Black Hole Information Paradox

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Abstract

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Contents

0 Introduction

0.1 Conventions

The signature of the metric used is (-, +, +, +). I will set $G = c = \hbar = k_B = 1$. But sometimes I will restore them.

0.2 Review of black holes in General Relativity

0.2.1 Schwarzschild black holes

In Schwarzschild coordinates (t, r, θ, ϕ) the Schwarzschild line element has the form

$$ds^2 = -\left(1 - \frac{r_\mathrm{s}}{r}\right) dt^2 + \left(1 - \frac{r_\mathrm{s}}{r}\right)^{-1} dr^2 + r^2 g_\Omega,$$

where $g_\Omega = \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$ is the metric on the two sphere and $r_s = 2M$. **Kruskal–Szekeres coordinates**: Replace t and r by a new timelike coordinate T and a new spacelike coordinate X:

$$T = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \sinh\left(\frac{t}{4GM}\right)$$

$$X = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \cosh\left(\frac{t}{4GM}\right)$$

$$ds^2 = -\frac{4r_\mathrm{s}^3}{r} e^{-\frac{r}{r_\mathrm{s}}} \left(dT^2 - dR^2\right) + r^2 \, g_\Omega$$

Note: $T^2 - R^2 = \left(1 - \dfrac{r}{r_\mathrm{s}}\right) e^{\frac{r}{r_\mathrm{s}}}$. It is regular at horizon and maximally extends to full spacetime.

Tortoise coordinate:$r^* = r + 2Mln(\frac{r-2M}{2M})$. As $r^* \to \infty$, $r \to \infty$ and as $r^* \to -\infty$, $r \to 2GM$.

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dv^2 + 2\,dv\,dr + r^2 d\Omega^2$$

where $v = t + r^*$.

0.2.2 Reissner–Nordström black holes

In spherical coordinates (t, r, θ, φ), the Reissner–Nordström line element is

$$ds^2 = -\left(1 - \frac{r_\mathrm{s}}{r} + \frac{r_\mathrm{Q}^2}{r^2}\right) c^2\,dt^2 + \left(1 - \frac{r_\mathrm{s}}{r} + \frac{r_\mathrm{Q}^2}{r^2}\right)^{-1} dr^2 + r^2\,g_\Omega$$

where $r_\mathrm{s} = \frac{2GM}{c^2}$, $r_Q^2 = \frac{Q^2 G}{4\pi\varepsilon_0 c^4}$

$$M = \frac{Q^2}{16\pi\varepsilon_0 G M_\mathrm{irr}} + M_\mathrm{irr}$$

0.2.3 Kerr–Newman black holes

In Boyer–Lindquist coordinates (t, r, θ, ϕ) the Kerr–Newman line element has the form

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr - Q^2}{r^2 + a^2\cos^2\theta}\right) dt^2 + \frac{r^2 + a^2\cos^2\theta}{r^2 - 2Mr + a^2 + Q^2} dr^2 \\ &+ (r^2 + a^2\cos^2\theta)\,d\theta^2 + \left(r^2 + a^2 + \frac{a^2(2Mr - Q^2)\sin^2\theta}{r^2 + a^2\cos^2\theta}\right)\sin^2\theta\,d\phi^2 \\ &- \frac{2a(2Mr - Q^2)\sin^2\theta}{r^2 + a^2\cos^2\theta}\,dt\,d\phi \end{aligned}$$

or

$$ds^2 = -\left(c\,dt - a\sin^2\theta\,d\phi\right)^2 \frac{\Delta}{\rho^2} + \left(\left(r^2 + a^2\right)d\phi - ac\,dt\right)^2 \frac{\sin^2\theta}{\rho^2} + \left(\frac{dr^2}{\Delta} + d\theta^2\right)\rho^2$$

where $a = \frac{J}{M}$, $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - r_s r + a^2 + r_Q^2$, $r_s = 2GM$ and $r_Q^2 = \frac{Q^2 G}{4\pi\epsilon_0 c^4}$

$$M_\mathrm{irr} = \frac{1}{2}\sqrt{2M^2 - r_Q^2 c^4/G^2 + 2M\sqrt{M^2 - (r_Q^2 + a^2)c^4/G^2}}$$

Inner and outer event horizon:

$$r_{\mathrm{H}}^{\pm} = r_{\pm} = \frac{r_{\mathrm{s}}}{2} \pm \sqrt{\frac{r_{\mathrm{s}}^2}{4} - a^2 - r_Q^2}$$

