Black Hole Information Paradox

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The four classical laws of black hole mechanics

No.	Laws of thermodynamics	Analogous black hole laws
0	T is same for systems in equilibrium	κ is constant on the Event Horizon fora stationary black hole.
1	$dE = TdS + \Omega\, dJ + \Phi\, dQ$	$dE = \frac{\kappa}{2\pi}\left(\frac{dA}{4}\right) + \Omega\, dJ + \Phi\, dQ$
2	$dS \geq 0$ where $S = k_{\mathrm{B}} \ln \Omega$	$dA \geq 0$ or $\frac{dA}{4} \geq 0$
3	$T = 0$ K cannot be attained	Extremal black holes with $\kappa = 0$ cannot exist. (also called as *Cosmic censorship hypothesis* and it forbids naked singularities)

Table: Analogy between the black hole laws and laws of thermodynamics

In inertial Minkowski space-times we have time transnational symmetry i.e. ∂_t is a Killing vector. So we can uniquely define positive- or "negative"(by that we mean $\partial_t f_k^* = +i\omega f_k^*$ where $\omega > 0$. This terminology was made historically when they considered ϕ as a wave function instead of a quantum field)-frequency modes and **the notion of a particle is Lorentz-invariant**. Since for a general spacetime there will not be any timelike Killing vector, we will not in general be able to find solutions to the wave equation that separate into time-dependent and space-dependent factors, and so cannot classify modes as positive- or "negative"-frequency. We can find a set of basis modes, but the problem is that there will generally be many such sets, with no way to prefer one over any others, and the notion of a vacuum or number operator will depend sensitively on which set we choose. We can easily define particles in QFT in curved spacetime if a spacetime has positive and "negative" frequency modes. In general we can't find them. For static space-times we can define them and this process will work. So it is not necessary that different observers in different places agree on the number of particles that are in a particular state.

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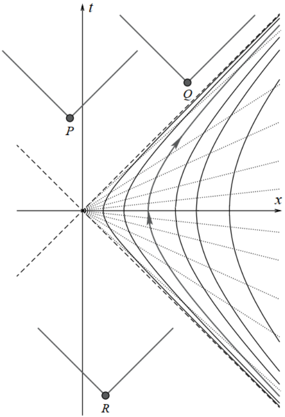
Rindler coordinates

$$t = \frac{1}{a} e^{a\xi} sinh(a\eta) \qquad x = \frac{1}{a} e^{a\xi} cosh(a\eta)$$

Notice that $x + t > 0$ and $x - t > 0$. So Rindler coordinates can only describe a part of the Minkowski space called the **Rindler Space or wedge**.

$$ds^2 = -dt^2 + dx^2 = e^{2a\xi}(-d\eta^2 + d\xi^2)$$

If $\xi = k$ for a particle, where k is a constant, then when observed in the Minkowski frame the particle has constant four acceleration **magnitude** $\alpha = \frac{a}{e^{ak}}$ (the 4 accleration is not constant) as $a^\mu = \frac{d^2x^\mu}{d\tau^2} = (\alpha sinh(\alpha t), \alpha cosh(\alpha t))$. η is related to the proper time τ of this particle by $\eta = \frac{\alpha}{a}\tau$. The locus of $x^2 = t^2$ is called the **Rindler horizon**.



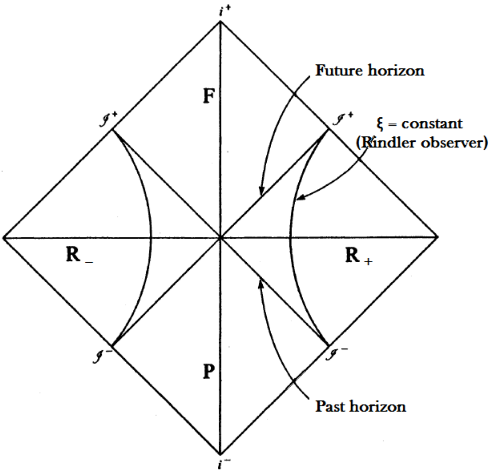


Figure: Rindler chart (*left*). The dashed lines are the Rindler horizons. Penrose-Carter diagram (*right*) of Rindler wedge in Minkowski space.

