Shudurshan Shah Sir
Doti Multiple Campus
Silgadhi Doti

1 UNIT

Limit and Continuity

EXERCISE – 1 (A)

1. Evaluate the following limits.

(a) $\lim\limits_{x \to 1} (2x + 3)$ (b) $\lim\limits_{x \to -7} (2x + 5)$

(c) $\lim\limits_{x \to 5} \dfrac{4}{x - 7}$ (d) $\lim\limits_{h \to 0} \dfrac{5}{\sqrt{2h + 1} + 1}$

Solution

(a) When $x = 1$, $2x + 3 = 2 \times 1 + 3 = 5$

$\therefore \lim\limits_{x \to 1} (2x + 3) = 5$

(b) When $x = -7$, $2x + 5 = 2(-7) + 5 = -9$

$\lim\limits_{x \to -7} (2x + 5) = -9$

(c) When $x = 5$, $\dfrac{4}{x - 7} = \dfrac{4}{5 - 7} = \dfrac{4}{-2} = -2$

$\lim\limits_{x \to 5} \dfrac{4}{x - 7} = -2$

(d) When $h = 0$, $\dfrac{5}{\sqrt{2h + 1} + 1} = \dfrac{5}{\sqrt{2 \times 0 + 1} + 1} = \dfrac{5}{\sqrt{1} + 1} = \dfrac{5}{2}$

2. Evaluate

(a) $\lim\limits_{x \to 0} \dfrac{7x^2 + 4x}{x}$ (b) $\lim\limits_{x \to 5} \dfrac{x^2 - 25}{x - 5}$

(c) $\lim\limits_{x \to 1} \dfrac{x^2 - 4x + 3}{x - 1}$ (d) $\lim\limits_{x \to 2} \dfrac{x^2 - 7x + 10}{x - 2}$

(e) $\lim\limits_{x \to 1} \dfrac{x^2 + x - 2}{x^2 - 1}$ (f) $\lim\limits_{x \to 1} \dfrac{x^2 - 4x + 3}{x^2 - 3x + 2}$

(g) $\lim\limits_{x \to -3} \dfrac{x + 3}{x^2 + 4x + 3}$ (h) $\lim\limits_{x \to 3} \left[\dfrac{1}{x - 3} - \dfrac{9}{x^3 - 3x^2}\right]$

Solution

(a) $\lim\limits_{x \to 0} \dfrac{7x^2 + 4x}{x}$ $\left[\dfrac{0}{0}\text{ form}\right]$

$\lim\limits_{x \to 0} \dfrac{x(7x + 4)}{x} = \lim\limits_{x \to 0} (7x + 4) = 7 \times 0 + 4 = 4$

(b) $\lim\limits_{x \to 5} \dfrac{x^2 - 25}{x - 5}$ $\left[\dfrac{0}{0}\text{ form}\right]$

$= \lim\limits_{x \to 5} \dfrac{(x + 5)(x - 5)}{(x - 5)} = \lim\limits_{x \to 5} (x + 5) = 5 + 5 = 10$

(c) $$\lim_{x \to 1} \frac{x^2 - 4x + 3}{x - 1} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 1} \frac{x^2 - 3x - x + 3}{(x - 1)} = \lim_{x \to 1} \frac{x(x - 3) - 1(x - 3)}{(x - 1)} = \lim_{x \to 1} (x - 3) = 1 - 3 = -2$$

(d) $$\lim_{x \to 2} \frac{x^2 - 7x + 10}{x - 2} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$\lim_{x \to 2} \frac{x^2 - 5x - 2x + 10}{x - 2} = \lim_{x \to 2} \frac{x(x - 5) - 2(x - 5)}{x - 2}$$

$$= \lim_{x \to 2} \frac{(x - 2)(x - 5)}{x - 2} = \lim_{x \to 2} (x - 5) = 2 - 5 = -3$$

(e) $$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - 1} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 1} \frac{x^2 + 2x - x - 2}{x^2 - 1^2} = \lim_{x \to 1} \frac{x(x + 2) - 1(x + 2)}{(x + 1)(x - 1)}$$

$$= \lim_{x \to 1} \frac{(x - 1)(x + 2)}{(x - 1)(x + 1)} = \lim_{x \to 1} \frac{x + 2}{x + 1} = \frac{1 + 2}{1 + 1} = \frac{3}{2}$$

(f) $$\lim_{x \to 1} \frac{x^2 - 4x + 3}{x^2 - 3x + 2} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 1} \frac{x^2 - 3x - x + 3}{x^2 - 2x - x + 2} = \lim_{x \to 1} \frac{x(x - 3) - 1(x - 3)}{x(x - 2) - 1(x - 2)} = \lim_{x \to 1} \frac{(x - 1)(x - 3)}{(x - 1)(x - 2)}$$

$$= \lim_{x \to 1} \frac{(x - 3)}{(x - 2)} = \frac{1 - 3}{1 - 2} = \frac{-2}{-1} = 2$$

(g) $$\lim_{x \to -3} \frac{x + 3}{x^2 + 4x + 3} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to -3} \frac{x + 3}{x^2 + 3x + x + 3} = \lim_{x \to -3} \frac{x + 3}{x(x + 3) + 1(x + 3)}$$

$$= \lim_{x \to -3} \frac{x + 3}{(x + 1)(x + 3)} = \lim_{x \to -3} \frac{1}{x + 1} = \frac{1}{-3 + 1} = -\frac{1}{2}$$

(h) $$\lim_{x \to 3} \left[\frac{1}{x - 3} - \frac{9}{x^3 - 3x^2}\right] \qquad (\infty - \infty \text{ form})$$

$$= \lim_{x \to 3} \frac{x^2 - 9}{(x - 3)x^2} = \lim_{x \to 3} \frac{(x + 3)(x - 3)}{x^2(x - 3)} = \lim_{x \to 3} \frac{x + 3}{x^2} = \frac{6}{9} = \frac{2}{3}$$

3. Evaluate

(a) $$\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9}$$

(b) $$\lim_{x \to 0} \frac{\sqrt{x + 4} - 2}{x}$$

(c) $$\lim_{x \to 0} \frac{7x}{\sqrt{3x + 4} - 2}$$

(d) $$\lim_{x \to 1} \frac{x - 1}{\sqrt{x + 3} - 2}$$

(e) $$\lim_{x \to 4} \frac{x^2 - 16}{\sqrt{3x + 4} - 4}$$

(f) $$\lim_{x \to -1} \frac{\sqrt{x^2 + 8} - 3}{x + 1}$$

(g) $$\lim_{x \to a} \frac{\sqrt{3a - x} - \sqrt{x + a}}{4(x - a)}$$

(h) $$\lim_{x \to a} \frac{\sqrt{2x} - \sqrt{3x - a}}{\sqrt{x} - \sqrt{a}}$$

(i) $$\lim_{x \to 2} \frac{x - \sqrt{8 - x^2}}{\sqrt{x^2 + 12} - 4}$$

Solution

(a) $$\lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 9} \frac{\sqrt{x} - 3}{x - 9} \times \frac{\sqrt{x} + 3}{\sqrt{x} + 3} = \lim_{x \to 9} \frac{x - 9}{(x - 9)(\sqrt{x} + 3)}$$

$$= \lim_{x \to 9} \frac{1}{\sqrt{x} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{3 + 3} = \frac{1}{6}$$

(b) $\lim_{x\to 0} \frac{\sqrt{x+4}-2}{x}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x\to 0} \frac{\sqrt{x+4}-2}{x} \times \frac{(\sqrt{x+4}+2)}{(\sqrt{x+4}+2)} = \lim_{x\to 0} \frac{(\sqrt{x+4})^2-2^2}{x(\sqrt{x+4}+2)} = \lim_{x\to 0} \frac{x+4-4}{x(\sqrt{x+4}+2)}$

$= \lim_{x\to 0} \frac{x}{x(\sqrt{x+4}+2)} = \lim_{x\to 0} \frac{1}{(\sqrt{x+4}+2)} = \frac{1}{(\sqrt{0+4}+2)} = \frac{1}{\sqrt{4}+2} = \frac{1}{2+2} = \frac{1}{4}$

(c) $\lim_{x\to 0} \frac{7x}{\sqrt{3x+4}-2}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x\to 0} \frac{7x}{\sqrt{3x+4}-2} \times \frac{\sqrt{3x+4}+2}{\sqrt{3x+4}+2} = \lim_{x\to 0} \frac{7x(\sqrt{3x+4}+2)}{3x+4-4} = \lim_{x\to 0} \frac{7(\sqrt{3x+4}+2)}{3}$

$= \frac{7}{3}(\sqrt{3\times 0+4}+2) = \frac{7}{3}(2+2) = \frac{28}{3}$

(d) $\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2}$ $\left[\frac{0}{0}\text{ form}\right]$

$\lim_{x\to 1} \frac{x-1}{\sqrt{x+3}-2} \times \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2} = \lim_{x\to 1} \frac{(x-1)(\sqrt{x+3}+2)}{x+3-4}$

$= \lim_{x\to 1} \frac{(x-1)(\sqrt{x+3}+2)}{x-1} = \lim_{x\to 1} (\sqrt{x+3}+2) = \sqrt{1+3}+2 = 2+2 = 4$

(e) $\lim_{x\to 4} \frac{x^2-16}{\sqrt{3x+4}-4}$

$= \lim_{x\to 4} \frac{(x-4)(x+4)}{\sqrt{3x+4}-4} \times \frac{\sqrt{3x+4}+4}{\sqrt{3x+4}-4} = \lim_{x\to 4} \frac{(x-4)(x+4)(\sqrt{3x+4}+4)}{3x+4-16}$

$= \lim_{x\to 4} \frac{(x-4)(x+4)(\sqrt{3x+4}+4)}{3(x-4)} = \lim_{x\to 4} \frac{(x+4)(\sqrt{3x+4}+4)}{3}$

$= \frac{(4+4)(\sqrt{3\times 4+4}+4)}{3} = \frac{8\times 8}{3} = \frac{64}{3}$

(f) $\lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim_{x\to -1} \frac{\sqrt{x^2+8}-3}{x+1} \times \frac{\sqrt{x^2+8}+3}{\sqrt{x^2+8}+3}$

$= \lim_{x\to -1} \frac{x^2+8-9}{(x+1)(\sqrt{x^2+8}+3)} = \lim_{x\to -1} \frac{x^2-1}{(x+1)(\sqrt{x^2+8}+3)}$

$= \lim_{x\to -1} \frac{(x+1)(x-1)}{(x+1)(\sqrt{x^2+8}+3)} = \lim_{x\to -1} \frac{x-1}{\sqrt{x^2+8}+3} = \frac{-1-1}{\sqrt{(-1)^2+8}+3} = \frac{-2}{3+3} = \frac{-1}{3}$

g. $\lim_{x\to a} \frac{\sqrt{3a-x}-\sqrt{x+a}}{4(x-a)}$

$= \lim_{x\to a} \frac{\sqrt{3a-x}-\sqrt{x+a}}{4(x-a)}$ $\left[\frac{0}{0}\right]$

$= \lim_{x\to a} \frac{3a-x-x-a}{4(x-a)(\sqrt{3x-x}+\sqrt{x+a})} = \lim_{x\to a} \frac{-2(x-a)}{4(x-a)(\sqrt{3a-x}+\sqrt{x+a})}$

$= \lim_{x\to a} \frac{-1}{2(\sqrt{3a-x}+\sqrt{x+a})} = \frac{-1}{2(\sqrt{3a-a}+\sqrt{a+a})} = \frac{-1}{2(\sqrt{2a}+\sqrt{2a})} = -\frac{1}{4\sqrt{2a}}$

h. $\lim_{x\to a} \frac{\sqrt{2x}-\sqrt{3x-a}}{\sqrt{x}-\sqrt{a}}$

$= \lim_{x\to a} \left[\frac{\sqrt{2x}-\sqrt{3x-a}}{\sqrt{x}-\sqrt{a}} \times \frac{\sqrt{2x}+\sqrt{3x-a}}{\sqrt{2x}+\sqrt{3x-a}} \times \frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}\right]$

$$= \lim_{x \to a} \left[\frac{2x-(3x-a)}{x-a} \times \frac{\sqrt{x}+\sqrt{a}}{\sqrt{2x}+\sqrt{3x-a}}\right] = \lim_{x \to a} \left[\frac{-x+a}{x-a} \times \frac{\sqrt{x}+\sqrt{a}}{\sqrt{2x}+\sqrt{3x-a}}\right]$$

$$= \lim_{x \to a} \left[\frac{-(x-a)}{x-a} \times \frac{\sqrt{x}+\sqrt{a}}{\sqrt{2x}+\sqrt{3x-a}}\right] = -\left(\frac{\sqrt{a}+\sqrt{a}}{\sqrt{2a}+\sqrt{2a}}\right) = -\frac{2\sqrt{a}}{2\sqrt{2a}} = -\frac{1}{\sqrt{2}}$$

i. $\lim_{x \to 2} \frac{x-\sqrt{8-x^2}}{\sqrt{x^2+12}-4}$ $\left[\frac{0}{0} \text{ form}\right]$

$$= \lim_{x \to 2} \frac{x-\sqrt{8-x^2}}{\sqrt{x^2+12}-4} \times \frac{x+\sqrt{8-x^2}}{x+\sqrt{8-x^2}} \times \frac{\sqrt{x^2+12}+4}{\sqrt{x^2+12}+4}$$

$$= \lim_{x \to 2} \frac{\{x^2-(8-x^2)\}\left(\sqrt{x^2+12}+4\right)}{(x^2+12-4^2)\left(x+\sqrt{8-x^2}\right)} = \lim_{x \to 2} \frac{(x^2-8+x^2)\left(\sqrt{x^2+12}+4\right)}{(x^2-4)\left(x+\sqrt{8-x^2}\right)}$$

$$= \lim_{x \to 2} \frac{2(x^2-4)\left(\sqrt{x^2+12}+4\right)}{(x^2-4)\left(x+\sqrt{8-x^2}\right)} = \frac{2\left(\sqrt{2^2+12}+4\right)}{2+\sqrt{8-2^2}} = \frac{16}{4} = 4$$

4. Evaluate

(a) $\lim_{x \to a} \frac{x^5-a^5}{x-a}$ (b) $\lim_{x \to 2} \frac{x^8-256}{x-2}$

(c) $\lim_{x \to a} \frac{x^{\frac{2}{3}}-a^{\frac{2}{3}}}{x-a}$ (d) $\lim_{x \to a} \frac{x^{\frac{5}{2}}-a^{\frac{5}{2}}}{x^{\frac{3}{2}}-a^{\frac{3}{2}}}$, $a > 0$

Solution

(a) $\lim_{x \to a} \frac{x^5-a^5}{x-a}$ $\left[\frac{0}{0} \text{ form}\right]$

$= 5a^{5-1}$ $\left[\because \lim_{x \to a} \frac{x^n-a^n}{x-a} = na^{n-1}\right]$

$= 5a^4$

(b) $\lim_{x \to 2} \frac{x^8-256}{x-2}$ $\left[\frac{0}{0} \text{ form}\right]$

$$= \lim_{x \to 2} \frac{x^8-2^8}{x-2} = \lim_{x \to 2} \frac{(x^4)^2-(2^4)^2}{x-2} = \lim_{x \to 2} \frac{(x^4+16) \times (x^4-16)}{x-2}$$

$$= \lim_{x \to 2} \frac{(x^4+16)(x^2+4)(x+2)(x-2)}{(x-2)} = \lim_{x \to 2} (x^4+16)(x^2+4)(x+2)$$

$= (2^4+16)(2^2+4)(2+2) = 32 \times 8 \times 4 = 1024$

Alternatively

$$\lim_{x \to 2} \frac{x^8-2^8}{x-2} = 8 \cdot 2^{8-1} = 1024$$

(c) $\lim_{x \to a} \frac{x^{\frac{2}{3}}-a^{\frac{2}{3}}}{x-a}$

$$= \frac{2}{3}a^{\frac{2}{3}-1} = \frac{2}{3}a^{-\frac{1}{3}} = \frac{2}{3a^{\frac{1}{3}}}$$

(d) $\lim_{x \to a} \frac{x^{5/2}-a^{5/2}}{x^{3/2}-a^{3/2}}$ $\left[\because \lim_{x \to a} \frac{x^n-a^n}{x-a} = na^{n-1}\right]$

$$= \frac{\lim_{x \to a} \frac{x^{5/2}-a^{5/2}}{x-a}}{\lim_{x \to a} \frac{x^{3/2}-a^{3/2}}{x-a}} = \frac{\frac{5}{2}a^{5/2-1}}{\frac{3}{2}a^{3/2-1}} = \frac{\frac{5}{2}a^{3/2}}{\frac{3}{2}a^{1/2}}$$

$$= \frac{5}{3}a$$

5. Compute

(a) $\lim\limits_{x \to \infty} \dfrac{x^2 + 7x + 3}{9x^2 + 7x + 2}$ (b) $\lim\limits_{x \to \infty} \dfrac{4x^2 + x + 1}{3x^2 + 2x + 1}$

(c) $\lim\limits_{x \to \infty} \dfrac{x^2 + x - 2}{4x^3 - 1}$ (d) $\lim\limits_{x \to \infty} (\sqrt{x+1} - \sqrt{x})$