Inner and outer ergosphere:

$$r_{\mathrm{E}}^{\pm} = \frac{r_{\mathrm{s}}}{2} \pm \sqrt{\frac{r_{\mathrm{s}}^2}{4} - a^2 \cos^2\theta - r_Q^2}$$

Area of outer event horizon: The 2D metric on the surface $t =$ constant and $r = r_+$ is

$$ds_+^2 = (r_+^2 + a^2\cos^2\theta)\, d\theta^2 + \left(r_+^2 + a^2 + \frac{a^2(2Mr_+ - Q^2)\sin^2\theta}{r_+^2 + a^2\cos^2\theta} \right) \sin^2\theta\, d\phi^2$$

$$\begin{aligned}
dA_+ &= \sqrt{\det g_+}\, d\theta\, d\phi \\
&= \sqrt{(r_+^2 + a^2\cos^2\theta)\left(r_+^2 + a^2 + \frac{a^2(2Mr_+ - Q^2)\sin^2\theta}{r_+^2 + a^2\cos^2\theta} \right)} \sin\theta\, d\theta\, d\phi \\
&= \sqrt{(r_+^2 + a^2\cos^2\theta)(r_+^2 + a^2) + a^2(2Mr_+ - Q^2)(1 - \cos^2\theta)} \sin\theta\, d\theta\, d\phi \\
&= \sqrt{(r_+^4 + a^2 r_+^2 + 2Ma^2 r_+ - a^2Q^2) + a^2(r_+^2 - 2Mr_+ + a^2 + Q^2)\cos^2\theta} \sin\theta\, d\theta\, d\phi.
\end{aligned}$$

Conveniently, the coefficient of $\cos^2\theta$ in the square root vanishes by the definition of r_+,

$$\Rightarrow dA_+ = (r_+^2 + a^2)\sin\theta\, d\theta\, d\phi$$

On integrating we get

$$A_+ = 4\pi(r_+^2 + a^2) = 4\pi\left(2M^2 - Q^2 + 2M\sqrt{M^2 - a^2 - Q^2} \right)$$

0.3 The four classical laws of black hole mechanics

Using General relativity and some other classically reasonable assumptions (which may not be correct in Quantum Gravity) the following laws can be proved.

No.	Laws of thermodynamics	Analogous black hole laws
0	T is same for systems in equilibrium	κ is constant on the Event Horizon fora stationary black hole.
1	$dE = TdS + \Omega\, dJ + \Phi\, dQ$	$dE = \frac{\kappa}{2\pi}\left(\frac{dA}{4}\right) + \Omega\, dJ + \Phi\, dQ$
2	$dS \geq 0$ where $S = k_{\mathrm{B}} \ln \Omega$	$dA \geq 0$ or $\frac{dA}{4} \geq 0$
3	$T = 0$ K cannot be attained	Extremal black holes with $\kappa = 0$ cannot exist. (also called as *Cosmic censorship hypothesis* and it forbids naked singularities)

Table 0.1: Analogy between the black hole laws and laws of thermodynamics

If $T = \frac{\kappa}{2\pi}$ and $S = \frac{dA}{4}$ then we can see that they are very similar to the laws of thermodynamics. (Of course here the exact coefficients seem arbitrary). We will see that these are not analogous but are the same phenomenon. Black holes although classically seem to be "not so random" (they can be completely characterized by very few parameters and do not seem to have large number of micro states) objects they have very high entropy. For a supermassive astrophysical black hole like the one at the center of the Milky Way, this is an enormous number, or order $(\frac{10^6 km}{l_p})^2 \sim 10^{88}$ (and there are billions of them). For comparison, the entropy of all baryons in the observable universe is around 10^{82}, and the entropy of the CMB is about 10^{89}.