An observer who is at the origin of the Rindler coordinates ($\xi = 0$) is called the **Rindler observer**. The magnitude of accleration of this observer is a and $\eta = \tau$.
This metric is independent of η so ∂_η is a Killing vector field in these coordinates. In Minkowski coordinates the vector ∂_η is

$$\begin{aligned}\partial_\eta &= \frac{\partial t}{\partial \eta}\partial_t + \frac{\partial x}{\partial \eta}\partial_x \\ &= a(x\partial_t + t\partial_x)\end{aligned}$$

It is the Killing field associated with a boost in the x direction. So time translation in the Rindler coordinates is a boost in the Minkowski frame. in regions II and III it is spacelike, while in region IV it is timelike but past-directed. Which is what we intuitively expect since as time passes in the Rindler frame the velocity of Rindler observer increases in the Minkowski frame. This Killing field naturally extends throughout the spacetime, in regions II and III it is spacelike, while in region IV it is timelike but past-directed.

Now we can similarly define **left Rindler space or wedge**.

$$t = -\frac{1}{a} e^{a\xi} sinh(a\eta) \qquad\qquad x = -\frac{1}{a} e^{a\xi} cosh(a\eta)$$

The metric is same as for the right wedge. Coordinates (η, ξ) cannot be used simultaneously in wedges right and left, because the range of these parameters are the same in each regions. The vector field $\partial\eta$ is a Killing vector field in wedges left and right, but is future pointing in the right wedge while **past pointing in the left one**. $-\partial_\eta$ is the future pointing timelike killing vector field in left wedge.

Intuitive explanation of Unruh effect

The future horizon and past or illusory horizon behave like an event horizon of a black hole and white hole respectively in the Einstein–Rosen bridge. In QFT often time negative and positive energy particles will be produced and will be annihilated. The product of their energy and life time is of the order of Planck's constant. But near the future horizon if such a particle pair is produced the negative energy particle may escape from the Rindler space and go to the outside Minkowski space and **it can be stable in the Minkowski space**. To understand this remember that energy is the component of four momentum corresponding to a future pointing timelike killing vector field. In Rindler space it corresponds to ∂_η and in Minkowski space it corresponds to ∂_t. So **a particle is said to be having negative energy in Rindler space if $P_\mu N^\mu$ is negative.** (where N^μ is the unit η vector which is (1,0) in (η,ξ)). A particle is said to be having negative energy in Minkowski space if $P_\mu T^\mu$ is negative. (where T^μ is the unit t vector).

$$dt = e^{a\xi}(cosh(a\eta)d\eta + sinh(a\eta)d\xi)$$

Near the horizon if $cosh(a\eta)d\eta + sinh(a\eta)d\xi > 0$ and $d\eta < 0$ for a particle then it is having -ve energy wrt Rindler space (or *equivalently* moving backward in time with +ve energy) but wrt Minkowski space it will have +ve energy and it will be stable after getting out of Rindler space. **The reverse process cannot occur because if the positive energy is escaped the negative energy particle cannot be stabilised in the Rindler space.**

The Unruh temperature is spatially inhomogeneous across the Rindler space but the Unruh state is in the equilibrium (unlike in Hawking effect). This is a direct consequence of Ehrenfest–Tolman effect.

