Quantum field theory

Kasi Reddy Sreeman Reddy, Roll number: 190070029

E-mail: sreeman@iitb.ac.in

Abstract: Quantum field theory notes. Assignment 1 is written in section 1.1. Assignment 2 is written in section 1.2.

Contents

1 Basics in field quantization

For a classical scalar field $\phi(x)$ the equations of motion are can be obtained using the principle of stationary action. Action is defined as:

$$\begin{aligned}\mathcal{S} &= \int \mathrm{d}^{D-1}x\mathrm{d}t\mathcal{L} \\ &= \int \mathrm{d}^{D-1}x\mathrm{d}t\left[\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right] \\ &= \int \mathrm{d}^{D-1}x\mathrm{d}t\left[\frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}\delta^{ij}\partial_i\phi\partial_j\phi - \frac{1}{2}m^2\phi^2\right],\end{aligned}$$

where $\mathcal{L}$ is the Lagrangian density. The Euler–Lagrange equations obtained by $d\mathcal{S} = 0$ for any $\mathcal{L}$ are

$$\frac{\partial\mathcal{L}}{\partial\varphi} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right).$$

for the above Lagrangian density they reduce to

$$(\partial^\mu\partial_\mu + m^2)\phi = (\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2)\phi = (\partial_t^2 - \nabla^2 + m^2)\phi = 0\ ,$$

To quantize classical fields there are 2 main ways called **canonical quantization formalism** and **path integral formalism**. Both the formalisms are important as some things are easier in the 1st and other in the 2nd.

1.1 Canonical quantization formalism

Quantum mechanics is obtained after we replace the canonical variables by operators and the Poisson brackets by commutators in classical mechanics.

Similarly we assume the follwing coomutation rules at **equal time**

$$[\phi\left(\vec{x},t\right),\pi\left(\vec{y},t\right)] = \mathrm{i}\delta^3(\vec{x}-\vec{y})$$

$$[\phi\left(\vec{x},t\right),\phi\left(\vec{y},t\right)] = [\pi\left(\vec{x},t\right),\pi\left(\vec{y},t\right)] = 0$$

here

$$\pi = \partial_t \phi$$

Hamiltonian density is defined as

$$\mathcal{H}(\phi,\pi,\mathbf{x},t) = \dot{\phi}\pi - \mathcal{L}(\phi,\nabla\phi,\partial\phi/\partial t,\mathbf{x},t)\,.$$

The Heisenberg equations of motion are

$$\dot{\hat{\phi}}(x) = \mathrm{i}[\hat{H},\hat{\phi}(x)]$$

$$\dot{\hat{\pi}}(x) = \mathrm{i}[\hat{H},\hat{\pi}(x)]$$

1.1.1 Fourier decomposition of the free scalar field

The four-dimensional analogue of the Fourier expansion of the field (if it is a classical field $\hat{a}(k)$ are constants but for quantum field it they are operators) ϕ takes the form

$$\hat{\phi}(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}^3k}{(2\pi)^{3/2}\sqrt{2\omega}}\left[\hat{a}(k)\mathrm{e}^{-\mathrm{i}k\cdot x} + \hat{a}^\dagger(k)\mathrm{e}^{\mathrm{i}k\cdot x}\right]$$

with a similar expansion for the **conjugate momentum** $\hat{\pi} = \dot{\hat{\phi}}$:

$$\hat{\pi}(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}^3k}{(2\pi)^{3/2}\sqrt{2\omega}}(-\mathrm{i}\omega)\left[\hat{a}(k)\mathrm{e}^{-\mathrm{i}k\cdot x} - \hat{a}^\dagger(k)\mathrm{e}^{\mathrm{i}k\cdot x}\right]$$

Here $k\cdot x$ is the four-dimensional dot product $k\cdot x = \omega t - \boldsymbol{k}\cdot\boldsymbol{x}$, and $\omega = +\left(\boldsymbol{k}^2+m^2\right)^{1/2}$. A **positive-frequency** solution of the field equation has as its coefficient the operator that **annihilates** ($\hat{a}(k)$) a particle in that single-particle wave-function. A **negative-frequency** solution of the field equation, being the Hermitian conjugate of a positive-frequency solution, has as its coefficient the operator that **creates** ($\hat{a}^\dagger(k)$) a particle in that positive-energy single-particle wave function.