(e) $\lim\limits_{x \to \infty} \sqrt{x}(\sqrt{x+1} - \sqrt{x})$ (f) $\lim\limits_{x \to \infty} \dfrac{\sqrt{x^2+1}}{x+1}$

Solution

(a) $\lim\limits_{x \to \infty} \dfrac{x^2 + 7x + 3}{9x^2 + 7x + 2}$ $\left[\dfrac{0}{0} \text{ form}\right]$

Diving both number and denominator by x^2

$$\lim_{x \to \infty} \left(\frac{\frac{x^2}{x^2} + \frac{7x}{x^2} + \frac{3}{x^2}}{\frac{9x^2}{x^2} + \frac{7x}{x^2} + \frac{2}{x^2}} \right) = \lim_{x \to \infty} \left(\frac{1 + \frac{7}{x} + \frac{3}{x^2}}{9 + \frac{7}{x} + \frac{2}{x^2}} \right) = \frac{1+0+0}{9+0+0} = \frac{1}{9}$$

(b) $\lim\limits_{x \to \infty} \dfrac{4x^2 + x + 1}{3x^2 + 2x + 1}$ $\left[\dfrac{0}{0} \text{ form}\right]$

$$= \lim_{x\to\infty} \frac{x^2\left(4 + \frac{1}{x} + \frac{1}{x^2}\right)}{x^2\left(3 + \frac{2}{x} + \frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{\left(4 + \frac{1}{x} + \frac{1}{x^2}\right)}{\left(3 + \frac{2}{x} + \frac{1}{x^2}\right)} = \frac{\left(4 + \frac{1}{\infty} + \frac{1}{\infty^2}\right)}{\left(3 + \frac{2}{\infty} + \frac{1}{\infty^2}\right)} = \frac{4+0+0}{3+0+0} = \frac{4}{3}$$

(c) $\lim\limits_{x \to \infty} \dfrac{x^2 + x - 2}{4x^3 - 1}$ $\left[\dfrac{0}{0} \text{ form}\right]$

Dividing both numerator and denominator by x^3

$$= \lim_{x\to\infty} \frac{\frac{x^2}{x^3} + \frac{x}{x^3} - \frac{2}{x^3}}{\frac{4x^3}{x^3} - \frac{1}{x^3}} = \lim_{x\to\infty} \frac{\frac{1}{x} + \frac{1}{x^2} - \frac{2}{x^3}}{4 - \frac{1}{x^3}} = \frac{\frac{1}{\infty} + \frac{1}{\infty^2} - \frac{2}{\infty^3}}{4 - \frac{1}{\infty^3}} = \frac{0+0-0}{4-0} = \frac{0}{4} = 0$$

(d) $\lim\limits_{x\to\infty} \sqrt{x+1} - \sqrt{x}$

$$= \lim_{x\to\infty} \frac{\sqrt{x+1} - \sqrt{x}}{\sqrt{x+1} + \sqrt{x}} \times (\sqrt{x+1} + \sqrt{x}) = \lim_{x\to\infty} \frac{x+1-x}{\sqrt{x+1} + \sqrt{x}} = \frac{1}{\infty} = 0$$

(e) $\lim\limits_{x\to\infty} \sqrt{x}(\sqrt{x+1} - \sqrt{x})$ ($\infty - \infty$ form)

$$= \lim_{x\to\infty} \frac{\sqrt{x}(\sqrt{x+1} - \sqrt{x})}{1} \times \frac{\sqrt{x+1} + \sqrt{x}}{\sqrt{x+1} + \sqrt{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{x}(x+1-x)}{\sqrt{x+1} + \sqrt{x}}$$

Dividing both numerator and denominator by $\sqrt{x}$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{\frac{x+1}{x}} + 1}$$

$$= \lim_{x\to\infty} \frac{1}{\sqrt{1 + \frac{1}{x}} + 1}$$

$$= \frac{1}{\sqrt{1 + \frac{1}{\infty}} + 1} = \frac{1}{\sqrt{1+0} + 1} = \frac{1}{2}$$

(f) $\lim_{x \to \infty} \frac{\sqrt{x^2+1}}{x+1}$ $\left[\frac{\infty}{\infty} \text{ form}\right]$

Dividing both the numerator and denominator by x

$$\lim_{x \to \infty} \left\{\frac{\frac{\sqrt{x^2+1}}{x}}{\frac{x+1}{x}}\right\} = \lim_{x \to \infty} \left\{\frac{\sqrt{\frac{x^2+1}{x^2}}}{\frac{x}{x}+\frac{1}{x}}\right\} = \lim_{x \to \infty} \frac{\sqrt{1+\frac{1}{x^2}}}{1+\frac{1}{x}} = \frac{\sqrt{1+\frac{1}{\infty}}}{1+\frac{1}{\infty}} = \frac{\sqrt{1+0}}{1+0} = 1$$

6. If $f(x) = \frac{ax+b}{x-5}$, $\lim_{x \to 0} f(x) = -1$ and $\lim_{x \to \infty} f(x) = 3$, find the value of $f(3)$.

Solution

Given, $\lim_{x \to 0} f(x) = -1$

$\lim_{x \to 0} \frac{ax+b}{x-5} = -1$

or, $\frac{a.0+b}{0-5} = -1$

$\therefore$ $b = 5$

Again, $\lim_{x \to \infty} f(x) = 3$

or, $\lim_{x \to \infty} \frac{ax+b}{x-5} = 3$

or, $\lim_{x \to \infty} \frac{x\left(a+\frac{b}{x}\right)}{x\left(1-\frac{5}{x}\right)} = 3$

or, $\frac{a+\frac{b}{\infty}}{1-\frac{5}{\infty}} = 3$

or, $\frac{a+0}{1-0} = 3$

$\therefore$ $a = 3$

Thus, $f(x) = \frac{ax+b}{x-5} = \frac{3x+5}{x-5}$

Now, $f(3) = \frac{3 \times 3+5}{3-5} = \frac{9+5}{-2} = \frac{14}{-2} = -7$

7. If $\lim_{x \to 1} \frac{x^4-1}{x-1} = \lim_{x \to k} \frac{x^3-k^3}{x^2-k^2}$, find the value of k.

Solution

Here $\lim_{x \to 1} \frac{x^4-1}{x-1} = \lim_{x \to k} \frac{x^3-k^3}{x^2-k^2}$

or, $\lim_{x \to 1} \frac{(x^2-1)(x^2+1)}{x-1} = \lim_{x \to k} \frac{(x-k)(x^2+kx+k^2)}{(x-k)(x+k)}$

or, $\lim_{x \to 1} \frac{(x-1)(x+1)(x^2+1)}{x-1} = \frac{k^2+k.k+k^2}{k+k}$

or, $(1+1)(1^2+1) = \frac{3k^2}{2k}$

or, $4 \times 2k = 3k^2$

$\therefore$ $k = \frac{8}{3}$

Objective Questions

1. Which of the following is not an indeterminate form?

(a) $\frac{0}{0}$ (b) $\frac{\infty}{\infty}$

(c) $\infty - \infty$ (d) $\infty + \infty$

Ans: d

Options a, b & c are indeterminate forms but d is not indeterminate form

2. Suppose $\lim_{x \to a} f(x) = -2$ and $\lim_{x \to a} g(x) = 3$. Then $\lim_{x \to a} f(x)\, g(x) =$

(a) -2 (b) 3

(c) -6 (d) 0

Ans: c

$$\lim_{x \to a} f(x)\, g(x) = \lim_{x \to a} f(x) . \lim_{x \to a} g(x) = (-2) \times 3 = -6$$

3. $\lim_{x \to 2} (4) =$

(a) 0 (b) 1

(c) 2 (d) 4

Ans: d

$$\lim_{x \to 2} (4) = 4$$

4. $\lim_{t \to 1} \frac{4}{t-9} =$

(a) 2 (b) -2

(c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

Ans: d

$$\lim_{x \to 1} \frac{4}{t-9} = \frac{4}{1-9} = \frac{4}{-8} = -\frac{1}{2}$$

5. $\lim_{x \to a} \frac{x^n - a^n}{x - a} =$

(a) a^{n-1} (b) na^n

(c) na^{n-1} (d) $(n-1)a^n$

Ans: c

Formula

6. $\lim_{x \to 3} \frac{x^2 - 9}{x - 3} =$

(a) 3 (b) 6

(c) 9 (d) 12

Ans: b

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x+3)}{(x-3)} = \lim_{x \to 3} (x+3) = 3 + 3 = 6$$

7. $\lim_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - 4x + 3} =$

(a) 1 (b) 2

(c) $\frac{1}{2}$ (d) $\frac{1}{4}$

Ans: b

$$\lim_{x \to 3} \frac{x^2 - 2x - 3}{x^2 - 4x + 3} = \lim_{x \to 3} \frac{x^2 - 3x + x - 3}{x^2 - 3x - x + 3} = \lim_{x \to 3} \frac{x(x-3) + 1(x-3)}{x(x-3) - 1(x-3)}$$

$$= \lim_{x \to 3} \frac{(x-3)(x+1)}{(x-3)(x-1)} = \frac{3+1}{3-1} = 2$$

8. $\lim_{x \to a} \dfrac{x^{\frac{5}{2}} - a^{\frac{5}{2}}}{x - a} =$

(a) $\frac{5}{2}a^{\frac{5}{2}}$ (b) $\frac{5}{2}a^{\frac{3}{2}}$

(c) $\frac{5}{2}a$ (d) $\frac{5}{2}\sqrt{a}$

Ans: b

$$\lim_{x \to a} \frac{x^{\frac{5}{2}} - a^{\frac{5}{2}}}{x - a} = \frac{5}{2}a^{\frac{5}{2}-1} = \frac{5}{2}a^{\frac{3}{2}}$$

9. $\lim_{x \to 0} \dfrac{\sqrt{x+1} - 1}{x} =$

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{1}{4}$ (d) $\frac{1}{8}$

Ans: a

$$\lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x}$$

$$= \lim_{x \to 0} \frac{\sqrt{x+1} - 1}{x} \times \frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1} = \lim_{x \to 0} \frac{x + 1 - 1}{x(\sqrt{x+1} + 1)} = \lim_{x \to 0} \frac{1}{\sqrt{x+1} + 1} = \frac{1}{2}$$

10. $\lim_{x \to 2} \dfrac{1 - \sqrt{3 - x}}{x - 2} =$

(a) 2 (b) $\frac{1}{2}$

(c) 1 (d) 4

Ans: b

$$\lim_{x \to 2} \frac{1 - \sqrt{3 - x}}{x - 2}$$

$$= \lim_{x \to 2} \frac{1 - \sqrt{3 - x}}{x - 2} \times \frac{1 + \sqrt{3 - x}}{1 + \sqrt{3 - x}} = \lim_{x \to 2} \frac{1 - 3 + x}{(x - 2)(1 + \sqrt{3 - x})}$$

$$= \lim_{x \to 2} \frac{x - 2}{(x - 2)(1 + \sqrt{3 - x})} = \frac{1}{1 + \sqrt{3 - 2}} = \frac{1}{2}$$

11. $\lim_{x \to \infty} \dfrac{3x^2 + 4x - 1}{4x^2 + 5} =$

(a) $\frac{3}{4}$ (b) $\frac{4}{3}$

(c) $\frac{3}{5}$ (d) $\frac{4}{5}$

Ans: a

$$\lim_{x \to \infty} \frac{3x^2 + 4x - 1}{4x^2 + 5} = \lim_{x \to \infty} \left(\frac{\frac{3x^2}{x^2} + \frac{4x}{x^2} - \frac{1}{x^2}}{\frac{4x^2}{x^2} + \frac{5}{x^2}} \right) = \lim_{x \to \infty} \left(\frac{3 + \frac{4}{x} - \frac{1}{x^2}}{4 + \frac{5}{x^2}} \right) = \frac{3 + 0 - 0}{4 + 0} = \frac{3}{4}$$

12. $\lim_{x \to \infty} \sqrt{x + a} - \sqrt{x} =$

(a) 0 (b) $\frac{a}{2}$

(c) 1 (d) a

Ans: *a*

$$\lim_{x \to \infty} \sqrt{x+a} - \sqrt{x} = \lim_{x \to a} \frac{\sqrt{x+a} - \sqrt{x}}{1} \times \frac{\sqrt{x+a} + \sqrt{x}}{\sqrt{x+a} + \sqrt{x}} = \lim_{x \to a} \frac{x+a-x}{\sqrt{x+a} + \sqrt{x}} = \frac{a}{\infty} = 0$$

13. $\lim_{x \to 2} \frac{x^n - 2^n}{x - 2} = 80$ then $n =$

(a) 2 (b) 3
(c) 4 (d) 5

Ans:

$$\lim_{x \to 2} \frac{x^n - 2^n}{x - 2} = 80$$

or, $n\, 2^{n-1} = 80$

or, $n\, 2^{n-1} = 5\, 2^{5-1}$

$\therefore$ $n = 5$

14. A function $f(x)$ is said to be an infinitesimal as $x \to a$ if $\lim_{x \to a} f(x) =$

(a) 0 (b) a
(c) 1 (d) doesn't exist

Ans: *a*

Definition

EXERCISE – 1 (B)

Evaluate the following limits (1 – 22):

1. $\lim_{x \to 0} \frac{\sin bx}{x}$

Solution

$$\lim_{x \to 0} \frac{\sin bx}{x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \left(\frac{\sin bx}{bx} \times b\right) = 1 \times b = b$$

2. $\lim_{x \to 0} \frac{\tan 5x}{x}$

Solution

$$\lim_{x \to 0} \frac{\tan 5x}{x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \left[\frac{\sin 5x}{x.\cos 5x}\right] = \lim_{x \to 0} \left[\frac{\sin 5x}{5x}.5.\frac{1}{\cos 5x}\right] = 1.5.\frac{1}{\cos 0} = 5$$

Alternative method

$$\lim_{x \to 0} \frac{\tan 5x}{x} = \lim_{x \to 0} \left(\frac{\tan 5x}{5x} \times 5\right) = 1 \times 5 = 5$$

3. $\lim_{x \to 0} \frac{\sin 4x}{\sin 5x}$

Solution

$$\lim_{x \to 0} \frac{\sin 4x}{\sin 5x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$\lim_{x \to 0} \left(\frac{\frac{\sin 4x}{x}}{\frac{\sin 5x}{x}}\right) = \lim_{x \to 0} \left(\frac{\frac{\sin 4x}{4x} \times 4}{\frac{\sin 5x}{5x} \times 5}\right) = \frac{1 \times 4}{1 \times 5} = \frac{4}{5}$$

4. $\lim\limits_{x \to 0} \dfrac{\tan mx}{\tan nx}$

Solution

$$\lim_{x \to 0} \frac{\tan mx}{\tan nx} \left[\frac{0}{0} \text{ form}\right] = \lim_{x \to 0} \left(\frac{\frac{\tan mx}{x}}{\frac{\tan nx}{x}}\right) = \lim_{x \to 0} \left(\frac{\frac{\tan mx}{mx} \times m}{\frac{\tan nx}{nx} \times n}\right) = \frac{1 \times m}{1 \times n} = \frac{m}{n}$$

5. $\lim\limits_{x \to 0} \dfrac{\tan ax}{\sin bx}$

Solution

$$\lim_{x \to 0} \frac{\tan ax}{\sin bx} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{\frac{\tan ax}{x}}{\frac{\sin bx}{x}} = \lim_{x \to 0} \frac{\frac{\tan ax}{ax} \times a}{\frac{\sin bx}{bx} \times b} = \frac{1 \times a}{1 \times b} = \frac{a}{b}$$

6. $\lim\limits_{x \to b} \dfrac{\tan (x - b)}{x^2 - b^2}$

Solution

$$\lim_{x \to b} \frac{\tan (x - b)}{x^2 - b^2} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to b} \frac{\tan (x - b)}{(x - b)(x + b)} = \lim_{x - b \to 0} \frac{\tan (x - b)}{x - b} \lim_{x \to b} \frac{1}{x + b}$$

$$= 1 \cdot \frac{1}{b + b} = \frac{1}{2b}$$

7. $\lim\limits_{x \to 0} \dfrac{\sin ax \cos bx}{\tan cx}$

Solution

$$\lim_{x \to 0} \frac{\sin ax \cos bx}{\tan cx} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \left(\frac{\frac{\sin ax}{x} . \cos bx}{\frac{\tan cx}{x}}\right) = \lim_{x \to 0} \left(\frac{\frac{\sin ax}{ax} . a. \cos bx}{\frac{\tan cx}{cx} . c}\right) = \frac{1. a. \cos 0}{1.c} = \frac{a}{c}$$

8. $\lim\limits_{x \to 0} \dfrac{\sin 2x + \sin 6x}{\sin 5x - \sin 3x}$

Solution

$$\lim_{x \to 0} \frac{\sin 2x + \sin 6x}{\sin 5x - \sin 3x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \left(\frac{\frac{\sin 2x + \sin 6x}{x}}{\frac{\sin 5x - \sin 3x}{x}}\right) = \lim_{x \to 0} \left(\frac{\frac{\sin 2x}{2x} . 2 + \frac{\sin 6x}{6x} . 6}{\frac{\sin 5x}{5x} . 5 - \frac{\sin 3x}{3x} . 3}\right) = \frac{1 \times 2 + 1 \times 6}{1 \times 5 - 1 \times 3} = \frac{8}{2} = 4$$

