# MA 109 Tutorial 3

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09-Dec-2020

# Q)8

(ii) Let $f'(x) = x + 1$. It satisfies all the given conditions. So, $f(x) = \frac{x^2}{2} + x + c$ satisfies.
(iii) Apply MVT(Mean Value Theorem) for $f'(x)$ on $[0, x]$ for some $x > 0$

$$\frac{f'(x) - f'(0)}{x - 0} = f''(c_1) \geq 0$$
$$\Rightarrow f'(x) \geq 1$$

Similarly doing for $f(x)$

$$\frac{f(x) - f(0)}{x - 0} = f'(c_2) \geq 1$$
$$\Rightarrow f(x) \geq f(0) + x$$
$$\Rightarrow f(100 - f(0)) \geq 100$$
$$\Rightarrow \forall x > 100 - f(0), f(x) > 100$$

So no such function exists.

# Q)8

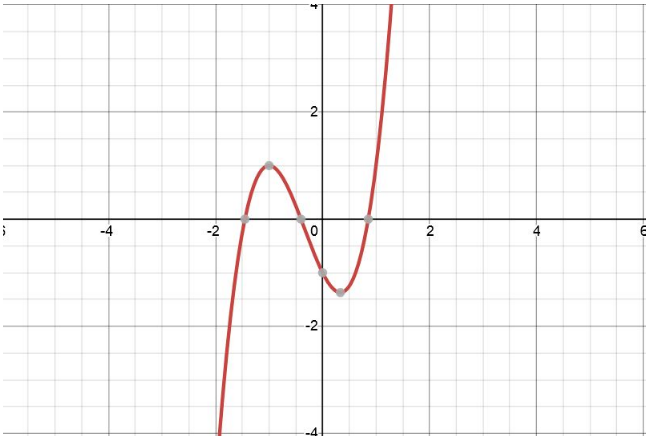
(iv) $f(x) = e^x$ satisfies all the conditions needed. You can find other solutions also. $g(x) = \dfrac{x + e^x}{2}$ also satisfies all the conditions needed. Here $g'(x) = \frac{1+e^x}{2}$, $g''(x) = \frac{e^x}{2}$

# Q)10(i)

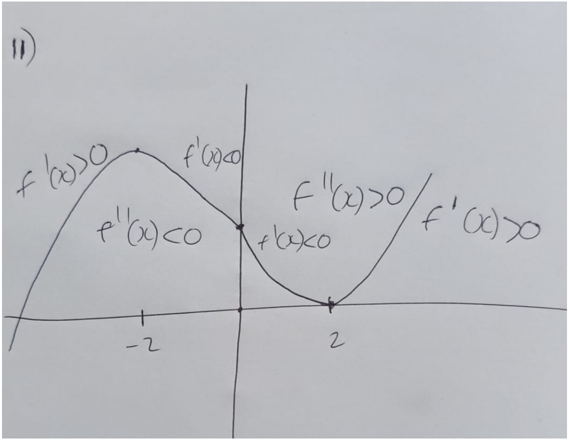
$$f(x) = 2x^3 + 2x^2 - 2x - 1$$
$$f'(x) = 6x^2 + 4x - 2x = 2(x+1)(3x-1)$$
$$f''(x) = 12x + 4$$

Observe that $f'(x) > 0$ in $(-\infty, -1) \cup (\frac{1}{3}, \infty)$, $f'(x) < 0$ in $(-1, \frac{1}{3})$ and $f'(-1) = f'(\frac{1}{3}) = 0$. So $f(x)$ has a local maximum at $x = -1$; and a local minimum at $x = \frac{1}{3}$. Also observe that $f''(x) = 12x + 4 > 0$ in $(-\frac{1}{3}, \infty)$, ¡0 in $(-\infty, -\frac{1}{3}$ and $f''(-\frac{1}{3}) = 0$ so $x = -\frac{1}{3}$ is a point of inflection. This function also doesn't have any asymptotes.

# Q)10(i)



# Q)11




Also $f(x) = \frac{3}{4}(\frac{x^3}{3} - 4x) + 4$ is a possible solution. Draw it.

# Q)1(ii)

Here $f(x) = arctan(x)$. From Taylor's theorem we can write that

$$f(x) = \sum_{r=0}^{n} \frac{f^{(r)(x_0)}}{r!}(x - x_0)^r + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x - x_0)^{n+1}$$

where $c_x \in (x_0, x) or \in (x, x_0)$. Here I wrote $c_x$ because it depends on $x$. $f^{(1)}(x) = \frac{1}{1+x^2}$, from there you can go on differentiating and get all derivatives. But to express them as $x$ and $n$ is not easy. But using the complex number $i$ we can simply write is as

$$f^n(x) = \frac{1}{2}(i(-1)^n(n-1)!)((x-i)^{-n} - (x+i)^{-n})$$

This form can be obtained by using $\frac{1}{1+x^2} = \frac{1}{2i}(\frac{1}{x-i} - \frac{1}{x+i})$ and differentiating. Although since this course considers only $\mathbb{R}$ the induction method is better.

We can also do this using integration also like $f' = \frac{1}{1+x^2}$ express it as a geometric series and integrate term by term. But why we can integrate term by term is beyond the syllabus of this course. It will be taught in a MA 403 topic called sequences and series of functions. If you are interested read about Uniform convergence. **But for this course you can integrate the series without justification.** Here $x_0 = 0$, so

$$R_n(x) = \frac{f^{(n+1)}(c_x)(x-0)^{n+1}}{(n+1)!}$$

# Q)2

You can write the function as $f(x) = (x-1)^3$. Clearly it is the Taylor series. You can also find derivatives and do. For any polynomial the Taylor series is exactly the polynomial itself.

# Q)4

Let us denote the partial sums of of the given series by $s_m(x)$. We would like to show that $|s_m(x) - s_n(x)|$ can be made arbitrarily small whenever m and n are sufficiently large. Without loss of generality assume that $m > n$. We see that

$$|s_m(x) - s_n(x)| = |\sum_{k=n+1}^{m} \frac{x^k}{k!}| \leq |\frac{x^n}{n!}| \left(\frac{1}{2} + \frac{1}{4} \cdots + \frac{1}{2^{m-n}}\right) \leq 2\frac{|x^2|}{n!}$$

.

If $N$ is made sufficiently large and $n > N$, the last expression can be made as small as we please.

# Q)5

The taylor series for $e^x = \sum_{r=0}^{\infty} \frac{x^n}{n!}$. So,

$$\frac{e^x}{x} = \sum_{r=0}^{\infty} \frac{x^{n-1}}{n!}$$

$$\int_a^b \frac{e^x}{x} dx = \int_a^b \left( \sum_{r=0}^{\infty} \frac{x^{n-1}}{n!} dx \right)$$

$$\int_a^b \frac{e^x}{x} dx = \sum_{r=0}^{\infty} \left( \int_a^b \frac{x^{n-1}}{n!} dx \right)$$

$$\int \frac{e^x}{x} dx = \sum_{r=0}^{\infty} \left( \int \frac{x^{n-1}}{n!} dx \right)$$

$$\int \frac{e^x}{x} dx = \sum_{r=0}^{\infty} \left( \int \frac{x^n}{(n)n!} dx \right) + c$$

Here the interchange of $\sum$ and $\int$ are assumed.