Superalgebras and supergroups

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1 Introduction (skippable)

Even though Group theory in its modern from was there from 1880 its importance in physics was only realised after 1915 due to a remarkable theorem proved by mathematician Emmy Noether.

Newton's laws can be expressed equivalently using a formalism based on the calculus of variations (developed by Newton, Euler and Lagrange) called

Lagrangian mechanics. In this formalism we have scalar quantity called the **Lagrangian**. A particle takes the path for which the action (defined as $\mathcal{S} = \displaystyle\int_{t_1}^{t_2} L dt$) is stationary (minimum or maximum). The good thing about this approach is that L is **always a scalar** but the equations of motion may be vectors (Newton's laws), Tensors of rank 2 (Einstein field equations), spinors (Dirac equation) etc. In the quantum theory also the Lagrangian formulation works but it is complicated by the fact that we replace the classical notion of a single, unique classical trajectory for a system with a sum, or functional integral, over an infinity of quantum-mechanically possible trajectories to compute a quantum amplitude.

Noether's theorem: If a system has a continuous symmetry property, then there are corresponding quantities whose values are conserved in time.

Here by symmetry we mean any transformation which will leave $\mathcal{L}$ invariant up to an addition of $\frac{dF}{dt}$. (it becomes $\partial_\mu F^\mu = \frac{\partial F^0}{\partial t} - \vec{\nabla} \cdot \vec{F}$ after considering Einstein's relativity. Here we used the Einstein summation convention)

The Lagrangian of the standard model (developed from 1950s to 1970s), which completely describes 3 out of the 4 forces (except gravity) in the nature, has the following symmetries and conserved quantities (total 22 conserved quantities):

Conservation Law	Respective Noether symmetry invariance		Number of dimensions	
Conservation of mass-energy	Time-translation invariance	Poincaré invariance	1	translation along time axis
Conservation of linear momentum	Space-translation invariance		3	translation along *x*,*y*,*z* directions
Conservation of angular momentum	Rotation invariance		3	rotation about *x*,*y*,*z* axes
Conservation of CM (center-of-momentum) velocity	Lorentz-boost invariance		3	Lorentz-boost along *x*,*y*,*z* directions
Conservation of electric charge	U(1) Gauge invariance		$1 \otimes 4$	scalar field (1D) in 4D spacetime (*x*,*y*,*z* + time evolution)
Conservation of color charge	SU(3) Gauge invariance		3	*r*,*g*,*b*
Conservation of weak isospin	$SU(2)_L$ Gauge invariance		1	weak charge
Conservation of probability	Probability invariance[1]		$1 \otimes 4$	total probability always = 1 in whole *x*,*y*,*z* space, during time evolution

Figure 1.1: Source:Wikipedia

The Poincaré group contains all spacetime symmetries. That is all transformations of $(ct, x, y, z) \to (ct', x', y', z')$ such that the following Minkowski line element is invariant

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

Lorentz group is a subgroup of Poincaré group which doesn't contain spacetime translations. Apart from Poincaré group the rest of the groups in Figure 1.1 are due **internal symmetries** (i.e they have nothing to do with spacetime and they are related to particles). But physicists in 1970s discovered a symmetry which is both a spacetime symmetry and an internal symmetry.

2 Super objects

The essential feature of all *super objects*, such as superalgebras, supergroups and supermatrices, is that they are graded. Grading is the generalisation of the following structure

$$\left.\begin{array}{rcl} \text{even integer} + \text{even integer} & = & \text{even integer,} \\ \text{even integer} + \text{odd integer} & = & \text{odd integer,} \\ \text{odd integer} + \text{odd integer} & = & \text{even integer.} \end{array}\right\} \tag{2.1}$$

This type of structure comes in many places. For example polynomials of odd degree and even degree under multiplication operator.

2.1 Graded vector space

An I-graded vector space (where I is any set) is a vector space V together with a decomposition into a direct sum of the form

$$V = \bigoplus_{i \in I} V_i$$

where each V_n is a vector space.

For a given i the elements of V_i are then called homogeneous elements of degree i.

Superdimension: It is the pair (p, q) where $dim(V_0) = p$ and $dim(V_1) = q$ as ordinary vector spaces. We simply write $dim(V) = p|q$.

2.1.1 Supervector space

A super vector space is a $\mathbb{Z}_2$-graded vector space with decomposition

$$V = V_0 \oplus V_1, \quad 0, 1 \in \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$$

Homogeneous elements of degree 0 are said to be **even** and of degree 1 are said to be **odd**.

