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MA 109: Calculus I

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Tutorial sheet 1: Sequences, limits, continuity, differentiability

Sequences

1. Using the (ϵ-N) definition of a limit, prove the following:

   (i) $\lim\limits_{n\to\infty} \dfrac{10}{n} = 0$

   (ii) $\lim\limits_{n\to\infty} \dfrac{5}{3n+1} = 0$

   (iii) $\lim\limits_{n\to\infty} \dfrac{n^{2/3}\sin(n!)}{n+1} = 0$

   (iv) $\lim\limits_{n\to\infty} \left(\dfrac{n}{n+1} - \dfrac{n+1}{n}\right) = 0$

2. Show that the following limits exist and find them:

   (i) $\lim\limits_{n\to\infty} \left(\dfrac{n}{n^2+1} + \dfrac{n}{n^2+2} + \cdots + \dfrac{n}{n^2+n}\right)$

   (ii) $\lim\limits_{n\to\infty} \left(\dfrac{n!}{n^n}\right)$

   (iii) $\lim\limits_{n\to\infty} \left(\dfrac{n^3+3n^2+1}{n^4+8n^2+2}\right)$

   (iv) $\lim\limits_{n\to\infty} (n)^{1/n}$

   (v) $\lim\limits_{n\to\infty} \left(\dfrac{\cos \pi\sqrt{n}}{n^2}\right)$

   (vi) $\lim\limits_{n\to\infty} \left(\sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)\right)$

3. Show that the following sequences are not convergent:

   (i) $\left\{\dfrac{n^2}{n+1}\right\}_{n\geq 1}$ (ii) $\left\{(-1)^n\left(\dfrac{1}{2} - \dfrac{1}{n}\right)\right\}_{n\geq 1}$

4. Determine whether the sequences are increasing or decreasing:

   (i) $\left\{\dfrac{n}{n^2+1}\right\}_{n\geq 1}$

   (ii) $\left\{\dfrac{2^n 3^n}{5^{n+1}}\right\}_{n\geq 1}$

   (iii) $\left\{\dfrac{1-n}{n^2}\right\}_{n\geq 2}$

5. Prove that the following sequences are convergent by showing that they are monotone and bounded. Also find their limits:

   (i) $a_1 = \dfrac{3}{2}, a_{n+1} = \dfrac{1}{2}\left(a_n + \dfrac{2}{a_n}\right) \quad \forall\, n \geq 1$

   (ii) $a_1 = \sqrt{2}, a_{n+1} = \sqrt{2+a_n}\ \forall\ \ n \geq 1$

   (iii) $a_1 = 2,\ a_{n+1} = 3 + \dfrac{a_n}{2}\ \forall\, n \geq 1$

2. Show that the following limits exist and find them:

$$\text{(iv)} \quad \lim_{n\to\infty} (n)^{1/n}$$

$a_1 = 1$

$a_2 = 1.414 = 1 + \underbrace{.414}_{\alpha_2}$

$a_n = n^{1/n} \geq 1 \quad (\forall n)$

$(n \geq 1^n = 1)$

$a_n = n^{1/n} = 1 + \alpha_n$

$n = (1+\alpha_n)^n \underset{\text{binomial}}{=} 1 + n\alpha_n + \frac{n(n-1)}{2}\alpha_n^2 + \dots$

$\geq \frac{n(n-1)}{2}\alpha_n^2$

So $\cancel{n} \geq \frac{\cancel{n}(n-1)}{2}\alpha_n^2$

Target: $a_n = 1 + \alpha_n$, prove $\alpha_n \to 0$

$\Rightarrow \alpha_n^2 \leq \frac{2}{n-1}$

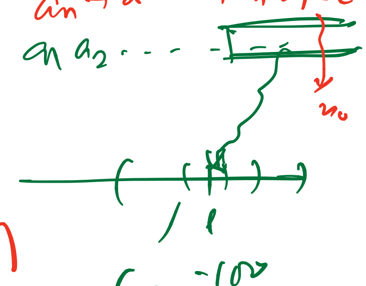
$0 \leq \alpha_n \leq \frac{\sqrt{2}}{\sqrt{n-1}}$, $0 \to 0$, $\frac{\sqrt{2}}{\sqrt{n-1}} \to 0$

