MA 109 Tutorial 3

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Q)8

(ii) Let $f'(x) = x + 1$. It satisfies all the given conditions. So, $f(x) = \frac{x^2}{2} + x + c$ satisfies.
(iii) Apply MVT(Mean Value Theorem) for $f'(x)$ on $[0, x]$ for some $x > 0$

$$\frac{f'(x) - f'(0)}{x - 0} = f''(c_1) \geq 0$$
$$\Rightarrow f'(x) \geq 1$$

Similarly doing for $f(x)$

$$\frac{f(x) - f(0)}{x - 0} = f'(c_2) \geq 1$$
$$\Rightarrow f(x) \geq f(0) + x$$
$$\Rightarrow f(100 - f(0)) \geq 100$$
$$\Rightarrow \forall x > 100 - f(0), f(x) > 100$$

So no such function exists.

Q)8

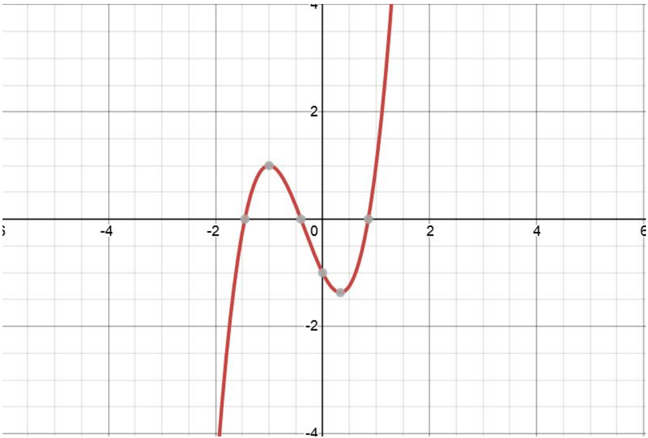
(iv) $f(x) = e^x$ satisfies all the conditions needed. You can find other solutions also. $g(x) = \dfrac{x + e^x}{2}$ also satisfies all the conditions needed. Here $g'(x) = \frac{1+e^x}{2}$, $g''(x) = \frac{e^x}{2}$

Q)10(i)

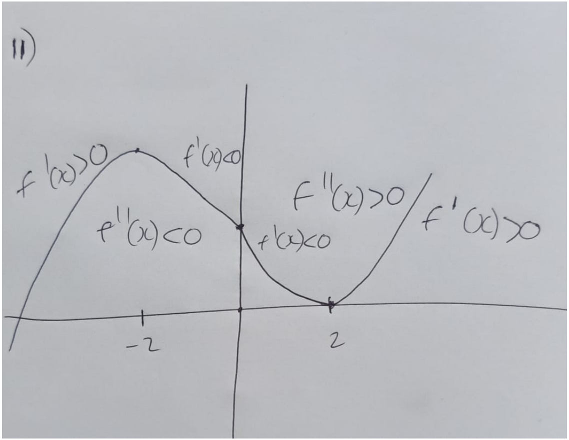
$$f(x) = 2x^3 + 2x^2 - 2x - 1$$
$$f'(x) = 6x^2 + 4x - 2x = 2(x+1)(3x-1)$$
$$f''(x) = 12x + 4$$

Observe that $f'(x) > 0$ in $(-\infty, -1) \cup (\frac{1}{3}, \infty)$, $f'(x) < 0$ in $(-1, \frac{1}{3})$ and $f'(-1) = f'(\frac{1}{3}) = 0$. So $f(x)$ has a local maximum at $x = -1$; and a local minimum at $x = \frac{1}{3}$. Also observe that $f''(x) = 12x + 4 > 0$ in $(-\frac{1}{3}, \infty)$, ¡0 in $(-\infty, -\frac{1}{3}$ and $f''(-\frac{1}{3}) = 0$ so $x = -\frac{1}{3}$ is a point of inflection. This function also doesn't have any asymptotes.

