Black Hole Information Paradox

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Abstract

This is a brief review of black hole thermodynamics and the information loss problem. Initially some useful concepts from general relativity, quantum field theory in curved spacetime and information theory are reviewed. Later Unruh effect, Hawking radiation and Page curve are discussed. Section 0.2 can be skipped.

Contents

0 Introduction

0.1 Conventions

The signature of the metric used is (-, +, +, +). I will generally set $c = \hbar = k_B = 1$. But sometimes I will restore them.

0.2 Review of black holes in General Relativity

0.2.1 Schwarzschild black holes

In Schwarzschild coordinates (t, r, θ, ϕ) the Schwarzschild line element has the form

$$ds^2 = -\left(1 - \frac{r_\mathrm{s}}{r}\right) dt^2 + \left(1 - \frac{r_\mathrm{s}}{r}\right)^{-1} dr^2 + r^2 g_\Omega,$$

where $g_\Omega = \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$ is the metric on the two sphere and $r_s = 2M$.
Kruskal–Szekeres coordinates: Replace t and r by a new timelike coordinate T and a new spacelike coordinate X:

$$T = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \sinh\left(\frac{t}{4GM}\right)$$

$$X = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \cosh\left(\frac{t}{4GM}\right)$$

$$ds^2 = -\frac{4r_s^3}{r} e^{-\frac{r}{r_s}} \left(dT^2 - dR^2\right) + r^2\, g_\Omega$$

Note: $T^2 - R^2 = \left(1 - \dfrac{r}{r_s}\right) e^{\frac{r}{r_s}}$. It is regular at horizon and maximally extends to full spacetime.
Tortoise coordinate:$r^* = r + 2Mln(\dfrac{r-2M}{2M})$. As $r^* \to \infty$, $r \to \infty$ and as $r^* \to -\infty$, $r \to 2GM$.

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dv^2 + 2\, dv\, dr + r^2 d\Omega^2$$

where $v = t + r^*$.

0.2.2 Reissner–Nordström black holes

In spherical coordinates (t, r, θ, φ), the Reissner–Nordström line element is

$$ds^2 = -\left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right) c^2\, dt^2 + \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right)^{-1} dr^2 + r^2\, g_\Omega$$

where $r_s = \dfrac{2GM}{c^2}$, $r_Q^2 = \dfrac{Q^2 G}{4\pi\varepsilon_0 c^4}$

$$M = \frac{Q^2}{16\pi\varepsilon_0 G M_{\text{irr}}} + M_{\text{irr}}$$

0.2.3 Kerr–Newman black holes

In Boyer–Lindquist coordinates (t, r, θ, ϕ) the Kerr–Newman line element has the form

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2Mr - Q^2}{r^2 + a^2\cos^2\theta}\right) dt^2 + \frac{r^2 + a^2\cos^2\theta}{r^2 - 2Mr + a^2 + Q^2} dr^2 \\ &+ (r^2 + a^2\cos^2\theta)\, d\theta^2 + \left(r^2 + a^2 + \frac{a^2(2Mr - Q^2)\sin^2\theta}{r^2 + a^2\cos^2\theta}\right)\sin^2\theta\, d\phi^2 \\ &- \frac{2a(2Mr - Q^2)\sin^2\theta}{r^2 + a^2\cos^2\theta}\, dt\, d\phi \end{aligned}$$

or

$$ds^2 = -\left(c\,dt - a\sin^2\theta\, d\phi\right)^2 \frac{\Delta}{\rho^2} + \left(\left(r^2 + a^2\right) d\phi - ac\, dt\right)^2 \frac{\sin^2\theta}{\rho^2} + \left(\frac{dr^2}{\Delta} + d\theta^2\right)\rho^2$$

where $a = \dfrac{J}{M}$, $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - r_s r + a^2 + r_Q^2$, $r_s = 2GM$ and $r_Q^2 = \dfrac{Q^2 G}{4\pi\epsilon_0 c^4}$

$$M_{\rm irr} = \frac{1}{2}\sqrt{2M^2 - r_Q^2 c^4/G^2 + 2M\sqrt{M^2 - (r_Q^2 + a^2)c^4/G^2}}$$

Inner and outer event horizon:

$$r_{\rm H}^{\pm} = r_{\pm} = \frac{r_{\rm s}}{2} \pm \sqrt{\frac{r_{\rm s}^2}{4} - a^2 - r_Q^2}$$

Inner and outer ergosphere:

$$r_{\rm E}^{\pm} = \frac{r_{\rm s}}{2} \pm \sqrt{\frac{r_{\rm s}^2}{4} - a^2\cos^2\theta - r_Q^2}$$

Area of outer event horizon: The 2D metric on the surface $t =$ constant and $r = r_+$ is

$$ds_+^2 = (r_+^2 + a^2\cos^2\theta)\, d\theta^2 + \left(r_+^2 + a^2 + \frac{a^2(2Mr_+ - Q^2)\sin^2\theta}{r_+^2 + a^2\cos^2\theta}\right)\sin^2\theta\, d\phi^2$$

$$\begin{aligned} dA_+ &= \sqrt{\det g_+}\, d\theta\, d\phi \\ &= \sqrt{(r_+^2 + a^2\cos^2\theta)\left(r_+^2 + a^2 + \frac{a^2(2Mr_+ - Q^2)\sin^2\theta}{r_+^2 + a^2\cos^2\theta}\right)}\sin\theta\, d\theta\, d\phi \\ &= \sqrt{(r_+^2 + a^2\cos^2\theta)(r_+^2 + a^2) + a^2(2Mr_+ - Q^2)(1 - \cos^2\theta)}\sin\theta\, d\theta\, d\phi \\ &= \sqrt{(r_+^4 + a^2 r_+^2 + 2Ma^2 r_+ - a^2 Q^2) + a^2(r_+^2 - 2Mr_+ + a^2 + Q^2)\cos^2\theta}\sin\theta\, d\theta\, d\phi. \end{aligned}$$

Conveniently, the coefficient of $\cos^2\theta$ in the square root vanishes by the definition of r_+,

$$\Rightarrow dA_+ = (r_+^2 + a^2)\sin\theta\, d\theta\, d\phi$$

On integrating we get

$$A_+ = 4\pi(r_+^2 + a^2) = 4\pi\left(2M^2 - Q^2 + 2M\sqrt{M^2 - a^2 - Q^2}\right)$$

0.2.4 Killing vectors and horizons

Killing vector fields or Killing vectors: A Killing vector K_μ satisfies $\nabla_{(\mu}K_{\nu)} = \frac{\nabla_\mu K_\nu + \nabla_\nu K_\mu}{2} = 0$ and along a geodesic $K_\mu p^\mu$ is conserved. K^μ generates an isometry; that is the geometry is invariant for an **infinitesimal** transformation along the direction of K^μ.
Killing horizons: A null hypersurface defined by the vanishing of the norm of a Killing vector field. Not all Killing horizons are event horizons but all event horizons are Killing horizons. $x^2 - t^2 = 0$ in Minkowski space is a Killing horizon for the Killing vector $x\partial_t + t\partial_x = (x, t)$ which corresponds to a Lorentz boost. In the Schwarzschild metric $(1, 0, 0, 0)$ is the Killing vector field and it's null horizon is Event horizon since the magnitude of the vector is $-\left(1 - \frac{2M}{r}\right)(1)^2$.

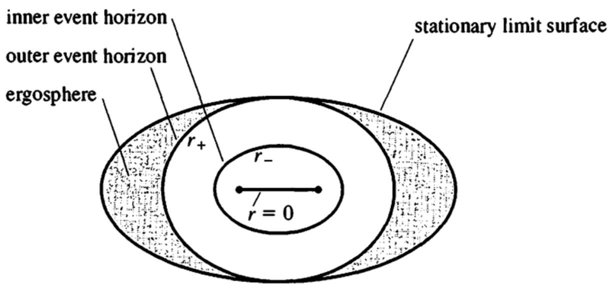

Figure 0.1: Horizon structure around the Kerr solution.