2.2 Associative superalgebra

A graded vector space V for which for every pair of elements a and b there exists a product ab that is also in V, and for which
(i) for all $a, b, a', b' \in$ V and $\mu, \lambda, \mu', \lambda'$ of the field of V (i.e. real or complex numbers as appropriate)

$$(\mu a + \mu' a')(\lambda b + \lambda' b') = \mu\lambda(ab) + \mu\lambda'(ab') + \mu'\lambda(a'b) + \mu'\lambda'(a'b')$$

(ii) for all $a, b, c \in \mathrm{V}$
$$(ab)c = a(bc)$$
and
(iii) the product satisfies the grading multiplication rule (2.1), is called an associative superalgebra.

2.2.1 Matrix associative superalgebras

Denoted by $\mathbf{M}(p/q; \mathbb{F})$ ($\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$). The product being taken to be ordinary matrix multiplication. The linearity and associative requirements of associative superalgebra are automatically satisfied. Suppose that p and q are any two positive integers, and consider a $(p+q) \times (p+q)$ matrix $\mathbf{M}$ with complex entries and with the partitioning

$$\mathbf{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$$

then define any matrix of the form

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix}$$

is said to be **even** and any matrix of the form

$$\begin{bmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{bmatrix}$$

is said to be **odd**. It can be seen that the (2.1) is satisfied, like for example even and odd gives odd case

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{C} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{AB} \\ \mathbf{DC} & \mathbf{0} \end{bmatrix}$$

2.2.2 Supercommutative associative superalgebra

An associative algebra is said to be supercommutative if

$$ba = (-1)^{(\deg a)(\deg b)} ab$$

for all homogeneous a and b of the algebra; that is, if

$$ba = \begin{cases} -ab \text{ if both } a \text{ and } b \text{ are odd} \\ ab \text{ otherwise} \end{cases}$$

deg is either 0 or 1. Sometimes it is also called as commutative associative superalgebra but this will be confusing since the less important superalgebras which are commutative in the normal sense are also called commutative associative superalgebra.

Any commutative algebra is a supercommutative algebra if given the **trivial gradation** (i.e. all elements are even). Grassmann algebras (also known as exterior algebras) are the most common examples of nontrivial supercommutative algebras.

2.2.3 Grassmann or exterior algebras

Grassmann algebras are particular examples of associative superalgebras that are very important. The first stage in the construction of such an algebra is to consider a set of L generators $\theta_1, \theta_2, \ldots, \theta_L$, which are assumed to have products $\theta_j\theta_k$ such that
(i) for all $j, k, l = 1, 2, \ldots, L$,

$$(\theta_j\theta_k)\,\theta_l = \theta_j\,(\theta_k\theta_l)$$

(ii) for all $i, j = 1, 2, \ldots, L$, and

$$\theta_i\theta_j = -\theta_j\theta_i$$

(iii) each non-zero product

$$\theta_{j_1}\theta_{j_2}\cdots\theta_{j_r}$$

involving r generators is linearly independent of products involving less than r generators. We now supplement the set of generator with an identity element denoted by 1, which commutes with any Grassmann number and gives itself. $\theta_i\theta_j$ is often written as $\theta_i \wedge \theta_j$.

It is easy to see from (ii) that $(\theta_j)^2 = 0$

The number of non-zero independent generators and their products (with 1 included) is 2^L, none of these products contains more than L factors, and, except for the identity 1, each one can be written in the form

$$\theta_{j_1}\theta_{j_2}\cdots\theta_{j_r}$$

$$1 \leq r \leq L$$

$$1 \leq j_1 < j_2 < \cdots < j_r \leq L$$

A general Grassmann number b will be the linear real or complex superposition of all the 2^L numbers. **Notation**:

$$B = \sum_{\mu} B_\mu \theta_\mu$$

here μ is a set. $\mu = \phi$ represents 1 and similarly $\mu = \{j_1, j_2, \cdots j_r\}$ represents $\theta_{j_1}\theta_{j_2}\cdots\theta_{j_r}$.

If the index set μ contains $N(\mu)$ integers then the corresponding element θ_μ may be said to be of level $N(\mu)$. θ_μ is even (odd) if $N(\mu)$ is even (odd).

$$BB' = \begin{cases} -B'B \text{ if } B \text{ and } B' \text{ are both odd} \\ \ \ B'B \text{ otherwise} \end{cases}$$

Notation:They are denoted by $\mathbb{F}\mathbf{B}_L$ ($\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$). The even and odd subset are denoted by $\mathbb{F}\mathbf{B}_{L0}$ and $\mathbb{F}\mathbf{B}_{L1}$.