Q)10(i)



Q)11



Also $f(x) = \frac{3}{4}(\frac{x^3}{3} - 4x) + 4$ is a possible solution. Draw it.

Q)1(ii)

Here $f(x) = arctan(x)$. From Taylor's theorem we can write that

$$f(x) = \sum_{r=0}^{n} \frac{f^{(r)(x_0)}}{r!}(x - x_0)^r + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x - x_0)^{n+1}$$

where $c_x \in (x_0, x) or \in (x, x_0)$. Here I wrote c_x because it depends on x. $f^{(1)}(x) = \frac{1}{1+x^2}$, from there you can go on differentiating and get all derivatives. But to express them as x and n is not easy. But using the complex number i we can simply write is as

$$f^n(x) = \frac{1}{2}(i(-1)^n(n-1)!)((x-i)^{-n} - (x+i)^{-n})$$

This form can be obtained by using $\frac{1}{1+x^2} = \frac{1}{2i}(\frac{1}{x-i} - \frac{1}{x+i})$ and differentiating. Although since this course considers only $\mathbb{R}$ the induction method is better.

We can also do this using integration also like $f' = \frac{1}{1+x^2}$ express it as a geometric series and integrate term by term.We get the Taylore series as

$$f(x) = \int (1 - x^2 + x^4 + \cdots)dx$$
$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} \cdots$$

But why we can integrate term by term is beyond the syllabus of this course. It will be taught in a MA 403 topic called sequences and series of functions. If you are interested read about Uniform convergence. **But for this course you can integrate the series without justification.** Here $x_0 = 0$, so

$$R_n(x) = \frac{f^{(n+1)}(c_x)(x-0)^{n+1}}{(n+1)!}$$

Q)2

You can write the function as $f(x) = (x-1)^3$. Clearly it is the Taylor series. You can also find derivatives(only the 3rd derivative is non zero) and do. In general for any polynomial the Taylor series is exactly the polynomial itself.

Q)4

Let us denote the partial sums of of the given series by $s_m(x) = \sum_{k=0}^{m} \frac{x^k}{k!}$. We would like to show that $|s_m(x) - s_n(x)|$ can be made arbitrarily small whenever m and n are sufficiently large. Without loss of generality assume that $m > n$. By applying $\frac{x^{n+1}}{(n+1)!} < \frac{x^n}{2(n)!}$ repeatedly we get $\frac{x^m}{(m)!} < \frac{x^n}{2^{m-n}(n)!}$. Now

$$|s_m(x) - s_n(x)| = \left| \sum_{k=n+1}^{m} \frac{x^k}{k!} \right| < \left| \frac{x^n}{n!} \right| \left(\frac{1}{2} + \frac{1}{4} \cdots + \frac{1}{2^{m-n}} \right) \leq \frac{|x^n|}{n!}$$

.

If N is made sufficiently large and since $n > N$, the last expression can be made as small as we please. That is we can find $N \in \mathbb{N}$ such that $\frac{|x^n|}{n!} < \frac{|x^N|}{N!} < \epsilon$

Q)5

The taylor series for $e^x = \sum_{r=0}^{\infty} \frac{x^n}{n!}$. So,

$$\frac{e^x}{x} = \sum_{r=0}^{\infty} \frac{x^{n-1}}{n!}$$

$$\int_a^b \frac{e^x}{x} dx = \int_a^b \left(\sum_{r=0}^{\infty} \frac{x^{n-1}}{n!} dx \right)$$

$$\int_a^b \frac{e^x}{x} dx = \sum_{r=0}^{\infty} \left(\int_a^b \frac{x^{n-1}}{n!} dx \right)$$

$$\int \frac{e^x}{x} dx = \sum_{r=0}^{\infty} \left(\int \frac{x^{n-1}}{n!} dx \right)$$

$$\int \frac{e^x}{x} dx = ln(x) + \sum_{r=1}^{\infty} \left(\frac{x^n}{(n)n!} dx \right) + c$$

Here the interchange of $\sum$ and $\int$ are assumed.