For the Kerr black hole the ergosphere is the Killing horizon for the Killing vector $K = \partial_t$ or $K^\mu = (1, 0, 0, 0)$. Inside the ergosphere the timelike vector is $\chi^\mu = K^\mu + \Omega_H R^\mu$ where $\Omega_H = \frac{a}{r_+^2 + a^2}$ is the angular velocity of the horizon and $R = \partial_\phi$. The outer horizon is the Killing horizon of χ^μ and inside outer horizon the timelike coordinate is radial towards the center.

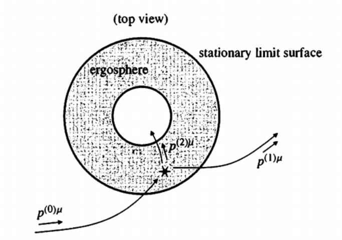

Figure 0.2: Penrose process.

$$M_{\text{irr}} = \sqrt{\frac{M^2 + \sqrt{M^4 - J^2c^2/G^2}}{2}} = \sqrt{\frac{A}{16\pi G^2}}$$

is the irreducible mass of the Kerr metric. Intuitively it is the rest mass energy of the black hole. The remaining energy is the rotational energy and can be extracted by Penrose process. Penrose process is a direct consequence of the fact that inside the ergo region moving forward in time means moving tangentially in the direction of rotation.

0.3 The 4 classical laws of black hole mechanics

Using General relativity and some other classically reasonable assumptions like energy conditions (which may not be correct in Quantum Gravity) the following laws can be proved.

No.	Laws of thermodynamics	Analogous black hole laws
0	T is same for systems in equilibrium	κ is constant on the Event Horizon for a stationary black hole.
1	$dE = TdS + \Omega\, dJ + \Phi\, dQ$	$dE = \frac{\kappa}{2\pi}\left(\frac{dA}{4}\right) + \Omega\, dJ + \Phi\, dQ$
2	$dS \geq 0$ where $S = k_{\text{B}} \ln \Omega$	$dA \geq 0$ or $\frac{dA}{4} \geq 0$
3	$T = 0$ K cannot be attained	Extremal black holes with $\kappa = 0$ cannot exist. (also called as *Cosmic censorship hypothesis* and it forbids naked singularities)

Table 0.1: Analogy between the black hole laws and laws of thermodynamics

If $T = \frac{\kappa}{2\pi}$ and $dS = \frac{dA}{4}$ then we can see that they are very similar to the laws of thermodynamics. (Of course here the exact coefficients seem arbitrary). We will see that these are not analogous but are the same phenomenon. Black holes although classically seem to be "not so random" (they can be completely characterized by very few parameters and do not seem to have large number of micro states) objects they have very high entropy. For a supermassive astrophysical black hole like the one at the center of the Milky Way, this is an enormous number, or order $(\frac{10^6 km}{l_p})^2 \sim 10^{88}$ (and there are billions of them). For comparison, the entropy of all baryons in the observable universe is around 10^{82}, and the entropy of the CMB is about 10^{89}.

0.3.1 Black Hole Heat Capacity

Note that adding energy to a Schwarzschild black hole increases its mass, which decreases its Hawking temperature. So it has **negative heats capacity**.

For the Reissner-Nordström metric

$$T = \frac{r_+ - r_-}{4\pi r_+^2} = \frac{1}{2\pi M}\frac{\sqrt{1-q^2}}{(1+\sqrt{1-q^2})^2}$$

we can see that $C := T\left(\frac{\partial S}{\partial T}\right)_Q = 2\pi M^2 \frac{\sqrt{1-q^2}(1+\sqrt{1-q^2})^2}{1-2\sqrt{1-q^2}}$, $q := Q/M$. If $|q| > \frac{\sqrt{3}}{2}$ it can be in **stable equilibrium** as specific heat will be **positive**. But practical black holes are almost always neutral and finding a black hole with this much charge is improbable.

For a Kerr-Newman black hole-

$$T = \frac{\kappa}{2\pi} = \frac{r_+ - r_-}{4\pi(r_+^2 + a^2)} = \frac{\sqrt{M^2 - Q^2 - J^2/M^2}}{2\pi(2M^2 - Q^2 + 2M\sqrt{M^2 - Q^2 - J^2/M^2})}$$

For a Kerr black hole-

$$C_J := T\left(\frac{\partial S}{\partial T}\right)_J = 2\pi r_+^2 \frac{(1-h^2)(h^2+1)^2}{3h^4 + 6h^2 - 1}$$

$$C_\Omega := T\left(\frac{\partial S}{\partial T}\right)_\Omega = -2\pi r_+^2 \frac{(1-h^2)}{h^2+1}$$

where $0 \le h = \frac{|a|}{r_+} \le 1$ C_J is positive only for $h > \sqrt{\frac{2}{3}\sqrt{3} - 1} = 0.3933$. Unlike large q black holes, there probably will be few black holes with large angular momentum. Only these black holes can be in **stable equilibrium**.

0.4 Quantum Field Theory in curved spacetime

Lagrange density of a scalar field in curved spacetime is (we should replace normal derivatives in QFT with covariant derivatives)

$$\mathcal{L} = \sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}m^2\phi^2\right)$$

In QFT we can express a massive scalar field using Fourier decomposition of the field as

$$\phi = \int d^3k(a_k f_k + a_k^\dagger f_k^*)$$

In inertial Minkowski space-times we have time transnational symmetry i.e. ∂_t is a Killing vector. So we can uniquely define positive- or "negative"(by that we mean $\partial_t f_k^* = +i\omega f_k^*$ where $\omega > 0$. This terminology was made historically when they considered ϕ as a wave function instead of a quantum field)-frequency modes and **the notion of a particle is Lorentz-invariant**. Since for a general spacetime there will not be any timelike Killing vector, we will not in general be able to find solutions to the wave equation that separate into time-dependent and space-dependent factors, and so cannot classify modes as positive- or "negative"-frequency. We can find a set of basis modes, but the problem is that there will generally be many such sets, with no way to prefer one over any others, and the notion of a vacuum or number operator will depend sensitively on which set we choose. We can easily define particles in QFT in curved spacetime if a spacetime has positive and "negative" frequency modes. In general we can't find them. **So it is not necessary that different observers in different places agree on the number of particles that are in a particular state**.

As we will see in the next chapter, we don't even have to include gravity, **even in flat spacetime** an accelerating observer will observe particles in the vacuum state of an inertial observer.