2.3 Supermatrix

Let R be a fixed superalgebra (assumed to be unital (i.e. has an identity) and associative). Often one requires R be supercommutative as well (for essentially the same reasons as in the ungraded case).

Let $p, q, r,$ and s be nonnegative integers. A supermatrix of dimension $(r|s)(p|q)$ is a matrix with entries in R that is partitioned into a 22 block structure

$$X = \begin{bmatrix} X_{00} & X_{01} \\ X_{10} & X_{11} \end{bmatrix}$$

with $r + s$ total rows and $p + q$ total columns.

Even supermatrix

$$\begin{bmatrix} \text{even} & \text{odd} \\ \text{odd} & \text{even} \end{bmatrix}$$

Odd supermatrix

$$\begin{bmatrix} \text{odd} & \text{even} \\ \text{even} & \text{odd} \end{bmatrix}$$

If X and Y are homogeneous with parities $|X|$ and $|Y|$ then XY is homogeneous with parity $|X| + |Y|$. That is, the product of two even or two odd supermatrices is even while the product of an even and odd supermatrix is odd.

Usually the superalgebra R is taken to be Grassmann algebras.

2.4 Supergeometry

2.4.1 Superspaces

Denoted by $\mathbb{F}\mathbf{B}_L^{m,n}$, it is a Grassmann generalization of $\mathbb{R}^m$. $\mathbb{F}\mathbf{B}_L^{m,n}$ consist of m copies of the even subspace $\mathbb{F}\mathbf{B}_{L0}$ and n copies of the odd subspace $\mathbb{F}\mathbf{B}_{L1}$ of the Grassmann algebra $\mathbb{F}\mathbf{B}_L$. (Here L may be taken to be either a finite positive integer or to be infinite.). The m copies of $\mathbb{F}\mathbf{B}_{L0}$ will be denoted by $X^1, X^2, \cdots, X^m$, and the n copies of $\mathbb{F}\mathbf{B}_{L1}$ will be indicated by $\theta^1, \theta^2, \cdots, \theta^m$. An even Grassmann number is of the form (here only even θ_μ are considered) (for $j = 1, 2, \ldots, m$)

$$X^j = \sum_\mu X^j_\mu \theta_\mu$$

Similarly, for each odd Grassmann element Θ^k (for $k = 1, 2, \ldots, n$)

$$\Theta^k = \sum_\mu \Theta^k_\mu \theta_\mu$$

It is convenient to make the notation more concise by regarding $X^1, X^2, \ldots, X^m$ as elements of an m-component quantity $\mathbf{X}$, and $\Theta^1, \Theta^2, \ldots, \Theta^n$ as elements of a n-component quantity $\boldsymbol{\Theta}$, and by defining $(\mathbf{X}; \boldsymbol{\Theta})$, a typical element of $\mathbb{R}\mathrm{B}^{m,n}_L$, by

$$(\mathbf{X}; \boldsymbol{\Theta}) = \left(X^1, X^2, \ldots, X^m, \Theta^1, \Theta^2, \ldots, \Theta^n\right)$$

For $\mathbb{R}\mathbf{B}^{m,n}_L$ the Euclidean norm may be defined by

$$\|(\mathbf{X}; \boldsymbol{\Theta})\| = \left\{ \sum_{j=1}^m \sum_\mu \left|X^j_\mu\right|^2 + \sum_{k=1}^n \sum_\mu \left|\Theta^k_\mu\right|^2 \right\}^{1/2}$$

and the absolute value norm may be defined by

$$\|(\mathbf{X}; \boldsymbol{\Theta})\| = \sum_{j=1}^m \left\|X^j\right\| + \sum_{k=1}^n \left\|\Theta^k\right\|$$

so that

$$\|(\mathbf{X}; \boldsymbol{\Theta})\| = \sum_{j=1}^m \sum_\mu \left|X^j_\mu\right| + \sum_{k=1}^n \sum_\mu \left|\Theta^k_\mu\right|.$$

With each of these choices of norm a metric d may be introduced by the prescription (it can be verified that these satisfy the normal metric properties)

$$d\left((\mathbf{X}; \boldsymbol{\Theta}), (\mathbf{X}'; \boldsymbol{\Theta}')\right) = \|(\mathbf{X}; \boldsymbol{\Theta}) - (\mathbf{X}'; \boldsymbol{\Theta}')\|$$

As $\mathbb{R}\mathbf{B}_{L0}$ and $\mathbb{R}\mathbf{B}_{L1}$ are both real vector spaces of dimension 2^{L-l}, $\mathbb{R}\mathbf{B}^{m,n}_L$ is a real vector space of dimension $(m+n)2^{L-l}$.