9. $\lim\limits_{x \to 0} \left(\dfrac{\tan 3x - 2x}{3x - \sin 2x}\right)$

Solution

$$\lim_{x \to 0} \frac{\tan 3x - 2x}{3x - \sin 2x} \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \left\{\frac{\frac{\tan 3x - 2x}{x}}{\frac{3x - \sin 2x}{x}}\right\} = \lim_{x \to 0} \left(\frac{\frac{\tan 3x}{x} - \frac{2x}{x}}{\frac{3x}{x} - \frac{\sin 2x}{x}}\right) = \lim_{x \to 0} \frac{\frac{\tan 3x}{3x} . 3 - 2}{3 - \frac{\sin 2x}{2} . 2} = \frac{1.3 - 2}{3 - 1.2} = \frac{1}{1} = 1$$

10. $\lim\limits_{x \to 0} \dfrac{1-\cos x}{x^2}$

Solution

$$\lim_{x \to 0} \frac{1-\cos x}{x^2} \left[\frac{0}{0} \text{ form}\right] = \lim_{x \to 0} \frac{2\sin^2 x}{x^2} = \lim_{x \to 0} 2\left(\frac{\sin x}{x}\right)^2 = 2 \times 1^2 = 2$$

11. $\lim\limits_{x \to 0} \dfrac{1-\cos 3x}{1-\cos 5x}$

Solution

$$\lim_{x \to 0} \frac{1-\cos 3x}{1-\cos 5x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{2\sin^2 \frac{3x}{3}}{2\sin^2 \frac{5x}{2}} \qquad \left(1-\cos A = 2\sin^2 \frac{A}{2}\right)$$

$$= \lim_{x \to 0} \left[\frac{\sin \frac{3x}{2}}{\sin \frac{5x}{2}}\right]^2 = \lim_{x \to 0} \left[\frac{\frac{\sin \frac{3x}{2}}{\frac{3x}{2}} \cdot \frac{3x}{2}}{\frac{\sin \frac{5x}{2}}{\frac{5x}{2}} \cdot \frac{5x}{2}}\right]^2 = \left(\frac{1 \cdot \frac{3}{2}}{1 \cdot \frac{5}{2}}\right)^2 = \frac{9}{25}$$

12. $\lim\limits_{x \to 0} \dfrac{\tan x - \sin x}{x^3}$

Solution

Here, $\lim\limits_{x \to 0} \dfrac{\tan x - \sin x}{x^3} \left[\dfrac{0}{0} \text{ form}\right]$

$$= \lim_{x \to 0} \frac{\left(\frac{\sin x}{\cos x} - \sin x\right)}{x^3} = \lim_{x \to 0} \frac{(\sin x - \sin x \cos x)}{x^3 \cos x} = \lim_{x \to 0} \frac{\sin x\,(1-\cos x)}{x^3 \cos x}$$

$$= \lim_{x \to 0} \frac{\sin x\, 2\sin^2 x/2}{x^3 \cos x} = \lim_{x \to 0} 2\, \frac{\sin x}{x} \cdot \frac{\sin^2 x/2}{x^2} \cdot \frac{1}{\cos x}$$

$$= \lim_{x \to 0} \frac{2\sin x}{x} \left(\frac{\sin \frac{x}{2}}{\frac{x}{2}}\right)^2 \times \frac{1}{4} \frac{1}{\cos x} = 2 \times 1 \times 1^2 \times \frac{1}{4} \times \frac{1}{1} \left[\because \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1\right]$$

$$= \frac{1}{2}$$

13. $\lim\limits_{x \to \pi/4} \dfrac{\sec^2 x - 2}{\tan x - 1}$

Solution

$$\lim_{x \to \pi/4} \frac{\sec^2 x - 2}{\tan x - 1} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to \pi/4} \frac{1+\tan^2 x - 2}{\tan x - 1} = \lim_{x \to \pi/4} \frac{\tan^2 x - 1}{\tan x - 1} = \lim_{x \to \pi/4} \frac{(\tan x + 1)(\tan x - 1)}{(\tan x - 1)}$$

$$= \lim_{x \to \pi/4} (\tan x + 1) = \tan \frac{\pi}{4} + 1 = 1 + 1 = 2$$

14. $\lim\limits_{x \to 0} \dfrac{\cos mx - \cos nx}{x^2}$

Solution

$$\lim_{x \to 0} \frac{\cos mx - \cos nx}{x^2} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{2 \sin \frac{(m+n)x}{2} \sin \frac{(n-m)x}{2}}{x^2} = \lim_{x \to 0} 2\left(\frac{\sin \frac{(m+n)x}{2}}{x} \cdot \frac{\sin \frac{(n-m)x}{2}}{x}\right)$$

$$= \lim_{x \to 0} 2\left\{\frac{\sin \frac{(m+n)x}{2}}{\frac{(m+n)x}{2}} \cdot \frac{m+n}{2} \cdot \frac{\sin \frac{(n-m)x}{2}}{\frac{(n-m)x}{2}} \cdot \frac{(n-m)x}{2}\right\} = 2.\ 1.\ \frac{m+n}{2}\ .1.\ \frac{n-m}{2} =$$

$$\frac{n^2 - m^2}{2}$$

15. $\lim\limits_{x \to y} \dfrac{\cos x - \cos y}{x - y}$

Solution

$$\lim_{x \to y} \frac{\cos x - \cos y}{x - y} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to y} \frac{-2 \sin \frac{x+y}{2} . \sin \frac{x-y}{2}}{x - y} = \lim_{x \to y} \left\{-2 \sin \left(\frac{x+y}{2}\right) . \frac{\sin \left(\frac{x-y}{2}\right)}{x - y}\right\}$$

$$= -2 \lim_{x \to y} \sin \left(\frac{x+y}{2}\right) \lim_{\frac{x-y}{2} \to 0} \left\{\frac{\sin \left(\frac{x-y}{2}\right)}{\frac{x-y}{2} .2}\right\} = -2.\ \sin \left(\frac{y+y}{2}\right).\ 1.\ \frac{1}{2}$$

$$= -\sin y$$

16. $\lim\limits_{x \to y} \dfrac{\sin x - \sin y}{x - y}$

Solution

$$\lim_{x \to y} \frac{\sin x - \sin y}{x - y} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to y} \frac{2 \cos \left(\frac{x+y}{2}\right) \sin \left(\frac{x-y}{2}\right)}{x - y} = \lim_{x \to y} \left\{2 \cos \left(\frac{x+y}{2}\right) . \frac{\sin \left(\frac{x-y}{2}\right)}{x - y}\right\}$$

$$= \lim_{x \to y} \left\{2 \cos \left(\frac{x+y}{2}\right) . \frac{\sin \left(\frac{x-y}{2}\right)}{\frac{x-y}{2} 2}\right\} = 2.\ \cos \left(\frac{y+y}{2}\right).\ 1.\ \frac{1}{2} = \cos y$$

17. $\lim\limits_{x \to \theta} \dfrac{x \tan \theta - \theta \tan x}{x - \theta}$

Solution

$$\lim_{x \to \theta} \frac{x \tan \theta - \theta \tan x}{x - \theta} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to \theta} \frac{x \tan \theta - \theta \tan \theta + \theta \tan\theta - \theta \tan x}{x - \theta} = \lim_{x \to \theta} \left\{\frac{(x - \theta) \tan \theta}{x - \theta} + \frac{\theta(\tan \theta - \tan x)}{x - \theta}\right\}$$

$$= \lim_{x \to \theta} \left\{\tan \theta + \frac{\theta}{x - \theta}\left(\frac{\sin \theta}{\cos \theta} - \frac{\sin x}{\cos x}\right)\right\}$$

$$= \lim_{x \to \theta} \left\{ \tan\theta + \frac{\theta}{x-\theta} \left(\frac{\sin\theta\cos x - \cos\theta\sin x}{\cos\theta\cos x} \right) \right\}$$

$$= \lim_{x \to \theta} \left\{ \tan\theta + \frac{\theta}{x-\theta} \cdot \frac{\sin(\theta - x)}{\cos\theta\cos x} \right\}$$

$$= \tan\theta - \lim_{x \to \theta} \left\{ \frac{\sin(\theta - x)}{\theta - x} \cdot \frac{\theta}{\cos\theta\cos x} \right\} = \tan\theta - 1 \cdot \frac{\theta}{\cos\theta\cos\theta} = \tan\theta - \theta\sec^2\theta$$

18. $\lim\limits_{x \to \theta} \dfrac{x\sin\theta - \theta\sin x}{x - \theta}$

Solution

$$\lim_{x \to \theta} \frac{x\sin\theta - \theta\sin x}{x - \theta} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x \to \theta} \frac{x\sin\theta - \theta\sin\theta + \theta\sin\theta - \theta\sin x}{x - \theta}$$

$$= \lim_{x \to \theta} \left\{ \frac{(x-\theta)\sin\theta}{(x-\theta)} - \frac{\theta}{x-\theta}(\sin x - \sin\theta) \right\}$$

$$= \lim_{x \to \theta} \left\{ \sin\theta - \frac{\theta}{x-\theta} \cdot 2\cos\left(\frac{x+\theta}{2}\right) \cdot \sin\left(\frac{x-\theta}{2}\right) \right\}$$

$$= \lim_{x \to \theta} \left\{ \sin\theta - 2\theta\cos\left(\frac{x+\theta}{2}\right) \cdot \frac{\sin\left(\frac{x-\theta}{2}\right)}{\frac{(x-\theta)}{2} \times 2} \right\} = \sin\theta - \theta\cos\left(\frac{\theta+\theta}{2}\right) \times 1$$

$$= \sin\theta - \theta\cos\theta$$

19. $\lim\limits_{x \to \theta} \dfrac{x\cos\theta - \theta\cos x}{x - \theta}$

Solution

$$\lim_{x \to \theta} \frac{x\cos\theta - \theta\cos x}{x - \theta} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x \to \theta} \frac{x\cos\theta - \theta\cos\theta + \theta\cos\theta - \theta\cos x}{x - \theta}$$

$$= \lim_{x \to \theta} \left\{ \frac{(x-\theta)\cos\theta}{x-\theta} + \frac{\theta}{x-\theta}(\cos\theta - \cos x) \right\}$$

$$= \lim_{x \to \theta} \left\{ \cos\theta + \frac{\theta}{x-\theta} 2\sin\left(\frac{x+\theta}{2}\right) \cdot \sin\left(\frac{x-\theta}{2}\right) \right\}$$

$$= \lim_{x \to \theta} \left\{ \cos\theta + 2\theta\sin\left(\frac{x+\theta}{2}\right) \frac{\sin\left(\frac{x-\theta}{2}\right)}{\frac{x-\theta}{2} \times 2} \right\} = \cos\theta + \theta\sin\theta.$$

20. $\lim\limits_{y \to 0} \dfrac{(x+y)\sec(x+y) - x\sec x}{y}$

Solution

$$\lim_{y \to 0} \frac{(x+y)\sec(x+y) - x\sec x}{y} \qquad \left(\frac{0}{0}\text{ from}\right)$$

$$= \lim_{y \to 0} \frac{x\sec(x+y) + y\sec(x+y) - x\sec x}{y}$$

$$= \lim_{y \to 0} \left[\sec(x+y) + \frac{x\{\sec(x+y) - \sec x\}}{y} \right]$$

$$= \sec x + \lim_{y \to 0} \frac{x \left\{ \frac{1}{\cos(x+y)} - \frac{1}{\cos x} \right\}}{y}$$

$$= \sec x + \lim_{y \to 0} \frac{x \{\cos x - \cos(x+y)\}}{y \cos x \cos(x+y)}$$

$$= \sec x + \lim_{y \to 0} \frac{2x \sin\left(\frac{x+y+x}{2}\right) \sin\left(\frac{x+y-x}{2}\right)}{y \cos x \cos(x+y)}$$

$$= \sec x + \lim_{y \to 0} \frac{2x \sin\left(\frac{2x+y}{2}\right)}{\cos x \cos(x+y)} \cdot \frac{\sin\frac{y}{2}}{\frac{y}{2} \times 2} = \sec x + \frac{x \sin x}{\cos x \cos x} = \sec x + x \tan x \sec x$$

$$= \sec x(1 + x \tan x)$$

21. $\lim_{x \to \pi/4} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$

Solution

$$\lim_{\theta \to \pi/4} \frac{\cos\theta - \sin\theta}{\theta - \frac{\pi}{4}} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{\theta \to \pi/4} \lim_{x \to \pi/4} \frac{\sqrt{2}\left(\frac{1}{\sqrt{2}} \cos\theta - \frac{1}{\sqrt{2}} \sin\theta\right)}{\theta - \frac{\pi}{4}}$$

$$= \lim_{\theta \to \pi/4} \frac{\sqrt{2}\left[\sin\frac{\pi}{4} \cos\theta - \cos\frac{\pi}{4} \sin\theta\right]}{\theta - \frac{\pi}{4}} = \lim_{\theta \to \pi/4} \frac{\sqrt{2} \sin\left(\frac{\pi}{4} - \theta\right)}{\theta - \frac{\pi}{4}}$$

$$= \lim_{\theta - \pi/4 \to 0} \frac{\sqrt{2} \sin\left(\frac{\pi}{4} - \theta\right)}{-\left(\frac{\pi}{4} - \theta\right)} = -\sqrt{2} \times 1 = -\sqrt{2}$$

22. $\lim_{x \to c} \frac{\sin x - \sin c}{\sqrt{x} - \sqrt{c}}$

Solution

$$\lim_{x \to c} \frac{\sin x - \sin c}{\sqrt{x} - \sqrt{c}} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to c} \frac{2 \cos\left(\frac{x+c}{2}\right) \sin\left(\frac{x-c}{2}\right)}{(\sqrt{x} - \sqrt{c})} \times \frac{\sqrt{x} + \sqrt{c}}{\sqrt{x} + \sqrt{x}}$$

$$= \lim_{x \to c} \frac{2 \cos\left(\frac{x+c}{2}\right) \sin\left(\frac{x-c}{2}\right)}{x - c} (\sqrt{x} + \sqrt{c})$$

$$= \lim_{x \to c} 2 \cos\left(\frac{x+c}{2}\right) (\sqrt{x} + \sqrt{c}) \lim_{\frac{x-c}{2} \to 0} \left(\frac{\sin\frac{x-c}{2}}{\frac{x-c}{2} \cdot 2} \right)$$

$$= 2 \cos\left(\frac{c+c}{2}\right) (\sqrt{c} + \sqrt{c}) \cdot 1 \cdot \frac{1}{2} = 2\sqrt{c} \cos c.$$

Evaluate (23 – 29)

23. $\lim\limits_{x\to 0} \dfrac{e^x - 1}{3x}$

Solution

$$\lim_{x\to 0} \frac{e^x - 1}{3x} = \lim_{x\to 0} \left(\frac{e^x - 1}{x} \cdot \frac{1}{3}\right) = 1 \cdot \frac{1}{3} = \frac{1}{3}$$

24. $\lim\limits_{x\to 0} \dfrac{e^{5x} - 1}{x \cdot 3^x}$

Solution

$$\lim_{x\to 0} \frac{e^{5x} - 1}{x \cdot 3^x} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \left(\frac{e^{5x} - 1}{5x} \times 5 \times \frac{1}{3^x}\right) = 1 \times 5 \times \frac{1}{3^0} = 5$$

25. $\lim\limits_{x\to 0} \dfrac{e^{2+x} - e^2}{x}$

Solution

$$\lim_{x\to 0} \frac{e^{2+x} - e^2}{x} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{e^2 . e^x - e^2}{x} = \lim_{x\to 0} \frac{e^2(e^x - 1)}{x} = e^2 \lim_{x\to 0} \frac{e^x - 1}{x} = e^2 . 1 = e^2$$

26. $\lim\limits_{x\to 0} \dfrac{\log_e(1+x)}{e^x - 1}$

Solution

$$\lim_{x\to 0} \frac{\log_e(1+x)}{e^x - 1} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \left(\frac{\frac{\log_e(1+x)}{x}}{\frac{e^x - 1}{x}}\right) = \frac{1}{1} = 1$$

27. $\lim\limits_{x\to 0} \dfrac{\log_e(1+5x)}{2x}$

Solution

$$\lim_{x\to 0} \frac{\log_e(1+5x)}{2x} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \left\{\frac{\log_e(1+5x)}{5x} \cdot \frac{5x}{2x}\right\} = 1 \cdot \frac{5}{2} = \frac{5}{2}$$

28. $\lim\limits_{x\to 0} \dfrac{a^x - b^x}{x}$

Solution

$$\lim_{x\to 0} \frac{a^x - b^x}{x} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x\to 0} \frac{a^x - b^x - 1 + 1}{x} = \lim_{x\to 0} \frac{a^x - 1 - (b^x - 1)}{x} = \lim_{x\to 0} \left(\frac{a^x - 1}{x} - \frac{b^x - 1}{x}\right)$$

$$= \log_e a - \log_e b = \log_e \left(\frac{a}{b}\right)$$

29. $\lim\limits_{x \to 0} \dfrac{3^{2x} - 2^{3x}}{x}$

Solution

$$\lim_{x \to 0} \frac{3^{2x} - 2^{3x}}{x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{3^{2x} - 1 + 1 - 2^{3x}}{x} = \lim_{x \to 0} \frac{3^{2x} - 1 - (2^{3x} - 1)}{x} = \lim_{x \to 0} \left\{ \frac{3^{2x} - 1}{2x} \times 2 - \frac{2^{3x} - 1}{3x} \times 3 \right\}$$

$$= 2 \log_e 3 - 3 \log_e 2 = \log_e 3^2 - \log_e 2^3 = \log_e 9 - \log_e 8 = \log_e \left(\frac{9}{8}\right)$$

