Black Hole Information Paradox

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Abstract

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Contents

0 Introduction

0.1 Conventions

The signature of the metric used is (-, +, +, +). We set $c = \hbar = k_B = 1$.

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0.2 The four classical laws of black hole mechanics

Using General relativity and some other classically reasonable assumptions (which may not be correct in Quantum Gravity) the following laws can be proved.

No.	Laws of thermodynamics	Analogous black hole laws
0	T is same for systems in equilibrium	κ is constant on the Event Horizon fora stationary black hole.
1	$dE = TdS + \Omega\, dJ + \Phi\, dQ$	$dE = \frac{\kappa}{2\pi}\left(\frac{dA}{4}\right) + \Omega\, dJ + \Phi\, dQ$
2	$dS \geq 0$ where $S = k_{\mathrm{B}} \ln \Omega$	$dA \geq 0$ or $\frac{dA}{4} \geq 0$
3	$T = 0$ K cannot be attained	Extremal black holes with $\kappa = 0$ cannot exist. (also called as *Cosmic censorship hypothesis* and it forbids naked singularities)

Table 0.1: Analogy between the black hole laws and laws of thermodynamics

If $T = \frac{\kappa}{2\pi}$ and $S = \frac{dA}{4}$ then we can see that they are very similar to the laws of thermodynamics. (Of course here the exact coefficients seem arbitrary). We will see that these are not analogous but are the same phenomenon. Black holes although classically seem to be "not so random" (they can be completely characterized by very few parameters and do not seem to have large number of micro states) objects they have very high entropy. For a supermassive astrophysical black hole like the one at the center of the Milky Way, this is an enormous number, or order $(\frac{10^6 km}{l_p})^2 \sim 10^{88}$ (and there are billions of them). For comparison, the entropy of all baryons in the observable universe is around 10^{82}, and the entropy of the CMB is about 10^{89}.

0.3 Quantum Field Theory in curved spacetime

Lagrange density of a scalar field in curved spacetime is (we should replace normal derivatives with covariant derivatives)

$$\mathcal{L} = \sqrt{-g}(-\frac{1}{2})g^{\mu\nu}\nabla_\mu\phi\nabla_{nu}\phi - \frac{1}{2}m^2\phi^2$$

In QFT we can express a massive scalar field using Fourier decomposition of the field as

$$\phi = \int d^3k(a_k f_k + a_k^\dagger f_k^*)$$

In inertial Minkowski space-times we have time transnational symmetry i.e. ∂_t is a Killing vector. So we can uniquely define positive- or "negative"(by that we mean $\partial_t f_k^* = +i\omega f_k^*$ where $\omega > 0$. This terminology was made historically when they considered ϕ as a wave function instead of a quantum field)-frequency modes and **the notion of a particle is Lorentz-invariant**. Since for a general spacetime there will not be any timelike Killing vector, we will not in general be able to find solutions to the wave equation that separate into time-dependent and space-dependent factors, and so cannot classify modes as positive- or "negative"-frequency. We can find a set of basis modes, but the problem is that there will generally be many such sets, with no way to prefer one over any others, and the notion of a vacuum or number operator will depend sensitively on which set we choose. We can easily define particles in QFT in curved spacetime if a spacetime has positive and "negative" frequency modes. In general we can't find them. For static space-times we can define them and this process will work. So it is not necessary that different observers in different places agree on the number of particles that are in a particular state.

As we will see in the next chapter, We don't even have to include gravity, **even in flat spacetime** an accelerating observer will observe particles in the vacuum state of an inertial observer.

0.3.1 Bogoliubov coefficients and their relations

1 Unruh effect using Rindler coordinates

1.1 Rindler coordinates and space

We will first write the Minkowski line element suppressed to two dimensions in the Minkowski coordinates (t, x) and **Rindler coordinates** (η, ξ) related by:

$$t = \frac{1}{a}e^{a\xi}sinh(a\eta) \qquad\qquad x = \frac{1}{a}e^{a\xi}cosh(a\eta)$$

Notice that $x+t > 0$ and $x-t > 0$. So Rindler coordinates can only describe a part of the Minkowski space called the **Rindler Space or wedge**.

$$ds^2 = -dt^2 + dx^2 = e^{2a\xi}(-d\eta^2 + d\xi^2)$$

If $\xi = k$ for a particle, where k is a constant, then when observed in the Minkowski frame the particle has constant four acceleration **magnitude** $\alpha = \frac{a}{e^{ak}}$ (the 4 accleration is not constant) as $a^\mu = \frac{d^2 x^\mu}{d\tau^2} = (\alpha sinh(\alpha t), \alpha cosh(\alpha t))$. η is related to the proper time τ of this particle by $\eta = \frac{\alpha}{a}\tau$. The locus of $x^2 = t^2$ is called the **Rindler horizon**.