0.4 Quantum Field Theory in curved spacetime

Lagrange density of a scalar field in curved spacetime is (we should replace normal derivatives with covariant derivatives)

$$\mathcal{L} = \sqrt{-g}(-\frac{1}{2})g^{\mu\nu}\nabla_{\mu}\phi\nabla_{nu}\phi - \frac{1}{2}m^2\phi^2$$

In QFT we can express a massive scalar field using Fourier decomposition of the field as

$$\phi = \int d^3k(a_k f_k + a_k^{\dagger} f_k^*)$$

In inertial Minkowski space-times we have time transnational symmetry i.e. ∂_t is a Killing vector. So we can uniquely define positive- or "negative"(by that we mean $\partial_t f_k^* = +i\omega f_k^*$ where $\omega > 0$. This terminology was made historically when they considered ϕ as a wave function instead of a quantum field)-frequency modes and **the notion of a particle is Lorentz-invariant**. Since for a general spacetime there will not be any timelike Killing vector, we will not in general be able to find solutions to the wave equation that separate into time-dependent and space-dependent factors, and so cannot classify modes as positive- or "negative"-frequency. We can find a set of basis modes, but the problem is that there will generally be many such sets, with no way to prefer one over any others, and the notion of a vacuum or number operator will depend sensitively on which set we choose. We can easily define particles in QFT in curved spacetime if a spacetime has positive and "negative" frequency modes. In general we can't find them. For static space-times we can define them and this process will work. So it is not necessary that different observers in different places agree on the number of particles that are in a particular state.

As we will see in the next chapter, We don't even have to include gravity, **even in flat spacetime** an accelerating observer will observe particles in the vacuum state of an inertial observer.

0.4.1 Bogoliubov coefficients and their relations

0.5 Information Theory

0.5.1 Shannon Entropy

Shannon entropy of a random variable is the average level of "information", "surprise", or "uncertainty" inherent in the variable's possible outcomes. For a discrete random variable X, with possible outcomes $x_1, ..., x_n$, which occur with probability $\mathrm{p}_i, ..., \mathrm{p}_n$, the entropy of X is formally defined as:

$$S = -\sum_i p_i \ln(p_i)$$

For simplicity take a random variable with 2 possible outcomes, 0 with probability p and $S1$ with probability $1-p$. The number of possible outcomes is

$$\frac{N!}{(pN)!((1-p)N)!} \approx \frac{N^N}{(pN)^{pN}((1-p)N)^{(1-p)N}} = 2^{NS}$$

where $S = -pln(p) - (1-p)ln(1-p)$ is the entropy of a single random variable and NS is the entropy of N such random variables. Since 2^{NS} different messages can be sent using N such random variables, the **number of bits** of information in N random variables is NS.

0.5.2 Qubits

A qubit is a two-state (or two-level) quantum-mechanical system similar to the spin states of a 1/2 spin particle. We denote the state analogous to $|\downarrow\rangle$ as $|0\rangle$ and $|\uparrow\rangle$ as $|1\rangle$. The Pauli operators denoted by X, Y and Z follow in this notation

$$
\begin{aligned}
X|a\rangle &= |a+1\rangle \\
Y|a\rangle &= i(-1)^{a+1}|a+1\rangle \\
Z|a\rangle &= (-1)^{a}|a\rangle
\end{aligned}
$$

Note that for $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, all one point correlations with X_i,Y_i and Z_i are zero but they have two-point functions $\langle\psi|X_1X_2|\psi\rangle = 1$, $\langle\psi|Y_1Y_2|\psi\rangle = -1$ and $\langle\psi|Z_1Z_2|\psi\rangle = 1$.

0.5.3 von Neumann/ Entanglement entropy

For a quantum-mechanical system described by a density matrix ρ, the **von Neumann entropy** is $S = -\operatorname{tr}(\rho \ln \rho)$ (ln denotes the matrix logarithm). For a **pure/ isolated bipartite state**

$$\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$$

$$\rho_{AB} = \sum_i p_i |\psi^i_{AB}\rangle\langle\psi^i_{AB}|$$

$$\mathcal{S}(\rho_A) = -\operatorname{Tr}[\rho_A \log \rho_A] = -\operatorname{Tr}[\rho_B \log \rho_B] = \mathcal{S}(\rho_B)$$

For $|\psi_{AB}\rangle = \sum_i \sqrt{p_i}|\psi^i_A\rangle \otimes |\psi^i_B\rangle$ we define

$$\rho_A = \sum_i p_i |\psi^i_A\rangle\langle\psi^i_A|,\ \rho_B = \sum_i p_i |\psi^i_B\rangle\langle\psi^i_B|$$

and $S_A = S_B = -\sum_i p_i \ln(p_i)$. Here $|\psi^i_{AB}\rangle = |\psi^i_A\rangle \otimes |\psi^i_B\rangle$ and The $|\psi^i_A\rangle$ and $|\psi^i_B\rangle$ **may not be bases** of $\mathcal{H}_A$ or $\mathcal{H}_B$, because there may not be enough of them but they are orthonormal. ρ_A is the reduced density matrix for subsystem A. The reason we defined ρ_A this way is because if we take an operator $O_A \otimes 1_B$ then