0.4.1 Bogoliubov coefficients and their relations

Any field configuration $\phi(x)$ solution to the Klein-Gordon equation of motion can be expanded in terms of the two bases,

$$\phi(x) = \sum_i \left(a_i f_i + a_i^\dagger f_i^*\right) = \sum_i \left(b_i g_i + b_i^\dagger g_i^*\right)$$

where basis modes are normalized with respect to the Klein-Gordon inner product, and where operators a_k, b_k in the expansion satisfy $\left[a_k, a_{k'}^\dagger\right] =$

$\delta(k-k')$ and $\left[b_p, b^\dagger_{p'}\right] = \delta(p-p')$. The modes g_i can be expressed in terms of the basis $\{f_j, f^*_j\}$,

$$g_i = \sum_j \left(A_{ij} f_j + B_{ij} f^*_j\right)$$

This relation between the two bases is called a **Bogoliubov transformation**, and the coefficients A, B inside the transformation are called the Bogoliubov coefficients. Above formula implies

$$g^*_i = \sum_j \left(B^*_{ij} f_j + A^*_{ij} f^*_j\right)$$

In matrix form

$$\begin{pmatrix} g \\ g^* \end{pmatrix} = M \begin{pmatrix} f \\ f^* \end{pmatrix} = \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} \begin{pmatrix} f \\ f^* \end{pmatrix}$$

$$AA^\dagger - BB^\dagger = 1.$$

$$AB^t - BA^t = 0$$

$$M^{-1} = \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}^{-1} = \begin{pmatrix} A^\dagger & -B^t \\ -B^\dagger & A^t \end{pmatrix}$$

$$\Rightarrow M^{-1}M = \begin{pmatrix} AA^\dagger - BB^\dagger & -AB^t + BA^t \\ B^*A^\dagger - A^*B^\dagger & -B^*B^t + A^*A^t \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The field expansion can be written in matrix form,

$$\phi = \begin{pmatrix} b & b^\dagger \end{pmatrix} \begin{pmatrix} g \\ g^* \end{pmatrix} = \begin{pmatrix} a & a^\dagger \end{pmatrix} \begin{pmatrix} f \\ f^* \end{pmatrix}$$

0.5 Information Theory

0.5.1 Shannon Entropy

Shannon entropy of a random variable is the average level of "information", "surprise", or "uncertainty" inherent in the variable's possible outcomes. For a discrete random variable X, with possible outcomes $x_1, ..., x_n$, which occur with probability $\mathrm{p}_i, ..., \mathrm{p}_n$, the entropy of X is formally defined as:

$$S = -\sum_i p_i \ln(p_i)$$

For simplicity take a random variable with 2 possible outcomes, 0 with probability p and $S1$ with probability $1-p$. The number of possible outcomes is

$$\frac{N!}{(pN)!((1-p)N)!} \approx \frac{N^N}{(pN)^{pN}((1-p)N)^{(1-p)N}} = 2^{NS}$$

where $S = -pln(p) - (1-p)ln(1-p)$ is the entropy of a single random variable and NS is the entropy of N such random variables. Since 2^{NS} different messages can be sent using N such random variables, the **number of bits** of information in N random variables is NS.

0.5.2 Qubits

A qubit is a two-state (or two-level) quantum-mechanical system similar to the spin states of a 1/2 spin particle. We denote the state analogous to $|\downarrow\rangle$ as $|0\rangle$ and $|\uparrow\rangle$ as $|1\rangle$. The Pauli operators denoted by X, Y and Z follow in this notation

$$\begin{aligned} X|a\rangle &= |a+1\rangle \\ Y|a\rangle &= i(-1)^{a+1}|a+1\rangle \\ Z|a\rangle &= (-1)^a|a\rangle \end{aligned}$$

Note that for $|\psi\rangle = \dfrac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, all one point correlations with X_i,Y_i and Z_i are zero but they have two-point functions $\langle\psi|X_1X_2|\psi\rangle = 1$, $\langle\psi|Y_1Y_2|\psi\rangle = -1$ and $\langle\psi|Z_1Z_2|\psi\rangle = 1$.

0.5.3 von Neumann/ Entanglement entropy

For a quantum-mechanical system described by a density matrix ρ, the **von Neumann entropy** is $S = -\operatorname{tr}(\rho \ln \rho)$ (ln denotes the matrix logarithm). For a **pure/ isolated bipartite state**

$$\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$$

$$\mathcal{S}(\rho_A) = -\operatorname{Tr}[\rho_A \log \rho_A] = -\operatorname{Tr}[\rho_B \log \rho_B] = \mathcal{S}(\rho_B)$$

For $|\psi_{AB}\rangle = \sum_i \sqrt{p_i}|\psi_A^i\rangle \otimes |\psi_B^i\rangle$ (i.e. we choose a basis in which $|\psi_{AB}\rangle$ expressed as $|A| \times |B|$ matrix will be diagnosed) we define

$$\rho_A = \sum_i p_i|\psi_A^i\rangle\langle\psi_A^i|,\ \rho_B = \sum_i p_i|\psi_B^i\rangle\langle\psi_B^i|$$

and $S_A = S_B = -\sum_i p_i \ln(p_i)$. Here $|\psi^i_{AB}\rangle = |\psi^i_A\rangle \otimes |\psi^i_B\rangle$ and The $|\psi^i_A\rangle$ and $|\psi^i_B\rangle$ **may not be bases** of $\mathcal{H}_A$ or $\mathcal{H}_B$, because there may not be enough of them but they are orthonormal. ρ_A is the reduced density matrix for subsystem A. The reason we defined ρ_A this way is because if we take an operator $O_A \otimes 1_B$ then

$$\begin{aligned}
\langle\psi_{AB}|O_A \otimes 1_B|\psi_{AB}\rangle &= \sum_{i,j} \sqrt{p_i p_j}\langle\psi^i_A|O_A|\psi^i_A\rangle\langle\psi^i_B|1_B|\psi^i_B\rangle \\
&= \sum_{i,j} \sqrt{p_i p_j}\langle\psi^i_A|O_A|\psi^i_A\rangle\delta_{ij} \\
&= \sum_i p_i\langle\psi^i_A|O_A|\psi^i_A\rangle \\
&= Tr_{\mathcal{H}_A}(\rho_A O_A)
\end{aligned}$$

We can also find ρ_A by doing partial trace over $\mathcal{H}_B$:

$$= \sum_k (\langle 1_A| \otimes \langle\psi^k_B|)\rho_{AB}(|1_A\rangle \otimes |\psi^k_B\rangle) = \sum_k (\langle 1_A| \otimes \langle\psi^k_B|)|\psi_{AB}\rangle\langle\psi_{AB}|(|1_A\rangle \otimes |\psi^k_B\rangle)$$

$$= \sum_k \left(\langle 1_A| \otimes \langle\psi^k_B| \left(\sum_i p_i|\psi^i_{AB}\rangle\langle\psi^i_{AB}| + \sum_{i\neq j}\sqrt{p_i p_j}|\psi^i_{AB}\rangle\langle\psi^j_{AB}|\right)|1_A\rangle \otimes |\psi^k_B\rangle\right)$$

$$= \langle 1_A|\sum_k \left(\sum_i p_i|\psi^i_A\rangle\langle\psi^i_A|\delta_{ik} + \sum_{i\neq j}\sqrt{p_i p_j}|\psi^i_A\rangle\langle\psi^j_A|\delta_{ik}\delta_{jk}\right)|1_A\rangle = \langle 1_A|(\sum_i p_i|\psi^i_A\rangle\langle\psi^i_A| + 0)|1_A\rangle$$

$$= \langle 1_A|\rho_A|1_A\rangle$$

When $p_i = \frac{1}{N}$ and $S = ln(n)$ it is called maximally entangled state. If A and B are entangled then $S_A > 0$ and if they are not entangled (i.e $|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$) then $S_A = 0$.

For the von Neumann entropy S, NS is the number of qubits of information one can store in an ensemble ρ with N subsystems. It is similar to Shannon entropy, for which instead of ensemble, qubits it will be random variable, bits.

Entanglement in the vacuum: The two-point correlation function is non zero for vacuum which indicates entanglement between points in the vacuum. For Minkowski vaccum $|\Omega\rangle$, $\langle\Omega|\phi(0,x)\phi(0,y)|\Omega\rangle$ scales like $\dfrac{1}{|x-y|^2}$ for $|x-y| << m^{-1}$ and as $e^{-m|x-y}$ for $|x-y| >> m^{-1}$. This is somewhat similar to the entanglement in the state $|\psi\rangle = \frac{1}{2}(|00\rangle + |11\rangle)$ which has zero 1 point correlations but non zero 2 point correlations with the Pauli operators.