Objective Questions

1. $\lim\limits_{x \to 0} \cos x =$

(a) 0 (b) 1

(c) $\frac{1}{2}$ (d) 2

Ans: b

$$\lim_{x \to 0} \cos x = \cos 0 = 1$$

2. $\lim\limits_{\theta \to 0} \dfrac{\tan \theta}{\theta} =$

(a) 0 (b) $\frac{1}{2}$

(c) $\frac{2}{3}$ (d) 1

Ans: d

$$\lim_{\theta \to 0} \frac{\tan \theta}{\theta} = \lim_{\theta \to 0} \frac{\sin \theta}{\cos\theta . \theta}$$

$$= \lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta} . \frac{1}{\cos\theta} \right)$$

$$= 1 . \frac{1}{\cos 0} = 1$$

3. $\lim\limits_{x \to 0} \dfrac{\sin x^o}{x} =$

(a) 1 (b) $\frac{\pi}{180}$

(c) $\frac{180}{\pi}$ (d) 0

Ans: b

$$\lim_{x \to 0} \frac{\sin x^o}{x}$$

$$= \lim_{x \to 0} \frac{\sin \frac{\pi x}{180}}{x} \qquad \left[\because 1^o = \frac{\pi^c}{180} \right]$$

$$= \lim_{x \to 0} \frac{\sin \frac{\pi x}{180}}{\frac{\pi x}{180}} \times \frac{\pi}{180} = \lim_{x \to 0} \left(\frac{\sin \frac{\pi x}{180}}{\frac{\pi x}{180}} \right) \times \frac{\pi}{180} = \frac{\pi}{180} \lim_{\frac{\pi x}{180} \to 0} \left(\frac{\sin \frac{\pi x}{180}}{\frac{\pi x}{180}} \right) = \frac{\pi}{180} . 1 = \frac{\pi}{180}$$

4. $\lim\limits_{x \to 0} \frac{\sin x^{\circ}}{x} =$

(a) 1 (b) $\frac{200}{\pi}$

(c) $\frac{\pi}{200}$ (d) 0

Ans: c

$$= \lim_{x \to 0} \frac{\sin\left(\frac{\pi x}{200}\right)}{\frac{\pi x}{200}} \times \frac{\pi}{200} = 1 \times \frac{\pi}{200} = \frac{\pi}{200}$$

5. $\lim\limits_{x \to 0} \frac{\sin ax}{\sin bx} =$

(a) $\frac{b}{a}$ (b) $\frac{a}{b}$

(c) a (d) b

Ans: b

$$\lim_{x \to 0} \frac{\sin ax}{\sin bx} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{\frac{\sin ax}{ax} \times ax}{\frac{\sin bx}{bx} \times bx} = \frac{1 \times a}{1 \times b} = \frac{a}{b}$$

6. $\lim\limits_{x \to 0} \frac{1 - \cos 2x}{x^2} =$

(a) 2 (b) $\frac{1}{2}$

(c) 3 (d) $\frac{1}{3}$

Ans: a

$$\lim_{x \to 0} \frac{1 - \cos 2x}{x^2} = \lim_{x \to 0} \frac{2 \sin^2 x}{x^2} = 2 \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 = 2 \times 1^2 = 2$$

7. **In $\lim\limits_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$, the angle θ is measured in**

(a) degree (b) grade

(c) radian (d) none

Ans: c

8. $\lim\limits_{x \to \pi/2} (\sec x - \tan x) =$

(a) 0 (b) 1

(c) 2 (d) 4

Ans: a

$$\lim_{x \to \pi/2} (\sec x - \tan x) \qquad [\infty - \infty \text{ form}]$$

$$\lim_{x \to \pi/2} \frac{(\sec x - \tan x)(\sec x + \tan x)}{\sec x + \tan x} = \lim_{x \to \pi/2} \frac{1}{\sec x + \tan x} = \frac{1}{\sec\frac{\pi}{2} + \tan\frac{\pi}{2}} = \frac{1}{\infty} = 0$$

9. $\lim_{x \to 0} \dfrac{\tan \alpha x}{\sin \beta x} =$

(a) α (b) β

(c) $\dfrac{\alpha}{\beta}$ (d) $\dfrac{\beta}{\alpha}$

Ans: c

$$\lim_{x \to 0} \frac{\tan \alpha x}{\sin \beta x} = \lim_{x \to 0} \left(\frac{\frac{\tan \alpha x}{\alpha x} \alpha x}{\frac{\tan \beta x}{\beta x} \beta x} \right) = \frac{1}{1} \frac{\alpha}{\beta} = \frac{\alpha}{\beta}$$

10. $\lim_{x \to \pi} \dfrac{1 + \cos x}{\tan^2 x} =$

(a) 2 (b) 1

(c) 0 (d) $\dfrac{1}{2}$

Ans: d

$$\lim_{x \to \pi} \frac{1 + \cos x}{\tan^2 x} = \lim_{x \to \pi} \frac{1 + \cos x}{\tan^2 x} \times \frac{1 - \cos x}{1 - \cos x} = \lim_{x \to \pi} \frac{1 - \cos^2 x}{\tan^2 x (1 - \cos x)}$$

$$= \lim_{x \to \pi} \frac{\sin^2 x}{\frac{\sin^2 x}{\cos^2 x}(1 - \cos x)} = \lim_{x \to \pi} \frac{\cos^2 x}{1 - \cos x} = \frac{\cos^2 \pi}{1 - \cos \pi} = \frac{(-1)^2}{1 - (-1)} = \frac{1}{2}$$

11. $\lim_{x \to 0} \dfrac{\sin x \cos x}{3x} =$

(a) 3 (b) 1

(c) $\dfrac{1}{3}$ (d) 0

Ans: c

$$\lim_{x \to 0} \frac{\sin x \cos x}{3x} = \lim_{x \to 0} \frac{\sin x \cos x}{3x} = \lim_{x \to 0} \left(\frac{\sin x}{x} \cdot \frac{1}{3} \cdot \cos x \right) = 1 . \frac{1}{3} . \cos 0 = \frac{1}{3}$$

12. $\lim_{x \to 0} \sin \dfrac{1}{x} =$

(a) 0 (b) 1

(c) -1 (d) doesn't exist

Ans: d

13. $\lim_{x \to 0} \dfrac{e^{2x} - 1}{x} =$

(a) 1 (b) 2

(c) $\dfrac{1}{2}$ (d) 0

Ans: b

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x} = \lim_{x \to 0} \frac{e^{2x} - 1}{2x} . 2 = 1 . 2 = 2$$

14. $\lim_{x \to 0} \dfrac{e^x - x - 1}{x} =$

(a) 0 (b) 1

(c) 2 (d) 4

Ans: a

$$\lim_{x \to 0} \frac{e^x - x - 1}{x} = \lim_{x \to 0} \left(\frac{e^x - 1}{x} - 1 \right) = 1 - 1 = 0$$

15. $\lim_{x \to 0} \frac{3^x - 1}{x} =$

(a) $\log_e 3$ (b) 3

(c) $\log_e \frac{1}{3}$ (d) 0

Ans: a

$$\lim_{x \to 0} \frac{3^x - 1}{x} = \log_e 3$$

16. $\lim_{x \to 1} \frac{\log x}{x - 1} =$

(a) 0 (b) $\frac{1}{2}$

(c) $\frac{1}{3}$ (d) 1

Ans: d

$$\lim_{x \to 1} \frac{\log x}{x - 1} = \lim_{x \to 1} \frac{\log_e (x - 1 + 1)}{x - 1} = \lim_{x - 1 \to 0} \frac{\log (1 + (x - 1))}{x - 1} = 1$$

17. $\lim_{x \to \infty} \frac{\sin x}{x} =$

(a) 1 (b) -1

(c) 0 (d) 2

Ans: c

Put $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$

Then, as $x \to \infty$, $y \to 0$

$$\lim_{y \to 0} \frac{\sin \frac{1}{y}}{\frac{1}{y}} = \lim_{y \to 0} y \sin \frac{1}{y} = 0$$

EXERCISE – 1 (C)

1. **Find the limits at the points specified.**

(a) $f(x) = \begin{cases} 4x + 2 & \text{for } x \geq 0 \\ 2 & \text{for } x < 0 \end{cases}$ at $x = 0$

(b) $f(x) = \begin{cases} 2x - 3 & \text{for } x \geq 1 \\ x & \text{for } x < 1 \end{cases}$ at $x = 1$

(c) $f(x) = \begin{cases} 7x + 1 & \text{for } x \geq 2 \\ x^2 + 11 & \text{for } x < 2 \end{cases}$ at $x = 2$

(d) $f(x) = \begin{cases} 3x - 1 & \text{for } x < 1 \\ 5 & \text{for } x = 1 \\ 2x & \text{for } x > 1 \end{cases}$ at $x = 1$

Solution

Let hand limit at $x = 0$ is

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2) = 2$$

Right hand limit at $x = 0$ is

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (4x + 2) = 4 \times 0 + 2 = 2$$

Here, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x)$

Thus $\lim_{x \to 0} f(x)$ exists and $\lim_{x \to 0} f(x) = 2$

(b) L.H.L $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x) = 1$

R.H.L $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x - 3) = 2 \times 1 - 3 = -1$

$\therefore$ L.H.L $\neq$ R.H.L

(c) L.H.L $= \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2 + 11) = 2^2 + 11 = 15$

R.H.L $= \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (7x + 1) = 7 \times 2 + 1 = 15$

$\therefore$ L.H.L = R.H.L

Hence the limit exists.

Thus, $\lim_{x \to 2} f(x) = 15$

(d) L.H.L $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (3x - 1) = 3 \times 1 - 1 = 2$

R.H.L $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2x = 2 \times 1 = 2$

$\therefore$ L.H.L = R.H.L. So, limit exists

Hence, $\lim_{x \to 1} f(x) = 2$.

2. Evaluate the following limits:

(a) $\lim_{x \to 3} |x - 3|$ (b) $\lim_{x \to 0} \frac{x}{|x|}$

Solution

(a) $\lim_{x \to 3} |x - 3|$

Let $f(x) = |x - 3|$

By definition, we have,

$$f(x) = |x - 3| = \begin{cases} x - 3 \text{ if } x - 3 \geq 0 \\ -(x - 3) \text{ if } x - 3 < 0 \end{cases}$$

$$\therefore \quad f(x) = \begin{cases} x - 3 \text{ if } x \geq 3 \\ 3 - x \text{ if } x < 3 \end{cases}$$

Then,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (3 - x) = 3 - 3 = 0$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x - 3) = 3 - 3 = 0$$

Here, $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x)$.

So, $\lim_{x \to 3} f(x)$ exists and $\lim_{x \to 3} f(x) = 0$

(b)

Let $f(x) = \frac{x}{|x|}$

By Definition, we have,

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

$$\text{Thus, } f(x) = \frac{x}{|x|} = \begin{cases} \frac{x}{x} \text{ if } x \geq 0 \\ \frac{x}{-x} \text{ if } x < 0 \end{cases}$$

Then, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{x}{-x} = -1$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{x}{x} = 1$

Since $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$, so

$\lim_{x \to 0} f(x)$ does not exist

Objective Questions

1. $\lim_{x \to a^-} f(x)$ is same as

(a) $\lim_{h \to 0} f(a - h)$ (b) $\lim_{h \to 0} f(a + h)$

(c) $\lim_{h \to 0} f(ah)$ (d) $\lim_{h \to 0} f(a \div h)$

Ans: a

2. $\lim_{x \to a^+} f(x)$ is same as

(a) $\lim_{h \to 0} f(a - h)$ (b) $\lim_{h \to 0} f(a + h)$

(c) $\lim_{h \to 0} f(ah)$ (d) $\lim_{h \to 0} f(a \div h)$

Ans: b

3. A function $f(x)$ is defined by $f(x) = \begin{cases} 2x + 1 & x \geq 1 \\ x + 4 & x < 1 \end{cases}$ then $\lim_{x \to 1^+} f(x) =$

(a) 5 (b) 1

(c) 0 (d) 3

Ans: d

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x + 1) = 2 \times 1 + 1 = 3$

4. A function $f(x)$ is defined by $f(x) = \begin{cases} 2x - 3 \text{ for } x \geq 3 \\ 3x - 2 \text{ for } x < 3 \end{cases}$ then $\lim_{x \to 3^-} f(x) =$

(a) 5 (b) 7

(c) 0 (d) 3

Ans: d

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (3x - 2) = 3 \times 3 - 2 = 7$

5. If $f(x) = \begin{cases} x + 1 & \text{if } x \geq 1 \\ 2x & \text{if } x < 1 \end{cases}$ then $\lim_{x \to 1} f(x) =$

(a) 1 (b) 2

(c) 4 (d) 8

Ans: b

$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2x = 2 \times 1 = 2$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x + 1) = 1 + 1 = 2$

$\lim_{x \to 1} f(x) = 2$

6. If $f(x) = \begin{cases} 2x + 3 \text{ if } x < 2 \\ 3x - 5 \text{ if } x \geq 2 \end{cases}$ then $\lim_{x \to 2} f(x) =$

(a) 7
(b) 1
(c) 0
(d) the limit doesn't exist

Ans: d

$$\text{L.H.L} = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x + 3) = 2 \times 2 + 3 = 7$$

$$\text{R.H.L} = \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x - 5) = 3 \times 2 - 5 = 1$$

$\therefore$ L.H.L $\neq$ R.H.L. So, limit does not exist

7. $\lim_{x \to 0} |x| =$

(a) 0
(b) 1
(c) 2
(d) the limit doesn't exist

Ans: a

Let $f(x) = |x|$

We have, $|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 2 \end{cases}$

$$\text{L.H.L} = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-x) = 0$$

$$\text{R.H.L} = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$$

$$\therefore \lim_{x \to 0} |x| = 0$$

8. $\lim_{x \to 1} \dfrac{x - 1}{|x - 1|} =$

(a) 0
(b) 1
(c) 2
(d) the limit doesn't exist

Ans: d

Let $f(x) = \dfrac{x - 1}{|x - 1|}$

We have $|x - 1| = \begin{cases} x - 1 \text{ if } x - 1 \geq 0 \\ -(x - 1) \text{ if } x - 1 < 0 \end{cases} = \begin{cases} x - 1 \text{ if } x \geq 1 \\ -(x - 1) \text{ if } x < 1 \end{cases}$

$$\text{L.H.L} = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{x - 1}{-(x - 1)} = -1$$

$$\text{R.H.L} = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{x - 1}{x - 1} = 1$$

$\therefore$ L.H.L. $\neq$ R.H.L

Thus limit does not exist.

9. $\lim_{x \to 1^-} \dfrac{1}{x - 1} =$

(a) ∞
(b) $-\infty$
(c) 0
(d) 1

Ans: b

$$\lim_{x \to 1^-} f(x) = \lim_{h \to 0} f(1 - h)$$

$$= \lim_{h \to 0} \frac{1}{(1 - h) - 1} = \lim_{h \to 0} -\frac{1}{h} = -\infty$$

EXERCISE 1 (D)

1. Test the continuity or discontinuity of the following functions at the points specified.

(a) $f(x) = x^3 + 3x + 2$ at $x = 1$ (b) $f(x) = 7 - x^2$ at $x = 0$

(c) $f(x) = \dfrac{1}{3x}$ at $x = 0$ (d) $f(x) = \dfrac{1}{1-x}$ at $x = 1$

(e) $f(x) = \dfrac{1}{x-3}$ at $x \neq 3$ (f) $f(x) = \dfrac{x^2-4}{x-2}$ at $x = 2$

(g) $f(x) = \dfrac{|x-1|}{x-1}$ at $x = 1$.

Solution

(a) Left hand limit at x = 1 is

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^3 + 3x + 2) = 1^3 + 3 \times 1 + 2 = 6$$

Right hand limit at x = 1 is

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^3 + 3x + 2) = 1^3 + 3 \times 1 + 2 = 6$$

Hence $\lim_{x \to 1^-} f(x)$ and $\lim_{x \to 1^+} f(x)$ are finite and equal.

So, $\lim_{x \to 1} f(x)$ exists and $\lim_{x \to 1} f(x) = 6$

Also, $f(1) = 1^3 + 3 \times 1 + 2 = 6$

$\therefore \lim_{x \to 1} f(x) = f(1)$. Hence f(x) is continuous at x = 1

(b) Left hand limit at x = 0 is

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (7 - x^2) = 7 - 0 = 7$$

Right hand limit at x = 0 is

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (7 - x^2) = 7 - 0 = 7$$

Hence, $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$ are finite and equal.

Thus, $\lim_{x \to 0^-} f(x) = 7$

Also, $f(0) = 7 - 0^2 = 7$

$\therefore \lim_{x \to 0} f(x) = f(0)$

Thus, f(x) is continuous at x = 0.