$$\begin{aligned}\langle \psi_{AB}|O_A \otimes 1_B|\psi_{AB}\rangle &= \sum_{i,j} \sqrt{p_i p_j}\langle \psi^i_A|O_A|\psi^i_A\rangle\langle \psi^i_B|1_B|\psi^i_B\rangle \\ &= \sum_{i,j} \sqrt{p_i p_j}\langle \psi^i_A|O_A|\psi^i_A\rangle\delta_{ij} \\ &= \sum_{i} p_i\langle \psi^i_A|O_A|\psi^i_A\rangle \\ &= Tr_{\mathcal{H}_A}(\rho_A O_A)\end{aligned}$$

We can also find ρ_A by doing partial trace over $\mathcal{H}_B$:

$$= \sum_k (\langle 1_A| \otimes \langle \psi^k_B|)\rho_{AB}(|1_A\rangle \otimes |\psi^k_B\rangle) = \sum_k (\langle 1_A| \otimes \langle \psi^k_B|)|\psi_{AB}\rangle\langle\psi_{AB}|(|1_A\rangle \otimes |\psi^k_B\rangle)$$
$$= \sum_k \left(\langle 1_A| \otimes \langle \psi^k_B| \left(\sum_i p_i|\psi^i_{AB}\rangle\langle\psi^i_{AB}| + \sum_{i\neq j}\sqrt{p_i p_j}|\psi^i_{AB}\rangle\langle\psi^j_{AB}|\right)|1_A\rangle \otimes |\psi^k_B\rangle\right)$$
$$= \sum_k \left(\sum_i p_i|\psi^i_A\rangle\langle\psi^i_A|\delta_{ik} + \sum_{i\neq j}\sqrt{p_i p_j}|\psi^i_A\rangle\langle\psi^j_A|\delta_{ik}\delta_{jk}\right) = \sum_i p_i|\psi^i_A\rangle\langle\psi^i_A| + 0$$
$$= \rho_A$$

When $p_i = \frac{1}{N}$ and $S = ln(n)$ it is called maximally entangled state. If A and B are entangled then $S_A > 0$ and if they are not entangled (i.e $|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$) then $S_A = 0$.
For the von Neumann entropy S, NS is the number of qubits of information one can store in an ensemble ρ with N subsystems. It is similar to Shannon entropy, for which instead of ensemble, qubits it will be random variable, bits.
Entanglement in the vacuum: The two-point correlation function is non zero for vacuum which indicates entanglement between points in the vacuum. For Minkowski vaccum $|\Omega\rangle$, $\langle\Omega|\phi(0,x)\phi(0,y)|\Omega\rangle$ scales like $\dfrac{1}{|x-y|^2}$ for $|x-y| << m^{-1}$ and as $e^{-m|x-y}$ for $|x-y| >> m^{-1}$. This is somewhat similar to the entanglement in the state $|\psi\rangle = \frac{1}{2}(|00\rangle + |11\rangle)$ which has zero 1 point correlations but non zero 2 point correlations with the Pauli operators.

1 Unruh effect using Rindler coordinates

1.1 Rindler coordinates and space

We will first write the Minkowski line element suppressed to two dimensions in the Minkowski coordinates (t, x) and **Rindler coordinates** (η, ξ) related by:

$$t = \frac{1}{a}e^{a\xi}sinh(a\eta) \qquad\qquad x = \frac{1}{a}e^{a\xi}cosh(a\eta)$$

Notice that $x+t > 0$ and $x-t > 0$. So Rindler coordinates can only describe a part of the Minkowski space called the **Rindler Space or wedge**.

$$ds^2 = -dt^2 + dx^2 = e^{2a\xi}(-d\eta^2 + d\xi^2)$$

If $\xi = k$ for a particle, where k is a constant, then when observed in the Minkowski frame the particle has constant four acceleration **magnitude** $\alpha = \frac{a}{e^{ak}}$ (the 4 accleration is not constant) as $a^\mu = \frac{d^2x^\mu}{d\tau^2} = (\alpha sinh(\alpha t), \alpha cosh(\alpha t))$. η is related to the proper time τ of this particle by $\eta = \frac{\alpha}{a}\tau$. The locus of $x^2 = t^2$ is called the **Rindler horizon**.