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Some comments on previous discussion

1) The temperature of a supermassive black hole is far lower than the cosmic microwave background, so even a completely isolated supermassive black hole will grow over time and not shrink. **The cooler object gets even cooler in this case**. In reality CMB cools down due to expansion and after a very long time CMB will have less temperature than the black hole and black hole stars radiating. **The hot object becomes even hotter**. If CMB and a supermassive black hole are in equilibrium it will be an **unstable equilibrium** in most cases. For the Reissner-Nordström metric

$$T = \frac{r_+ - r_-}{4\pi r_+^2} = \frac{1}{2\pi M}\frac{\sqrt{1-q^2}}{(1+\sqrt{1-q^2})^2}$$

we can see that $C := T\left(\frac{\partial S}{\partial T}\right)_Q = 2\pi M^2 \frac{\sqrt{1-q^2}(1+\sqrt{1-q^2})^2}{1-2\sqrt{1-q^2}}$. If $|q| > \frac{\sqrt{3}}{2}$ it can be in **stable equilibrium**. But practical black holes are almost always neutral and finding a black hole with this much charge is improbable.

For a Kerr-Newman black hole-

$$T = \frac{\kappa}{2\pi} = \frac{r_+ - r_-}{4\pi(r_+^2 + a^2)} = \frac{\sqrt{M^2 - Q^2 - J^2/M^2}}{2\pi(2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2 - J^2/M^2})}$$

$$C_J := T\left(\frac{\partial S}{\partial T}\right)_J = 2\pi r_+^2 \frac{(1-h^2)(h^2+1)^2}{3h^4 + 6h^2 - 1}$$

$$C_\Omega := T\left(\frac{\partial S}{\partial T}\right)_\Omega = -2\pi r_+^2 \frac{(1-h^2)}{h^2+1}$$

where $0 \leq h = \frac{|a|}{r_+} \leq 1$ C_J is positive only for $h > \sqrt{\frac{2}{3}\sqrt{3} - 1} = 0.3933$.
Unlike large q black holes, there probably will be few black holes with large angular momentum. Only these black holes can be in **stable equilibrium** with CMB.
Although majority researchers think that once a black hole has higher temperature than CMB it will keep on radiating and will evaporate, there is a minority which thinks it will stop evaporating and gives a Planck size stable remnant.

2) **Killing vector fields or Killing vectors**: A Killing vector K_μ satisfies $\nabla_{(\mu}K_{\nu)} = \frac{\nabla_\mu K_\nu + \nabla_\nu K_\mu}{2} = 0$ and along a geodesic $K_\mu p^\mu$ is conserved. K^μ generates an isometry; that is the geometry is invariant for an **infinitesimal** transformation along the direction of K^μ.

Killing horizons:A null hypersurface defined by the vanishing of the norm of a Killing vector field. Not all Killing horizons are event horizons but all event horizons are Killing horizons. $x^2 - t^2 = 0$ in Minkowski space is a Killing horizon for the Killing vector $x\partial_t + t\partial_x = (x, t)$ which corresponds to a Lorentz boost. In the Schwarzschild metric $(1, 0, 0, 0)$ is the Killing vector field and it's null horizon is Event horizon since the magnitude of the vector is $-\left(1 - \frac{2M}{r}\right)$.

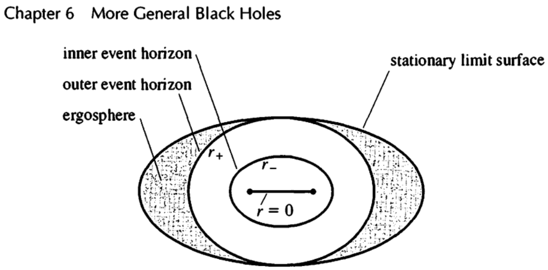

Figure: Horizon structure around the Kerr solution.

