MA 109 Tutorial 5

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Q)2

If $f : D \in \mathbb{R}^n \to \mathbb{R}$ be a function. Then the main difference between a level curve and a contour line is that level curve is a subset of $\mathbb{R}^n$ but contour line is a subset of $\mathbb{R}^{n+1}$.

(ii) For $c < 0$ the level curve and contour lines are null set. For $c = 0$ the level curve is the singleton set $\{(0,0)\}$ and contour line is the singleton set $\{(0,0,0)\}$. For $c > 0$ the level curve is the circle with center $(0,0)$, radius $\sqrt{c}$ and lies in $\mathbb{R}^2$ and contour line is the circle with center $(0,0,c)$, radius $\sqrt{c}$, parallel to the $x-y$ plane and lies in $\mathbb{R}^3$.

(iii)Here if $c \neq 0$ then the level curve is a hyperbola in $\mathbb{R}^2$ and the contour is a hyperbola which is parallel to the $x-y$ plane in $\mathbb{R}^3$. If $c = 0$ instead of parabola it will be a pair of straight lines.

Q)4

(i) We know that $(x_n, y_n) \to (x_0, y_0) \Leftrightarrow x_n \to x_0$ and $y_n \to y_0$. Let the given function be $h(x, y) = f(x) + g(y)$. For any arbitrary sequence $(x_n, y_n) \to (x_0, y_0)$

$$\lim_{n\to\infty} h(x_n, y_n) = \lim_{x_n\to x_0} f(x_n) \pm \lim_{y_n\to y_0} g(y_n)$$
$$\Rightarrow \lim_{(x_n,y_n)\to(x_0,y_0)} h(x, y) = f(x_0) \pm g(y_0)$$

(ii) Similar to above we can take any arbitrary sequence $(x_n, y_n) \to (x_0, y_0)$, and let $h(x_n, y_n) = f(x_n)g(y_n)$ and apply limit and it will also be continuous.

(iii) and (iv) Observe that $max\{f(x), g(y)\} = \frac{f(x)+g(y)}{2} + \left|\frac{f(x)-g(y)}{2}\right|$ and $min\{f(x), g(y)\} = \frac{f(x)+g(y)}{2} - \left|\frac{f(x)-g(y)}{2}\right|$. For any continuous function $f(x, y)$, $|f(x, y)|$ is also continuous. Because if $f(x_0, y_0) > 0$ or < 0 then it is clearly continuous at that point. If $f(x_0, y_0) = 0$ then also it is continuous at that point because for $-f(x_0, y_0) \leq |f(x_0, y_0)| \leq f(x_0, y_0)$ applying sandwich theorem we can say that it is continuous at this point. Now using this point and (i) we can say that
$max\{f(x), g(y)\} = \frac{f(x)+g(y)}{2} + \left|\frac{f(x)-g(y)}{2}\right|$ and
$min\{f(x), g(y)\} = \frac{f(x)+g(y)}{2} - \left|\frac{f(x)-g(y)}{2}\right|$ are continuous.

Q)6(ii)

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h}$$
$$= \lim_{h \to 0} \frac{\frac{sin^2(h)}{|h|}}{h}$$

the right hand limit is 1 and left hand limit is -1. So it doesn't exist. We get the exact same limit for $f_y(0,0)$. So it also doesn't exist.

Q)8

We know that $(x_n, y_n) \to (x_0, y_0) \Leftrightarrow x_n \to x_0 and y_n \to y_0$, we also have seen in previous tutorials that $\lim_{x\to 0} xsin(\frac{1}{x}) = 0$ using sandwich theorem. By using both we can say that it is continuous. Let $\mathbf{v} = (v_x, v_y)$ be a unit vector. For $\mathbf{x} = (0, 0)$ and if $v_x, v_y \neq 0$

$$\nabla_{\mathbf{v}} f(\mathbf{x}) = \lim_{h\to 0} \frac{f(\mathbf{x} + h\mathbf{v}) - f(\mathbf{x})}{h}$$

$$\Rightarrow \nabla_{\mathbf{v}} f(0, 0) = \lim_{h\to 0} \frac{hv_x sin(\frac{1}{hv_x}) + hv_y sin(\frac{1}{hv_y})}{h}$$

this limit doesn't exist as we can get different values by taking different sequences. Similarly f_x,f_y also don't exist. So, none of the partial derivatives exist.