(c) left hand limit at x = 0 is

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{3x} = -\infty$$

i.e $\lim_{x \to 0^-} f(x)$ does not exist

So, f(x) is not continuous at x = 0.

Alternative method

$$f(0) = \frac{1}{3 \times 0} = \infty$$

i.e. f(0) does not exist

So, f(x) is not continuous at x = 0.

(d) Here, $f(x) = \frac{1}{1-x}$

Functional value $= f(1) = \frac{1}{1-1} = \frac{1}{0} = \infty$

i.e. f(1) does not exist.
So, f(x) is not continuous at x = 1.

(e) Here, $f(x) = \frac{1}{x-3}$, $x \neq 3$.

Let x = a where $a \neq 3$.

Left hand limit $= \lim_{x \to a^-} f(x) = \lim_{x \to a^-} \frac{1}{x-3} = \frac{1}{a-3}$

Right hand limit $= \lim_{x \to a^+} f(x) = \lim_{x \to a^+} \frac{1}{x-3} = \frac{1}{a-3}$

Functional value $= f(a) = \frac{1}{a-3}$

$\therefore$ L.H.L = R.H.L = f(a) at $x = a \neq 3$.
So, f(x) is continuous at $x \neq 3$.

(f) Left hand limit at x = 2 is

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2-4}{x-2} = \lim_{x \to 2^-} \frac{(x-2)(x+2)}{x-2} = 2 + 2 = 4$$

Right hand limit at x = 2 is

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x^2-4}{x-2} = \lim_{x \to 2^+} \frac{(x-2)(x+2)}{(x-2)} = 2 + 2 = 4$$

$\therefore$ L.H.L = R.H.L i.e. $\lim_{x \to 2} f(x)$ exists.

Functional value $= f(2) = \frac{2^2-4}{2-2} = \frac{0}{0}$

which is an indeterminate form.
Hence f(x) is discontinuous at x = 2

(g) $f(1) = \frac{|1-1|}{1-1} = \frac{0}{0}$ which is an indeterminate form.

So, f(x) is discontinuous at x = 1

2. Are the following functions continuous at the points mentioned?

(a) $f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{when } x \neq 2 \\ 4 & \text{when } x = 2 \end{cases}$ at $x = 2$

(b) $f(x) = \begin{cases} \frac{x^2-3x}{x-3} & x \neq 3 \\ 3 & x = 3 \end{cases}$ at $x = 3$

(c) $f(x) = \begin{cases} \frac{x^2-x-6}{x-3} & \text{if } x \neq 3 \\ 3 & \text{if } x = 3 \end{cases}$ at $x = 3$

Solution

(a) Left hand limit at x = 2 is

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2-4}{x-2} = \lim_{x \to 2^-} \frac{(x-1)(x+2)}{x-2} = \lim_{x \to 2^-} (x+2) = 2 + 2 = 4.$$

Right hand limit at x = 2 is

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x^2-4}{x-2} = \lim_{x \to 2^+} \frac{(x-2)(x+2)}{x-2} = \lim_{x \to 2^+} (x+2) = 2 + 2 = 4$$

Hence $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$ are finite and equal.

So $\lim_{x \to 2}$ f(x) exists and $\lim_{x \to 2} f(x) = 4$

Also, f(2) = 4

$$\lim_{x \to 2} f(x) = f(2)$$

Hence f(x) is continuous at x = 2

(b) L.H.L $= \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{x^2 - 3x}{x - 3} = \lim_{x \to 3^-} \frac{x(x-3)}{x-3} = \lim_{x \to 3^-} (x) = 3$

R.H.L. $= \lim_{x \to 3^+} f(x) = = \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \frac{x^2 - 3x}{x - 3} = \lim_{x \to 3^+} \frac{x(x-3)}{x-3}$

$= \lim_{x \to 3^+} (x) = 3$

And, f(3) = 3

$\therefore$ L.H.L = R.H.L = f(3)

So, f(x) is continuous at x = 3

(c) L.H.L. $= \lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{x^2 - x - 6}{x - 3}$

$= \lim_{x \to 3^-} \frac{x^2 - 3x + 2x - 6}{x - 3} = \lim_{x \to 3^-} \frac{x(x-3) + 2(x-3)}{x - 3} = \lim_{x \to 3^-} \frac{(x-3)(x+2)}{x-3}$

R.H.L. $= \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \frac{x^2 - x - 6}{x - 3}$

$= \lim_{x \to 3^+} \frac{x^2 - 3x + 2x - 6}{x - 3} = \lim_{x \to 3^+} \frac{x(x-3) + 2(x-3)}{x - 3} = \lim_{x \to 3^+} \frac{(x-3)(x+2)}{x-3}$

= 3 + 2 = 5

And, f(3) = 3

Here, L.H.L = R.H.L ≠ f(3)

So, f(x) is not continuous at x = 3

3. **Discuss the continuity of the function at the points specified.**

(a) $f(x) = \begin{cases} 2 - x^2 & \text{for } x \le 1 \\ x & \text{for } x > 1 \end{cases}$ at $x = 1$

(b) $f(x) = \begin{cases} 3x^2 + 5 & \text{for } x \ge 2 \\ 2x + 11 & \text{for } x < 2 \end{cases}$ at $x = 2$

(c) $f(x) = \begin{cases} 2x - 1 & \text{for } x < 2 \\ 3 & \text{for } x = 2 \\ x + 1 & \text{for } x > 2 \end{cases}$ at $x = 2$

(d) $f(x) = \begin{cases} 3 + 2x & \text{for } -\frac{3}{2} \le x < 0 \\ 3 - 2x & \text{for } 0 \le x < \frac{3}{2} \\ -3 - 2x & \text{for } x \ge \frac{3}{2} \end{cases}$ (i) at $x = 0$ (ii) at $x = \frac{3}{2}$

Solution

(a) L.H.L $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2 - x^2) = 2 - 1^2 = 1$

R.H.L. $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x) = 1$

Functional value = f(1) = $2 - 1^2 = 1$

$\therefore \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$

So, f(x) is continuous at x = 1.

(b) L.H.L. = $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x + 11) = 2 \times 2 + 11 = 15$

R.H.L. = $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (3x^2 + 5) = 3 \times 2^2 + 5 = 17$

Here, $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$.

So, f(x) is discontinuous at x = 2.

(c) When $x = 2$, f(2) = 3

Again, $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2x - 1) = 2 \times 2 - 1 = 3$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} x + 1 = 2 + 1 = 3$

$\therefore \lim_{x \to 2} f(x) = 3$

Here, $\lim_{x \to 2} f(x) = f(3) = 3$

So, the given function f(x) is continuous at $x = 2$.

(d) At x = 0

L.H.L = $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (3 + 2x) = 3 + 2 \times 0 = 3$

R.H.L. = $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (3 - 2x) = 3 - 2 \times 0 = 3$

Functional value f(0) = $3 - 2 \times 0 = 3$

$\therefore$ L.H.L = R.H.L = functional value

So, f(x) is continuous at x = 0.

At $x = \frac{3}{2}$

L.H.L. = $\lim_{x \to 3/2^-} f(x) = \lim_{x \to 3/2^-} (3 - 2x) = 3 - 2 \times \frac{3}{2} = 0$

R.H.L. = $\lim_{x \to 3/2^+} f(x) = \lim_{x \to 3/2^+} (-3 - 2x) = -3 - 2 \times \frac{3}{2} = -6.$

$\therefore$ L.H.L $\neq$ R.H.L

So, f(x) is discontinuous at $x = \frac{3}{2}$.

4. Find the points of discontinuity of the following functions:

(a) $f(x) = \dfrac{x + 5}{x - 4}$ **(b)** $f(x) = \dfrac{x^2}{x^2 - 3x + 2}$

(a) Here, $f(x) = \dfrac{x + 5}{x - 4}$

The function f(x) will not be defined and hence discontinuous at the points where the denominator is 0.

i.e. $x - 4 = 0$

$\therefore$ $x = 4$

(b) Here, $f(x) = \dfrac{x^2}{x^2 - 3x + 2}$

The function f(x) will not be defined and hence discontinuous at the points where the denominator is 0.

i.e. $x^2 - 3x + 2 = 0$

or, $x^2 - 2x - x + 2 = 0$

or, $x(x - 2) - 1(x - 2) = 0$

or, $x(x - 2) - 1(x - 2) = 0$

or, $(x - 1)(x - 2) = 0$

$\therefore$ $x = 1, 2$

5. Determine the value of the constant so that the given function is continuous at the point mentioned.

(a) $f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & \text{if } x \neq 3 \\ k & \text{if } x = 3 \end{cases}$ at $x = 3$

(b) $f(x) = \begin{cases} 2x-1 & \text{if } x < 2 \\ k & \text{if } x = 0 \\ x+1 & \text{if } x > 2 \end{cases}$ at $x = 2$

(c) $f(x) = \begin{cases} 2ax+3 & \text{if } x < 1 \\ 1-ax^2 & \text{if } x \geq 1 \end{cases}$ at $x = 1$

Solution

(a) When $x = 3$, $f(3) = k$

and $\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3} \dfrac{x^2-9}{x-3}$

$= \lim\limits_{x\to 3} \dfrac{(x+3)(x-3)}{x-3} = \lim\limits_{x\to 3} (x+3) = 3+3 = 6$

$\therefore$ Since $f(x)$ is continuous at $x = 3$, so

So, $\lim\limits_{x\to 3} f(x) = f(3)$

or, $3 = k$

$\therefore$ $k = 6$

(b) When $x = 2$, $f(2) = k$

$\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2} 2x-1 \quad 4-1 = 3$

Since $f(x)$ is continuous at $x = 2$, so, $\lim\limits_{x\to 2} f(x) = f(2)$

or, $3 = k$

$\therefore$ $k = 3$

(c) Since $f(x)$ is continuous, so

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1} f(x) = f(1)$

i.e. $\lim\limits_{x\to 1^-} (2ax+3) = \lim\limits_{x\to 1^+} (1-ax^2)$

or, $2a+3 = 1-a$

or, $3a = -2$

$\therefore$ $a = -\dfrac{2}{3}$.

6. (a) A function $f(x)$ is defined as follows

$f(x) = \begin{cases} 2x+3 & \text{for } x < 1 \\ 4 & \text{for } x < 1 \\ 6x-1 & \text{for } x < 1 \end{cases}$

Is the function $f(x)$ continuous at $x = 1$? If not, state how can you make it continuous at $x = 1$.

(b) Let a function $f(x)$ be defined by

$f(x) = \begin{cases} 2-x^2 & \text{for } x < 2 \\ 3 & \text{for } x = 2 \\ x-4 & \text{for } x > 2 \end{cases}$

Verify that the limit of the function exists at $x = 2$. Is the function continuous at $x = 2$? State how you can make it continuous.

Solution

(a) L.H.L. $= \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (2x + 3) = 2 \times 1 + 3 = 5$

R.H.L. $= \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (6x - 1) = 6 \times 1 - 1 = 5$

Functional value $f(1) = 4$

$\therefore \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) \neq f(1)$

So, $f(x)$ is discontinuous at $x = 1$. The function $f(x)$ can be made continuous at $x = 1$ by redefining as follows;

$$f(x) = \begin{cases} 2x + 3 & \text{for } x < 1 \\ 5 & \text{for } x = 1 \\ 6x - 1 & \text{for } x > 1 \end{cases}$$

(b) L.H.L. $= \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (2 - x^2) = 2 - 2^2 = -2$

R.H.L. $= \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x - 4) = 2 - 4 = -2$

Functional value $= f(2) = 3$

$\therefore \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) \neq f(2)$

Hence, $f(x)$ is discontinuous at $x = 2$.

The function can be made continuous by redefining as follows:

$$f(x) = \begin{cases} 2 - x^2 & \text{for } x < 2 \\ -2 & \text{for } x = 2 \\ x - 4 & \text{for } x > 2 \end{cases}$$

Objective Questions

1. A function $f(x)$ is said to have removable discontinuity at $x = a$ if

(a) $\lim_{x \to a} f(x)$ does not exist (b) $f(a)$ does not exist

(c) $\lim_{x \to a} f(x) \neq f(a)$ (d) $\lim_{x \to a} f(x) = f(a)$

Ans: c (definition)

2. The function $y = \dfrac{x^2 - 4}{x - 2}$ is discontinuous at

(a) $x = 2$ (b) $x = -2$

(c) $x = 1$ (d) $x = -1$

Ans: a

The function becomes undefined and hence discontinuous at the point where denominator is zero

i.e. $x - 2 = 0$

$\Rightarrow x = 2$

3. If the function $f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & x \neq 3 \\ k & x = 3 \end{cases}$ is continuous at $x = 3$ then $k =$

(a) 3 (b) 6

(c) 9 (d) 12

Ans: b

If $f(x)$ is continuous at $x = 3$ then

$\lim_{x \to 3} f(x) = f(3)$

or, $\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3} = k$

or. $\lim_{x \to 3} \frac{(x-3)(x+3)}{x-3} = k$

or. $3 + 3 = k$

$\therefore$ $k = 6$

4. If the function $f(x) = \begin{cases} 2x - 1 & x < 2 \\ a & x = 2 \\ x + 1 & x > 2 \end{cases}$ is continuous at $x = 2$ then $a =$

(a) 1 (b) 2

(c) 0 (d) 3

Ans: d

If f(x) is continuous at x = 2 then

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$

or. $\lim_{x \to 2^-} (2x - 1) = \lim_{x \to 2^+} (x + 1) = a$

or. $3 = 3 = a$

$\therefore$ $a = 3$

5. If f and g are two continuous functions then which of the following is true?

(a) $f + g$ is continuous function.

(b) $f - g$ is continuous function.

(c) fg is continuous function.

(d) all of the above.

Ans: d

6. The function $f(x) = \sin\frac{1}{x}$ at $x = 0$ has

(a) jump discontinuity (b) oscillating discontinuity

(c) infinite discontinuity (d) removable discontinuity

Ans: b

7. A function f is continuous at a left end point $x = a$ of its domain if

(a) $\lim_{x \to a^-} f(x) = f(a)$ (b) $\lim_{x \to a^+} f(x) = f(a)$

(c) $\lim_{x \to a^+} f(x)$ does not exist (d) $f(a)$ does not exist

Ans: b (definition)

8. What point is the function $f(x) = \frac{1}{x-2} - 3x$ continuous?

(a) all x except $x = 2$. (b) all x except $x = -2$.

(c) all x except $x = 3$. (d) all x.

Ans: a

The function is discontinuous at $x - 2 = 0 \Rightarrow x = 2$

So, the function f(x) is continuous for all x except x = 2

OOO

2

UNIT

Differentiation

EXERCISE– 2 (A)

1. Find $\frac{dy}{dx}$ from first principle (or by definition).

(a) $2x + 5$ (b) $x^2 + x$ (c) $(x - 1)^2$

(d) $2x^2 + 3x + 1$ (e) $x + \frac{1}{x}$ (f) $\frac{ax + b}{x}$

(g) $\frac{1}{x + 2}$ (h) $\frac{1}{2 - 3x}$ (i) $\sqrt{x}$

(j) $\sqrt{2x + 3}$ (k) $x + \sqrt{x}$ (l) $\frac{1}{\sqrt{x}}$

(m) $\sqrt{\frac{1}{1 - x}}$ (n) $\frac{1}{\sqrt{2x - 5}}$

Solution

(a) Let $y = 2x + 5$

Again let Δx and Δy be the small increments in x and y respectively.

