

MA 109 Tutorial 5

Kasi Reddy Sreeman Reddy

2nd year physics student

<http://iamsreeman.github.io/MA109>

IIT Bombay

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Q)2

If $f : D \in \mathbb{R}^n \rightarrow \mathbb{R}$ be a function. Then the main difference between a level curve and a contour line is that level curve is a subset of \mathbb{R}^n but contour line is a subset of \mathbb{R}^{n+1} .

(ii) For $c < 0$ the level curve and contour lines are null set. For $c = 0$ the level curve is the singleton set $\{(0, 0)\}$ and contour line is the singleton set $\{(0, 0, 0)\}$. For $c > 0$ the level curve is the circle with center $(0, 0)$, radius \sqrt{c} and lies in \mathbb{R}^2 and contour line is the circle with center $(0, 0, c)$, radius \sqrt{c} , parallel to the $x - y$ plane and lies in \mathbb{R}^3 .

(iii) Here if $c \neq 0$ then the level curve is a hyperbola in \mathbb{R}^2 and the contour is a hyperbola which is parallel to the $x - y$ plane in \mathbb{R}^3 . If $c = 0$ instead of parabola it will be a pair of straight lines.



Q)4

(i) We know that $(x_n, y_n) \rightarrow (x_0, y_0) \Leftrightarrow x_n \rightarrow x_0$ and $y_n \rightarrow y_0$. Let the given function be $h(x, y) = f(x) + g(y)$. For any arbitrary sequence $(x_n, y_n) \rightarrow (x_0, y_0)$

$$\begin{aligned} \lim_{n \rightarrow \infty} h(x_n, y_n) &= \lim_{x_n \rightarrow x_0} f(x_n) \pm \lim_{y_n \rightarrow y_0} g(y_n) \\ \Rightarrow \lim_{(x_n, y_n) \rightarrow (x_0, y_0)} h(x, y) &= f(x_0) \pm g(y_0) \end{aligned}$$

(ii) Similar to above we can take any arbitrary sequence $(x_n, y_n) \rightarrow (x_0, y_0)$, and let $h(x_n, y_n) = f(x_n)g(y_n)$ and apply limit and it will also be continuous.



(iii) and (iv) Observe that $\max\{f(x), g(y)\} = \frac{f(x)+g(y)}{2} + \left| \frac{f(x)-g(y)}{2} \right|$ and $\min\{f(x), g(y)\} = \frac{f(x)+g(y)}{2} - \left| \frac{f(x)-g(y)}{2} \right|$. For any continuous function $f(x, y)$, $|f(x, y)|$ is also continuous. Because if $f(x_0, y_0) > 0$ or < 0 then it is clearly continuous at that point. If $f(x_0, y_0) = 0$ then also it is continuous at that point because for $-f(x_0, y_0) \leq |f(x_0, y_0)| \leq f(x_0, y_0)$ applying sandwich theorem we can say that it is continuous at this point. Now using this point and (i) we can say that

$$\max\{f(x), g(y)\} = \frac{f(x)+g(y)}{2} + \left| \frac{f(x)-g(y)}{2} \right| \text{ and}$$

$$\min\{f(x), g(y)\} = \frac{f(x)+g(y)}{2} - \left| \frac{f(x)-g(y)}{2} \right| \text{ are continuous.}$$



Q)6(ii)

$$\begin{aligned} f_x(0,0) &= \lim_{h \rightarrow 0} \frac{f(h,0) - f(0,0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{\sin^2(h)}{|h|}}{h} \end{aligned}$$

the right hand limit is 1 and left hand limit is -1. So it doesn't exist. We get the exact same limit for $f_y(0,0)$. So it also doesn't exist.



Q)8

We know that $(x_n, y_n) \rightarrow (x_0, y_0) \Leftrightarrow x_n \rightarrow x_0 \text{ and } y_n \rightarrow y_0$, we also have seen in previous tutorials that $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0$ using sandwich theorem. By using both we can say that it is continuous. Let $\mathbf{v} = (v_x, v_y)$ be a unit vector. For $\mathbf{x} = (0, 0)$ and if $v_x, v_y \neq 0$

$$\begin{aligned}\nabla_{\mathbf{v}} f(\mathbf{x}) &= \lim_{h \rightarrow 0} \frac{f(\mathbf{x} + h\mathbf{v}) - f(\mathbf{x})}{h} \\ \Rightarrow \nabla_{\mathbf{v}} f(0, 0) &= \lim_{h \rightarrow 0} \frac{h v_x \sin\left(\frac{1}{h v_x}\right) + h v_y \sin\left(\frac{1}{h v_y}\right)}{h}\end{aligned}$$

this limit doesn't exist as we can get different values by taking different sequences. Similarly f_x, f_y also don't exist. So, none of the partial derivatives exist.