1 Unruh effect using Rindler coordinates

1.1 Rindler coordinates and space

We will first write the Minkowski line element suppressed to two dimensions in the Minkowski coordinates (t, x) and **Rindler coordinates** (η, ξ) related by:

$$t = \frac{1}{a}e^{a\xi}sinh(a\eta) \qquad\qquad x = \frac{1}{a}e^{a\xi}cosh(a\eta)$$

Notice that $x + t > 0$ and $x - t > 0$. So Rindler coordinates can **only describe a part of the Minkowski space** called the **Rindler Space or wedge**.

$$ds^2 = -dt^2 + dx^2 = e^{2a\xi}(-d\eta^2 + d\xi^2)$$

If $\xi = k$ for a particle, where k is a constant, then when observed in the Minkowski frame the particle has constant four acceleration **magnitude** $\alpha = \frac{a}{e^{ak}}$ (the 4 accleration is not constant) as $a^\mu = \frac{d^2x^\mu}{d\tau^2} = (\alpha sinh(\alpha t), \alpha cosh(\alpha t))$. η is related to the proper time τ of this particle by $\eta = \frac{\alpha}{a}\tau$. The locus of $x^2 = t^2$ is called the **Rindler horizon**.

An observer who is at the origin of the Rindler coordinates ($\eta = \xi = 0$) is called the **Rindler observer**. The magnitude of accleration of this observer is a and $\eta = \tau$.

This metric is independent of η so ∂_η is a Killing vector field in these coordinates. In Minkowski coordinates the vector ∂_η is

$$\begin{aligned}\partial_\eta &= \frac{\partial t}{\partial \eta}\partial_t + \frac{\partial x}{\partial \eta}\partial_x \\ &= a(x\partial_t + t\partial_x)\end{aligned}$$

It is the Killing field associated with a boost in the x direction. So time translation in the Rindler coordinates is a boost in the Minkowski frame. in regions II and Ill it is spacelike, while in region IV it is timelike but past-directed. Which is what we intuitively expect since as time passes in the Rindler frame the velocity of Rindler observer increases in the Minkowski frame. This Killing field naturally extends throughout the spacetime, in regions II and III it is spacelike, while in region IV it is timelike but past-directed.

Now we can similarly define **left Rindler space or wedge**.

$$t = -\frac{1}{a}e^{a\xi}sinh(a\eta) \qquad\qquad x = -\frac{1}{a}e^{a\xi}cosh(a\eta)$$

The metric is same as for the right wedge. Coordinates (η, ξ) cannot be used simultaneously in wedges right and left, because the range of these parameters are the same in each regions. The vector field ∂_η is a Killing vector field in wedges left and right, but is future pointing in the right wedge while **past pointing in the left one**. $-\partial_\eta$ is the future pointing timelike killing vector field in left wedge.

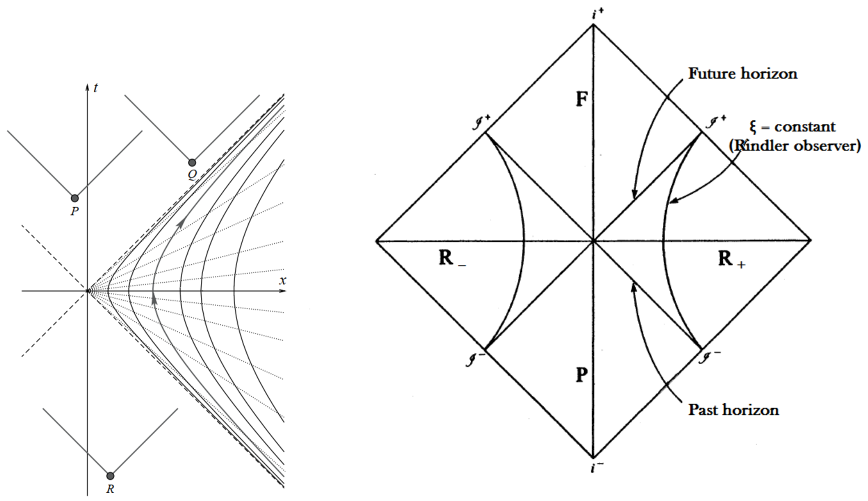

Figure 1.1: Rindler chart (*left*). The dashed lines are the Rindler horizons. Penrose-Carter diagram (*right*) of Rindler wedge in Minkowski space.

1.2 Intuitive explanation of Unruh effect

The future horizon and past or illusory horizon behave like an event horizon of a black hole and white hole respectively in the Einstein–Rosen bridge. In QFT often negative and positive energy particles will be produced and will be annihilated. The product of their energy and life time is of the order of Planck's constant. But near the future horizon if such a particle pair is produced the negative energy particle may escape from the Rindler space and go to the outside Minkowski space and **it can be stable in the Minkowski space**. To understand this remember that energy is the component of four momentum corresponding to a future pointing timelike killing vector field. In Rindler space it corresponds to ∂_η and in Minkowski space it corresponds to ∂_t. So **a particle is said to be having negative energy in Rindler space if $P_\mu N^\mu$ is negative.** (where N^μ is the unit η vector which is (1,0) in (η,ξ)). A particle is said to be having negative energy in Minkowski space if $P_\mu T^\mu$ is negative. (where T^μ is the unit t vector).

$$dt = e^{a\xi}(cosh(a\eta)d\eta + sinh(a\eta)d\xi)$$

Near the horizon if $cosh(a\eta)d\eta + sinh(a\eta)d\xi > 0$ and $d\eta < 0$ for a particle then it is having -ve energy wrt Rindler space (or *equivalently* moving backward in time with +ve energy) but wrt Minkowski space it will have +ve energy and it will be stable after getting out of Rindler space. **The reverse process cannot occur because if the positive energy is escaped the negative energy particle cannot be stabilised in the Rindler space.**

1.3 Unruh effect

In Minkowski spacetime, the equation of motion is $\Box\phi = (-\partial_t^2 + \partial_x^2)\phi = 0$ and admits plane waves solutions,

$$f_k = \frac{1}{\sqrt{4\pi\omega}} e^{ik_\mu x^\mu}, \qquad k^\mu = (\omega, k)$$
$$\partial_t f_k = -i\omega f_k \qquad \partial_t f_k^* = +i\omega f_k^*$$

where f_k^* are "negative" frequency modes. After canonical quantization, any field configuration ϕ solution to the equation of motion can be expanded in terms of $\{f_k,\, f_k^*\}$,

$$\phi = \sum_k (a_k f_k + a_k^\dagger f_k^*)$$

The vacuum is defined by state ${}^M|0\rangle$ such that $a_k{}^M|0\rangle, \forall k$.

In the right Rindler wedge, the equation of motion reads $\Box\phi = e^{-2a\xi}(-\partial_\eta^2 + \partial_\xi^2)\phi = 0$ and admits plane wave solutions,

$$g_k^R = \frac{1}{\sqrt{4\pi\omega}} e^{ik_\mu x^\mu}, \qquad x^\mu = (\eta, \xi)$$
$$\partial_t g_k^R = -i\omega g_k^R \qquad \partial_t g_k^{R*} = +i\omega g_k^{R*}$$

Positive frequency modes are defined with respect the the Killing vector field ∂_η. So g_k^R is positive frequency mode and g_k^{R*} is "negative" frequency mode. After canonical quantization, any field configuration ϕ solution to the equation of motion can be expanded in terms of $\{g_k^R,\, g_k^{R*}\}$,

$$\phi = \sum_k (b_k g_k^R + b_k^\dagger g_k^{R*})$$

The vacuum is defined by state ${}^R|0\rangle$ such that $a_k{}^R|0\rangle, \forall k$.