Then, $y + \Delta y = 2(x + \Delta x) + 5$

or, $\Delta y = 2x + 2\Delta x + 5 - y$

or, $\Delta y = 2x + 2\Delta x + 5 - 2x - 5$

or, $\Delta y = 2\Delta x$

or, $\frac{\Delta y}{\Delta x} = 2$

Taking limit $\Delta x \to 0$ on both sides, we get,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2)$$

$\therefore \frac{dy}{dx} = 2.$

(b) Let $y = x^2 + x$

Also let Δx and Δy be small increments in x and y respectively. Then,

$y + \Delta y = (x + \Delta x)^2 + (x + \Delta x)$

or $\Delta y = x^2 + 2x \,.\, \Delta x + (\Delta x)^2 + x + \Delta x - x^2 - x$

or $\Delta y = 2x \,.\, \Delta x + (\Delta x)^2 + \Delta x$

or $\Delta y = \Delta x\,(2x + \Delta x + 1)$

or $\frac{\Delta y}{\Delta x} = 2x + \Delta x + 1$

Taking limit $\Delta x \to 0$ on both sides, we get,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x + 1)$$

$\frac{dy}{dx} = 2x + 1$

(c) Let, $y = (x-1)^2 = x^2 - 2x + 1$

Let Δx & Δy be the small increments of x and y respectively then

$$y + \Delta y = (x + \Delta x)^2 - 2x - 2\Delta x + 1$$

or, $\Delta y = x^2 + 2x\Delta x + (\Delta x)^2 - 2x - 2\Delta x + 1 - y$

or, $\Delta y = x^2 + 2x\Delta x + (\Delta x)^2 - 2x - 2\Delta x + 1 - x^2 + 2x - 1$

or, $\Delta y = \Delta x\,(2x + \Delta x - 2)$

Dividing both sides by Δx, we get

$$\frac{\Delta y}{\Delta x} = \frac{\Delta x}{\Delta x}(2x + \Delta x - 2)$$

Now, taking limit as $\Delta x \to 0$ on both sides, we get

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x - 2)}{\Delta x}$$

$\therefore \quad \frac{dy}{dx} = 2x - 2$

(d) Let $y = 2x^2 + 3x + 1$

Let Δx and Δy be the small increments in x and y respectively. Then

$$y + \Delta y = 2(x + \Delta x)^2 + 3(x + \Delta x) + 1$$

Now, Subtracting (i) from (ii), we get

$$y + \Delta y - y = 2\{x^2 + 2x\cdot\Delta x + (\Delta x)^2\} + 3(x + \Delta x) + 1 - (2x^2 + 3x + 1)$$

or, $\Delta y = 2x^2 + 4x\cdot\Delta x + 2(\Delta x)^2 + 3x + 3\Delta x + 1 - 2x^2 - 3x - 1$

or, $\Delta y = 4x\cdot\Delta x + 2(\Delta x)^2 + 3\Delta x$

or, $\Delta y = \Delta x\,(4x + 2\Delta x + 3)$

Dividing both sides by Δx, we get

or, $\frac{\Delta y}{\Delta x} = (4x + 2\Delta x + 3)$

Now, taking limit as $\Delta x \to 0$ on both sides, we get,

$$\lim_{\Delta \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta \to 0} (4x + 2\Delta x + 3)$$

$\therefore \quad \frac{dy}{dx} = 4x + 3$

(e) Let, $y = x + \frac{1}{x}$

Let Δx & Δy be the small increments of x and y respectively

or, $y + \Delta y = x + \Delta x + \frac{1}{x + \Delta x}$

or, $\Delta y = x + \Delta x + \frac{1}{x + \Delta x} - x - \frac{1}{x}$

or, $\Delta y = \Delta x + \frac{1}{x + \Delta x} - \frac{1}{x}$

or, $\Delta y = \frac{\Delta x\,(x + \Delta x)\,x + x - (x + \Delta x)}{(x + \Delta x)x}$

or, $\Delta y = \frac{\Delta x\,(x^2 + x\Delta x) - \Delta x}{x\,(x + \Delta x)}$

or, $\Delta y = \frac{\Delta x\,(x^2 + x\,\Delta x - 1)}{x\,(x + \Delta x)}$

Dividing both sides by Δx

$$\frac{\Delta y}{\Delta x} = \frac{(x^2 + x\,\Delta x - 1)}{x\,(x + \Delta x)}$$

Now, taking limite as $\Delta x \to 0$ on both sides, we get

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{(x^2 + x\,\Delta x - 1)}{x\,(x + \Delta x)}$$

$\therefore \quad \frac{dy}{dx} = \frac{x^2 - 1}{x^2} = 1 - \frac{1}{x^2}$

(f) Let $\frac{ax+b}{x} = a + \frac{b}{x}$

Let Δx and Δy be the small increments of x and y respectively

$$y + \Delta y = a + \frac{b}{x + \Delta x}$$

or, $$\Delta y = a + \frac{b}{x + \Delta x} - a - \frac{b}{x}$$

or, $$\Delta y = \frac{bx - bx - b\Delta x}{x(x + \Delta x)}$$

or, $$\frac{\Delta y}{\Delta x} = \frac{-b}{x(x + \Delta x)}$$

Taking limit $\Delta x \to 0$ on both sides,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{-b}{x(x + \Delta x)}$$

$$\therefore \frac{dy}{dx} = \frac{-b}{x^2}$$

(g) Let, $y = \frac{1}{x+2}$

Let Δx & Δy be the small increments of x and y respectively

$$\therefore \quad y + \Delta y = \frac{1}{x + \Delta x + 2}$$

or, $$\Delta y = \frac{1}{x + 2 + \Delta x} - \frac{1}{x + 2}$$

or, $$\Delta y = \frac{x + 2 - x - 2 - \Delta x}{(x + 2)(x + 2 + \Delta x)}$$

or, $$\frac{\Delta y}{\Delta x} = \frac{-\Delta x}{\Delta x(x + 2)(x + 2 + \Delta x)}$$

Now, taking limit as $\Delta x \to 0$ on both sides, we get

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{-1}{(x + 2)(x + 2 + \Delta x)}$$

$$\therefore \quad \frac{dy}{dx} = \frac{-1}{(x + 2)^2}$$

(h) Let $y = \frac{1}{2 - 3x}$

Also, let Δx and Δy be the small increments in x and y respectively. Then,

$$y + \Delta y = \frac{1}{2 - 3(x + \Delta x)}$$

or, $$\Delta y = \frac{1}{2 - 3x - 3\Delta x} - \frac{1}{2 - 3x}$$

$$= \frac{2 - 3x - (2 - 3x - 3\Delta x)}{(2 - 3x - 3\Delta x)(2 - 3x)}$$

$$= \frac{2 - 3x - 2 + 3x + 3\Delta x}{(2 - 3x - 3\Delta x)(2 - 3x)}$$

$$\frac{\Delta y}{\Delta x} = \frac{3}{(2 - 3x - 3\Delta x)(2 - 3x)}$$

Taking limit $\Delta x \to$ on both sides, we get

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{3}{(2 - 3x - 3\Delta x)(2 - 3x)}$$

$$\frac{dy}{dx} = \frac{3}{(2 - 3x)(2 - 3x)} = \frac{3}{(2 - 3x)^2}$$

(i) Let $y = \sqrt{x}$

Also, let Δx and Δy be the small increments in x and y respectively. Then

$$y + \Delta y = \sqrt{x + \Delta x}$$

or, $$\Delta y = \sqrt{x + \Delta x} - y$$

or, $$\Delta y = \sqrt{x + \Delta x} - \sqrt{x}$$

or, $$\Delta y = \frac{(\sqrt{x + \Delta x} - \sqrt{x})(\sqrt{x + \Delta x} + \sqrt{x})}{(\sqrt{x + \Delta x} + \sqrt{x})}$$

or, $$\Delta y = \frac{x + \Delta x - x}{\sqrt{x + \Delta x} + \sqrt{x}}$$

or, $$\Delta y = \frac{\Delta x}{\sqrt{x + \Delta x} + \sqrt{x}}$$

or, $$\frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}}$$

By definition of derivative, we have

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{\sqrt{x + 0} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$\therefore \frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

(j) Let $y = \sqrt{2x + 3}$

Also let Δx and Δy be the small increment in x and y respectively.

Then $y + \Delta y = \sqrt{2(x + \Delta x) + 3}$

or, $$\Delta y = \sqrt{2x + 2\Delta x + 3} - \sqrt{2x + 3}$$

or, $$\Delta y = \frac{\sqrt{2x + 2\Delta x + 3} - \sqrt{2x + 3}}{1} \times \frac{\sqrt{2x + 2\Delta x + 3} + \sqrt{2x + 3}}{\sqrt{2x + 2\Delta x + 3} + \sqrt{2x + 3}}$$

or, $$\Delta y = \frac{2x + 2\Delta x + 3 - 2x - 3}{\sqrt{2x + 2\Delta x + 3} + \sqrt{2x + 3}}$$

or, $$\frac{\Delta y}{\Delta x} = \frac{2}{\sqrt{2x + 2\Delta x + 3} + \sqrt{2x + 3}}$$

Taking limit $\Delta x \to 0$ on both sides, we get

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left(\frac{2}{\sqrt{2x + 2\Delta x + 3} + \sqrt{2x + 3}} \right)$$

$$\frac{dy}{dx} = \left(\frac{2}{\sqrt{2x + 3} + \sqrt{2x + 3}} \right) = \frac{2}{2(\sqrt{2x + 3})} = \frac{1}{\sqrt{2x + 3}}$$

(k) Let $y = x + \sqrt{x}$

Also, let Δx and Δy be the small increment in x and y respectively.

Then,

$$y + \Delta y = x + \Delta x + \sqrt{x + \Delta x}$$

or, $$\Delta y = x + \Delta x + \sqrt{x + \Delta x} - x - \sqrt{x}$$

or, $$\Delta y = \Delta x + \frac{\sqrt{x + \Delta x} - \sqrt{x}}{1} \times \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}}$$

or, $$\Delta y = \Delta x + \frac{x + \Delta x - x}{\sqrt{x + \Delta x} + \sqrt{x}}$$

or, $$\Delta y = \Delta x \left(1 + \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} \right)$$

or, $$\frac{\Delta y}{\Delta x} = 1 + \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}}$$

Taking limit $\Delta x \to 0$ on both sides, we get

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x}+\sqrt{x}}$$

$$\frac{dy}{dx} = 1 + \frac{1}{\sqrt{x+0}+\sqrt{x}} = 1 + \frac{1}{2\sqrt{x}}$$

(l) Let, $y = \frac{1}{\sqrt{x}}$

Also, let Δx and Δy be the small increments in x and y respectively. Then,

$$y + \Delta y = \frac{1}{\sqrt{x+\Delta x}}$$

or, $$\Delta y = \frac{1}{\sqrt{x+\Delta x}} - \frac{1}{\sqrt{x}}$$

or, $$\Delta y = \frac{\sqrt{x}-\sqrt{x+\Delta x}}{\sqrt{x+\Delta x}\sqrt{x}} \times \frac{\sqrt{x}+\sqrt{x+\Delta x}}{\sqrt{x}+\sqrt{x+\Delta x}}$$

or, $$\Delta y = \frac{x-x-\Delta x}{\sqrt{x+\Delta x}\sqrt{x}(\sqrt{x}+\sqrt{x+\Delta x})}$$

or, $$\frac{\Delta y}{\Delta x} = \frac{-1}{\sqrt{x+\Delta x}\sqrt{x}.(\sqrt{x}+\sqrt{x+\Delta x})}$$

Taking limit $\Delta x \to 0$ on both sides, we get

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left\{\frac{-1}{\sqrt{x+\Delta x}\sqrt{x}(\sqrt{x}+\sqrt{x+\Delta x})}\right\}$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{x}\cdot\sqrt{x}(\sqrt{x}+\sqrt{x})} = \frac{-1}{x\cdot 2\sqrt{x}} = \frac{-1}{2x^{3/2}}$$

(m) Let $y = \sqrt{\frac{1}{1-x}} = \frac{1}{\sqrt{1-x}}$

Also, let Δx and Δy be the small increments in x and y respectively. Then

$$y + \Delta y = \frac{1}{\sqrt{1-(x+\Delta x)}}$$

or, $$\Delta y = \frac{1}{\sqrt{1-x-\Delta x}} - \frac{1}{\sqrt{1-x}}$$

or, $$\Delta y = \frac{\sqrt{1-x}-\sqrt{1-x-\Delta x}}{\sqrt{1-x-\Delta x}\sqrt{1-x}} \times \frac{\sqrt{1-x}+\sqrt{1-x-\Delta x}}{\sqrt{1-x}+\sqrt{1-x-\Delta x}}$$

or, $$\Delta y = \frac{1-x-(1-x-\Delta x)}{\sqrt{1-x-\Delta x}\sqrt{1-x}\left(\sqrt{1-x}+\sqrt{1-x-\Delta x}\right)}$$

or, $$\Delta y = \frac{1-x-1+x+\Delta x}{\sqrt{1-x-\Delta x}\sqrt{1-x}\left(\sqrt{1-x}+\sqrt{1-x-\Delta x}\right)}$$

or, $$\Delta y = \frac{\Delta x}{\sqrt{1-x-\Delta x}\sqrt{1-x}\left(\sqrt{1-x}+\sqrt{1-x-\Delta x}\right)}$$

or, $$\frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{1-x-\Delta x}\sqrt{1-x}\left(\sqrt{1-x}+\sqrt{1-x-\Delta x}\right)}$$

Taking limit $\Delta x \to 0$ on both sides, we get,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{1-x-\Delta x}\sqrt{1-x}\left(\sqrt{1-x}+\sqrt{1-x-\Delta x}\right)}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x}\sqrt{1-x}\left(\sqrt{1-x}+\sqrt{1-x}\right)} = \frac{1}{(1-x)\,2\,(\sqrt{1-x})} = \frac{1}{2(1-x)^{3/2}}$$

(n) Let $y = \frac{1}{\sqrt{2x-5}}$

Also let Δx and Δy be the small increments in x and y respectively. Then,

$$y + \Delta y = \frac{1}{\sqrt{2(x+\Delta x)-5}}$$

$$\Delta y = \frac{1}{\sqrt{2x+2\Delta x-5}} - \frac{1}{\sqrt{2x-5}}$$

$$= \frac{\sqrt{2x-5}-\sqrt{2x+2\Delta x-5}}{\sqrt{2x+2\Delta x-5}\sqrt{2x-5}}$$

$$= \frac{\sqrt{2x-5}-\sqrt{2x+2\Delta x-5}}{\sqrt{2x+2\Delta x-5}\sqrt{2x-5}} \times \frac{\sqrt{2x-5}+\sqrt{2x+2\Delta x-5}}{\sqrt{2x-5}+\sqrt{2x+2\Delta x-5}}$$

$$= \frac{2x-5-2x-2\Delta x+5}{\sqrt{2x+2\Delta x-5}\sqrt{2x-5}\left(\sqrt{2x-5}+\sqrt{2x+2\Delta x-5}\right)}$$

$$\frac{\Delta y}{\Delta x} = \frac{-2}{\sqrt{2x+2\Delta x-5}\sqrt{2x-5}\left(\sqrt{2x-5}+\sqrt{2x+2\Delta x-5}\right)}$$

Taking limit $\Delta x \to 0$ on both sides, we get

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}$$

$$\frac{dy}{dx} = \frac{-2}{\sqrt{2x-5}\sqrt{2x-5}\left(\sqrt{2x-5}+\sqrt{2x-5}\right)} = \frac{-2}{(2x-5).\,2\sqrt{2x-5}} = -\frac{1}{(2x-5)^{3/2}}$$

2. Find the differential coefficient of

(a) x^6 (b) $2x^{1/2}$

(c) $\frac{6}{\sqrt[3]{x^2}}$ (d) $7x^{10} + 5x^3 - 6$

(e) $ax^3 + bx^2 + cx + d$ (f) $x^4 - 4 + x^{-2} - 5x^{-5}$

Solution

(a) Let, $y = x^6$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}x^6$$

or, $$\frac{dy}{dx} = 6x^{6-1} = 6x^5$$

$$\therefore \frac{dy}{dx} = 6x^5$$

(b) $y = 2x^{1/2}$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = 2 \times \frac{1}{2}x^{-1/2}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{x}}$$

(c) $y = 6x^{-2/3}$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = 6 \times \left(\frac{-2}{3}\right)x^{-5/3}$$

$$\frac{dy}{dx} = \frac{-4}{x^{5/3}}$$

(d) $y = 7x^{10} + 5x^3 - 6$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = \frac{d}{dx}(7x^{10} + 5x^3 - 6)$$

$$\frac{dy}{dx} = 7\times 10x^{10-1} + 5\times 3x^{3-2} - 0$$

$$\frac{dy}{dx} = 70x^9 + 15x^2$$

(e) $y = ax^3 + bx^2 + cx + d$

Differentiating both sides with respect to x, we get

$$\therefore \quad \frac{dy}{dx} = 3ax^2 + 2bx + c$$

(f) $y = x^4 - 4 + x^{-2} - 5x^{-5}$

Differentiating both sides with respect to x, we get

$$\therefore \quad \frac{dy}{dx} = 4x^3 - 2x^{-3} + 25x^{-6}$$

3. Find the differential coefficient of

(a) $y = (x^2 + 5x)(3x^2 - x)$ (b) $y = (x^2 + 1)(x^8 + 2)$

(c) $y = (x^2 + 7)(x^2 + 10)$ (d) $y = (x^3 - 3x^2 + 4)(4x^5 + x^2 - 1)$

Solution

(a) $y = (x^2 + 5x)(3x^2 - x)$

Differentiating both sides with respect to 'x', we get

$$\frac{dy}{dx} = (x^2 + 5x)\frac{d}{dx}(3x^2 - x) + (3x^2 - x)\frac{d}{dx}(x^2 + 5x)$$

or, $\frac{dy}{dx} = (x^2 + 5x)(6x - 1) + (3x^2 - x)(2x + 5)$

or, $\frac{dy}{dx} = 6x^3 - x^2 + 30x^2 - 5x + 6x^3 + 15x^2 - 2x^2 - 5x = 12x^3 + 42x^2 - 10x$