For the Kerr black hole the ergosphere is the Killing horizon for the Killing vector $K = \partial_t$ or $K^\mu = (1, 0, 0, 0)$. Inside the ergosphere the timelike vector is $\chi^\mu = K^\mu + \Omega_H R^\mu$ where $\Omega_H = \frac{a}{r_+^2 + a^2}$ is the angular velocity of the horizon and $R = \partial_\phi$. The outer horizon is the Killing horizon of χ^μ and inside outer horizon the timelike coordinate is radial towards the center.

$$M_{\text{irr}} = \sqrt{\frac{M^2 + \sqrt{M^4 - J^2c^2/G^2}}{2}} = \sqrt{\frac{A}{16\pi G^2}}$$

is the irreducible mass of the Kerr metric. Intuitively it is the rest mass energy of the black hole. The remaining energy is the rotational energy and can be extracted by Penrose process. Penrose process is a direct consequence of the fact that inside the ergo region moving forward in time is same as moving tangentially in the direction of rotation.

$$dE = \frac{\kappa}{2\pi}\left(\frac{dA}{4}\right) + \Omega\, dJ + \Phi\, dQ$$

3) *What is energy in the above equation?*
I said last time how change in the energy of black hole can be found by adding the energy obtained by the dot product of Killing Vector field and 4-momentum. We can generalise this is to the total black hole and integrate the t component of the 4 momentum (since we want energy as observed by outside observer). This energy is called Komar mass. **Komar mass/energy**: $m = \int_V (2T_{ab} - Tg_{ab})\, u^a \xi^b dV$ where V is the volume being integrated over u^a is a unit time-like vector and $\xi^b \xi^b$ is a Killing vector associated with the time coordinate.
The final answer coincides with the parameter M for Schwarzschild and Kerr metrics. For a Kerr metric the energy will be M and for Kerr-Newman metric it will be $M - \frac{Q^2}{2r} - \frac{Q^2(r^2+a^2)}{2ar^2} tan^{-1}(\frac{a}{r})$.

Within the ADM Hamiltonian formulation of General Relativity there is another energy called ADM energy. ADM mass can be defined iff the space time is asymptotically Minkowski metric and is defined as a certain boundary integral on a 2-sphere at $r \to \infty$. If $g_{\mu\nu}$ is time-independent at infinity and it's asymptotic value is the Mikowski metric then ADM energy $=$ Komar mass. If it doesn't have a timelike Killing vector field then ADM energy is not conserved. (more info: arXiv:2101.12570 discusses 8 different masses). We always work in a stationary black hole and by
No-hair theorem: Stationary, asymptotically flat black hole solutions to general relativity coupled to electromagnetism that are nonsingular outside the event horizon are fully characterized by the parameters of M, Q and P (magnetic charge), and J.
and the fact that most astrophysical black holes are nearly neutral, we can say that Kerr metric is a very general metric and these 2 definitions coincide with M.

4) I took for granted that -ve energy particle outside the event horizon will be attracted towards the black hole. This is like in Newtonian gravity where a +ve mass will attract a -ve mass but the -ve mass will repel the +ve mass and both will be accelerating towards the same direction. Sir Hermann Bondi showed that in GR also it is similar.

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Entanglement\von Neumann entanglement entropy

For a quantum-mechanical system described by a density matrix ρ, the **von Neumann entropy** is $S = -\operatorname{tr}(\rho \ln \rho)$ (ln denotes the matrix logarithm). For a **pure bipartite state** $\rho_{AB} = |\Psi\rangle\langle\Psi|_{AB}$,

$$\mathcal{S}(\rho_A) = -\operatorname{Tr}[\rho_A \log \rho_A] = -\operatorname{Tr}[\rho_B \log \rho_B] = \mathcal{S}(\rho_B)$$

Because for $|\psi_{AB}\rangle = \sum_i \sqrt{p_i}|\psi_A^i\rangle \otimes |\psi_B^i\rangle$ we have

$$\rho_{AB} = \sum_i p_i |\psi^i_{AB}\rangle\langle\Psi^i_{AB}|,\ \rho_A = \sum_i p_i |\psi^i_A\rangle\langle\Psi^i_A|,\ \rho_B = \sum_i p_i |\psi^i_B\rangle\langle\Psi^i_B|$$

and $S_A = S_B = -\sum_i p_i \ln(p_i)$. Here $|\psi^i_{AB}\rangle = |\psi^i_A\rangle \otimes |\psi^i_B\rangle$. ρ_A is the reduced density matrix for subsystem A. If A and B are entangled then $S_A > 0$ and if they are not entangled (i.e $|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$) then $S_A = 0$.