$\lim a_n = 1 + \lim \alpha_n = 1$

$|\alpha_n| \leq \frac{\sqrt{2}}{\sqrt{n-1}}$

$a_n \to a$ $\quad |a_n - a| < \epsilon$

$a_n \downarrow 1$



$\epsilon > 0$

$|\alpha_n| \leq \frac{\sqrt{2}}{\sqrt{n-1}} < \epsilon$

$\Downarrow$

$|\alpha_n - 0| = |\alpha_n|$

$\epsilon = 10^{-100}$

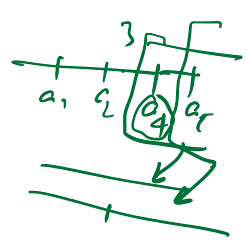
$\frac{2}{n-1} < \epsilon^2$

$\Rightarrow n - 1 > \frac{2}{\epsilon^2}$

$\Rightarrow n > 1 + 2/\epsilon^2$

$n_0 = 1 + \left[\frac{2}{\epsilon^2}\right] + 1 \in \mathbb{N}$

$n \geq n_0$



3. Show that the following sequences are not convergent:

(ii) $\left\{(-1)^n\left(\frac{1}{2}-\frac{1}{n}\right)\right\}_{n\geq 1}$

Assume the seq is convergent. So the seq will be Cauchy. We will prove by contradiction.

$\mathbb{R}$ ✓ Convergent $\Leftrightarrow$ Cauchy

$|a_n - a_m| < \epsilon \leftarrow$ Arb

$\forall n,m \geq n_0$

For proving false, it is enough that it does not hold for some ϵ.

$\epsilon = 1/4 \rightsquigarrow n_0 \in \mathbb{N}$

$n = 4n_0$, $m = 4n_0+1$.

$a_{4n_0} = \frac{1}{2} - \frac{1}{4n_0}$, $a_{4n_0+1} = -\frac{1}{2} + \frac{1}{4n_0+1}$

$|a_{4n_0} - a_{4n_0+1}| = \left|1 - \frac{1}{4n_0} - \frac{1}{4n_0+1}\right|$

$\frac{1}{4n_0} \leq \frac{1}{4}$, $\frac{1}{4n_0+1} \leq \frac{1}{4}$, $\frac{1}{4n_0} + \frac{1}{4n_0+1} \leq \frac{1}{4} + \frac{2}{4} = 1/2$

$\left|1 - \frac{1}{4n_0} - \frac{1}{4n_0+1}\right| \geq 1 - 1/2 = 1/2 > 1/4$

Contradicts to Cauchyness.

5. Prove that the following sequences are convergent by showing that they are monotone and bounded. Also find their limits:

$$\text{(iii)}\quad a_1 = 2,\ a_{n+1} = 3 + \frac{a_n}{2}\ \forall\, n \geq 1$$

$a_{n+1} \geq a_n$

$\leq$

$a_1 = 2, \quad a_2 = 3 + 2/2 = 4$

$a_{n+1} = 3 + \frac{a_n}{2} \quad \forall n \geq 1.$

$a_{n+1} - a_n = \cancel{3} + \frac{a_n}{2} - \left(\cancel{3} + \frac{a_{n-1}}{2}\right) = \frac{a_n - a_{n-1}}{2}$ $n = n-1$

$= \frac{1}{2}\left[\frac{a_{n-1} - a_{n-2}}{2}\right]$

$a_{n+1} - a_n = \frac{a_n - a_{n-1}}{2}$

$= \frac{a_{n-1} - a_{n-2}}{4}$

$a_n - a_{n-1} = \frac{a_{n-1} - a_{n-2}}{2}$

$\vdots$

$= \frac{a_2 - a_1}{2^{n-1}} = \frac{2}{2^{n-1}} > 0 \quad \forall n \geq 1$

$a_{n+1} - a_n \geq 0$

$\Rightarrow \quad a_{n+1} \geq a_n$

$\Rightarrow \quad \boxed{\text{Monotone increasing.}}$

$a_{n+1} - a_1 = (a_{n+1} - a_n) + (a_n - a_{n-1}) + (a_{n-1} - a_{n-2}) + \cdots + (a_2 - a_1)$

$= \frac{2}{2^{n-1}} + \frac{2}{2^{n-2}} + \cdots + \frac{2}{1} \quad \forall n \geq 1.$

$= 2 \cdot \frac{1 - 1/2^n}{1 - 1/2} = 4 \cdot (1 - 1/2^n)$

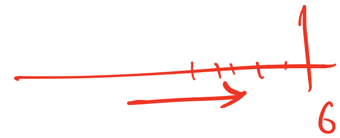
$\leq 4 \quad \left(\because 1 - \frac{1}{2^n} \leq 1\right)$

$a_{n+1} - a_1 \leq 4$

$a_{n+1} \leq 4 + a_1 \leq 4 + 2 = 6 \quad \forall n$

$\Rightarrow \quad a_{n+1} \leq 6 \quad \forall n.$

$\Rightarrow \quad \boxed{a_n \leq 6 \quad \forall n.}$



So given seq^n is convergent. So it has a limit, say ℓ.