An observer who is at the origin of the Rindler coordinates ($\eta = \xi = 0$) is called the **Rindler observer**. The magnitude of accleration of this observer is a and $\eta = \tau$.

This metric is independent of η so ∂_η is a Killing vector field in these coordinates. In Minkowski coordinates the vector ∂_η is

$$\begin{aligned}\partial_\eta &= \frac{\partial t}{\partial \eta}\partial_t + \frac{\partial x}{\partial \eta}\partial_x \\ &= a(x\partial_t + t\partial_x)\end{aligned}$$

It is the Killing field associated with a boost in the x direction. So time translation in the Rindler coordinates is a boost in the Minkowski frame. in regions II and Ill it is spacelike, while in region IV it is timelike but past-directed. Which is what we intuitively expect since as time passes in the Rindler frame the velocity of Rindler observer increases in the Minkowski frame. This Killing field naturally extends throughout the spacetime, in regions II and III it is spacelike, while in region IV it is timelike but past-directed.

Now we can similarly define **left Rindler space or wedge**.

$$t = -\frac{1}{a}e^{a\xi}sinh(a\eta) \qquad\qquad x = -\frac{1}{a}e^{a\xi}cosh(a\eta)$$

The metric is same as for the right wedge. Coordinates (η, ξ) cannot be used simultaneously in wedges right and left, because the range of these parameters are the same in each regions. The vector field ∂_η is a Killing vector field in wedges left and right, but is future pointing in the right wedge while **past pointing in the left one**. $-\partial_\eta$ is the future pointing timelike killing vector field in left wedge.

1.2 Intuitive explanation of Unruh effect

The future horizon and past or illusory horizon behave like an event horizon of a black hole and white hole respectively in the Einstein–Rosen bridge. In QFT often time negative and positive energy particles will be produced and will be annihilated. The product of their energy and life time is of the order of Planck's constant. But near the future horizon if such a particle pair is produced the negative energy particle may escape from the Rindler space and go to the outside Minkowski space and **it can be stable in the Minkowski space**. To understand this remember that energy is the component of four momentum corresponding to a future pointing timelike killing vector field. In Rindler space it corresponds to ∂_η and in Minkowski space it corresponds to ∂_t. So **a particle is said to be having negative energy in Rindler space if $P_\mu N^\mu$ is negative.** (where N^μ is the unit η vector which is (1,0) in (η,ξ)). A particle is said to be having negative energy in Minkowski space if $P_\mu T^\mu$ is negative. (where T^μ is the unit t vector).

$$dt = e^{a\xi}(cosh(a\eta)d\eta + sinh(a\eta)d\xi)$$

Near the horizon if $cosh(a\eta)d\eta + sinh(a\eta)d\xi > 0$ and $d\eta < 0$ for a particle then it is having -ve energy wrt Rindler space (or *equivalently* moving backward in time with +ve energy) but wrt Minkowski space it will have +ve energy and it will be stable after getting out of Rindler space. **The reverse process cannot occur because if the positive energy is escaped the negative energy particle cannot be stabilised in the Rindler space.**

1.3 Unruh effect

In Minkowski spacetime, the equation of motion is $\Box\phi = (-\partial_t^2 + \partial_x^2)\phi = 0$ and admits plane waves solutions,

$$f_k = \frac{1}{\sqrt{4\pi\omega}} e^{ik_\mu x^\mu}, \qquad k^\mu = (\omega, k)$$

$$\partial_t f_k = -i\omega f_k \qquad \partial_t f_k^* = +i\omega f_k^*$$

where f_k^* are "negative" frequency modes. After canonical quantization, any field configuration ϕ solution to the equation of motion can be expanded in terms of $\{f_k,\, f_k^*\}$,

$$\phi = \sum_k (a_k f_k + a_k^\dagger f_k^*)$$

The vacuum is defined by state $^M|0\rangle$ such that $a_k {}^M|0\rangle, \forall k$.