We defined g_k^R in the right Rindler wedge. Now we generalise it to

$$g_k^R = \begin{cases} \frac{1}{\sqrt{4\pi\omega}} e^{ik_\mu x^\mu} & \text{in right Rindler wedge} \\ 0 & \text{in left Rindler wedge} \end{cases}$$

Similarly,

$$g_k^L = \begin{cases} 0 & \text{in right Rindler wedge} \\ \frac{1}{\sqrt{4\pi\omega}} e^{ik_\mu x^\mu} & \text{in left Rindler wedge} \end{cases}$$

g_k^L are positive frequency modes with respect to Killing vector field -∂_η. $\{g_k^R$, g_k^{R*}, g_k^L, $g_k^{L*}\}$ and $\{f_k$, $f_k^*\}$ are complete set of modes for the Minkowski space, and thus there are two possible modes expansions for any field configuration solution to the equation of motion,

$$\phi = \sum_k (b_k g_k^R + c_k g_k^L + b_k^\dagger g_k^{R*} + c_k^\dagger g_k^{L*})$$
$$\phi = \sum_k (a_k f_k + a_k^\dagger f_k^*)$$

Now we like to find ${}^M\langle 0| {}^R N_k {}^M|0\rangle$. That is the number of particles with momentum k in the Minkowski vacuum state observed from the Rindler frame.

Using Bogoliubov transformation

$$g_k^R(u) = \int d\omega' (A_{\omega\omega'} f_{\omega'} + B_{\omega\omega'} f_{\omega'}^*)$$

where $u = t - x$. Since $f_k = \frac{1}{\sqrt{4\pi\omega}} e^{ik_\mu x^\mu}$ we can write $f_k(u) = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega' u}$.

By definition inverse Fourier transform of $g_k^R(u)$ is $\tilde{g}_\omega(\omega') = \int_\infty^\infty du e^{i\omega' u} g_k^R(u)$. Now we can find that

$$A_{\omega\omega'} = \sqrt{\frac{\omega'}{\pi}} \tilde{g}_\omega(\omega') \qquad\qquad B_{\omega\omega'} = \sqrt{\frac{\omega'}{\pi}} \tilde{g}_\omega(-\omega')$$

The inverse Fourier transform has the property $\tilde{g}_\omega(-\omega') = -e^{-\frac{\omega\pi}{a}} \tilde{g}_\omega(\omega')$. Using that we get $A_{\omega\omega'} = -e^{-\frac{\omega\pi}{a}} B_{\omega\omega'}$. We also know that

$$AA^\dagger - BB^\dagger = 1 \implies |A|^2 - |B|^2 = 1 \implies |B|^2 = \frac{1}{e^{\overline{\;a\;}}{}^{2\pi\omega} - 1}$$

$$
\begin{aligned}
{}^{M}\langle 0|\,{}^{R}N_k\,{}^{M}|0\rangle &= {}^{M}\langle 0|(b_k^\dagger)(b_k)\,{}^{M}|0\rangle \\
&= {}^{M}\langle 0|(-B_{kp}a_P + A_{kp}^* a_p^\dagger)(A_{kq}^* a_q - B_{kq}^* a_q^\dagger)\,{}^{M}|0\rangle \\
&= {}^{M}\langle 0|(B_{kq}B_{kp}^*\delta_{qp})\,{}^{M}|0\rangle \\
&= |B|^2_{kk} = |B|^2_{\omega\omega} = \frac{1}{e^{\frac{2\pi\omega}{a}} - 1}
\end{aligned}
$$

That is the number of particle with momentum k and energy $\omega = k$ observed by the Rindler observer are $\frac{1}{e^{\frac{\omega}{\frac{a}{2\pi}}} - 1}$. Which is exactly the number of particle with momentum k observed for a black-body with $T = \frac{a}{2\pi}$ according to the Planck formula. This is called as **Unruh effect**.

The Unruh temperature is spatially inhomogeneous across the Rindler space but the Unruh state is in the equilibrium (unlike in Hawking effect). This is a direct consequence of Ehrenfest–Tolman effect.

1.4 The Rindler decomposition and entanglement

We know that ground state can be obtained by simply acting on any generic state $|\chi\rangle$ with e^{-HT} if $\langle\Omega|\chi\rangle \neq 0$.

$$
\begin{aligned}
|\Omega\rangle &= \lim_{T\to\infty} \frac{1}{\langle\Omega|\chi\rangle} e^{-HT}|\chi\rangle \\
\Rightarrow \langle\phi|\Omega\rangle &= \lim_{T\to\infty} \frac{1}{\langle\Omega|\chi\rangle} \langle\phi|e^{-HT}|\chi\rangle
\end{aligned}
$$

In the Euclidean path integral formalism, this means that we can compute this wave functional as

$$
\langle\phi|\Omega\rangle \propto \int_{\hat\phi(t_E=-\infty)=0}^{\hat\phi(t_E=0)=\phi} \mathcal{D}\hat\phi e^{-S_E}
$$

where S_E is the Euclidean action, obtained from the usual one by analytic continuation $t \to -it_E$. For the free massive scalar field it is

$$
S_E(\hat\phi) = \frac{1}{2}\int d^3x dt_E[(\partial_{t_E}\hat\phi)^2 + (\vec\nabla\hat\phi)^2 + m^2\hat\phi^2]
$$

Note that the sign of the time derivative component is different compared to the usual Lagrangian. Using the Wick rotation technique of path integrals of QFT and the fact that Lorentz boost in the Minkowski space is equal to 4 dimensional rotation in Euclidean space we can obtain

$$\langle \phi_L \phi_R | \Omega \rangle \propto \sum_i e^{-\pi \omega_i} \langle \phi_L | i* \rangle_L \langle \phi_R | i \rangle_R$$

$$\Rightarrow |\Omega\rangle = \frac{1}{\sqrt{Z}} \sum_i e^{-\pi \omega_i} |i*\rangle_L |i\rangle_R]$$

where z can be found by normalisation, now we can find the reduced density matrix for Right wedge subsystem. Which will be

$$\rho_R = \frac{1}{z} \sum_i e^{-2\pi\omega_i} |i\rangle_R \langle i|_R$$

The above density matrix is a thermal density matrix and it has temperature $T = \frac{1}{2\pi}$ (here $a = 1$ otherwise $T = \frac{a}{2\pi}$). We can also see that the left and right subsystems are highly entangled.
Since they are entangled $\rho \neq \rho_L \otimes \rho_R$.
But we can imagine a system $\rho = \rho_L \otimes \rho_R$ (there is still entanglement present between points in each wedge but the 2 wedges are uncorrelated. So a Rindler observer far from horizon on either side will not observe a difference) and then the typical difference between neighboring fields on either side should be of the order of $\frac{1}{\epsilon}$ (otherwise the correlations will still be large but we need to make the 2 point correlations as zero) where ϵ is a UV length cutoff, so we have

$$\partial_x \phi|_{x=0} \propto \frac{1}{\epsilon^2}$$

The gradient term in the Hamiltonian then produces the dominant contribution

$$H \approx dx \int d^2y (\partial_x \phi)^2 \propto \epsilon \int d^2y \frac{1}{\epsilon^2} = \frac{A}{\epsilon^3}$$

As ϵ is small this energy will be large and this phenomenon is analogous to the **firewall in black holes**.

2 Hawking radiation

We can calculate Hawking radiation temperature using Bogoliubov transformation similar to how we found Unruh temperature. But we can also simply use the equivalence principle and find the Hawking radiation temperature from Unruh temperature.