$$a_{n+1} = 3 + \frac{a_n}{2}$$

$$\Rightarrow \lim_{n \to \infty} a_{n+1} = 3 + \lim_{n \to \infty} \frac{a_n}{2}$$

$$\Rightarrow \quad \ell = 3 + \frac{1}{2}\ell$$

$$\Rightarrow \quad \ell - \frac{\ell}{2} = 3$$

$$\Rightarrow \quad \ell/2 = 3 \Rightarrow \boxed{\ell = 6}$$

$$\boxed{\lim_{n \to \infty} a_n = 6}$$

6. If $\lim_{n\to\infty} a_n = L$, find the following: $\lim_{n\to\infty} a_{n+1}$, $\lim_{n\to\infty} |a_n|$.

7. If $\lim_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| \geq \frac{|L|}{2}, \ \ \forall\, n \geq n_0.$$

8. If $a_n \geq 0$ and $\lim_{n\to\infty} a_n = 0$, show that $\lim_{n\to\infty} a_n^{1/2} = 0$. State and prove a corresponding result if $a_n \to L > 0$.

9. For given sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, prove or disprove the following:

   (i) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent.

   (ii) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent and $\{b_n\}_{n\geq 1}$ is bounded.

10. Show that a sequence $\{a_n\}_{n\geq 1}$ is convergent iff both the subsequences $\{a_{2n}\}_{n\geq 1}$ and $\{a_{2n+1}\}_{n\geq 1}$ are convergent to the same limit.

Limits of functions of a real variable, continuity, differentiability

11. Let $f, g : (a, b) \to \mathbb{R}$ be functions and suppose that $\lim_{x\to c} f(x) = 0$ for $c \in [a, b]$. Prove or disprove the following statements.

   (i) $\lim_{x\to c}[f(x)g(x)] = 0$.

   (ii) $\lim_{x\to c}[f(x)g(x)] = 0$, if g is bounded.

   (iii) $\lim_{x\to c}[f(x)g(x)] = 0$, if $\lim_{x\to c} g(x)$ exists.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $\lim_{x\to\alpha} f(x)$ exists for some $\alpha \in \mathbb{R}$. Show that

$$\lim_{h\to 0}[f(\alpha + h) - f(\alpha - h)] = 0.$$

   Analyze the converse.

13. Discuss the continuity of the following functions:

   (i) $f(x) = \sin\dfrac{1}{x}$, if $x \neq 0$ and $f(0) = 0$

   (ii) $f(x) = x\sin\dfrac{1}{x}$, if $x \neq 0$ and $f(0) = 0$

   (iii) $f(x) = \begin{cases} \dfrac{x}{[x]} & \text{if } \ 1 \leq x < 2, \\ 1 & \text{if } \ x = 2, \\ \sqrt{6-x} & \text{if } \ 2 < x \leq 3. \end{cases}$

14. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, show that f is continuous at every $c \in \mathbb{R}$.

7. If $\lim_{n\to\infty} a_n = L \neq 0$, show that there exists $n_0 \in \mathbb{N}$ such that

$$|a_n| \geq \frac{|L|}{2}, \quad \forall n \geq n_0.$$

$|a_n| \to |L|$

$\epsilon = \frac{|L|}{2}$

given $\lim_{n\to\infty} a_n = L$, for $\epsilon > 0$

$|a_n - L| \leq \epsilon$ for some $n \geq n_0$

$$\big||a_n| - |L|\big| \leq |a_n - L| \leq \epsilon = |L|/2$$

$\forall n \geq n_0$

$$\Rightarrow \quad \big||a_n| - |L|\big| \leq |L|/2$$

$$\Rightarrow \quad -\frac{|L|}{2} \leq |a_n| - |L| \leq |L|/2$$

$$\Rightarrow \quad |L| - \frac{|L|}{2} \leq |a_n| \leq |L| + |L|/2$$

$$\Rightarrow \quad \underbrace{|L|/2 \leq |a_n|}_{\forall n \geq n_0} \leq \frac{3|L|}{2}$$

9. For given sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$, prove or disprove the following:

(i) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent.

(ii) $\{a_n b_n\}_{n\geq 1}$ is convergent, if $\{a_n\}_{n\geq 1}$ is convergent and $\{b_n\}_{n\geq 1}$ is bounded.