$$\hat{n}(k) = \hat{a}^\dagger(k)\hat{a}^\dagger(k)$$

The one-particle state with momentum k is given by the creation operator acting on the vacuum state (here the **normalisation** constant is chosen to satisfy **Lorentz invariance**)

$$|\mathbf{k}\rangle = \sqrt{(2\pi)^3 2\omega_k}\hat{a}(k)|0\rangle$$

The Hamiltonian for the scalar field is found to be

$$\hat{H}_{\mathrm{KG}}=\int \mathrm{d}^3 x \hat{\mathcal{H}}_{\mathrm{KG}}=\int_{-\infty}^{\infty} \mathrm{d}^3 x \frac{1}{2}\left[\hat{\pi}^2+\nabla \hat{\phi} \cdot \nabla \hat{\phi}+m^2 \hat{\phi}^2\right]$$

and this can be expressed in terms of the $\hat{a}$'s and the $\hat{a}^{\dagger}$'s using the expansion for $\hat{\phi}$ and $\hat{\pi}$ and the commutator

$$\left[\hat{a}(k), \hat{a}^{\dagger}\left(k^{\prime}\right)\right]=\delta^3\left(\boldsymbol{k}-\boldsymbol{k}^{\prime}\right)$$

with all others vanishing. The result is, as expected,

$$\hat{H}_{\mathrm{KG}}=\frac{1}{2} \int \mathrm{d}^3 \boldsymbol{k}\left[\hat{a}^{\dagger}(k) \hat{a}(k)+\hat{a}(k) \hat{a}^{\dagger}(k)\right] \omega$$

and to remove the zero-point energy we use **Wick ordering** (operators rearranged with all creation operators on the left).

$$\hat{H}_{\mathrm{KG}}=\int \mathrm{d}^3 \boldsymbol{k} \hat{a}^{\dagger}(k) \hat{a}(k) \omega$$

$$\vec{P}=-\int d^3 x \pi \vec{\nabla} \phi=\int d^3 p \vec{p} a_{\vec{p}}^{\dagger} a_{\vec{p}}$$

Properties of creation and annihilation operators:

$$a_{\mathbf{p}}=\frac{i}{\sqrt{(2 \pi)^3} \sqrt{2 \omega_{\mathbf{p}}}} \int d^3 \mathbf{x}\left[\Pi(\mathbf{x})-i \omega_{\mathbf{p}} \phi(\mathbf{x})\right] e^{-i \mathbf{p} \cdot \mathbf{x}}$$

$$a_{\mathbf{k}}^{\dagger}=\frac{-i}{\sqrt{(2 \pi)^3} \sqrt{2 \omega_{\mathbf{k}}}} \int d^3 \mathbf{y}\left[\Pi(\mathbf{y})+i \omega_{\mathbf{k}} \phi(\mathbf{y})\right] e^{i \mathbf{k} \cdot \mathbf{y}}$$

$$\left[\hat{H}, a_{\mathbf{p}}^{\dagger}\right]=\omega_{\mathbf{p}} a_{\mathbf{p}}^{\dagger}$$

$$\left[\hat{H}, a_{\mathbf{p}}\right]=-\omega_{\mathbf{p}} a_{\mathbf{p}}$$

$$\overbrace{[H,[H,[\cdots[H}^{n \text { times }}, a_{\mathbf{p}}] \cdots]]]=\left(-\omega_{\mathbf{p}}\right)^n a$$

$$\overbrace{[\hat{H},[\hat{H},[\cdots[\hat{H}}^{n \text { times }}, a_{\mathbf{p}}^{\dagger}] \cdots]]]=\left(\omega_{\mathbf{p}}\right)^n a$$

$$\hat{H} a_{\mathbf{p}}=a_{\mathbf{p}}\left(\hat{H}-E_{\mathbf{p}}\right) \hat{H}^n a_{\mathbf{p}}=a_{\mathbf{p}}\left(\hat{H}-E_{\mathbf{p}}\right)^n$$

$$e^{i \hat{H} t} a_{\mathbf{p}} e^{-i \hat{H} t}=a_{\mathbf{p}} e^{-i E_{\mathbf{p}} t}, e^{i \hat{H} t} a_{\mathbf{p}}^{\dagger} e^{-i \hat{H} t}=a_{\mathbf{p}}^{\dagger} e^{i E_{\mathbf{p}} t}$$

