Chapter 1.2
Propositional Logic

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Propositional Logic

- Statements are sometimes called *propositions*
- The wffs also called *propositional wffs*
- In this section, we learn **how to derive conclusions** from formal logic based on given statements
- The formal system that uses propositional wffs:
  - Propositional logic
  - Statement logic
  - Or, propositional calculus

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Propositional Logic

- **Definition of Argument:**
  - An argument is a *sequence of statements* in which the conjunction of the initial statements (called the **premises/hypotheses**) is said to imply the final statement (called the **conclusion**).
- An argument can be presented in symbolic form as

$$(P_1 \land P_2 \land \ldots \land P_n) \to Q$$

  - where $P_1, P_2, \ldots, P_n$ are given statements, called **hypotheses**
  - and Q is the **conclusion**.

Valid Argument

- What is a valid argument? Different forms:
  - When can Q be *logically deduced from* $P_1, P_2, \ldots, P_n$?
  - When is Q a *logical conclusion from* $P_1, P_2, \ldots, P_n$?
  - When does $P_1, P_2, \ldots, P_n$ *logically imply* Q ?
  - When does Q *follow logically from* $P_1, P_2, \ldots, P_n$?
- Informal answer: **Whenever the truth of hypotheses leads to the conclusion**
- We need to *focus on the relationship* of the **conclusion to the hypotheses** and not just any knowledge we might have about the conclusion Q.

Valid Argument Example

- Consider two hypotheses:
  - P_1: **If** George Washington was the first president of the United States, **then** John Adams was the first vice president.
  - P_2 : George Washington was the first president of the United States.
- Conclusion:
  - Q: John Adams was the first vice president.
- Symbolic representation
  - $(A \rightarrow B) \wedge A \rightarrow B$
    P1 P2 Q
  - **Intrinsically True**

A	B	A→B	(A→B) ∧ A	(A→B) ∧ A→B
T	T	T	T	**T**
T	F	F	F	**T**
F	T	T	F	**T**
F	F	T	F	**T**

Valid Argument

- **Definition of valid argument**:
  - The propositional wff $P_1 \wedge P_2 \wedge \ldots \wedge P_n \rightarrow Q$ is a **valid argument** when it is a **tautology**
- How to arrive at a valid argument?
  - **Truths Tables**
  - **Using a proof sequence**

Proof Sequence

- Definition of Proof Sequence
  - A *sequence of wffs* in which,
  - each *wff is either a hypothesis* or,
  - *the result* of applying one of the formal system's *derivation rules* to *earlier wffs in the sequence*.

Derivation Rules

- **Derivation Rules:**
  - To test whether $P_1 \land P_2 \land \ldots \land P_n \to Q$ is tautology
  - We can use **derivation rules** which **manipulates wffs in a truth preserving manner**
  - Equivalence Rules & Inference Rules
- **Equivalence Rules**
  - Allows individual wffs to be **rewritten**
  - Truth preserving rules
- **Inference Rules**
  - Allows **new wffs to be derived**
  - Work only in one direction

Equivalence Rules

- **Certain pairs of wffs are equivalent**, hence one can be **substituted** for the other with no change to truth values.
- The set of equivalence rules are summarized here:
  - Let P, Q, and R be wffs

Expression	Equivalent to	**Abbreviation for rule**
$P \vee Q$ $P \wedge Q$	$Q \vee P$ $Q \wedge P$	**Commutative: comm**
$(P \vee Q) \vee R$ $(P \wedge Q) \wedge R$	$P \vee (Q \vee R)$ $P \wedge (Q \wedge R)$	**Associative: ass**
$(P \vee Q)'$ $(P \wedge Q)'$	$P' \wedge Q'$ $P' \vee Q'$	**De-Morgan's Laws: De-Morgan**
$P \rightarrow Q$	$P' \vee Q$	**Implication: imp**
P	$(P')'$	**Double Negation: dn**
$P \leftrightarrow Q$	$(P \rightarrow Q) \wedge (Q \rightarrow P)$	**Equivalence: equ**

Inference Rules

- Allow us to add to the proof sequence a **new wff** that matches the last part of the rule pattern, if one or more wffs that match the *first part of the rule already exist in the proof sequence*.

From (first part, exists already)	**Can Derive (last part)**	**Abbreviation for rule**
$P, P \rightarrow Q$	Q	**Modus Ponens: mp**
$P \rightarrow Q, Q'$	P'	**Modus Tollens: mt**
P, Q	$P \wedge Q$	**Conjunction: con**
$P \wedge Q$	P, Q	**Simplification: sim**
P	$P \vee Q$	**Addition: add**

- Note: Inference rules ***do not*** work in both directions, unlike equivalence rules.