For the von Neumann entropy S, NS is the number of qubits of information one can store in an ensemble ρ with N subsystems. Similar to Shannon entropy for which instead of ensemble, qubits it will be random variable, bits.

Entanglement in the vacuum: The two-point correlation function is non zero for vacuum which indicates entanglement between points in the vacuum. $\langle\Omega|\phi(0,x)\phi(0,y)|\Omega\rangle$ scales like $\dfrac{1}{|x-y|^2}$ for $|x-y| << m^{-1}$ and as $e^{-m|x-y}$ for $|x-y| >> m^{-1}$. This is somewhat similar to the entanglement in the state $|\psi\rangle = \frac{1}{2}(|00\rangle + |11\rangle)$ which has zero 1 point correlations but non zero 2 point correlations with the Pauli operators.

If we assume that the left and right Rindler wedges are not entangled then $\rho = \rho_L \otimes \rho_R$ then the typical difference between neighboring fields on either side is of order the typical field fluctuation, which is given by $\frac{1}{\epsilon}$ where ϵ is a UV length cutoff, so we have

$$\partial_x \phi|_{x=0} \propto \frac{1}{\epsilon^2}$$

The gradient term in the Hamiltonian then produces the dominant contribution

$$H \approx dx \int d^2y (\partial_x \phi)^2 \propto \epsilon \int d^2y \frac{1}{\epsilon^2} = \frac{A}{\epsilon^3}$$

As ϵ is small this energy will be large and this phenomenon is called **firewall**. By using Euclidean path integrals we can obtain the value of

$$|\Omega\rangle = \otimes_{\omega,k} [\sqrt{1 - e^{-2\pi\omega}} \sum_n e^{\pi\omega n} |n\rangle_{L,\omega(-k)} |n\rangle_{R,\omega(-k)}]$$

which clearly shows the entanglement between the right and left Rindler wedge.

Why massless particles dominate Hawking radiation?

We can find the radial geodesic equation classically and imagine it as if it is in some 1D motion in a potential function. Similarly we can solve the Schrodinger equation with this effective 1D potential for approximate answers.

$$-\frac{d^2}{dr_*^2}\Psi_{\omega l} + V(r)\Psi_{\omega l} = \omega^2\Psi_{\omega l}$$

where $V(r) = \frac{r-1}{r^3}(m^2r^2 + l(l+1) + \frac{1}{r})$ and as $r \to \infty$ $V(r) \to m^2$ which is **barrier**. If m is large, this means that any modes whose energy ω is of order the Schwarzschild radius will be confined very near the horizon. For $m = o$ there is a barrier is at $r = \frac{3}{2}$ and the height is of order l^2. So if gravitons exist (as string theory predicts but unlike loop quantum gravity) they will be the 2nd most dominating particles since they will have spin 2. Although gluons have $m = 0$ and spin 1 they are not as probable as photons due to color confinement.

Bibiligraphy

D. Harlow, "Jerusalem lectures on black holes and quantum information," *Rev. Mod. Phys.*, vol. 88, p. 015002, Feb 2016. [Online]. Available: https://link.aps.org/doi/10.1103/RevModPhys.88.015002

T. Hartman, "Lectures on quantum gravity and black holes," *Cornell University*, 2015. [Online]. Available: http://www.hartmanhep.net/topics2015/gravity-lectures.pdf

P.-H. Lambert, "Introduction to Black Hole Evaporation," *PoS*, vol. Modave2013, p. 001, 2013. [Online]. Available: https://pos.sissa.it/201/001