Q)10

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \text{ iff } \forall\epsilon > 0 \exists\delta > 0 \text{ such that}$$

$$(x,y) \in D_f, 0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta \Rightarrow |f(x,y) - L| < \epsilon$$

Here for $L = 0$ $\delta = \epsilon$ works $\forall\epsilon > 0$ as $|f(x,y) - L| = \sqrt{x^2 + y^2}$. So it is continuous at $(0,0)$. Let $\mathbf{v} = (v_x, v_y)$ be a unit vector. For $\mathbf{x} = (0,0)$ and if $v_y \neq 0$

$$\nabla_{\mathbf{v}} f(\mathbf{x}) = \lim_{h\to 0} \frac{f(\mathbf{x} + h\mathbf{v}) - f(\mathbf{x})}{h}$$

$$\Rightarrow \nabla_{\mathbf{v}} f(0,0) = \lim_{h\to 0} \frac{hv_y\sqrt{h^2(1)}}{h|hv_y|} = \lim_{h\to 0} \frac{hv_y|h|}{h|h||v_y|} = \frac{v_y}{|v_y|}$$

If $v_y = 0 \Rightarrow v_x = \pm 1 \Rightarrow \nabla_{\mathbf{v}} f(\mathbf{x}) = \frac{0-0}{h} = 0$

From above we get $f_x(0,0) = 0, f_y(0,0) = 1$ if it differentiable at $(0,0)$ then $\exists \alpha, \beta$ such that

$$\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x_0, y_0) - \alpha h - \beta k}{\sqrt{h^2+k^2}} = 0$$

and α, β coincides with the x, y directional derivatives. Here

$$\lim_{(h,k)\to(0,0)} \frac{\frac{k}{|k|}\sqrt{h^2+k^2} - k}{\sqrt{h^2+k^2}} = 0$$

It should be 0 along all sequences converging to (0,0). Take a sequence along the line $x = y$ approaching from the 1st quadrant. The limit will be $\frac{\sqrt{2}-1}{\sqrt{2}-1}$, so it is not differentiable.

Some important points

1. Differentiable at (x_0, y_0).
2. $\nabla_{\mathbf{v}} f(\mathbf{x}) = (f_x(x_0, y_0), f_y(x_0, y_0)) \cdot \mathbf{v}$ is true for all unit vectors $\mathbf{v}$.
3. The directional derivative exists for all $\mathbf{v}$.
4. f_x and f_y exist.
5. f is continuous (x_0, y_0)

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \not\Rightarrow (v), (i) \Rightarrow (v), (iii) \not\Rightarrow (v)$$

$$(v) \not\Rightarrow (iv) \not\Rightarrow (iii) \not\Rightarrow (ii) \not\Rightarrow (i)$$

The previous question(Q)10) has already proved that $(iii) \not\Rightarrow (i)$ and $(iii) \not\Rightarrow (ii)$. The following function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x, y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

is not continuous at $(0,0)$ (approach the origin along $y=x$ and $y=x^2$ this gives two different values, so discontinuous). Since (i) $\Rightarrow$ (v) it is also not differentiable. But all partial derivatives exist and are given by (as always $\mathbf{v}=(v_x,v_y)$ is a unit vector)

$$\nabla_{\mathbf{v}} f(x,y)=\begin{cases}\dfrac{v_x^2}{v_y} & v_y\neq 0\\ 0 & v_y=0\end{cases}$$

This example proves that $(iii)\not\Rightarrow(ii)$, $(iii)\not\Rightarrow(i)$ and $(iv)\not\Rightarrow(v)$ and even $(iii)\not\Rightarrow(v)$. Define a new function as $f:\mathbb{R}^2\to\mathbb{R}$ and

$$f(x,y)=\begin{cases}\dfrac{x^3y}{x^4+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

Here except (i) all other are true. This shows that $(ii)\not\Rightarrow(i)$.