2.1 Rindler decomposition

We know that in tortoise coordinate $r^* = r + 2Mln(\frac{r-2M}{2M})$

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r}\right) dv^2 + 2\, dv\, dr + r^2 d\Omega^2 \\ &= -\left(1 - \frac{2M}{r}\right) (dt^2 + dr^{*2} + 2dtdr^*) + 2dtdr + 2dr^*dr + r^2 d\Omega^2 \\ &= -\left(1 - \frac{2M}{r}\right) (dt^2 - dr^{*2}) + r^2 d\Omega^2 \end{aligned}$$

If $r \approx 2M$ then

$$r^* - 2M = r - 2M + 2Mln(\frac{r-2M}{2M}) \approx 2Mln(\frac{r-2M}{2M}) \approx 2Mln(\frac{r-2M}{r})$$

$$\Rightarrow ds^2 = e^{\frac{r^*-2M}{2M}}(-dt^2 + dr^{*2}) + r^2 d\Omega^2$$

neglecting the angular part and comparing with Rindler metric

$$ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2)$$

we can see that $a \to \frac{1}{4M} = \kappa$, which implies the temperature

$$T = \frac{a}{2\pi} \to \frac{1}{8\pi M} = \frac{\kappa}{2\pi}$$

Here we are used **the equivalence principle**. Locally we can't detect whether we are accelerating or in a gravitational field.
Since $\frac{dS}{dE} = \frac{1}{T} = 8\pi M = 8\pi E$, $A = 4\pi r_s^2$ and using $S(E=0) = 0$ we get

$$S = \frac{A}{4}$$

2.2 Schwarzschild modes

Klein-Gordon equation in Schwarzschild metric is

$$(\Box - m^2)\phi = (\nabla^\nu \nabla_\nu - m^2)\phi = 0$$

and $\nabla^\nu \nabla_\nu = \frac{1}{\sqrt{-g}} \partial_\mu(\sqrt{-g} g^{\mu\nu} \partial_\nu)$. Now substituting $\phi(t, r_*, \theta, \phi) = Y_{lm}(\theta, \phi) e^{-i\omega t} \chi_l(r_*, t)$ as a solution we get

$$\frac{d^2}{dt^2}\chi_l(r_*, t) - \frac{d^2}{dr_*^2}\chi_l(r_*, t) + V_l(r_*)\chi_l(r_*, t) = 0$$

where $V_l(r_*) = (1 - \frac{2M}{r})(m^2 + \dfrac{l(l+1)}{r^2} + \frac{2M}{r^3})$ and as $r \to \infty$ $V(r) \to m^2$ (here m is mass not magnetic quantum number) which is a **barrier**. If m is large, then that modes will be confined very near the horizon. For $m = 0$ there is a barrier at $r = \frac{3}{2}$ and the height and is of order l^2. If m and l are 0 then $\frac{2M}{r^3}$ will go to zero even faster and due to this **photons dominate Hawking radiation**. So massless particles will dominate Hawking radiation.

2.3 The information problem

If black hole never radiated like we expected in classical general relativity, then we can say information is conserved and is stored inside the EH. But after Hawking radiation was discovered people had to chose among these 3 options:
1)Information loss: Black evaporates completely and information is lost.
2)Planck-sized remnant: The evaporation stops, and the Planck-sized remnant contains the information.
3)Unitary evaporation: BH (Bekenstein-Hawking or Black Hole) entropy is correct only in a coarse-grained sense; the Hawking radiation does not actually come out in a mixed state. The information is carried out in subtle correlations between the Hawking photons, and the final state of the evaporation is a pure state of the radiation field. Because it is a complicated state any small subsystem looks thermal.

2.4 The Euclidean black hole

Let $t \to -it_E$

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 g_\Omega,$$

From now on in this subsection we take $r_s = 1$

$$ds^2 = \left(1 - \frac{1}{r}\right) dt_E^2 + \left(\frac{r}{r-1}\right) dr^2 + r^2 g_\Omega$$

Let us define $d\rho = \sqrt{\dfrac{r}{r-1}}dr$ and if $r \approx 1$ then $d\rho \approx \sqrt{\frac{1}{r-1}}dr$

$$\Rightarrow r \approx 1 + \frac{\rho^2}{4}$$

$$ds^2 = d\rho^2 + \frac{1}{4}\rho^2 dt_E^2 + d\Omega^2$$

Neglecting the angular part we can see that it is exactly similar to 2d Euclidean metric in spherical coordinates $ds^2 = d\rho^2 + \rho^2 d\theta^2$. This implies that $\frac{t_E}{2}$ like θ has period 2π. So, t_E has a period of 4π. We know that: Wick rotation also relates a quantum field theory at a finite inverse temperature β to a statistical-mechanical model with the imaginary time coordinate t_E being periodic with period β.

So, $\beta = \frac{1}{T} = 4\pi \Rightarrow T = \frac{1}{4\pi} = \frac{\kappa}{2\pi}$, since $\kappa = \frac{1}{4GM} = \frac{1}{2}$. This is in agreement with our earlier answer. We can also calculate the BH entropy using only Euclidean path integrals but it is more complex than calculating Hawking temperature.

3 Unitary evaporation

Until now we assumed some background spacetime as fixed and used quantum field theory in that spacetime. But in general there will be **back-reaction** caused by the Hawking radiation on the geometry after significant amount of it was radiated. To completely describe it we need a fully fledged quantum gravity theory. But to get some qualitative idea about the evaporation of black holes we can proceed by assuming some well established principles. Assuming that quantum gravity (and in turn black hole evaporation) is unitary is natural since quantum field theories are unitary.

In unitary theories S-matrix is very important since it is the limit $t_i \to -\infty$ and $t_f \to \infty$ of the unitary time translation operator. If we multiply the initial state of an object (long before it fell into black hole) with S-matrix we will get the final state of the Hawking radiation. So, the probability of finding an "out" state $|\chi\rangle$ given an "in" state $|\psi\rangle$ is, $P(\chi \mid \psi) = |\langle\chi|S|\psi\rangle|^2$. The BFSS supergravity matrix model is a unitary model based on the ideas of S-matrix.

3.1 The Page curve

We consider the case of infalling matter which is initially in a **pure state**. At any time R denotes the radiation that was already radiated. BH denotes the black hole. This gives a tensor product decomposition of the Hilbert space

$$\mathcal{H} = \mathcal{H}_R \otimes \mathcal{H}_{BH}$$

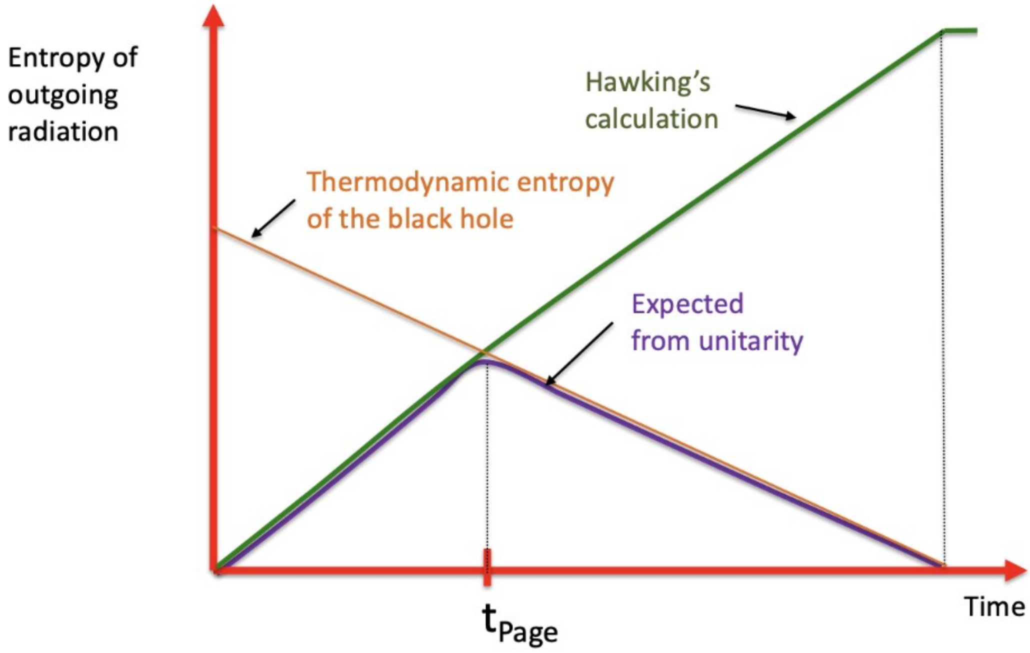

Figure 3.1: Page Curve is the non monotonic curve. Here the other two curves are **coarse grained** but the page curve is **fine grained**. Here $t = 0$ is when the black hole is formed.