$$e^{-i \hat{\mathbf{P}} \cdot \mathbf{x}} a_{\mathbf{p}} e^{i \hat{\mathbf{P}} \cdot \mathbf{x}}=a_{\mathbf{p}} e^{i \mathbf{p} \cdot \mathbf{x}}$$

$$e^{-i \hat{\mathbf{P}} \cdot \mathbf{x}} a_{\mathbf{p}}^{\dagger} e^{i \hat{\mathbf{P}} \cdot \mathbf{x}}=a_{\mathbf{p}}^{\dagger} e^{-i \mathbf{p} \cdot \mathbf{x}}$$

$$\phi(x)=e^{i(\hat{H} t-\hat{\mathbf{P}} \cdot \mathbf{x})} \phi(0) e^{-i(\hat{H} t-\hat{\mathbf{P}} \cdot \mathbf{x})}=e^{i P \cdot x} \phi(0) e^{-i P \cdot x}$$

1.1.2 Feynman propagator

Definition:

$$\begin{aligned} \mathrm{i}\Delta\left(x_1-x_2\right) &\equiv \langle 0\left|T\left(\phi\left(x_1\right)\phi\left(x_2\right)\right)\right|0\rangle \\ &= \theta\left(t_1-t_2\right)\langle 0\left|\phi\left(x_1\right)\phi\left(x_2\right)\right|0\rangle+\theta\left(t_2-t_1\right)\langle 0\left|\phi\left(x_2\right)\phi\left(x_1\right)\right|0\rangle \\ &= \int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{\mathrm{i}}{k^2-\mu^2+\mathrm{i}\varepsilon}\exp\left\{\mathrm{i}k\cdot\left(x_1-x_2\right)\right\} \end{aligned}$$

Here $\theta(x)$ is the Heaviside step function defined by

$$\theta(x) := \mathbf{1}_{x>0}$$

Quartic interaction example:

$$\mathcal{L}(\varphi)=\frac{1}{2}[\partial^\mu\varphi\partial_\mu\varphi-m^2\varphi^2]-\frac{\lambda}{4!}\varphi^4$$

$$\hat{\mathcal{H}}=\frac{1}{2}\left[\hat{\pi}^2+\nabla\hat{\phi}\cdot\nabla\hat{\phi}+m^2\hat{\phi}^2\right]+\frac{\lambda}{4!}\varphi^4$$

Let $\hat{\mathcal{H}}=\hat{\mathcal{H}}_0+\hat{\mathcal{H}}'$ with $\hat{\mathcal{H}}'=\frac{\lambda}{4!}\varphi^4$

1.1.3 Dynamical pictures

If $H_\mathrm{S}=H_{0,\mathrm{S}}+H_{1,\mathrm{S}}$,

Evolution	Picture (v · t · e)		
of:	Heisenberg	**Interaction**	Schrödinger
Ket state	constant	$\lvert\psi_\mathrm{I}(t)\rangle=e^{iH_{0,\mathrm{S}}\,t/\hbar}\lvert\psi_\mathrm{S}(t)\rangle$	$\lvert\psi_\mathrm{S}(t)\rangle=e^{-iH_\mathrm{S}\,t/\hbar}\lvert\psi_\mathrm{S}(0)\rangle$
Observable	$A_\mathrm{H}(t)=e^{iH_\mathrm{S}\,t/\hbar}A_\mathrm{S}e^{-iH_\mathrm{S}\,t/\hbar}$	$A_\mathrm{I}(t)=e^{iH_{0,\mathrm{S}}\,t/\hbar}A_\mathrm{S}e^{-iH_{0,\mathrm{S}}\,t/\hbar}$	constant
Density matrix	constant	$\rho_\mathrm{I}(t)=e^{iH_{0,\mathrm{S}}\,t/\hbar}\rho_\mathrm{S}(t)e^{-iH_{0,\mathrm{S}}\,t/\hbar}$	$\rho_\mathrm{S}(t)=e^{-iH_\mathrm{S}\,t/\hbar}\rho_\mathrm{S}(0)e^{iH_\mathrm{S}\,t/\hbar}$