An observer who is at the origin of the Rindler coordinates ($\eta = \xi = 0$) is called the **Rindler observer**. The magnitude of accleration of this observer is a and $\eta = \tau$.

This metric is independent of η so ∂_η is a Killing vector field in these coordinates. In Minkowski coordinates the vector ∂_η is

$$\begin{aligned}\partial_\eta &= \frac{\partial t}{\partial \eta}\partial_t + \frac{\partial x}{\partial \eta}\partial_x \\ &= a(x\partial_t + t\partial_x)\end{aligned}$$

It is the Killing field associated with a boost in the x direction. So time translation in the Rindler coordinates is a boost in the Minkowski frame. in regions II and IIl it is spacelike, while in region IV it is timelike but past-directed. Which is what we intuitively expect since as time passes in the Rindler frame the velocity of Rindler observer increases in the Minkowski frame. This Killing field naturally extends throughout the spacetime, in regions II and III it is spacelike, while in region IV it is timelike but past-directed.

Now we can similarly define **left Rindler space or wedge**.

$$t = -\frac{1}{a}e^{a\xi}sinh(a\eta) \qquad\qquad x = -\frac{1}{a}e^{a\xi}cosh(a\eta)$$

The metric is same as for the right wedge. Coordinates (η, ξ) cannot be used simultaneously in wedges right and left, because the range of these parameters are the same in each regions. The vector field ∂_η is a Killing vector field in wedges left and right, but is future pointing in the right wedge while **past pointing in the left one**. $-\partial_\eta$ is the future pointing timelike killing vector field in left wedge.

1.2 Intuitive explanation of Unruh effect

The future horizon and past or illusory horizon behave like an event horizon of a black hole and white hole respectively in the Einstein–Rosen bridge. In QFT often time negative and positive energy particles will be produced and will be annihilated. The product of their energy and life time is of the order of Planck's constant. But near the future horizon if such a particle pair is produced the negative energy particle may escape from the Rindler space and go to the outside Minkowski space and **it can be stable in the Minkowski space**. To understand this remember that energy is the component of four momentum corresponding to a future pointing timelike killing vector field. In Rindler space it corresponds to ∂_η and in Minkowski space it corresponds to ∂_t. So **a particle is said to be having negative energy in Rindler space if $P_\mu N^\mu$ is negative.** (where N^μ is the unit η vector which is (1,0) in (η,ξ)). A particle is said to be having negative energy in Minkowski space if $P_\mu T^\mu$ is negative. (where T^μ is the unit t vector).

$$dt = e^{a\xi}(cosh(a\eta)d\eta + sinh(a\eta)d\xi)$$

Near the horizon if $cosh(a\eta)d\eta + sinh(a\eta)d\xi > 0$ and $d\eta < 0$ for a particle then it is having -ve energy wrt Rindler space (or *equivalently* moving backward in time with +ve energy) but wrt Minkowski space it will have +ve energy and it will be stable after getting out of Rindler space. **The reverse process cannot occur because if the positive energy is escaped the negative energy particle cannot be stabilised in the Rindler space.**

1.3 Unruh effect

In Minkowski spacetime, the equation of motion is $\Box\phi = (-\partial_t^2 + \partial_x^2)\phi = 0$ and admits plane waves solutions,

$$f_k = \frac{1}{\sqrt{4\pi\omega}} e^{ik_\mu x^\mu}, \qquad k^\mu = (\omega, k)$$
$$\partial_t f_k = -i\omega f_k \qquad \partial_t f_k^* = +i\omega f_k^*$$

where f_k^* are "negative" frequency modes. After canonical quantization, any field configuration ϕ solution to the equation of motion can be expanded in terms of $\{f_k,\, f_k^*\}$,

$$\phi = \sum_k (a_k f_k + a_k^\dagger f_k^*)$$

The vacuum is defined by state ${}^M|0\rangle$ such that $a_k {}^M|0\rangle, \forall k$.

In the right Rindler wedge, the equation of motion reads $\Box\phi = e^{-2a\xi}(-\partial_\eta^2 + \partial_\xi^2)\phi = 0$ and admits plane wave solutions,

$$g_k^R = \frac{1}{\sqrt{4\pi\omega}} e^{ik_\mu x^\mu}, \qquad x^\mu = (\eta, \xi)$$
$$\partial_t g_k^R = -i\omega g_k^R \qquad \partial_t g_k^{R*} = +i\omega g_k^{R*}$$

Positive frequency modes are defined with respect the the Killing vector field ∂_η. So g_k^R is positive frequency mode and g_k^{R*} is "negative" frequency mode. After canonical quantization, any field configuration ϕ solution to the equation of motion can be expanded in terms of $\{g_k^R,\, g_k^{R*}\}$,

$$\phi = \sum_k (b_k g_k^R + b_k^\dagger g_k^{R*})$$

The vacuum is defined by state ${}^R|0\rangle$ such that $a_k{}^R|0\rangle$,$\forall k$.

