MA 109 Tutorial 2

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Q)13. (ii)

$f(x) = xsin\frac{1}{x}$; if $x \neq 0$ and $f(0) = 0$ $\forall x \neq 0$ $x, sin\frac{1}{x}$ are continuous at x. So their product is also continuous. At $x = 0$ x is continuous, but $sin\frac{1}{x}$ is not continuous. We also know that $|f(x)| \leq |x|$ since $|sin(\frac{1}{x})| \leq 1$.
Let $L = 0$ we have to show that ,$\forall \epsilon > 0 \exists \delta > 0$ such that

$$|x - 0| < \delta \Rightarrow |f(x) - 0| < \epsilon$$

We can observe that $\delta = \epsilon$ works $\forall \epsilon$ since

$$|x - 0| < \epsilon \Rightarrow |f(x) - L| = |xsin(\frac{1}{x})| \leq |x| < \epsilon$$

$$\lim_{x \to 0} f(x) = f(0) = 0$$

So it is continuous at $x = 0$. So it is $\forall x \in \mathbb{R}$. You can also do this question by using Sandwich theorem since $-|x| \leq xsin(\frac{1}{x}) \leq |x|$

Q)15

$f(x) = x^2 sin\frac{1}{x}$; if $x \neq 0$ and $f(0) = 0$

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{h^2 sin(\frac{1}{h})}{h} \\ &= \lim_{h\to 0} hsin(\frac{1}{h}) \\ &= 0 \end{aligned}$$

$\forall x \neq 0$ $f'(x) = 2xsin(\frac{1}{x}) - cos(\frac{1}{x})$;$f'(0) = 0$. Let $x_n = \frac{1}{2n\pi}$, $y_n = \frac{1}{(2n+0.5)\pi}$. $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$. So $f'(x)$ is discontinuous at $x = 0$.

Q)18

$y = 0 \Rightarrow f(x) = f(x)f(0)\ \forall x$. Assume that $f(x)$ is not identically equal to zero. Then $\exists x$ such that $f(x) \neq 0 \Rightarrow f(0) = 1$.

$$\frac{f(x+h) - f(x)}{h} = \frac{f(x)f(h) - f(x)f(0)}{h}$$
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \left(\lim_{h \to 0} \frac{f(h) - f(0)}{h} \right) f(x)$$

So if $f'(0)$ exists, $f'(x)$ also exists $\forall x \in \mathbb{R}$.

$$\Rightarrow f'(x) = f'(0)f(x)$$

Even if $f(x) = 0 \forall x$, $f'(x) = f'(0)f(x)$ is still true.

Q)7

$(i) \Rightarrow (ii)$

Let $\delta > 0$ be such that $(c-\delta, c+\delta) \subseteq (a,b)$. And let $\alpha = f'(c)$, then

$$\epsilon_1(h) = \frac{f(c+h) - f(c) - \alpha h}{h} \text{ if } h \neq 0$$

and $\epsilon_1(0) = 0$. Check that $\lim_{h\to 0} \epsilon_1(h) = f'(c) - \alpha = 0$. So this function satisfies all the properties we need.

$(ii) \Rightarrow (iii)$

$$\begin{aligned}\lim_{h\to 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} &= \lim_{h\to 0} |\epsilon_1(h)| \\ &= 0\end{aligned}$$

Here I used the fact $\lim_{x\to c} f(x) = 0 \Leftrightarrow \lim_{x\to c} |f(x)| = 0$(it is a consequence of $||f(x)| - L| = |f(x) - L|$ for $L = 0$)

$(iii) \Rightarrow (i)$

$$\begin{aligned}\lim_{h\to 0}\frac{|f(c+h)-f(c)-\alpha h|}{|h|} &= 0\\ \Rightarrow \lim_{h\to 0}\frac{f(c+h)-f(c)-\alpha h}{h} &= 0\\ \Rightarrow \lim_{h\to 0}\frac{f(c+h)-f(c)}{h} &= \alpha\\ \Rightarrow f'(c) &= \alpha\end{aligned}$$

So $f(x)$ is differentiable at $x = c$. So all (i), (ii) and (iii) are equivalent.

Q)10

Let $g : [0,1] \to \mathbb{R}$ be defined as $g(x) = f(x) - x$. A point is fixed if $f(x) = x$ at the point. If $f(0) = 0$ or $f(1) = 1$, clearly the function has a fixed point. If $f(0) \neq 0$ and $f(1) \neq 1$ then $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0$. So by IVT(Intermediate Value Theorem) we can say that $\exists c \in (0,1)$ such that $g(c) = 0 \Rightarrow f(c) = c$.

Q)3

By IVT(Intermediate Value Theorem) we can say that atleast one such an x_0 exists such that $f(x_0) = 0$. Let $x_1 \neq x_0$ be such that $f(x_1) = 0$, then by Rolle's theorem we can say that $\exists c \in (x_0, x_1)$ such that $f'(c) = 0 \Rightarrow$ contradiction(since $f'(x) \neq 0 \forall x \in (a, b)$). So there is a unique $x \in (a, b)$ such that $f(x) = 0$.

Q)5

If $a = b$ then $|sin(a) - sin(b)| \leq |a - b|$ holds true. If $a \neq b$, assume without loss of generality that $a < b$(as the case with $a > b$ will be similar) From MVT(Mean Value Theorem) we can say that $\exists c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$
$$\Rightarrow cos(c) = \frac{sin(b) - sin(a)}{b - a}$$
$$\Rightarrow |\frac{sin(a) - sin(b)}{a - b}| = |cos(c)| \leq 1$$
$$\Rightarrow |sin(a) - sin(b)| \leq |a - b|$$