Now we can guess qualitatively the graph as shown in 3.1. Initially the back-reaction will be negligible, so the entropy of radiation as predicted by Hawking should be nearly correct. But after sometime it will decrease since in the end it should again give 0 entropy as finally it will be a pure state again. The **fine grained** entropy of the $R + BH$ system is always constant and 0.

Page time: It is defined as the time at which $S_{BH}^{\{course\}} = S_R^{\{course\}}$.

Note that $S_R = S_{BH}$ and $S_{R+BH} = 0$ at all times (these are fine grained) from the properties of von Neumann entropy.

At the beginning of the evaporation process the radiation that comes out is entangled with the remaining black hole. But eventually it must start coming out entangled with the earlier radiation, since eventually the final state of the radiation must be pure. It is only once we are past the Page time that we can think of the quantum information about the initial state as having started to come out.

Information paradox will be considered as solved only if a theory can exactly calculate the Page curve. As of now it is still considered as an open problem.

Now we state a theorem that supports the idea that the final pure state will be very near to a mixed state.

3.2 Page's theorem

In the bipartite system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, without loss of generality we can take $|A| \leq |B|$, where $|X|$ denotes the dimensionality of the subsystem X.

Page's theorem then says that a randomly chosen pure state in $\mathcal{H}_{AB}$ is likely to be very close to maximally entangled as long as $\dfrac{|A|}{|B|} << 1$

A random state can be defined as

$$|\psi(U)\rangle \equiv U\,|\psi_0\rangle$$

$$\rho(U) \equiv U\,|\psi_0\rangle\,\langle\psi_0| U^\dagger$$

where U is a random unitary matrix

$$\|M\|_1 \equiv \mathrm{tr}\,\sqrt{M^\dagger M}$$

3.2.1 Unitary integrals

$$\int dU = 1$$

$$\int dU U_{ij} U^\dagger_{kl} = \frac{1}{N}\delta_{il}\delta_{jk}$$

$$\begin{aligned}\int dU U_{ij} U_{kl} U^\dagger_{mn} U^\dagger_{op} &= \frac{1}{N^2-1}\left(\delta_{in}\delta_{kp}\delta_{jm}\delta_{lo} + \delta_{ip}\delta_{kn}\delta_{jo}\delta_{lm}\right) \\ &\quad - \frac{1}{N\left(N^2-1\right)}\left(\delta_{in}\delta_{kp}\delta_{jo}\delta_{lm} + \delta_{ip}\delta_{kn}\delta_{jm}\delta_{lo}\right)\end{aligned}$$

The integral **vanishes** for any polynomial where the number of U's does not equal the number of $U\ U^\dagger$

Above properties can be rigorously formulated (the coefficients of delta functions are Weingarten functions). But we will try to understand **intuitively** here. Observe the 2nd integral, the matrix elements of U are random complex numbers. Since the arguments are also random,

$$\int U_{ij} U^\dagger_{kl} = \int |U_{ij}||U_{kl}| e^{i(\theta_{ij} - \theta_{lk})}$$

it is non zero only if $\theta_{ij} = \theta_{lk}$. In that case it becomes $\int |U_{ij}|^2$. By symmetry, all elements of U should have the same average. Since $\sum_i |U_{ij}|^2 = 1$ for all j, they should all have $1/N$ average.

3.2.2 Matrix norms

The trace norm distance between two density matrices ρ and σ is then defined as $\|\rho - \sigma\|_1$.

$$\|M\|_1 \equiv \operatorname{tr} \sqrt{M^\dagger M}$$

We will also be interested in the L_2 norm, defined as

$$\|M\|_2 \equiv \sqrt{\operatorname{tr} M^\dagger M}$$

We can show that these obey

$$\|M\|_2 \leq \|M\|_1 \leq \sqrt{N}\|M\|_2$$

for any operator M, where N is the dimensionality of the Hilbert space. Proof can be found at https://math.stackexchange.com/a/2293792/789844. Equivalently we can define them as

$$\|A\|_p = \max_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

where the p-norm for vectors is defined as $\|\mathbf{x}\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$.

One version of Page's theorem is:

THEOREM: For any bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, we have

$$\int dU \left\| \rho_A(U) - \frac{I_A}{|A|} \right\|_1 \leq \sqrt{\frac{|A|^2 - 1}{|A||B| + 1}}$$

For intuition we can slightly weaken the bound by just writing $\sqrt{\frac{|A|}{|B|}}$ on the right hand side. Page's theorem then says that once $|B|$ is significantly larger than $|A|$, **the typical deviation** of ρ_A from the maximally mixed state **is extremely small**.

Proof:

$$\begin{aligned}
\left(\int dU \left\|\rho_A(U) - \frac{I_A}{|A|}\right\|_1\right)^2 &\le \int dU \left(\left\|\rho_A(U) - \frac{I_A}{|A|}\right\|_1\right)^2 \\
&\le |A| \int dU \left(\left\|\rho_A(U) - \frac{I_A}{|A|}\right\|_2\right)^2 \\
&\le |A| \int dU \,\mathrm{tr}\left(\left(\rho_A(U) - \frac{I_A}{|A|}\right)^\dagger \left(\rho_A(U) - \frac{I_A}{|A|}\right)\right) \\
&\le |A| \int dU \,\mathrm{tr}\left(\left(\rho_A(U) - \frac{I_A}{|A|}\right) \left(\rho_A(U) - \frac{I_A}{|A|}\right)\right) \\
&\le |A| \int dU \,\mathrm{tr}\left(\rho_A(U)^2 + \frac{I_A}{|A|^2} - 2\frac{\rho_A(U)}{|A|}\right) \\
&\le |A| \int dU \left(\mathrm{tr}\left(\rho_A(U)^2\right) + \frac{|A|}{|A|^2} - 2\frac{1}{|A|}\right) \\
&\le |A| \int dU \left(\mathrm{tr}\left(\rho_A(U)^2\right) - \frac{1}{|A|}\right) \\
&\le \left(|A| \int dU \,\mathrm{tr}\left(\rho_A(U)^2\right)\right) - 1
\end{aligned}$$

the first inequality following from Jensen's inequality (i.e if $\int_X \mathrm{d}x = 1$ $\varphi\left(\int_X f(x)\,\mathrm{d}x\right) \le \int_X \varphi(f(x))\,\mathrm{d}x$) and the second following from the norm inequalities.