In the right Rindler wedge, the equation of motion reads $\Box\phi = e^{-2a\xi}(-\partial_\eta^2 + \partial_\xi^2)\phi = 0$ and admits plane wave solutions,

$$g_k^R = \frac{1}{\sqrt{4\pi\omega}} e^{ik_\mu x^\mu}, \qquad x^\mu = (\eta, \xi)$$
$$\partial_t g_k^R = -i\omega g_k^R \qquad \partial_t g_k^{R*} = +i\omega g_k^{R*}$$

Positive frequency modes are defined with respect the the Killing vector field ∂_η. So g_k^R is positive frequency mode and g_k^{R*} is "negative" frequency mode. After canonical quantization, any field configuration ϕ solution to the equation of motion can be expanded in terms of $\{g_k^R,\, g_k^{R*}\}$,

$$\phi = \sum_k (b_k g_k^R + b_k^\dagger g_k^{R*})$$

The vacuum is defined by state $^R|0\rangle$ such that $a_k {}^R|0\rangle, \forall k$.

We defined g_k^R in the right Rindler wedge. Now we generalise it to

$$g_k^R = \begin{cases} \frac{1}{\sqrt{4\pi\omega}} e^{ik_\mu x^\mu} & \text{in right Rindler wedge} \\ 0 & \text{in left Rindler wedge} \end{cases}$$

Similarly,

$$g_k^L = \begin{cases} 0 & \text{in right Rindler wedge} \\ \frac{1}{\sqrt{4\pi\omega}} e^{ik_\mu x^\mu} & \text{in left Rindler wedge} \end{cases}$$

g_k^L are positive frequency modes with respect to Killing vector field -∂_η. $\{g_k^R,\, g_k^{R*}, g_k^L,\, g_k^{L*}\}$ and $\{f_k,\, f_k^*\}$ are complete set of modes for the Minkowski space, and thus there are two possible modes expansions for any field configuration solution to the equation of motion,

$$\phi = \sum_k (b_k g_k^R + c_k g_k^L + b_k^\dagger g_k^{R*} + c_k^\dagger g_k^{L*})$$
$$\phi = \sum_k (a_k f_k + a_k^\dagger f_k^*)$$

Now we like to find ${}^{M}\langle 0|\,{}^{R}N_k\,{}^{M}|0\rangle$. That is the number of particles with momentum k in the Minkowski vacuum state observed from the Rindler frame.

Using Bogoliubov transformation

$$g_k^R(u) = \int d\omega'(A_{\omega\omega'}f_{\omega'} + B_{\omega\omega'}f^*_{\omega'})$$

where $u = t - x$. Since $f_k = \frac{1}{\sqrt{4\pi\omega}}e^{ik_\mu x^\mu}$ we can write $f_k(u) = \frac{1}{\sqrt{4\pi\omega}}e^{-i\omega' u}$.

By definition inverse Fourier transform of $g_k^R(u)$ is $\tilde{g}_\omega(\omega') = \int_{\inf}^{\inf} du e^{i\omega' u} g_k^R(u)$. Now we can find that

$$A_{\omega\omega'} = \sqrt{\frac{\omega'}{\pi}}\tilde{g}_\omega(\omega') \qquad\qquad B_{\omega\omega'} = \sqrt{\frac{\omega'}{\pi}}\tilde{g}_\omega(-\omega')$$

The inverse Fourier transform has the property $\tilde{g}_\omega(-\omega') = -e^{-\frac{\omega\pi}{a}}\tilde{g}_\omega(\omega')$. Using that we get $A_{\omega\omega'} = -e^{-\frac{\omega\pi}{a}}B_{\omega\omega'}$. We also know that

$$AA^\dagger - BB^\dagger = 1 \implies |A|^2 - |B|^2 = 1 \implies |B|^2 = \frac{1}{e^{\frac{2\pi\omega}{a}} - 1}$$

$$\begin{aligned}
{}^{M}\langle 0|\,{}^{R}N_k\,{}^{M}|0\rangle &= {}^{M}\langle 0|(b_k^\dagger)(b_k)^{M}|0\rangle \\
&= {}^{M}\langle 0|(-B_{kp}a_P + A^*_{kp}a_p^\dagger)(A^*_{kq}a_q - B^*_{kq}a_q^\dagger)^{M}|0\rangle \\
&= {}^{M}\langle 0|(B_{kq}B^*_{kp}\delta_{qp})^{M}|0\rangle \\
&= |B|^2_{kk} = |B|^2_{\omega\omega} = \frac{1}{e^{\frac{2\pi\omega}{a}} - 1}
\end{aligned}$$

That is the number of particle with momentum k and energy $\omega = k$ observed by the Rindler observer are $\dfrac{1}{e^{\frac{\omega}{\frac{a}{2\pi}}} - 1}$. Which is exactly the number of particle with momentum k observed for a black-body with $T = \dfrac{a}{2\pi}$ according to the Planck formula. This is called as **Unruh effect**.