Examples of the rules

- Example for using **equivalence rule** in a proof sequence:
  - **Simplify (A′ V B′) V C**

1. (A′ V B′) V C	
2. (A Λ B)′ V C	1, De Morgan
3. (A Λ B) → C	2, imp

- Example of using **inference rule**
  - If it is bright and sunny today (P), then I will wear my sunglasses. (Q) (P→Q)

Modus Ponens

It is bright and sunny today. (P)

Therefore, I will wear my sunglasses. (Q)

Modus Tollens

I will not wear my sunglasses. (Q')

Therefore, it is not (bright and sunny) today. Therefore, it is not bright or not sunny today. (P')

Examples

- Ex. 12) Suppose that $A \rightarrow (B \wedge C)$ and A are two hypotheses of an argument. The following is a proof sequence:

1. $A \rightarrow (B \wedge C)$	hyp
2. A	hyp
3. $B \wedge C$	**1,2, mp**

- Ex. 13) Suppose that $(A \rightarrow B) \vee C$ and A are two hypotheses of an argument. The following is a proof sequence:

1. $(A \rightarrow B) \vee C$	hyp
2. A	hyp

Can we apply mp?

Deduction Method

- To prove an argument of the form

  $P_1 \land P_2 \land \ldots \land P_n \to (R \to Q)$
- Deduction method allows for the **use of R as an additional hypothesis** and thus prove

  $P_1 \land P_2 \land \ldots \land P_n \land R \to Q$
- Prove $(A \to B) \land (B \to C) \to (A \to C)$

  Using deduction method, prove $(A \to B) \land (B \to C) \land A \to C$
  1. *A → B* *hyp*
  2. *B → C* *hyp*
  3. *A* *hyp*
  4. *B* *1,3 mp*
  5. *C* *2,4 mp*
- Called Rule of **Hypothetical Syllogism or hs** in short. (table 1.14)
- Many such other rules can be derived from existing rules which thus provide **easier and faster proofs**.

Additional Inference Rules

- These additional rules can be derived by the previous rules.

From	Can Derive	Name / Abbreviation
$P \to Q$, $Q \to R$	$P \to R$	**Hypothetical syllogism- hs**
$P \lor Q$, P'	Q	**Disjunctive syllogism- ds**
$P \to Q$	$Q' \to P'$	**Contraposition- cont**
$Q' \to P'$	$P \to Q$	**Contraposition- cont**
P	$P \land P$	**Self-reference - self**
$P \lor P$	P	**Self-reference - self**
$(P \land Q) \to R$	$P \to (Q \to R)$	**Exportation - exp**
P, P'	Q	**Inconsistency - inc**
$P \land (Q \lor R)$	$(P \land Q) \lor (P \land R)$	**Distributive - dist**
$P \lor (Q \land R)$	$(P \lor Q) \land (P \lor R)$	**Distributive - dist**

Proofs of Inference Rules

- **Prove that $(P \rightarrow Q) \rightarrow (Q' \rightarrow P')$** is a valid argument (called Contraposition – cont).
  - $(P \rightarrow Q) \wedge Q' \rightarrow P'$ Deduction Method
  - *Directly follows from Modus Tollens (mt)*
- **Prove $P \wedge P' \rightarrow Q$ (called Inconsistency - inc)**

1.	P	hyp
2.	P'	hyp
3.	$P \vee Q$	1, add
4.	$Q \vee P$	3, comm
5.	$(Q')' \vee P$	4, dn
6.	$Q' \rightarrow P$	5, imp
7.	$(Q')'$	2, 6, mt
8.	Q	7, dn

Proofs using Propositional Logic

- Prove the argument: $A \wedge (B \rightarrow C) \wedge [(A \wedge B) \rightarrow (D \vee C')] \wedge B \rightarrow D$

First, write down all the hypotheses.

1. A
2. $B \rightarrow C$
3. $(A \wedge B) \rightarrow (D \vee C')$
4. B

Use the inference and equivalence rules to get at the conclusion D.

5.	C	2,4, mp
6.	$A \wedge B$	1,4, con
7.	$D \vee C'$	3,6, mp
8.	$C' \vee D$	7, comm
9.	$C \rightarrow D$	8, imp
10.	D	5,9 mp

The idea is to keep focused on the result and sometimes it is very easy to go down a longer path than necessary.

