

MA 109 Tutorial 2

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Q)13. (ii)

$f(x) = x \sin \frac{1}{x}$; if $x \neq 0$ and $f(0) = 0 \forall x \neq 0, x, \sin \frac{1}{x}$ are continuous at x .

So their product is also continuous. At $x = 0$ x is continuous, but $\sin \frac{1}{x}$ is

not continuous. We also know that $|f(x)| \leq |x|$ since $|\sin(\frac{1}{x})| \leq 1$.

Let $L = 0$ we have to show that $\forall \epsilon > 0 \exists \delta > 0$ such that

$$|x - 0| < \delta \Rightarrow |f(x) - 0| < \epsilon$$

We can observe that $\delta = \epsilon$ works $\forall \epsilon$ since

$$|x - 0| < \epsilon \Rightarrow |f(x) - L| = |x \sin(\frac{1}{x})| \leq |x| < \epsilon$$

$$\lim_{x \rightarrow 0} f(x) = f(0) = 0$$

So it is continuous at $x = 0$. So it is $\forall x \in \mathbb{R}$. You can also do this question by using Sandwich theorem since $-|x| \leq x \sin(\frac{1}{x}) \leq |x|$



Q)15

$$f(x) = x^2 \sin \frac{1}{x}; \text{ if } x \neq 0 \text{ and } f(0) = 0$$

$$\begin{aligned} f'(0) &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} \\ &= \lim_{h \rightarrow 0} h \sin\left(\frac{1}{h}\right) \\ &= 0 \end{aligned}$$

$$\forall x \neq 0 \quad f'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right); f'(0) = 0. \text{ Let } x_n = \frac{1}{2n\pi},$$

$$y_n = \frac{1}{(2n+0.5)\pi}. \lim_{n \rightarrow \infty} f(x_n) \neq \lim_{n \rightarrow \infty} f(y_n). \text{ So } f'(x) \text{ is discontinuous at } x = 0.$$



Q)18

$y = 0 \Rightarrow f(x) = f(x)f(0) \forall x$. Assume that $f(x)$ is not identically equal to zero. Then $\exists x$ such that $f(x) \neq 0 \Rightarrow f(0) = 1$.

$$\frac{f(x+h) - f(x)}{h} = \frac{f(x)f(h) - f(x)f(0)}{h}$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \left(\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} \right) f(x)$$

So if $f'(0)$ exists, $f'(x)$ also exists $\forall x \in \mathbb{R}$.

$$\Rightarrow f'(x) = f'(0)f(x)$$

Even if $f(x) = 0 \forall x$, $f'(x) = f'(0)f(x)$ is still true.



Q)7

(i) \Rightarrow (ii)

Let $\delta > 0$ be such that $(c - \delta, c + \delta) \subseteq (a, b)$. And let $\alpha = f'(c)$, then

$$\epsilon_1(h) = \frac{f(c+h) - f(c) - \alpha h}{h} \text{ if } h \neq 0$$

and $\epsilon_1(0) = 0$. Check that $\lim_{h \rightarrow 0} \epsilon_1(h) = f'(c) - \alpha = 0$. So this function satisfies all the properties we need.

(ii) \Rightarrow (iii)

$$\begin{aligned} \lim_{h \rightarrow 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} &= \lim_{h \rightarrow 0} |\epsilon_1(h)| \\ &= 0 \end{aligned}$$

Here I used the fact $\lim_{x \rightarrow c} f(x) = 0 \Leftrightarrow \lim_{x \rightarrow c} |f(x)| = 0$ (it is a consequence of $||f(x)| - L| = |f(x) - L|$ for $L = 0$)



(iii) \Rightarrow (i)

$$\begin{aligned}
 \lim_{h \rightarrow 0} \frac{|f(c+h) - f(c) - \alpha h|}{|h|} &= 0 \\
 \Rightarrow \lim_{h \rightarrow 0} \frac{f(c+h) - f(c) - \alpha h}{h} &= 0 \\
 \Rightarrow \lim_{h \rightarrow 0} \frac{f(c+h) - f(c) -}{h} &= \alpha \\
 \Rightarrow f'(c) &= \alpha
 \end{aligned}$$

So $f(x)$ is differentiable at $x = c$ So all (i), (ii) and (iii) are equivalent.



Q)10

Let $g : [0, 1] \rightarrow \mathbb{R}$ be defined as $g(x) = f(x) - x$. A point is fixed if $f(x) = x$ at the point. If $f(0) = 0$ or $f(1) = 1$, clearly the function has a fixed point. If $f(0) \neq 0$ and $f(1) \neq 1$ then $g(0) = f(0) > 0$ and $g(1) = f(1) - 1 < 0$. So by IVT (Intermediate Value Theorem) we can say that $\exists c \in (0, 1)$ such that $g(c) = 0 \Rightarrow f(c) = c$.



Q)3

By IVT(Intermediate Value Theorem) we can say that atleast one such an x_0 exists such that $f(x_0) = 0$. Let $x_1 \neq x_0$ be such that $f(x_1) = 0$, then by Rolle's theorem we can say that $\exists c \in (x_0, x_1)$ such that $f'(c) = 0 \Rightarrow$ contradiction(since $f'(x) \neq 0 \forall x \in (a, b)$). So there is a unique $x \in (a, b)$ such that $f(x) = 0$.



Q)5

If $a = b$ then $|\sin(a) - \sin(b)| \leq |a - b|$ holds true. If $a \neq b$, assume without loss of generality that $a < b$ (as the case with $a > b$ will be similar). From MVT (Mean Value Theorem) we can say that $\exists c \in (a, b)$ such that

$$\begin{aligned} f'(c) &= \frac{f(b) - f(a)}{b - a} \\ \Rightarrow \cos(c) &= \frac{\sin(b) - \sin(a)}{b - a} \\ \Rightarrow \left| \frac{\sin(a) - \sin(b)}{a - b} \right| &= |\cos(c)| \leq 1 \\ \Rightarrow |\sin(a) - \sin(b)| &\leq |a - b| \end{aligned}$$