Heisenberg Picture

$$\frac{d}{dt}A_\mathrm{H}(t)=\frac{i}{\hbar}[H_\mathrm{H},A_\mathrm{H}(t)]+\left(\frac{\partial A_\mathrm{S}}{\partial t}\right)_\mathrm{H}$$

Interaction Picture

$$i\hbar\frac{d}{dt}\lvert\psi_\mathrm{I}(t)\rangle=H_{1,\mathrm{I}}(t)\lvert\psi_\mathrm{I}(t)\rangle,$$

$$i\hbar\frac{d}{dt}A_\mathrm{I}(t)=[A_\mathrm{I}(t),H_{0,\mathrm{S}}].$$

$$i\hbar\frac{d}{dt}\rho_\mathrm{I}(t)=[H_{1,\mathrm{I}}(t),\rho_\mathrm{I}(t)],$$

Note that quantum fields are operator fields. So they also **transform like operators**. In the previous case $H_{1,\mathrm{S}}$ was written as $\hat{\mathcal{H}}'$.

Now observing the 1st in the above 3 equations we can define the unitary operator

$$\phi^I(\mathbf{x},t) = U(t,0)\,\phi(\mathbf{x},t)U^{-1}(t,0)$$

$$\Rightarrow \mathrm{i}\frac{\partial}{\partial t}U(t,t_0) = H'^{\mathrm{I}}(t)U(t,t_0)$$

Now **naively** we might guess the solution as

$$U(t,t_0) = \exp\left(-i\int_{t_0}^{t} H'^{I}\left(\tilde{t}\right)d\tilde{t}\right)$$

but the Hamiltonians at different times won't commute and because of that when we substitute it in the above equation it won't satisfy. So the correct answer is

$$U(t,t_0) = T\,\left(\exp\left(-i\int_{t_0}^{t} H'^{I}\left(\tilde{t}\right)d\tilde{t}\right)\right)$$

$$\Rightarrow U(t,t_0) = 1+\sum_{p=1}^{\infty}\frac{(-i)^p}{p!}\int_{t_0}^{t} d^4x_1\int_{t_0}^{t} d^4x_2\cdots\int_{t_0}^{t} d^4x_p T(\mathcal{H}'^{I}(x_1)\mathcal{H}'^{I}(x_2)\ldots\mathcal{H}'^{I}(x_p))$$

Using the following facts we can easily check that the above equation satisfies.

1. Inside the time ordering operator the Hamiltonians at different times will commute.

2. Partial derivative commutes with time ordering operator.

1.1.4 Green's functions in $\lambda\phi^4$ theory

Green's function is defined by

$$G^{(n)}(x_1,\ldots,x_n) = \langle 0\,|T\left(\phi\left(x_1\right),\ldots,\phi\left(x_n\right)\right)|\,0\rangle$$

They are useful in finding S-matrix elements. Using the fact that inside Time ordering operator they commute we get

$$\begin{aligned} G^{(n)}(x,\ldots,x_n) = \langle 0|U^{-1}(t,0)T\left(\phi^{\mathrm{I}}(x_1)\ldots\phi^{\mathrm{I}}(x_n)\right. \\ \left.\times\exp\left[-\mathrm{i}\int_{-t}^{t}\mathrm{d}t' H'^{\mathrm{I}}\left(t'\right)\right]\right)U(-t,0)|0\rangle \end{aligned}$$

Now substituting the expression obtained earlier for the unitary operator we get

$$\begin{aligned} G^{(n)}(x_1,\ldots,x_n) = \sum_{p=0}^{\infty}\frac{(-\mathrm{i})^p}{p!}\int_{-\infty}^{\infty}\mathrm{d}^4y_1,\ldots,\mathrm{d}^4y_p\langle 0|T\left(\phi^{\mathrm{I}}(x_1),\ldots,\phi^{\mathrm{I}}(x_n)\right. \\ \left.\times\mathscr{H}'\left(\phi^{\mathrm{I}}(y_1)\right)\ldots\mathscr{H}'\left(\phi^{\mathrm{I}}(y_p)\right)|0\rangle_c. \end{aligned}$$