Q)10

$\lim_{(x,y) \rightarrow (x_0,y_0)} f(x,y) = L$ iff $\forall \epsilon > 0 \exists \delta > 0$ such that

$$(x,y) \in D_f, 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta \Rightarrow |f(x,y) - L| < \epsilon$$

Here for $L = 0$ $\delta = \epsilon$ works $\forall \epsilon > 0$ as $|f(x,y) - L| = \sqrt{x^2 + y^2}$. So it is continuous at $(0,0)$. Let $\mathbf{v} = (v_x, v_y)$ be a unit vector. For $\mathbf{x} = (0,0)$ and if $v_y \neq 0$

$$\begin{aligned} \nabla_{\mathbf{v}} f(\mathbf{x}) &= \lim_{h \rightarrow 0} \frac{f(\mathbf{x} + h\mathbf{v}) - f(\mathbf{x})}{h} \\ \Rightarrow \nabla_{\mathbf{v}} f(0,0) &= \lim_{h \rightarrow 0} \frac{h v_y \sqrt{h^2(1)}}{h |h v_y|} = \lim_{h \rightarrow 0} \frac{h v_y |h|}{h |h| |v_y|} = \frac{v_y}{|v_y|} \end{aligned}$$

$$\text{If } v_y = 0 \Rightarrow v_x = \pm 1 \Rightarrow \nabla_{\mathbf{v}} f(\mathbf{x}) = \frac{0-0}{h} = 0$$



From above we get $f_x(0,0) = 0, f_y(0,0) = 1$ if it differentiable at $(0,0)$ then $\exists \alpha, \beta$ such that

$$\lim_{(h,k) \rightarrow (0,0)} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0) - \alpha h - \beta k}{\sqrt{h^2 + k^2}} = 0$$

and α, β coincides with the x, y directional derivatives. Here

$$\lim_{(h,k) \rightarrow (0,0)} \frac{\frac{k}{|k|} \sqrt{h^2 + k^2} - k}{\sqrt{h^2 + k^2}} = 0$$

It should be 0 along all sequences converging to $(0,0)$. Take a sequence along the line $x = y$ approaching from the 1st quadrant. The limit will be $\frac{\sqrt{2}-1}{\sqrt{2}-1}$, so it is not differentiable.



Some important points

- ① Differentiable at (x_0, y_0) .
- ② $\nabla_{\mathbf{v}} f(\mathbf{x}) = (f_x(x_0, y_0), f_y(x_0, y_0)) \cdot \mathbf{v}$ is true for all unit vectors \mathbf{v} .
- ③ The directional derivative exists for all \mathbf{v} .
- ④ f_x and f_y exist.
- ⑤ f is continuous (x_0, y_0)

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \nRightarrow (v), (i) \Rightarrow (v), (iii) \nRightarrow (v)$$

$$(v) \nRightarrow (iv) \nRightarrow (iii) \nRightarrow (ii) \nRightarrow (i)$$

The previous question(Q)10) has already proved that $(iii) \nRightarrow (i)$ and $(iii) \nRightarrow (ii)$. The following function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined as

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$



is not continuous at $(0,0)$ (approach the origin along $y = x$ and $y = x^2$ this gives two different values, so discontinuous). Since $(i) \Rightarrow (v)$ it is also not differentiable. But all partial derivatives exist and are given by (as always $\mathbf{v} = (v_x, v_y)$ is a unit vector)

$$\nabla_{\mathbf{v}} f(x, y) = \begin{cases} \frac{v_x^2}{v_y} & v_y \neq 0 \\ 0 & v_y = 0 \end{cases}$$

This example proves that $(iii) \nRightarrow (ii)$, $(iii) \nRightarrow (i)$ and $(iv) \nRightarrow (v)$ and even $(iii) \nRightarrow (v)$.