To calculate $\int dU \,\mathrm{tr}\left(\rho_A(U)^2\right)$ in the last step, consider each index to U_{ij} to be a double index: $i = (i_A, i_B)$, and $j = (j_A, j_B)$. The Haar-random state by

$$|\psi(U)\rangle \equiv U\,|\psi_0\rangle$$

(from now let us take that $|\psi_0\rangle$ is a vector in the basis with the only non-zero component being indice $(0,0)$ although it is independent of $|\psi_0\rangle$), that is, $|\psi\rangle = \sum U_{(i_A,i_B),(0,0)}|i_A, i_B\rangle$. Then, using $\rho_A(U) = \mathrm{Tr}_B\rho(U)$

$$\mathrm{tr}(\rho_A(U)^2) = \sum U_{(i_A,i_B),(0,0)} U^\dagger_{(0,0),(j_A,i_B)} U_{(j_A,k_B),(0,0)} U^\dagger_{(0,0),(i_A,k_B)}$$

$$\Rightarrow \mathrm{tr}(\rho_A(U)^2) = \sum U_{(i_A,i_B),(0,0)} U_{(j_A,k_B),(0,0)} U^\dagger_{(0,0),(j_A,i_B)} U^\dagger_{(0,0),(i_A,k_B)}$$

Now we can use Page's theorem to justify the proposed form of the Page Curve. We assume that black hole evaporation is such a complex process that the pure state of R and BH together is random, up to the basic constraints imposed by energy conservation and causality. So, pure final radiation states look like exact thermal states in the coarse grained sense.

Now $\int dU \mathrm{tr}(\rho_A(U)^2)$ will be

$$
\begin{aligned}
=&\sum \frac{1}{(|A||B|)^2-1}\left(\delta_{(i_A,i_B),(j_A,i_B)}\delta_{(j_A,k_B),(i_A,k_B)}+\delta_{(i_A,i_B),(i_A,k_B)}\delta_{(j_A,k_B),(j_A,i_B)}\right)\\
&-\frac{1}{(|A||B|)\left((|A||B|)^2-1\right)}\left(\delta_{(i_A,i_B),(j_A,i_B)}\delta_{(j_A,k_B),(i_A,k_B)}+\delta_{(i_A,i_B),(i_A,k_B)}\delta_{(j_A,k_B),(j_A,i_B)}\right)
\end{aligned}
$$

$$
\begin{aligned}
\int dU \mathrm{tr}(\rho_A(U)^2) =& \frac{1}{(|A||B|)^2-1}\left(|A||B|^2+|A|^2|B|\right)\\
&-\frac{1}{(|A||B|)\left((|A||B|)^2-1\right)}\left(|A||B|^2+|A|^2|B|\right)\\
=&\frac{|A||B|\left(|A|+|B|\right)}{(|A||B|)^2-1}\left(1-\frac{1}{(|A||B|)}\right)\\
=&\frac{|A|+|B|}{|A||B|+1}
\end{aligned}
$$

$$
\left(\int dU\left\|\rho_A(U)-\frac{I_A}{|A|}\right\|_1\right)^2\leq\left(|A|\int dU\,\mathrm{tr}\left(\rho_A(U)^2\right)\right)-1
$$

In the beginning the radiation is almost a maximally mixed state. In the end the black hole is almost a maximally mixed state.

3.2.3 Average Entropy of a Subsystem

Let $\Delta\rho_A \equiv \rho_A - \frac{I_A}{|A|}$ then,

$$
\begin{aligned}
S_A &= \mathrm{Tr}\left[\rho_A \log \rho_A\right]\\
&= \mathrm{Tr}\left[\left(\frac{I_A}{|A|}+\Delta\rho_A\right)\log\left(\frac{I_A+|A|\Delta\rho_A}{|A|}\right)\right]\\
&= \mathrm{Tr}\left[\left(\frac{I_A}{|A|}+\Delta\rho_A\right)\left(-\log|A|+|A|\Delta\rho_A-\frac{1}{2}|A|^2\Delta\rho_A^2+\dots\right)\right]
\end{aligned}
$$

the average entropy of a subsystem is defined as $\int dU S_A$

$$
\begin{aligned}
\int dU S_A &= -\int dU\,\mathrm{Tr}\,\rho_A\log\rho_A\\
&=\int dU\,\mathrm{Tr}\left[\left(\frac{I_A}{|A|}+\Delta\rho_A\right)\left(\log|A|-|A|\Delta\rho_A+\frac{1}{2}|A|^2\Delta\rho_A^2+\dots\right)\right]\\
&=\log|A|-\frac{|A|}{2}\int dU\,\mathrm{Tr}\,\Delta\rho_A^2+\dots\\
&=\log|A|-\frac{1}{2}\frac{|A|}{|B|}+\dots,
\end{aligned}
$$

3.3 Black hole complementarity

Susskind and Thorlacius proposed that the information (when an object falls into the event horizon) is both reflected at the event horizon and passes through the event horizon and cannot escape. Susskind and Thorlacius pointed out (and which was later refined by Hayden and Preskill for more stringent cases) that no single observer can see both copies. They interpreted this as being consistent with the no-cloning theorem.

4 Prelude to AdS/CFT correspondence applications

Consider the thought experiment experiment of an object of linear size L, energy E and entropy S which is at a proper distance of order L of the horizon of a black hole and now fell into the black hole. From 2nd law of thermodynamics we can say that

$$S < \Delta S_{bh}$$

$$\Delta S_{bh} = \Delta\left(4\pi G^2 M^2\right) \approx 8\pi G^2 M \Delta M = 8\pi G^2 (\frac{r_s}{2G})(\frac{LE}{r_s}) \propto LE$$

since $\Delta M = \frac{LE}{r_s}$. Let $\Delta S_{bh} = CLE$ for some C.

$$\Rightarrow S < CLE$$

Notice that G is not there in the final argument, so this phenomenon can be understood without invoking gravity. This argument is somewhat vague, but it has been recently made precise. Casini [arXiv:0804.2182] proved that a precise version of the bound holds in any relativistic quantum field theory.

In another thought experiment Susskind considered a stationary object (which is not a black hole) of entropy S and energy E, which is contained in a sphere of area A. Let M be the mass of black hole whose ares is A. Then we can collapse a spherical shell of matter onto this object whose energy is $M - E$ to form a black hole. From 2nd law of thermodynamics we can say that

$$S \leq \frac{A}{4G}$$

this is the **holographic entropy bound**. The holographic entropy bound is saying that the number of degrees of freedom in spacetime is much less than we expected.

These and some other thought experiments led 't Hooft and Susskind to a conjecture called "holographic principle": that a true theory of quantum gravity must be dual to a theory with one fewer dimensions than naively expected.

Later Juan Maldacena conjectured the AdS/CFT correspondence or duality, which is the best known example for the holographic principle. Gravity in AdS spacetime (i.e. spacetime with constant negative scalar curvature) is conjectured to be dual to a conformal field theory (i.e. a quantum field theory that is invariant under conformal transformations: transformations that preserve angles locally) in 1 fewer dimensions. It is understood how to calculate some quantities from the corresponding quantities from the dual theory. For example the metric $g_{\mu\nu}$ in AdS corresponds to the energy momentum tensor $T_{\mu\nu}$ of the CFT. If the CFT has global symmetries then by Noether's theorem it must have conserved currents of dimension $d-1$, and these are dual to gauge fields in the bulk.

These are only some cases which are verified but this doesn't prove that everything we want to know from AdS can be understood from CFT. But **to prove that this conjecture is true** we need to first understand nonperturbative formulation of string theory. But it is not known in general how to define string theory nonperturbatively. If this conjecture is true then even if string theory is experimentally falsified it will be still be very useful because strongly coupled quantum field theories are dual to weakly coupled classical gravity and this provides a new approach to calculating observables in these strongly coupled quantum field theories.

Apart from its importance in quantum gravity, AdS/CFT correspondence has also been very important to resolve the black hole information paradox. It also increased the confidence that black hole evaporation is unitary since quantum gravity in AdS is unitary as it is dual to a unitary CFT theory.

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