The Unruh temperature is spatially inhomogeneous across the Rindler space but the Unruh state is in the equilibrium (unlike in Hawking effect). This is a direct consequence of Ehrenfest–Tolman effect.

1.4 The Rindler decomposition and entanglement

We know that ground state can be obtained by simply acting on any generic state $|\chi\rangle$ with e^{-HT} if $\langle\Omega|\chi\rangle \neq 0$.

$$|\Omega\rangle = \lim_{T\to\infty} \frac{1}{\langle\Omega|\chi\rangle} e^{-HT}|\chi\rangle$$
$$\Rightarrow \langle\phi|\Omega\rangle = \lim_{T\to\infty} \frac{1}{\langle\Omega|\chi\rangle} \langle\phi|e^{-HT}|\chi\rangle$$

In the Euclidean path integral formalism, this means that we can compute this wave functional as

$$\langle\phi|\Omega\rangle \propto \int_{\hat{\phi}(t_E=-\infty)=0}^{\hat{\phi}(t_E=0)=\phi} \mathcal{D}\hat{\phi} e^{-S_E}$$

where S_E is the Euclidean action, obtained from the usual one by analytic continuation $t \to -it_E$. For the free massive scalar field it is

$$S_E(\hat{\phi}) = \frac{1}{2}\int d^3x dt_E[(\partial_{t_E}\hat{\phi})^2 + (\vec{\nabla}\hat{\phi})^2 + m^2\hat{\phi}^2]$$

Note that the sign of the time derivative component is different compared to the usual Lagrangian. Using the Wick rotation technique of path integrals of QFT and the fact that Lorentz boost in the Minkowski space is equal to 4 dimensional rotation in Euclidean space we can obtain

$$\langle\phi_L\phi_R|\Omega\rangle \propto \sum_i e^{-\pi\omega_i}\langle\phi_L|i*\rangle_L\langle\phi_R|i\rangle_R$$

$$\Rightarrow |\Omega\rangle = \frac{1}{\sqrt{Z}}\sum_i e^{-\pi\omega_i}|i*\rangle_L|i\rangle_R]$$

where z can be found by normalisation, now we can find the reduced density matrix for Right wedge subsystem. Which will be

$$\rho_R = \frac{1}{z}\sum_i e^{-2\pi\omega_i}|i\rangle_R\langle i|_R$$

The above density matrix is a thermal density matrix and it has temperature $T = \frac{1}{2\pi}$ (here $a = 1$ otherwise $T = \frac{a}{2\pi}$). We can also see that the left and right subsystems are highly entangled.
Since they are entangled $\rho \neq \rho_L \otimes \rho_R$.
But we can imagine a system $\rho = \rho_L \otimes \rho_R$ (there is still entanglement present

between points in each wedge but the 2 wedges are uncorrelated. So a Rindler observer far from horizon on either side will not observe a difference) and then the typical difference between neighboring fields on either side should be of the order of $\frac{1}{\epsilon}$ (otherwise the correlations will still be large but we need to make the 2 point correlations as zero) where ϵ is a UV length cutoff, so we have

$$\partial_x \phi|_{x=0} \propto \frac{1}{\epsilon^2}$$

The gradient term in the Hamiltonian then produces the dominant contribution

$$H \approx dx \int d^2y (\partial_x \phi)^2 \propto \epsilon \int d^2y \frac{1}{\epsilon^2} = \frac{A}{\epsilon^3}$$

As ϵ is small this energy will be large and this phenomenon is analogous to the **firewall in black holes**.

2 Hawking effect

2.1 Intuition using Penrose process

2.2 Rindler decomposition

We know that in tortoise coordinate $r^* = r + 2Mln(\frac{r-2M}{2M})$

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r}\right) dv^2 + 2\, dv\, dr + r^2 d\Omega^2 \\ &= -\left(1 - \frac{2M}{r}\right) (dt^2 + dr^{*2} + 2dtdr^*) + 2dtdr + 2dr^* dr + r^2 d\Omega^2 \\ &= -\left(1 - \frac{2M}{r}\right) (dt^2 - dr^{*2}) + r^2 d\Omega^2 \end{aligned}$$