More Proofs

➢ **$(A \wedge B)' \wedge (C' \wedge A)' \wedge (C \wedge B')' \to A'$ is a valid argument**

1.	$(A \wedge B)'$	hyp
2.	$(C' \wedge A)'$	hyp
3.	$(C \wedge B')'$	hyp
4.	$A' \vee B'$	1, De Morgan
5.	$B' \vee A'$	4, comm
6.	$B \to A'$	5, imp
7.	$(C')' \vee A'$	2, De Morgan
8.	$C' \to A'$	7, imp
9.	$C' \vee (B')'$	3, De Morgan
10.	$(B')' \vee C'$	9, comm
11.	$B' \to C'$	10, imp
12.	$B' \to A'$	8, 11, hs
13.	$(B \to A') \wedge (B' \to A')$	6, 12, con

Not done yet!!

➢ **At this point, we have now to prove that**

$(B \to A') \wedge (B' \to A') \to A'$

➢ Proof sequence

1.	$B \to A'$	hyp
2.	$B' \to A'$	hyp
3.	$A \to B'$	1, cont
4.	$A \to A'$	3, 2, hs
5.	$A' \vee A'$	4, imp
6.	A'	5, self

Verbal Arguments

- Russia was a superior power, and either France was not strong or Napoleon made an error. Napoleon did not make an error. If the army did not fail, then France was strong. **Hence** the army failed and Russia was a superior power.
- Converting it to a propositional form using letters A, B, C and D

A: Russia was a superior power	
B: France was strong	B′: France was not strong
C: Napoleon made an error	C′: Napoleon did not make an error
D: The army failed	D′: The army did not fail

- Combining the statements using logic

$(A \wedge (B' \vee C))$	hypothesis
C'	hypothesis
$(D' \rightarrow B)$	hypothesis
$(D \wedge A)$	conclusion

- Combining them, the propositional form is

  $\mathbf{(A \wedge (B' \vee C)) \wedge C' \wedge (D' \rightarrow B) \rightarrow (D \wedge A)}$

Proving Verbal Argument

- **Prove** $\mathbf{(A \wedge (B' \vee C)) \wedge C' \wedge (D' \rightarrow B) \rightarrow (D \wedge A)}$
- **Proof sequence**

1.	$A \wedge (B' \vee C)$	hyp
2.	C'	hyp
3.	$D' \rightarrow B$	hyp
4.	A	1, sim
5.	$B' \vee C$	1, sim
6.	$C \vee B'$	5, comm
7.	B'	2, 6, ds
8.	$B' \rightarrow (D')'$	3, cont
9.	$(D')'$	7, 8, mp
10.	D	9, dn
11.	$D \wedge A$	4, 10 , con

Home/Class Exercise - 1

- Prove the following arguments
  - **(A' → B') ∧ (A → C) → (B → C)**

Deduction method: **(A'->B') ^ (A->C) ^ B -> C**

1. (A'->B')	hyp,	1. (A'->B')	hyp,	1. (A'->B')	hyp,
2. (A->C)	hyp,	2. (A->C)	hyp,	2. (A->C)	hyp,
3. B	hyp,	3. B	hyp,	3. B	hyp,
4. (A')'	1,3 mt,	4. B -> A	1, cont,	4. B -> A	1, cont,
5. A	4 dn,	5. A	3,4 mp,	5. B -> C	2,4 hs,
6. C	2,5 mp	6. C	2,5 mp	6. C	3,5 mp
Approach #1		*Approach #2*		*Approach #3*	

Home/Class Exercise - 2

- If the program is efficient, it executes quickly. Either the program is efficient, or it has a bug. However, the program does not execute quickly. Therefore, it has a bug. (use letters E, Q, B)

(E->Q) ^ E v B ^ Q' -> B		*Another Proof Sequence*	
1. E->Q	hyp,	1. E->Q	hyp,
2. E v B	hyp,	2. E v B	hyp,
3. Q'	hyp,	3. Q'	hyp,
4. E' V Q	1 imp,	4. E'	1,3 mt,
5. Q V E'	4, comm	5. (E')' v B	2, dn
6. E'	3,5 ds,	6. E' -> B	5, imp,
7. B	2,6 ds	7. B	4,6 mp

Home/Class Exercise - 3

- The crop is good, but there is not enough water. If there is a lot of rain or not a lot of sun, then there is enough water. Therefore, the crop is good and there is a lot of sun. (use letters C, W, R, S)

(C ^ W') ^ (R v S' -> W) -> (C ^ S)

1. C ^ W'	hyp,	6. R' ^ (S')'	5 DM,
2. (R v S') -> W	hyp,	7. R'^S	6 dn,
3. W'	1 sim,	8. S	7 sim,
4. W' -> (R V S')'	2 cont,	9. C	1 sim,
5. (R V S')'	3,4 mp,	10. C^S	8,9 con

Discussion