Now look at the 1st order term for $\mathscr{H}' = \frac{\lambda}{4!}\phi^4$

$$G_1^{(4)}(x_1,\ldots,x_4) = -\frac{i\lambda}{4!}\int\mathrm{d}^4y\left\langle 0\middle|T\left(\phi^{\mathrm{I}}(x_1),\ldots,\phi^{\mathrm{I}}(x_4)\left[\phi^{\mathrm{I}}(y)\right]^4\right)\middle|0\right\rangle$$

If we order the creation (to the left) and annihilation (to the right) operators then they act on the respective side vacuum state to give 0 and those terms will be eliminated. We then use the Wick's theorem to obtain

$$\begin{aligned}G_1^{(4)}\left(x_1,\ldots,x_4\right)=&(-\mathrm{i}\lambda)\int \mathrm{d}^4 y\left\langle 0\left|T\left(\phi^{\mathrm{I}}\left(x_1\right)\phi^{\mathrm{I}}(y)\right)\right|0\right\rangle\left\langle 0\left|T\left(\phi^{\mathrm{I}}\left(x_2\right)\phi^{\mathrm{I}}(y)\right)\right|0\right\rangle\\&\left\langle 0\left|T\left(\phi^{\mathrm{I}}\left(x_3\right)\phi^{\mathrm{I}}(y)\right)\right|0\right\rangle\left\langle 0\left|T\left(\phi^{\mathrm{I}}\left(x_4\right)\phi^{\mathrm{I}}(y)\right)\right|0\right\rangle\end{aligned}$$

this can be expressed as the following Feynman diagram

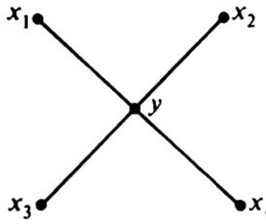

FIG. 1.1. Graphic representation of eqn (1.33).

Similarly we can draw Feynman diagrams for higher order terms also.

1.1.5 Feynman rules of $\lambda\phi^4$ theory

Perturbation theory gives us the following important rules named after Richard Feynman. For $\mathscr{H}_1 = \frac{\lambda}{4!}\phi^4$ interaction, we have the following rules for calculating the N-point amputated Green's function.

1. Each field in the n-point Euclidean Green's function is represented by an external line (half-edge) in the graph. Draw all possible connected, topologically distinct, graphs with N external lines.

2. Each vertex is represented by a factor $-i\lambda$.

3. If there are internal momentums l that not fixed by momentum conservation at each vertex, perform an integration $\int \dfrac{d^4 l}{(2\pi)^4}$ for each such l.

4. The result is divided by a symmetry factor S, which is the number of ways the lines and vertices of the graph can be rearranged without changing its connectivity.

1.2 Path integral formalism

Compared to the canonical quantization approach Path integral formalism the advantage of exhibiting a closer relationship to the classical dynamical description.

1.2.1 Quantum mechanics

The transition matrix element corresponding to the overlap between initial and final stages is

$$\left\langle q^{\prime}; t^{\prime} \mid q; t\right\rangle=\left\langle q^{\prime}\left|\mathrm{e}^{-\mathrm{i} H\left(t^{\prime}-t\right)}\right| q\right\rangle$$

here $\mid q;t\rangle$ is the state in Heisenberg picture defined by $\mid q;t\rangle = e^{iHt} \mid q(t)\rangle$ where $\mid q(t)\rangle$ is in the Schrodinger picture. Now if we divide the interval (t',t) into n segments with space $\delta t = (t'-t)/n$ and by inserting complete sets of eigenstates of Q in between we get

$$\left\langle q' \left| \mathrm{e}^{-\mathrm{i}H(t'-t)} \right| q \right\rangle = \int \mathrm{d}q_1 \ldots \mathrm{d}q_{n-1} \left\langle q' \left| \mathrm{e}^{-\mathrm{i}H\delta t} \right| q_{n-1} \right\rangle \left\langle q_{n-1} \left| \mathrm{e}^{-\mathrm{i}H\delta t} \right| q_{n-2} \right\rangle \ldots \\ \times \ldots \left\langle q_1 \left| \mathrm{e}^{-\mathrm{i}H\delta t} \right| q \right\rangle$$