We defined g_k^R in the right Rindler wedge. Now we generalise it to

$$g_k^R = \begin{cases} \frac{1}{\sqrt{4\pi\omega}} e^{ik_\mu x^\mu} & \text{in right Rindler wedge} \\ 0 & \text{in left Rindler wedge} \end{cases}$$

Similarly,

$$g_k^L = \begin{cases} 0 & \text{in right Rindler wedge} \\ \frac{1}{\sqrt{4\pi\omega}} e^{ik_\mu x^\mu} & \text{in left Rindler wedge} \end{cases}$$

g_k^L are positive frequency modes with respect to Killing vector field -∂_η. $\{g_k^R$, g_k^{R*}, g_k^L, $g_k^{L*}\}$ and $\{f_k,\, f_k^*\}$ are complete set of modes for the Minkowski space, and thus there are two possible modes expansions for any field configuration solution to the equation of motion,

$$\phi = \sum_k (b_k g_k^R + c_k g_k^L + b_k^\dagger g_k^{R*} + c_k^\dagger g_k^{L*})$$
$$\phi = \sum_k (a_k f_k + a_k^\dagger f_k^*)$$

Now we like to find ${}^M\langle 0|\,{}^R N_k\,{}^M|0\rangle$. That is the number of particles with momentum k in the Minkowski vacuum state observed from the Rindler frame.

Using Bogoliubov transformation

$$g_k^R(u) = \int d\omega' (A_{\omega\omega'} f_{\omega'} + B_{\omega\omega'} f_{\omega'}^*)$$

where $u = t - x$. Since $f_k = \frac{1}{\sqrt{4\pi\omega}} e^{ik_\mu x^\mu}$ we can write $f_k(u) = \frac{1}{\sqrt{4\pi\omega}} e^{-i\omega' u}$.

By definition inverse Fourier transform of $g_k^R(u)$ is $\tilde{g}_\omega(\omega') = \int_{\text{inf}}^{\text{inf}} du e^{i\omega' u} g_k^R(u)$. Now we can find that

$$A_{\omega\omega'} = \sqrt{\frac{\omega'}{\pi}}\tilde{g}_\omega(\omega') \qquad\qquad B_{\omega\omega'} = \sqrt{\frac{\omega'}{\pi}}\tilde{g}_\omega(-\omega')$$

The inverse Fourier transform has the property $\tilde{g}_\omega(-\omega') = -e^{-\frac{\omega\pi}{a}}\tilde{g}_\omega(\omega')$. Using that we get $A_{\omega\omega'} = -e^{-\frac{\omega\pi}{a}}B_{\omega\omega'}$. We also know that

$$AA^\dagger - BB^\dagger = 1 \implies |A|^2 - |B|^2 = 1 \implies |B|^2 = \frac{1}{e^{\frac{2\pi\omega}{a}} - 1}$$

$$\begin{aligned}
{}^M\langle 0|\,{}^R N_k\,{}^M|0\rangle &= {}^M\langle 0|(b_k^\dagger)(b_k)\,{}^M|0\rangle \\
&= {}^M\langle 0|(-B_{kp}a_P + A_{kp}^* a_p^\dagger)(A_{kq}^* a_q - B_{kq}^* a_q^\dagger)\,{}^M|0\rangle \\
&= {}^M\langle 0|(B_{kq}B_{kp}^*\delta_{qp})\,{}^M|0\rangle \\
&= |B|_{kk}^2 = |B|_{\omega\omega}^2 = \frac{1}{e^{\frac{2\pi\omega}{a}} - 1}
\end{aligned}$$

That is the number of particle with momentum k and energy $\omega = k$ observed by the Rindler observer are $\frac{1}{e^{\frac{\omega}{\frac{a}{2\pi}}} - 1}$. Which is exactly the number of particle with momentum k observed for a black-body with $T = \frac{a}{2\pi}$ according to the Planck formula. This is called as **Unruh effect**.

The Unruh temperature is spatially inhomogeneous across the Rindler space but the Unruh state is in the equilibrium (unlike in Hawking effect). This is a direct consequence of Ehrenfest–Tolman effect.

2 Hawking effect

2.1 Intuition using Penrose process

2.2 Hawking radiation

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