(b) $y = (x^2 + 1)(x^8 + 2)$

or, $y = x^{10} + 2x^2 + x^8 + 2$

$= x^{10} + x^8 + 2x^2 + 2$

Differentiating both sides with respect to 'x', we get

$$\therefore \quad \frac{dy}{dx} = 10x^9 + 8x^7 + 4x$$

(c) $y = (x^2 + 7)(x^2 + 10)$

Differentiating both sides with respect to 'x', we get

or, $\frac{dy}{dx} = (x^2 + 7)\frac{d}{dx}(x^2 + 10) + (x^2 + 10)\frac{d}{dx}(x^2 + 7)$

$$\frac{dy}{dx} = (x^2 + 7)\,2x + (x^2 + 10)\,2x$$

$$\frac{dy}{dx} = 2x^3 + 14x + 2x^3 + 20x$$

$$\therefore \quad \frac{dy}{dx} = 4x^3 + 34x$$

(d) $y = (x^3 - 2x^2 + 4)(4x^5 + x^2 - 1)$

Differentiating both sides with respect to 'x', we get

$$\frac{dy}{dx} = (x^3 - 3x^2 + 4)\frac{d}{dx}(4x^5 + x^2 - 1) + (4x^5 + x^2 - 1)\frac{d}{dx}(x^3 - 3x^2 + 4)$$

$$\frac{dy}{dx} = (x^3 - 3x^2 + 4)(20x^4 + 2x) + (4x^5 + x^2 - 1)(3x^2 - 6x)$$

$$\frac{dy}{dx} = 20x^7 - 60x^6 + 80x^4 + 2x^4 - 6x^3 + 8x + 12x^7 + 3x^4 - 3x^2 - 24x^6 - 6x^3 + 6x$$

$$\frac{dy}{dx} = 32x^7 - 84x^6 + 85x^4 - 12x^3 - 3x^2 + 14x$$

4. Find the derivative of

(a) $y = \dfrac{x^2}{x-1}$ (b) $y = \dfrac{4x}{x^2+1}$ (c) $y = \dfrac{3x^2-2}{x^2+7}$

(d) $y = \dfrac{x^4+1}{x^2+1}$ (e) $y = \dfrac{x^3+3x+1}{x^2+1}$

Solution

(a) $y = \dfrac{x^2}{x-1}$

Differentiating both sides with respect to 'x', we get

$$\frac{dy}{dx} = \frac{(x-1)\dfrac{d}{dx}(x^2) - x^2\dfrac{d}{dx}(x-1)}{(x-1)^2}$$

$$\frac{dy}{dx} = \frac{(x-1)\,2x - x^2}{(x-1)^2}$$

or, $$\frac{dy}{dx} = \frac{2x^2 - 2x - x^2}{(x-1)^2}$$

$\therefore$ $$\frac{dy}{dx} = \frac{x^2 - 2x}{(x-1)^2}$$

(b) $y = \dfrac{4x}{x^2+1}$

Differentiating both sides with respect to 'x', we get

$$\frac{dy}{dx} = \frac{(x^2+1)\dfrac{d(4x)}{dx} - 4x\dfrac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

or, $$\frac{dy}{dx} = \frac{(x^2+1)\,4 - 4x \times 2x}{(x^2+1)^2}$$

or, $$\frac{dy}{dx} = \frac{4x^2 + 4 - 8x^2}{(x^2+1)^2}$$

$\therefore$ $$\frac{dy}{dx} = \frac{-4x^2+4}{(x^2+1)^2} = \frac{4(1-x^2)}{(x^2+1)^2}$$

(c) $y = \dfrac{3x^2-2}{x^2+7}$

Differentiating both sides with respect to 'x', we get

$$\frac{dy}{dx} = \frac{(x^2+7)\dfrac{d\,(3x^2-2)}{dx}\,(3x^2-2)\dfrac{d\,(x^2+7)}{dx}}{(x^2+7)^2}$$

or, $$\frac{dy}{dx} = \frac{(x^2+7)\,6x - (3x^2-2)\,2x}{(x^2+7)^2}$$

or, $$\frac{dy}{dx} = \frac{6x^3 + 42x - 6x^3 + 4x}{(x^2+7)^2}$$

$\therefore$ $$\frac{dy}{dx} = \frac{46x}{(x^2+7)^2}$$

(d) $y = \dfrac{x^4+1}{x^2+1}$

Differentiating both sides with respect to 'x', we get

$$\frac{dy}{dx} = \frac{(x^2+1)\dfrac{d\,(x^4+1)}{dx} - (x^4+1)\dfrac{d\,(x^2+1)}{dx}}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{(x^2+1)\,4x^3 - (x^4+1)\,2x}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{4x^5 + 4x^3 - 2x^5 - 2x}{(x^2+1)^2}$$

$\therefore$ $$\frac{dy}{dx} = \frac{2x^5 + 4x^3 - 2x}{(x^2+1)^2}$$

(e) $y = \dfrac{x^3 + 3x + 1}{x^2 + 1}$

Differentiating both sides with respect to 'x', we get

$$\frac{dy}{dx} = \frac{(x^2+1)\frac{d}{dx}(x^3+3x+1) - (x^3+3x+1)\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$\frac{dy}{dx} = \frac{(x^2+1)(3x^2+3) - (x^3+3x+1)\,2x}{(x^2+1)^2}$$

or, $$\frac{dy}{dx} = \frac{3x^2+3+3x^4+3x^2-2x^4-6x^2-2x}{(x^2+1)^2}$$

$\therefore$ $$\frac{dy}{dx} = \frac{x^4-2x+3}{(x^2+1)^2}$$

5. Find the derivatives of

(a) $y = (x+1)^6$ (b) $y = (4x^3 - 5x^2 + 1)^4$

(c) $y = \sqrt{ax^2 + bx + c}$ (d) $y = \dfrac{1}{\sqrt{a^2 - x^2}}$

(e) $y = \sqrt{\dfrac{1+x}{1-x}}$ (f) $x\sqrt{1 + x^2}$

Solution

(a) $y = (x+1)^6$

Differentiating both sides with respect to 'x', we get

or, $$\frac{dy}{dx} = \frac{d}{dx}\{(x+1)^6\}$$

or, $$\frac{dy}{dx} = \frac{d\{(x+1)^6\}}{d(x+12)}\ \frac{d(x+1)}{dx}$$

or, $$\frac{dy}{dx} = 6(x+1)^5\cdot 1$$

$$\frac{dy}{dx} = 6(x+1)^5$$

(b) $y = (4x^3 - 5x^2 + 1)^4$

Differentiating both sides with respect to 'x', we get

$$\frac{dy}{dx} = \frac{d(4x^3-5x^2+1)^4}{d(4x^3-5x^2+1)}\cdot\frac{d(4x^3-5x^2+1)}{dx}$$

$$\frac{dy}{dx} = 4(4x^3-5x^2+1)^3 \times (12x^2 - 10x) = 8x(4x^3-5x^2+1)^3(6x-5)$$

(c) $y = \sqrt{ax^2 + bx + c}$

or, $y = (ax^2 + bx + c)^{1/2}$

Differentiating both sides with respect to 'x', we get

$$\frac{dy}{dx} = \frac{d(ax^2+bx+C)^{1/2}}{d(ax^2+bx+C)}\cdot\frac{d(ax^2\ bx+C)}{dx}$$

or, $$\frac{dy}{dx} = \frac{1}{2}(ax^2+bx+C)^{-1/2}(2ax+b)$$

$$\frac{dy}{dx} = \frac{2ax+b}{2\sqrt{ax^2+bx+c}}$$

(d) $y = \dfrac{1}{\sqrt{a^2-x^2}} = (a^2-x^2)^{-1/2}$

Differentiating both sides with respect to 'x', we get

or, $$\frac{dy}{dx} = \frac{d}{dx}(a^2-x^2)^{-1/2}$$

or, $$\frac{dy}{dx} = \frac{d(a^2-x^2)^{-1/2}}{d(a^2-x^2)} \times \frac{d(a^2-x^2)}{dx}$$

or $$\frac{dy}{dx} = -\frac{1}{2}(a^2-x^2)^{-3/2} \times (-2x)$$

$\therefore$ $$\frac{dy}{dx} = \frac{x}{(a^2-x^2)^{3/2}}$$

(e) $y = \left(\frac{1+x}{1-x}\right)^{1/2}$

Differentiating both sides with respect to 'x', we get

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\left(\frac{1+x}{1-x}\right)^{1/2}\right\}$$

$$\frac{dy}{dx} = \frac{d\left\{\left(\frac{1+x}{1-x}\right)\right\}}{d\left(\frac{1+x}{1-x}\right)} \times \frac{d\left(\frac{1+x}{1-x}\right)}{dx}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{-1/2}\left\{\frac{(1-x)\frac{d}{dx}(1+x)+(1+x)\frac{d}{dx}(1-x)}{(1-x)^2}\right\}$$

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{1-x}{1+x}}\left\{\frac{1-x+1+x}{(1-x)^2}\right\}$$

$$\frac{dy}{dx} = \frac{1}{(1-x)^2}\sqrt{\frac{1-x}{1+x}} = \frac{1}{(1-x)^{3/2}\sqrt{1+x}}$$

(f) $y = x\sqrt{1+x^2} = x(1+x^2)^{1/2}$

Differentiating both sides with respect to 'x', we get

$$\frac{dy}{dx} = x\frac{d}{dx}\{(1+x^2)^{1/2} + (1+x^2)^{1/2}\frac{dx}{dx}$$

$$\frac{dy}{dx} = x \times \frac{1}{2}(1+x^2)^{-1/2} \times 2x + \sqrt{1+x^2} \times 1$$

$$\therefore \quad \frac{dy}{dx} = \frac{x^2}{\sqrt{1+x^2}} + \sqrt{1+x^2} = \frac{x^2+1+x^2}{\sqrt{1+x^2}} = \frac{2x^2+1}{\sqrt{1+x^2}}$$

6. **Find derivatives of**

(a) $y = \frac{1}{\sqrt{x+b}-\sqrt{x}}$ (b) $y = \frac{1}{\sqrt{3x-2}-\sqrt{3x-5}}$

(c) $y = \frac{1}{\sqrt{x+a}-\sqrt{x+b}}$

Solution

(a) $y = \frac{1}{\sqrt{x+b}-\sqrt{x}}$

or, $y = \frac{1}{\sqrt{x+b}-\sqrt{x}} \times \frac{\sqrt{x+b}+\sqrt{x}}{\sqrt{x+b}+\sqrt{x}}$

or, $y = \frac{\sqrt{x+b}+\sqrt{x}}{x+b-x}$

Differentiating both sides with respect to 'x', we get

$$\frac{dy}{dx} = \frac{1}{b}\left[\frac{d}{dx}(x+b)^{1/2} + \frac{d}{dx}(x^{1/2})\right]$$

$$= \frac{1}{b}\left[\frac{1}{2}(x+b)^{-1/2} + \frac{1}{2}(x^{-1/2})\right]$$

$$\frac{dy}{dx} = \frac{1}{2b}\left[\frac{1}{\sqrt{x+b}} + \frac{1}{\sqrt{x}}\right]$$

(b) $y = \frac{\sqrt{3x-2}+\sqrt{3x-9}}{3x-2-3x+5}$

$$y = \frac{\sqrt{3x-2}+\sqrt{3x-9}}{(\sqrt{3x-2}-\sqrt{3x-9})(\sqrt{3x-2}+\sqrt{3x-9})}$$

or, $y = \frac{1}{3}[(3x-2)^{1/2} + (3x-5)^{1/2}]$

Differentiating both sides with respect to 'x' we get

or, $\frac{dy}{dx} = \frac{1}{3}\left[\frac{3}{2\sqrt{3x-2}} + \frac{3}{2\sqrt{3x-5}}\right] = \frac{1}{2}\left[\frac{1}{\sqrt{3x-2}} + \frac{1}{\sqrt{3x-5}}\right]$

(e) $y = \frac{1}{\sqrt{x+a} - \sqrt{x+b}}$

$y = \frac{1}{\sqrt{x+a} - \sqrt{x+b}} \times \frac{\sqrt{x+a} + \sqrt{x+b}}{\sqrt{x+a} + \sqrt{x+b}}$

$y = \frac{\sqrt{x+a} + \sqrt{x+b}}{x+a-x-b} = \frac{1}{a-b}\{(x+a)^{1/2} + (x+b)^{1/2}$

Differentiating both sides with respect to 'x', we get

$\frac{dy}{dx} = \frac{1}{a-b}\left\{\frac{d}{dx}(x+a)^{1/2} + \frac{d}{dx}(x+b)^{1/2}\right\} = \frac{1}{a-b}\left\{\frac{1}{2}(x+a)^{-1} + \frac{1}{2}(x+b)^{-1/2}\right\}$

$= \frac{1}{2(a-b)}\left[\frac{1}{\sqrt{x+a}} + \frac{1}{\sqrt{x+b}}\right]$

7. **Find $\frac{dy}{dx}$ if**

(a) $x = at^2, y = 2at$

(b) $x = t^2 - 1, y = t^4 - 1$

(c) $y = z^3 + 2z + 1,\ x = z^2 + 2$

(d) $x = t + \frac{1}{t}$ **and** $y = t - \frac{1}{t}$

(e) $x = \frac{3at}{1+t^2}, y = \frac{3at^2}{1+t^2}$

Solution

(a) $x = a\,t^2, y = 2\,at$

$\frac{dx}{dt} = 2\,at,\ \frac{dy}{dt} = 2a$

Now, $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2a \times \frac{1}{2at} = \frac{1}{t}$

(b) $x = t^2 - 1, y = t^4 - 1$

$\frac{dx}{dt} = 2t, \frac{dy}{dt} = 4t^3$

Now,

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

or, $\frac{dy}{dx} = 4t^3 \times \frac{1}{2t}$

$\frac{dy}{dx} = 2t^2$

(c) $y = z^3 + 2z + 1, x = z^2 + 2$

$\frac{dy}{dz} = 3z^2 + 2, \frac{dx}{dz} = 2z$

Now, $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$

$\therefore \frac{dy}{dx} = \frac{3z^2 + 2}{2z}$

(d) $x = t + t^{-1}$ and $y = t - t^{-1}$

$\frac{dx}{dt} = 1 - \frac{1}{t^2}$ and $\frac{dy}{dt} = 1 + \frac{1}{t^2}$

$\frac{dx}{dt} = \frac{t^2 - 1}{t^2}$ and $\frac{dy}{dt} = \frac{t^2 + 1}{t^2}$

Now,

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{1+t^2}{t^2} \times \frac{t^2}{t^2-1}$$

$$\frac{dy}{dx} = \frac{t^2+1}{t^2-1}$$

(e) $x = \frac{3\,at}{1+t^2}, y = \frac{3at^2}{1+t^2}$

Here, $x = \frac{3\,at}{1+t^2}$

Differentiating with respect to 't', we get

$$\frac{dx}{dt} = \frac{(1+t^2)\frac{d}{dt}(3at) - 3at\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$\frac{dx}{dt} = \frac{(1+t^2)\,3a - 3at \times 2t}{(1+t^2)^2}$$

or, $$\frac{dx}{dt} = \frac{3a + 3at^2 - 6at^2}{(1+t^2)^2}$$

or $$\frac{dx}{dt} = \frac{3a - 3at^2}{(1+t^2)^2}$$

$\therefore$ $$\frac{dx}{dt} = \frac{3a\,(1-t^2)}{(1+t^2)^2}$$

And $y = \frac{3at^2}{1+t^2}$

Differentiating with respect to 't', we get

$$\frac{dy}{dt} = \frac{(1+t^2)\frac{d}{dt}(3at^2) - 3at^2\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{(1+t^2)\,6at - 3at^2 \times 2t}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{6at + 6at^3 - 6at^3}{(1+t^2)^2}$$

$$\frac{dy}{dt} = \frac{6\,at}{(1+t^2)^2}$$

Now, $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{6\,at}{(1+t^2)^2} \times \frac{(1+t^2)^2}{3a\,(1-t^2)}$$

$\therefore$ $$\frac{dy}{dx} = \frac{2t}{1-t^2}$$

8. Find the derivative of y w.r.t. 'x' in each of the following.

(a) $x^2 + y^2 = 1$ (b) $y^2 = 4ax$ (c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(d) $x^3 + y^3 = 3\,axy$ (e) $(x^2 + y^2)^2 = xy$

Solution

(a) $x^2 + y^2 = 1$

Differentiating both sides with respect to 'x', we get

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d\,(1)}{dx}$$

or, $$2x + \frac{d}{dy}(y^2)\,.\,\frac{dy}{dx} = 0$$

or, $$2y\frac{dy}{dx} = -\,2x$$

$\therefore$ $$\frac{dy}{dx} = -\,\frac{x}{y}$$

(b) $y^2 = 4ax$

Differentiating both sides with respect to 'x', we get

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(4ax)$$

or, $$\frac{d}{dy}(y^2)\frac{dy}{dx} = 4a$$

or, $$2y\frac{dy}{dx} = 4a$$

$$\therefore \frac{dy}{dx} = \frac{2a}{y}$$

(c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Differentiating both sides with respect to 'x', we get

$$\frac{1}{a^2}\frac{d}{dx}(x^2) + \frac{1}{b^2}\frac{d}{dx}(y^2) = \frac{d}{dx}(1)$$

or, $$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

or, $$\frac{2y}{b^2}\frac{dy}{dx} = \frac{-2x}{a^2}$$

or, $$\frac{dy}{dx} = -\frac{2x/a^2}{2y/b^2} = \frac{-b^2x}{a^2y}$$

(d) $x^3 + y^3 = 3axy$

Differentiating both sides with respect to 'x', we get

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) = 3a\frac{d}{dx}(xy)$$

or $$3x^2 + \frac{d}{dx}(y^3)\frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y\frac{dx}{dx}\right]$$

or, $$3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y\right]$$

or, $$x^2 + y^2\frac{dy}{dx} = ax\frac{dy}{dx} + ay$$

or, $$(y^2 - ax)\frac{dy}{dx} = ay - x^2$$

$$\therefore \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}.$$

(e) $(x^2 + y^2)^2 = xy$

Differentiating both sides with respect to 'x', we get

$$\frac{d}{dx}\{(x^2 + y^2)^2\} = \frac{d}{dx}(xy)$$

or, $$\frac{d(x^2 + y^2)^2}{d(x^2 + y^2)}\frac{d}{dx}(x^2 + y^2) = \left[x\frac{dy}{dx} + y\frac{dx}{dx}\right]$$

$$2(x^2 + y^2)\left[2x + 2y\frac{dy}{dx}\right] = \left[x\frac{dy}{dx} + y\right]$$

or, $$4x(x^2 + y^2) + 4y(x^2 + y^2)\frac{dy}{dx} = x\frac{dy}{dx} + y$$

or, $$(4x^2y + 4y^3 - x)\frac{dy}{dx} = y - 4x^3 - 4xy^2$$

$$\therefore \frac{dy}{dx} = \frac{y - 4x^3 - 4xy^2}{4x^2y + 4y^3 - x}$$

Objective Questions

1. $\frac{d}{dx}(5) =$

(a) 5 (b) -5

(c) 0 (d) 1

Ans: c

$$\frac{d}{dx}(5) = 0$$

2. If $y = 2x^3$ then $\frac{dy}{dx} =$

(a) $2x^3$ (b) $6x^2$

(c) $9x^2$ (d) $12x$

Ans: b

$$\frac{dy}{dx} = \frac{d}{dx}(2x^3) = 2\frac{d}{dx}(x^3) = 2 \cdot 3x^2 = 6x^2$$