2.4.2 Supermanifolds

Just as an analytic manifold of dimension m is obtained by patching together subspaces each of which is homeomorphic to some open set of $\mathbb{R}^m$, a supermanifold of even dimension m and odd dimension n may be set up as a topological space whose subspaces are each homeomorphic to some open set of $\mathbb{R}\mathbf{B}_L^{m,n}$.

2.5 Linear Lie supergroup

Just like how a Lie group is a group that is also a differentiable manifold, Lie supergroup is a supergroup that is also a differentiable supermanifold.
Linear Lie supergroup of even dimension m and odd dimension n: Let $G_s(\mathbb{C}\mathbf{B}_L)$ be a set of even $(p/q)\times(p/q)$ supermatrices $\mathbf{G}$ with entries in $\mathbb{C}\mathbf{B}_L$, that is,

$$\boldsymbol{G} = \begin{bmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{R} & \mathbf{S} \end{bmatrix}$$

where $\mathbf{P}$ is a $p\times p$ matrix with elements in $\mathbb{C}\mathbb{B}_{L0}, \mathbf{Q}$ is a $p\times q$ matrix with elements in $\mathbb{C}\mathbf{B}_{L1}, \mathbf{R}$ is a $q\times p$ matrix with elements in $\mathbb{C}\mathbf{B}_{L1}$ and $\mathbf{S}$ is a $q\times q$ matrix with elements in $\mathbb{C}\mathbf{B}_{L0}$. This set is said to form a linear Lie supergroup of even dimension m and odd dimension n if it satisfies the following conditions (A)(D):

(A) $G_{\mathrm{s}}(\mathbb{C}\mathrm{B}_L)$ must be a group with supermatrix multiplication providing the group multiplication operation and $\mathbf{1}_{p+q}$ being the identity of the group.

(B) With the absolute value norm, there must exist a $\delta>0$ such that every element $\mathbf{G}$ of $G_s(\mathbb{C}\mathbf{B}_L)$ lying in the sphere $\mathbf{M}_\delta$ of radius δ centred on the identity $\mathbf{1}_{p+q}$ of $G_{\mathrm{s}}(\mathbb{C}\mathbb{B}_L)$ can be parametrized by an element $(\mathbf{X};\boldsymbol{\Theta}) = (X^1, X^2, \ldots, X^m, \Theta^1, \Theta^2, \ldots, \Theta^n)$ of $\mathbb{R}\mathrm{B}_L^{m.n}$ (no two sets of points corresponding to the same element $\mathbf{G}$ of $G_{\mathrm{s}}(\mathbb{C}\mathbf{B}_L)$), the identity being parametrized by $(\mathbf{X};\boldsymbol{\Theta}) = (\mathbf{0},\mathbf{0})$, that is, by

$$X^1 = X^2 = \cdots = X^m = \Theta^1 = \Theta^2 = \cdots = \Theta^n = 0$$

Here the sphere $\mathbf{M}_\delta$ consists of all $\mathbf{G}$ of $G_{\mathrm{s}}(\mathbb{C}\mathbb{B}_L)$ such that

$$d(\mathbf{M}, \mathbf{1}_{p+q}) < \delta$$

From this assumption we can show that, every element of $G_{\mathrm{s}}(\mathbb{C}\mathrm{B}_L)$ lying in $\mathbf{M}_\delta$ corresponds to one and only one point of $\mathbb{R}\mathbf{B}_L^{m,n}$, the identity corresponding to the origin $(0,0)$ of $\mathbb{R}\mathbf{B}_L^{m,n}$. Moreover, no point in $\mathbb{R}\mathbf{B}_L^{m,n}$ corresponds to more than one element G in $\mathbf{M}_\delta$.

(C) There must exist an $\eta > 0$ such that every point $(\mathbf{X};\mathbf{\Theta})$ of the sphere S_η of radius η centred on the origin $(\mathbf{0},\mathbf{0})$ of $\mathbb{R}\mathrm{B}_L^{m,n}$ corresponds to some element $\mathbf{G}$ in $\mathbf{M}_\delta$.

Let R_η be the set of elements G of $G_\mathrm{s}\left(CB_L\right)$ so obtained. Then R_η is a subset of M_δ, and there is a one-to-one correspondence between the elements of $G_\mathbf{s}\left(\mathbb{C}B_L\right)$ in $\mathbf{R}_\eta$ and the points of $\mathbb{R}\mathbf{B}_L^{m,n}$ lying in the sphere S_η. Let $\mathbf{G}(\mathbf{X};\mathbf{\Theta})$ denote the element of $\mathbf{R}_\eta$ that corresponds to the point $(\mathbf{X};\mathbf{\Theta})$ of S_η.

