2 using mathematical induction?**

Example 4 (Number of Unique Handshakes)

- **To prove**: P(n) = 1 + 2 + 3 + … (n-1) = n(n-1)/2, for any positive integer ‘n’
- **Basis Step:** Show P(1) is true.
  - Replacing n = 1 in the equation for P(n)
  - 0 = 1(1-1)/2 = 0. Thus, P(1) is true
- **Inductive Hypothesis:** Assume P(k) is true for an arbitrary positive integer ‘k’
  - That is, P(k) = 1 + 2 + 3 + … + (k-1) = k(k-1)/2 is true
- **To prove P(k+1) is true.**
  - That is, Prove P(k+1) = 1 + 2 + 3 + … + k = ((k+1)(k))/2
  - Start expanding P(k+1) from the left-hand side (LHS)

    1 + 2 + 3 + … + k = (1 + 2 + 3 + … + (k – 1)) + k
    = k(k-1)/2 + k (using inductive hypothesis)
    = (k(k-1) + 2k)/2
    = k(k – 1 + 2)/2
    = k(k+1)/2
  - This verifies P(k+1)
- This proves that P(n) is true for any positive integer n

Discussion