Now substituting the Hamiltonian operator $H(P,Q) = \dfrac{P^2}{2m} + V(Q)$ and by inserting complete sets of eigenstates of P we get

$$\left\langle q' \left| \mathrm{e}^{-\mathrm{i}H(t'-t)} \right| q \right\rangle \simeq \int \left(\frac{\mathrm{d}p_1}{2\pi}\right) \ldots \left(\frac{\mathrm{d}p_n}{2\pi}\right) \int \mathrm{d}q_1 \ldots \mathrm{d}q_{n-1} \\ \times \exp\left\{ \mathrm{i} \sum_{i=1}^{n} \left[p_i \left(q_i - q_{i-1}\right) - \delta t H \left(p_i, \frac{q_i + q_{i-1}}{2}\right)\right]\right\}$$

Note that the above equation the RHS contains p, q and not P, Q. As $n \to \infty$ it becomes

$$\left\langle q' \left| \mathrm{e}^{-\mathrm{i}H(t'-t)} \right| q \right\rangle = \int \left[\frac{\mathrm{d}p\ \mathrm{d}q}{2\pi}\right] \exp\left\{ \mathrm{i} \int_t^{t'} \mathrm{d}t [p\dot{q} - H(p,q)]\right\}$$

If we now integrate the dp_i integrals using the Gaussian integral formula and applying $n \to \infty$ we get

$$\left\langle q'; t' \mid q; t \right\rangle = N \int [\mathrm{d}q] \exp\left\{ \mathrm{i} \int_t^{t'} L(q,\dot{q}) \mathrm{d}\tau \right\}$$

1.2.2 Field theory

To go from QM to QFT we replace as given below

$$\prod_{i=1}^{N} [\ \mathrm{d}q_i\ \mathrm{d}p_i] \to [\mathrm{d}\phi(x) \mathrm{d}\pi(x)]$$

$$L\left(q_i, \dot{q}_i\right), H\left(q_i, p_i\right) \to \int \mathrm{d}^3 x \mathscr{L}\left(\phi, \partial_\mu \phi\right), \int \mathrm{d}^3 x \mathscr{H}(\phi, \pi)$$

The connected Green's function is defined by

$$G^{(n)}\left(\bar{x}_1, \ldots, \bar{x}_n\right) = \left[\frac{1}{W_{\mathrm{E}}[J]} \frac{\delta^n W_{\mathrm{E}}[J]}{\delta J\left(\bar{x}_1\right) \ldots \delta J\left(\bar{x}_n\right)}\right]\Bigg|_{J=0}$$

$$G^{(n)}\left(x_1, \ldots, x_n\right) = \left[\frac{\delta^n lnW[J]}{\delta J\left(x_1\right) \ldots \delta J\left(x_n\right)}\right]\Bigg|_{J=0}$$

$$W[J] = \int [\mathrm{d}\phi] \exp\left\{ \mathrm{i} \int \mathrm{d}^4 x [\mathscr{L}(\phi(x)) + J(x)\phi(x)]\right\}$$

1.2.3 $\lambda\phi^4$ theory

In ϕ^4 or $-\frac{\lambda}{4!}\phi^4$ theory let $W[J]$ be the vacuum-to-vacuum transition amplitude in the presence of an external source $J(x)$ or the generating functional.

If $\lambda = 0$ (i.e. free field) then let the generating functional be denoted by $W_0[J]$ and it is given by

$$W_0[J] = \exp\left[\frac{1}{2}\int \mathrm{d}^4x\ \mathrm{d}^4y J(x)\Delta(x,y)J(y)\right].$$

For $\lambda \neq 0$ it is

$$W[J] = \int [\mathrm{d}\phi] \exp\left\{\mathrm{i}\int \mathrm{d}^4x[\mathscr{L}_0(\phi(x)) - \frac{\lambda}{4!}\phi^4 + J(x)\phi(x)]\right\}$$

here $\mathscr{L}_I = -\frac{\lambda}{4!}\phi^4$

$$W[J] = \int [\mathrm{d}\phi] \exp\left\{\mathrm{i}\int \mathrm{d}^4x[\mathscr{L}_0(\phi(x)) + J(x)\phi(x)]\right\}(\sum_{n=0}^{\infty}\frac{(-\frac{\lambda}{4!}\phi^4)^n}{n!})$$

$$\Rightarrow W[J] = \left[exp\int d^4x\left(-\frac{\lambda}{4!}\left(\frac{\delta}{\delta J}\right)^4\right)\right]W_0[J],$$

and we can expand it as

$$W[J] = W_0[J](1 + \lambda\omega_1[J] + \lambda^2\omega_2[J] + \ldots.)$$

By equating the above equation to the exponential expansion we get

$$\omega_1[J] = -\frac{1}{4!}W_0^{-1}[J]\left\{\left[\mathrm{d}^4x\left[\frac{\delta}{\delta J(x)}\right]^4\right\}W_0[J]$$