(i) $a_n = 1/n \to 0$ $\quad b_n = n^2$ $\quad a_n b_n = n$

not convergent

FALSE

(ii) $a_n = \frac{1}{2} - \frac{1}{n}$ $\quad \forall\, n \geq 1$

$b_n = (-1)^n$ $\quad \forall\, n \geq 1$

$a_n b_n = (-1)\left(\frac{1}{2} - \frac{1}{n}\right)$ $\qquad$ (3. ii)

we discuss

FALSE.

11. Let $f, g : (a, b) \to \mathbb{R}$ be functions and suppose that $\lim_{x\to c} f(x) = 0$ for $c \in [a, b]$. Prove or disprove the following statements.

(i) $\lim\limits_{x\to c}[f(x)g(x)] = 0$.

(ii) $\lim\limits_{x\to c}[f(x)g(x)] = 0$, if g is bounded.

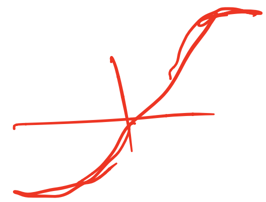
(iii) $\lim\limits_{x\to c}[f(x)g(x)] = 0$, if $\lim\limits_{x\to c} g(x)$ exists.

(i) $f(x) = x-a$, $g(x) = \frac{1}{x-a}$ defined on (a,b)

$\lim\limits_{x\to a} f(x) = 0$

$\lim\limits_{x\to a} f(x)g(x) = 1 \neq 0$

FALSE.



(ii) $|g(x)| \leq M$

$-M \leq g(x) \leq M$

$-M f(x) \leq f(x)g(x) \leq f(x) M$

Taking limit: $-M f(x) \to 0$, $f(x) M \to 0$

by sandwich $\lim f(x) g(x) = 0$

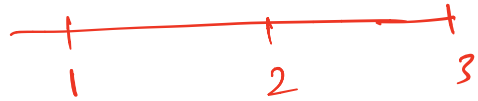
(iii) You can use product rule because both limit exists.

$\boxed{\lim\limits_{x\to 0^+} \frac{1}{x} = \infty}$

Ans $= 0$

13. Discuss the continuity of the following functions:

(iii) $f(x) = \begin{cases} \dfrac{x}{[x]} & \text{if } 1 \leq x < 2, \\ 1 & \text{if } x = 2, \\ \sqrt{6-x} & \text{if } 2 < x \leq 3. \end{cases}$

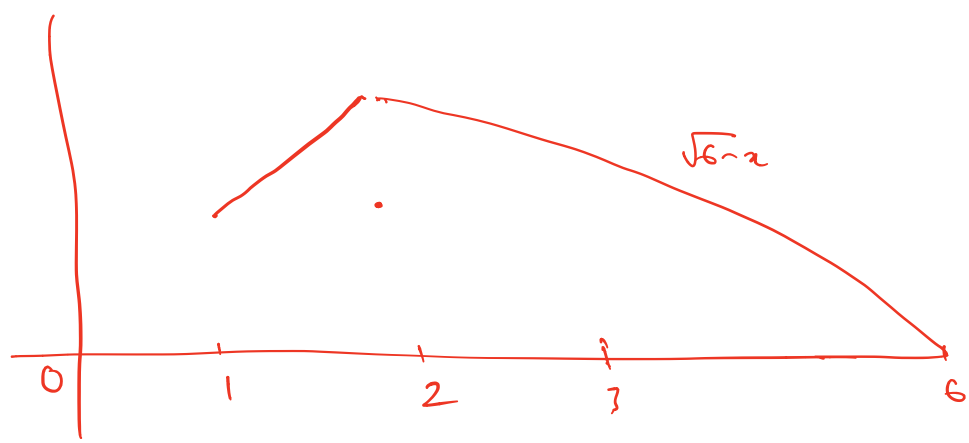


$[x] = 1$ when $1 \leq x < 2$

$f(x) = x$ when $1 \leq x < 2$

$f(2) = 1$

$f(x) = \sqrt{6-x}$ when $2 < x \leq 3$



$[1, 2)$ $\downarrow$ cont

$(2, 3]$ $\downarrow$ continuous

at 2 $\quad \lim_{x \to 2^-} f(x) = 2$, $\lim_{x \to 2^+} f(x) = 2$

but $f(2) = 1 \neq 2$

So $f(x)$ is not cont at 2

15. Let $f(x) = x^2 \sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that f is differentiable on $\mathbb{R}$. Is f' a continuous function?

16. Let $f : (a, b) \to \mathbb{R}$ be a function such that

$$|f(x+h) - f(x)| \leq C|h|^\alpha$$

for all $x, x+h \in (a, b)$, where C is a constant and $\alpha > 1$. Show that f is differentiable on (a, b) and compute $f'(x)$ for $x \in (a, b)$.