If $r \approx 2M$ then $r^* - 2M = r - 2M + 2Mln(\frac{r-2M}{2M}) \approx 2Mln(\frac{r-2M}{2M})$

$$\Rightarrow ds^2 = e^{\frac{r^* - 2M}{2M}} (-dt^2 + dr^{*2}) + r^2 d\Omega^2$$

neglecting the angular part and comparing with Rindler metric

$$ds^2 = e^{2a\xi} (-d\eta^2 + d\xi^2)$$

we can see that $a \to \frac{1}{4M} = \kappa$, which implies the temperature

$$T = \frac{a}{2\pi} \to \frac{1}{8\pi M} = \frac{\kappa}{2\pi}$$

Since $\frac{dS}{dE} = \frac{1}{T} = 8\pi M = 8\pi E$, $A = 4\pi r_s^2$ and using $S(E=0) = 0$ we get

$$S = \frac{A}{4}$$

2.3 Schwarzschild modes

Klein-Gordon equation in Schwarzschild metric is

$$(\Box - m^2)\phi = (\nabla^\nu \nabla_\nu - m^2)\phi = 0$$

and $\nabla^\nu \nabla_\nu = \dfrac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu)$. Now substituting $\phi(t, r_*, \theta, \phi) = Y_{lm}(\theta, \phi) e^{-i\omega t} \chi_l(r_*, t)$ as a solution we get

$$\frac{d^2}{dt^2}\chi_l(r_*, t) - \frac{d^2}{dr_*^2}\chi_l(r_*, t) + V_l(r_*)\chi_l(r_*, t) = 0$$

where $V_l(r_*) = (1 - \frac{2M}{r})(m^2 + \dfrac{l(l+1)}{r^2} + \frac{2M}{r^3})$ and as $r \to \infty$ $V(r) \to m^2$ (here m is mass not magnetic quantum number) which is a **barrier**. If m is large, then that modes will be confined very near the horizon. For $m = 0$ there is a barrier at $r = \frac{3}{2}$ and the height and is of order l^2. If m and l are 0 then $\frac{2M}{r^3}$ will go to zero even faster and due to this **photons dominate Hawking radiation**. So massless particles will dominate Hawking radiation.

2.4 The information problem

If black hole never radiated like we expected in classical general relativity, then we can say information is conserved and is stored inside the EH. But after Hawking radiation was discovered people had to chose among these 3 options:
1)Information loss: Black evaporates completely and information is lost.
2)Planck-sized remnant: The evaporation stops, and the Planck-sized remnant contains the information.
3)Unitary evaporation: BH (Bekenstein-Hawking or Black Hole) entropy is correct only in a coarse-grained sense; the Hawking radiation does not actually come out in a mixed state. The information is carried out in subtle correlations between the Hawking photons, and the final state of the evaporation is a pure state of the radiation field. Because it is a complicated state any small subsystem looks thermal, justifying the approximate validity of if we don't look at too many photons at once.

2.5 The Euclidean black hole

Let $t \to -it_E$

$$ds^2 = -\left(1 - \frac{r_{\mathrm{s}}}{r}\right) dt^2 + \left(1 - \frac{r_{\mathrm{s}}}{r}\right)^{-1} dr^2 + r^2 g_\Omega,$$

From now on in this subsection we take $r_s = 1$

$$ds^2 = \left(1 - \frac{1}{r}\right) dt_E^2 + \left(\frac{r}{r-1}\right) dr^2 + r^2 g_\Omega$$

Let us define $d\rho = \sqrt{\frac{r}{r-1}}dr$ and if $r \approx 1$ then $d\rho \approx \sqrt{\frac{1}{r-1}}dr \Rightarrow$

$$r \approx 1 + \frac{\rho^2}{4}$$

$$ds^2 = d\rho^2 + \frac{1}{4}\rho^2 dt_E^2 + d\Omega^2$$

Neglecting the angular part we can see that it is exactly similar to 2d Euclidean metric in spherical coordinates $ds^2 = d\rho^2 + \rho^2 d\theta^2$. This implies that $\frac{t_E}{2}$ like θ has period 2π. So, t_E has a period of 4π. We know that: Wick rotation also relates a quantum field theory at a finite inverse temperature β to a statistical-mechanical model with the imaginary time coordinate t_E being periodic with period β.

So, $\beta = \frac{1}{T} = 4\pi$. This is in agreement with our earlier answer. We can also calculate the BH entropy using only Euclidean path integrals but it is more complex than calculating Hawking temperature.

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