Now note that

$$\begin{aligned}
\frac{\delta}{\delta J(x)}W_0[J] &= \frac{\delta}{\delta J(x)}\exp\frac{1}{2}\int d^4y\int d^4z J(y)\Delta(y,z)J(z) \\
&= \frac{1}{2}W_0[J]\frac{\delta}{\delta J(x)}\int d^4y\int d^4z J(y)\Delta(y,z)J(z) \\
&= \frac{1}{2}W_0[J]\int d^4y\int d^4z\left(\frac{\delta J(y)}{\delta J(x)}\Delta(y,z)J(z) + J(y)\Delta(y,z)\frac{\delta J(z)}{\delta J(x)}\right) \\
&= \frac{1}{2}W_0[J]\int d^4y\int d^4z\left(\delta^{(4)}(y-x)\Delta(y,z)J(z) + J(y)\Delta(y,z)\delta^{(4)}(z-x)\right) \\
&= W_0[J]\int d^4w_1\Delta(x,w_1)J(w_1)
\end{aligned}$$

in the above equation x is not a dummy variable (but we will later integrate over it) but w_1 is a dummy variable.

$$\begin{aligned}
\frac{\delta^2}{\delta J(x)^2}W_0[J] &= \frac{\delta}{\delta J(x)}\left(W_0[J]\int d^4w_1\Delta(x,w_1)J(w_1)\right) \\
&= W_0[J]\int d^4w_2\Delta(x,w_2)J(w_2)\left(\int d^4w_1\Delta(x,w_1)J(w_1)\right) + W_0[J]\Delta(x,x) \\
&= W_0[J]\int d^4w_1 d^4w_2\Delta(x,w_1)\Delta(x,w_2)J(w_1)J(w_2) + W_0[J]\Delta(x,x)
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow \frac{\delta^4}{\delta J(x)^4} W_0[J] &= \frac{\delta^2}{\delta J(x)^2}\left(W_0[J]\int d^4w_1 d^4w_2 \Delta(x,w_1)\Delta(x,w_2)J(w_1)J(w_2) + W_0[J]\Delta(x,x)\right)\\
&= \frac{\delta^2}{\delta J(x)^2}\left(W_0[J]\int d^4w_1 d^4w_2 \Delta(x,w_1)\Delta(x,w_2)J(w_1)J(w_2)\right) + \Delta(x,x)\frac{\delta^2}{\delta J(x)^2}W_0[J]\\
&= \frac{\delta^2}{\delta J(x)^2}\left(W_0[J]\right)\int d^4w_1 d^4w_2 \Delta(x,w_1)\Delta(x,w_2)J(w_1)J(w_2)\\
&\quad +2\frac{\delta}{\delta J(x)}W_0[J]\frac{\delta}{\delta J(x)}\left(\int d^4w_1 d^4w_2 \Delta(x,w_1)\Delta(x,w_2)J(w_1)J(w_2)\right)\\
&\quad +W_0[J]\frac{\delta^2}{\delta J(x)^2}\left(\int d^4w_1 d^4w_2 \Delta(x,w_1)\Delta(x,w_2)J(w_1)J(w_2)\right)\\
&\quad +\Delta(x,x)\frac{\delta^2}{\delta J(x)^2}W_0[J]\\
&= \left(W_0[J]\int d^4w_3 d^4w_4 \Delta(x,w_3)\Delta(x,w_4)J(w_3)J(w_4) + W_0[J]\Delta(x,x)\right)\\
&\quad \times\int d^4w_1 d^4w_2 \Delta(x,w_1)\Delta(x,w_2)J(w_1)J(w_2)\\
&\quad +2(W_0[J]\int d^4w_1 \Delta(x,w_1)J(w_1))\left(2\int d^4w_3 \Delta(x,x)\Delta(x,w_3)J(w_3)\right)\\
&\quad +W_0[J]\left(\Delta(x,x)^2\right)\\
&\quad +\Delta(x,x)\frac{\delta^2}{\delta J(x)^2}W_0[J]
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \frac{\delta^4}{\delta J(x)^4} W_0[J] &= W_0[J]\int d^4w_1 d^4w_2 d^4w_3 d^4w_4 \Delta(x,w_1)\Delta(x,w_2)\Delta(x,w_3)\Delta(x,w_4)J(w_1)J(w_2)J(w_3)J(w_4)\\
&\quad +W_0[J]\Delta(x,x)\int d^4w_1 d^4w_2 \Delta(x,w_1)\Delta(x,w_2)J(w_1)J(w_2)\\
&\quad +4W_0[J]\Delta(x,x)\int d^4w_1 d^4w_3 \Delta(x,w_1)\Delta(x,w_3)J(w_1)J(w_3)\\
&\quad +W_0[J]\left(\Delta(x,x)^2\right)\\
&\quad +\Delta(x,x)(W_0[J]\int d^4w_1 d^4w_2 \Delta(x,w_1)\Delta(x,w_2)J(w_1)J(w_2) + W_0[J]\Delta(x,x))
\end{aligned}
$$