17. If $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$, then show that

$$\lim_{h \to 0^+} \frac{f(c+h) - f(c-h)}{2h}$$

exists and equals $f'(c)$. Is the converse true ? [Hint: Consider $f(x) = |x|$.]

18. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy

$$f(x+y) = f(x)f(y) \text{ for all } x, y \in \mathbb{R}.$$

If f is differentiable at 0, then show that f is differentiable at every $c \in \mathbb{R}$ and $f'(c) = f'(0)f(c)$.

19. Using the theorem on derivative of inverse function, compute the derivative of (i) $\cos^{-1} x, \ -1 < x < 1$. (ii) $\operatorname{cosec}^{-1} x, \ |x| > 1$.

20. Compute $\dfrac{dy}{dx}$, given

$$y = f\left(\frac{2x-1}{x+1}\right) \text{ and } f'(x) = \sin(x^2).$$

Supplement

1. A sequence $\{a_n\}_{n \geq 1}$ is said to be Cauchy if for any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon, \ \forall\, m, n \geq n_0$. In other words, if we choose n_0 large enough, we can make sure that the elements of a Cauchy sequence are close to each other as we want beyond n_0. One can show that a sequence in $\mathbb{R}$ is convergent if and only if it is Cauchy. To show that a convergent sequence in $\mathbb{R}$ is Cauchy is easy. To show that every Cauchy sequence in $\mathbb{R}$ converges is harder, and moreover, involves making a precise definition of the set of real numbers. Sets in which every Cauchy sequence converges are called *complete*. Thus the set of real numbers is complete.

2. To prove that a sequence $\{a_n\}_{n \geq 1}$ is convergent to a limit L, one needs to first guess what this limit L might be and then verify the required property. However the concept of 'Cauchyness' of a sequence is an intrinsic property, that is, we can decide whether a sequence is Cauchy by examining the sequence itself. There is no need to guess what the limit might be.

3. In problem 5(i), we defined

$$a_1 = \frac{3}{2}, \quad a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n}) \ \forall\, n \geq 1.$$

The sequence $\{a_n\}_{n\geq 1}$ is a monotonically decreasing sequence of rational numbers which is bounded below. However, it cannot converge to a rational (why?). This exhibits the need to enlarge the concept of numbers beyond rational numbers. The sequence $\{a_n\}_{n\geq 1}$ converges to $\sqrt{2}$ and its elements a_n's are used to find a rational approximation (in computing machines) of $\sqrt{2}$.

Optional Exercises:

1. Show that the function f in Question 14 satisfies $f(kx) = kf(x)$, for all $k \in \mathbb{R}$.

2. Show that in Question 18, f has a derivative of every order on $\mathbb{R}$.

3. Construct an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is continuous everywhere and is differentiable everywhere except at 2 points.

4. Let $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$ Show that f is discontinuous at every $c \in \mathbb{R}$.

5. Let $g(x) = \begin{cases} x, & \text{if } x \text{ is rational,} \\ 1-x, & \text{if } x \text{ is irrational.} \end{cases}$ Show that g is continuous only at $c = 1/2$.

6. Let $f : (a,b) \to \mathbb{R}$ and $c \in (a,b)$ be such that $\lim\limits_{x \to c} f(x) > \alpha$. Prove that there exists some $\delta > 0$ such that
$$f(c+h) > \alpha \text{ for all } 0 < |h| < \delta.$$

7. Let $f : (a,b) \to \mathbb{R}$ and $c \in (a,b)$. Show that the following are equivalent:
   (i) f is differentiable at c.
   (ii) There exist $\delta > 0$ and a function $\epsilon_1 : (-\delta, \delta) \to \mathbb{R}$ such that $\lim_{h \to 0} \epsilon_1(h) = 0$ and
   $$f(c+h) = f(c) + \alpha h + h\epsilon_1(h) \text{ for all } h \in (-\delta, \delta).$$
   (iii) There exists $\alpha \in \mathbb{R}$ such that
   $$\lim_{h \to 0} \left(\frac{|f(c+h) - f(c) - \alpha h|}{|h|} \right) = 0.$$

8. Suppose f is a function that satisfies the equation $f(x+y) = f(x) + f(y) + x^2y + xy^2$ for all real numbers x and y. Suppose also that
$$\lim_{x \to 0} \frac{f(x)}{x} = 1.$$
Find $f(0), f'(0), f'(x)$.

9. Suppose f is a function with the property that $|f(x)| \leq x^2$ for all $x \in \mathbb{R}$. Show that $f(0) = 0$ and $f'(0) = 0$.

10. Show that any continuous function $f : [0,1] \to [0,1]$ has a fixed point.