(D) Each of the matrix elements of $\mathbf{G}(\mathbf{X};\mathbf{\Theta})$ must be a superanalytic function of $(\mathbf{X};\mathbf{\Theta})$ on S_n

Theorem: If L is finite then the linear Lie supergroup $G_\mathrm{s}\left(\mathbb{CB}_L\right)$ with m even dimensions and n odd dimensions is an ordinary linear Lie group of order $(m+n)2^{L-1}$, the **basis elements** of its corresponding real Lie algebra $L_\mathbf{s}\left(\mathbb{CB}_L\right)$ being the $m2^{L-1}$ even $(p/q)\times(p/q)$ supermatrices

$$\mathbf{M}_\mu^j = \left.\frac{\partial\mathbf{G}(\mathbf{X};\mathbf{\Theta})}{\partial X_\mu^j}\right|_{\mathbf{X}=0,\mathbf{\Theta}=0}$$

(for $j = 1, 2, \ldots, m$ and all even index sets μ), together with the $n2^{L-1}$ even $(p/q)\times(p/q)$ supermatrices

$$\mathbf{N}_\mu^k = \left.\frac{\partial\mathbf{G}(\mathbf{X};\Theta)}{\partial\Theta_\mu^k}\right|_{\mathbf{X}=\mathbf{0},\mathbf{\Theta}=\mathbf{0}}$$

(for $k = 1, 2, \ldots, m$ and all odd index sets μ).

3 Applications to physics

All odd elements we saw correspond to a class of particles called fermions and even mathematical elements correspond to bosons. Bosons follow a certain commutation relations in quantum field theory. Fermions follow similar but anti-commutation relations.

Poincaré group is the group of symmetry transformations of $(ct, x, y, z) \to (ct', x', y', z')$ such that the following Minkowski line element is invariant

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

The Poincaré algebra is given by the commutation relations:

$$[P_\mu, P_\nu] = 0$$

$$\frac{1}{i}\,[M_{\mu\nu}, P_\rho] = \eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu$$

$$\frac{1}{i}\,[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}\,,$$

where P is the generator of spacetime translations, M is the generator of Lorentz transformations, and $=\eta$ is the $(+,-,-,-)$ Minkowski metric.

3.1 Poincaré superalgebra and supergroup

Apart from the above commutation relations Poincaré superalgebras have following extra relations

$$\begin{aligned}
[M^{\mu\nu}, Q_\alpha] &= \frac{1}{2}(\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta \\
[Q_\alpha, P^\mu] &= 0 \\
\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} &= 2(\sigma^\mu)_{\alpha\dot{\beta}}P_\mu
\end{aligned}$$

Here $\sigma^{\mu\nu} = \frac{i}{2}\,[\gamma^\mu, \gamma^\nu]$ and γ^μ are the Dirac matrices which satisfy $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.

Here supercharge (also called Grassmann variables or Grassmann directions) Q is a generator of supersymmetry transformations.

The supersymmetry generators transform bosonic particles into fermionic ones and vice versa (so they are internal symmetries), but the anticommutator of two such transformations yields a translation in spacetime. (so they are also spacetime symmetries). This tell us that supersymmetry is **both an internal symmetry and a spacetime symmetry**. Poincaré symmetry is a space time symmetry. The rest of the symmetries in Figure 1.1 are internal symmetries.

The Haag–Łopuszański–Sohnius theorem demonstrates that supersymmetry is **the only way spacetime and internal symmetries can be combined consistently**.

Supersymmetry predicts a **superpartner** to all the known particles. For each boson (fermion) there is corresponding as of now undetected fermion (boson).

The most popular approach to quantize gravity is called string theory. Unlike other approaches it tries to solve many other problems apart from quantizing gravity. It also tries to unify all the four laws of physics into a single theory. This theory needs supersymmetry to be present in nature. Although there is no experimental evidence that nature has this symmetry people are hoping that nature has this supersymmetry, since it greatly reduces the complexity of many unsolved questions.

Falsifying supersymmetry is hard. Naively trying to create them might require any amount of energy between just above the maximum energy we probed at the LHC to all the way up to the Planck scale **which requires a particle accelerator as big as the Milky Way galaxy**. People are hoping that it can be found at slightly higher energies that what is probed at LHC now. People are also trying to find creative ways to falsify it using realistically achievable accelerators.

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