$$
\begin{aligned}
&= W_0[J]\int d^4w_1d^4w_2d^4w_3d^4w_4\Delta(x,w_1)\Delta(x,w_2)\Delta(x,w_3)\Delta(x,w_4)J(w_1)J(w_2)J(w_3)J(w_4)\\
&\quad +5W_0[J]\Delta(x,x)\int d^4w_1d^4w_2\Delta(x,w_1)\Delta(x,w_2)J(w_1)J(w_2)\\
&\quad +W_0[J]\left(\Delta(x,x)^2\right)\\
&\quad +W_0[J]\Delta(x,x)\int d^4w_1d^4w_2\Delta(x,w_1)\Delta(x,w_2)J(w_1)J(w_2)+W_0[J]\Delta(x,x)^2\\
&= W_0[J]\int d^4w_1d^4w_2d^4w_3d^4w_4\Delta(x,w_1)\Delta(x,w_2)\Delta(x,w_3)\Delta(x,w_4)J(w_1)J(w_2)J(w_3)J(w_4)\\
&\quad +6W_0[J]\Delta(x,x)\int d^4w_1d^4w_2\Delta(x,w_1)\Delta(x,w_2)J(w_1)J(w_2)\\
&\quad +2W_0[J]\Delta(x,x)^2
\end{aligned}
$$

The term 6 can now be written as 3!. We know make the notation compact by removing the integrations over w_i and neglecting the term independent of w_i

$$
\begin{aligned}
\Rightarrow \frac{\delta^4}{\delta J(x)^4}W_0[J] &= W_0[J]\Delta(x,w_1)\Delta(x,w_2)\Delta(x,w_3)\Delta(x,w_4)J(w_1)J(w_2)J(w_3)J(w_4)\\
&\quad +3!W_0[J]\Delta(x,x)\Delta(x,w_1)\Delta(x,w_2)J(w_1)J(w_2)
\end{aligned}
$$

replacing the dummy variables $w_i \rightarrow y_i$ and substituting in the formula for $\omega_1[J]$ and cancelling $W_0[J]^{-1}W_0[J]$ we get we get

$$
\begin{aligned}
\Rightarrow \omega_1[J] = -\frac{1}{4!}&\left[\Delta\left(x,y_1\right)\Delta\left(x,y_2\right)\Delta\left(x,y_3\right)\Delta\left(x,y_4\right)J\left(y_1\right)J\left(y_2\right)J\left(y_3\right)J\left(y_4\right)\right.\\
&\left.+3!\Delta\left(x,y_1\right)\Delta\left(x,y_2\right)\Delta(x,x)J\left(y_1\right)J\left(y_2\right)\right].
\end{aligned}
$$

FIG. 1.4. Graphic representation of ω_1 in eqn (1.87).

In the above picture we can see the two terms of $\omega_1[J]$.

References

[1] TA-PEI CHENG and LING -FONG LI, Gauge theory of elementary particle physics.