3. If $f(x) = 4 - x^2$ then $f'(-3) =$

(a) -6 (b) 6

(c) -3 (d) 3

Ans: b

$f(x) = 4 - x^2$, $f'(x) = -2x$

$\therefore \quad f'(-3) = -2(-3) = 6$

4. If $y = t - \frac{1}{t}$ then $\frac{dy}{dt} =$

(a) $1 + \frac{1}{t^2}$ (b) $1 - \frac{1}{t^2}$

(c) $1 + \frac{1}{t}$ (d) $1 - \frac{1}{t}$

Ans: a

$y = t - t^{-1}$. Then, $\frac{dy}{dt}$ $1 - (-1)t^{-2} = 1 + \frac{1}{t^2}$

5. $\frac{d}{dx}(\sqrt{x}) =$

(a) $\sqrt{x}$ (b) $\frac{1}{\sqrt{x}}$

(c) $2\sqrt{x}$ (d) $\frac{1}{2\sqrt{x}}$

Ans: d

$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}\left((x)^{\frac{1}{2}}\right) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{\frac{-1}{2}} = \frac{1}{2\sqrt{x}}$$

6. $\frac{d}{dx}(x^n) =$

(a) nx^n (b) nx^{n-1}

(c) $(n-1)x^n$ (d) $(n-1)x^{n-1}$

Ans: b (Formula)

7. If u and v are differentiable at x, then $\frac{d}{dx}(u \cdot v) =$

(a) $u\frac{d}{dx}(v) + v\frac{d}{dx}(u)$
(b) $\frac{d(u)}{dx} \cdot \frac{d(v)}{dx}$
(c) $\dfrac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$
(d) $u\frac{d}{dx}(v) - v\frac{d}{dx}(u)$

Ans: a (formula for product rule)

8. $\frac{d}{dx}\left(\frac{1}{x}\right) =$

(a) $\frac{1}{x}$
(b) $-\frac{1}{x}$
(c) $\frac{1}{x^2}$
(d) $-\frac{1}{x^2}$

Ans: d

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -1 \cdot x^{-1-1} = -\frac{1}{x^2}$$

9. $\frac{d}{dx}\{(2x+1)^6\}$

(a) $6(2x+1)^6$
(b) $6(2x+1)^5$
(c) $12(2x+1)^6$
(d) $12(2x+1)^5$

Ans: d

$$\frac{d}{dx}(2x = 1)^6 = \frac{d(2x+1)^6}{d(2x+1)} \cdot \frac{d(2x+1)}{dx} = 6(2x+1)^5 \cdot 2 = 12\,(2x+1)^5$$

10. $\frac{d}{dx}\left(\frac{x}{2x-1}\right) =$

(a) $\frac{1}{(2x-1)^2}$
(b) $-\frac{1}{(2x-1)^2}$
(c) $\frac{x}{(2x-1)^2}$
(d) $-\frac{x}{(2x-1)^2}$

Ans: b

$$\frac{d}{dx}\left(\frac{x}{2x-1}\right) = \frac{(2x-1)\frac{d}{dx}(x) - x\frac{d}{dx}(2x-1)}{(2x-1)^2} = \frac{(2x-1)\cdot 1 - x\cdot 2}{(2x-1)^2} = \frac{2x-1-2x}{(2x-1)^2}$$

$$= -\frac{1}{(2x-1)^2}$$

11. If $y = \frac{1}{\sqrt{x^2-a^2}}$ then $\frac{dy}{dx} =$

(a) $\frac{x}{(x^2-a^2)^{\frac{3}{2}}}$
(b) $\frac{2x}{(x^2-a^2)^{\frac{3}{2}}}$
(c) $-\frac{2x}{(x^2-a^2)^{\frac{3}{2}}}$
(d) $-\frac{x}{(x^2-a^2)^{\frac{3}{2}}}$

Ans: d

$$\frac{d}{dx}\left(\frac{1}{\sqrt{x^2-a^2}}\right) = \frac{d}{dx}(x^2-a^2)^{\frac{-1}{2}} = \frac{d(x^2-a^2)^{\frac{-1}{2}}}{d(x^2-a^2)} \cdot \frac{d(x^2-a^2)}{dx} = -\frac{1}{2}(x^2-a^2)^{\frac{-3}{2}} \cdot 2x$$

$$= -\frac{x}{(x^2-a^2)^{\frac{3}{2}}}$$

12. If $y = t^3 + 2t + 1$ and $x = t^2 + 2$ then $\frac{dy}{dx} =$

(a) $\frac{2t}{3t^2 + 2}$ (b) $2t(3t^2 + 2)$

(c) $\frac{3t^2 + 2}{2t}$ (d) $3t^2 + 2$

Ans: c

$$y = t^3 + 2t + 1,\ x = t^2 + 2$$

$$\frac{dy}{dt} = 3t^2 + 2,\ \frac{dx}{dt} = 2t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 + 2}{2t}$$

13. If $x^2 + y^2 = 4$ then $\frac{dy}{dx} =$

(a) $-\frac{x}{y}$ (b) $\frac{x}{y}$

(c) $-\frac{y}{x}$ (d) $\frac{y}{x}$

Ans: a

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(u)$$

or, $$2x + 2y\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y}$$

14. Derivative of a function $y = f(x)$ gives
(a) slope of the tangent
(b) ratio of change in the value of the function to the change in the independent variable
(c) the rate at which the value of y changes with respect to x
(d) all of the above

Ans: d

EXERCISE 2 (B)

1. Find from definition the derivative of

(a) $\sin 3x$ (b) $\cos(3x - 4)$

(c) $\tan\frac{3x}{2}$ (d) $\sin^2 x$

(e) $\sqrt{\sin 2x}$ (f) $\sqrt{\sec x}$

Solution

(a) Let $y = \sin 3x$

Again, let Δx and Δy be the small increments in x and y respectively. Then,

$$y + \Delta y = \sin 3(x + \Delta x)$$

or, $$\Delta y = \sin(3x + 3\Delta x) - \sin 3x$$

or, $$\Delta y = 2\cos\left(\frac{3x + 3\Delta x + 3x}{2}\right)\sin\left(\frac{3x + 3\Delta x - 3x}{2}\right)$$

or, $$\Delta y = 2\cos\left(3x + \frac{3\Delta x}{2}\right)\cdot\sin\left(\frac{3\Delta x}{2}\right)$$

Dividing both sides by Δx

$$\frac{\Delta y}{\Delta x} = \frac{2\cos\left(3x + \frac{3\Delta x}{2}\right)\sin\left(\frac{3\Delta x}{2}\right)}{\Delta x}$$

Taking limit $\Delta x \to 0$ on both sides, we get,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\cos\left(3x + \frac{3\Delta x}{2}\right)\sin\left(\frac{3\Delta x}{2}\right)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[2\cos\left(3x + \frac{3\Delta x}{2}\right) \cdot \frac{\sin\frac{3\Delta x}{2}}{\frac{3\Delta x}{2}} \times \frac{3}{2}\right] = \cos 3x \cdot 3 = 3\cos 3x$$

(b) Let $y = \cos(3x - 4)$

Again, let Δx and Δy be the small increments in x and y respectively.

Then,

$$y + \Delta y = \cos\{3(x + \Delta x) - 4\}$$

or, $\Delta y = \cos(3x + 3\Delta x - 4) - \cos(3x - 4)$

or, $\Delta y = -2\sin\left(\frac{3x + 3\Delta x - 4 + 3x - 4}{2}\right) \cdot \sin\left(\frac{3x + 3\Delta x - 4 - 3x + 4}{2}\right)$

or, $\Delta y = -2\sin\left(3x - 4 + \frac{3\Delta x}{2}\right)\sin\left(\frac{3\Delta x}{2}\right)$

Dividing both sides by Δx,

$$\frac{\Delta y}{\Delta x} = \frac{-2\sin\left(3x - 4 + \frac{3\Delta x}{2}\right)\sin\left(\frac{3\Delta x}{2}\right)}{\Delta x}$$

Taking limit $\Delta x \to 0$ on both sides, we get,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{-2\sin\left(3x - 4 + \frac{3\Delta x}{2}\right)\sin\left(\frac{3\Delta x}{2}\right)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0}\left[-2\sin\left(3x - 4 + \frac{3\Delta x}{2}\right) \cdot \frac{\sin\left(\frac{3\Delta x}{2}\right)}{\frac{3\Delta x}{2}} \times \frac{3}{2}\right]$$

$$= -2\sin(3x - 4) \cdot 1 \cdot \frac{3}{2}$$

$$= -3\sin(3x - 4)$$

(c) Let $y = \tan\frac{3x}{2}$

Again, let Δx and Δy be the small increments in x and y respectively.

Then, $y + \Delta y = \tan\frac{3}{2}(x + \Delta x)$

or, $\Delta y = \tan\frac{3x + 3\Delta x}{2} - \tan\left(\frac{3x}{2}\right)$

or, $\Delta y = \frac{\sin\left(\frac{3x + 3\Delta x}{2}\right)}{\cos\left(\frac{3x + 3\Delta x}{2}\right)} - \frac{\sin\left(\frac{3x}{2}\right)}{\cos\left(\frac{3x}{2}\right)}$

or, $\Delta y = \frac{\sin\left(\frac{3x + 3\Delta x}{2}\right) \cdot \cos\left(\frac{3x}{2}\right) - \cos\left(\frac{3x + 3\Delta x}{2}\right) \cdot \sin\left(\frac{3x}{2}\right)}{\cos\left(\frac{3x + 3\Delta x}{2}\right) \cdot \cos\left(\frac{3x}{2}\right)}$

Dividing both sides by Δx, we get,

$$\frac{\Delta y}{\Delta x} = \frac{\sin\left(\dfrac{3x + 3\Delta x}{2} - \dfrac{3x}{2}\right)}{\Delta x}$$

Taking limit $\Delta x \to 0$ on both sides, we get,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sin\left(\dfrac{3\Delta x}{2}\right)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\sin\dfrac{3\Delta x}{2}}{\dfrac{3\Delta x}{2}} \times \frac{3}{2} \times \frac{1}{\cos\left(\dfrac{3x + 3\Delta x}{2}\right) \cdot \cos\dfrac{3x}{2}}$$

$$= 1 \cdot \frac{3}{2} \times \frac{1}{\cos\dfrac{3x}{2} \cdot \cos\dfrac{3x}{2}} = \frac{3}{2}\sec^2\frac{3x}{2}.$$

(d) Let $y = \sin^2 x$

Again, let Δx and Δy be the small increments in x and y respectively.

Then, $y + \Delta y = \sin^2(x + \Delta x)$

or, $\Delta y = \sin^2(x + \Delta x) - \sin^2 x$

or, $\Delta y = (\sin(x + \Delta x) + \sin x)(\sin(x + \Delta x) - \sin x)$

or, $\Delta y = [\sin(x + \Delta x) + \sin x]\, 2\cos\left(\dfrac{\Delta + \Delta x + x}{2}\right) \cdot \sin\left(\dfrac{\Delta + \Delta x - x}{2}\right)$

or, $\Delta y = 2\{\sin(x + \Delta x) + \sin x\} \cos\left(x + \dfrac{\Delta x}{2}\right) \cdot \sin\dfrac{\Delta x}{2}$

Dividing both sides by Δx

$$\frac{\Delta y}{\Delta x} = \frac{2\{\sin(x + \Delta x) + \sin x\} \cos\left(x + \dfrac{\Delta x}{2}\right) \cdot \sin\dfrac{\Delta x}{2}}{\Delta x}$$

Taking limit $\Delta x \to 0$ on both sides,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\{\sin(x + \Delta x) + \sin x\} \cos\left(x + \dfrac{\Delta x}{2}\right) \cdot \sin\dfrac{\Delta x}{2}}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \left\{ 2\{\sin(x + \Delta x) + \sin x\} \cdot \cos\left(x + \frac{\Delta x}{2}\right) \cdot \frac{\sin\dfrac{\Delta x}{2}}{\dfrac{\Delta x}{2}} \cdot 2 \right\}$$

$$= 2 \cdot (\sin x + \sin x) \cdot \cos x \cdot 1 \cdot \frac{1}{2}$$

$$= (\sin x + \sin x) \cdot \cos x = 2\sin x \cos x = \sin 2x$$

(e) Let $y = \sqrt{\sin 2x}$

Let, Δx be the small increments in x and Δy be the corresponding increments in y,

then $y + \Delta y = \sqrt{\sin 2(x + \Delta x)} = \sqrt{\sin(2x + 2\Delta x)}$

or. $\Delta y = \sqrt{\sin(2x + 2\Delta x)} - y$

$\therefore$ $\Delta y = \sqrt{\sin(2x + 2\Delta x)} - \sqrt{\sin 2x}$

or, $\Delta y = \left(\sqrt{\sin(2x + 2\Delta x)} - \sqrt{\sin 2x}\right) \times \dfrac{\sqrt{\sin(2x + 2\Delta x)} + \sqrt{\sin 2x}}{\sqrt{\sin(2x + 2\Delta x)} + \sqrt{\sin 2x}}$

$$= \frac{\sin(2x + 2\Delta x) - \sin 2x}{\sqrt{\sin(2x + 2\Delta x)} + \sqrt{\sin 2x}} = \frac{2\cos\left(\dfrac{2x + 2\Delta x + 2x}{2}\right)\sin\left(\dfrac{2x + 2\Delta x - 2x}{2}\right)}{\sqrt{\sin(2x + 2\Delta x)} + \sqrt{\sin 2x}}$$

$$\frac{2\cos\left(\frac{4x+2\Delta x}{2}\right)\sin\Delta x}{\sqrt{\sin(2x+2\Delta x)}+\sqrt{\sin 2x}} = \frac{2\cos(2x+\Delta x)\ \sin\Delta x}{\sqrt{\sin(2x+2\Delta x)}+\sqrt{\sin 2x}}$$

$$\frac{\Delta y}{\Delta x} = \frac{2\cos(2x+\Delta x)}{\left(\sqrt{\sin(2x+2\Delta x)}+\sqrt{\sin 2x}\right)} \times \left(\frac{\sin\Delta x}{\Delta x}\right)$$

$$= 2\cos(2x+\Delta x) \times \frac{1}{\sqrt{\sin(2x+2\Delta x)}+\sqrt{\sin 2x}} \times \left(\frac{\sin\Delta x}{\Delta x}\right)$$

By the definition, we know that

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta x}\right)$$

$$= \lim_{\Delta x\to 0} 2\cos(2x+\Delta x) \times \frac{1}{\sqrt{\sin(2x+2\Delta x)}+\sqrt{\sin 2x}} \times \left(\frac{\sin\Delta x}{\Delta x}\right)$$

$$= 2\cos 2x \times \frac{1}{\sqrt{\sin 2x}+\sqrt{\sin 2x}} \times 1 = \frac{2\cos 2x}{2\sqrt{\sin 2x}} = \frac{\cos 2x}{\sqrt{\sin 2x}}$$

$$\therefore \frac{d}{dx}\left(\sqrt{\sin 2x}\right) = \frac{\cos 2x}{\sqrt{\sin 2x}}.$$

(f) Let $y = \sqrt{\sec x}$

Again, let Δx and Δy be the small increments in x and y respectively.

Then, $y + \Delta y = \sqrt{\sec(x+\Delta x)}$

or, $\Delta y = \sqrt{\sec(x+\Delta x)} - \sqrt{\sec x}$

or, $\Delta y = \dfrac{\sec(x+\Delta x) - \sec x}{\sqrt{\sec(x+\Delta x)} - \sqrt{\sec x}}$

or, $\Delta y = \dfrac{\dfrac{1}{\cos(x+\Delta x)} - \dfrac{1}{\cos x}}{\sqrt{\sec(x+\Delta x)} - \sqrt{\sec x}}$

or, $\Delta y = \dfrac{\cos x - \cos(x+\Delta x)}{\cos(x+\Delta x)\cos x\left(\sqrt{\sec(x+\Delta x)} - \sqrt{\sec x}\right)}$

or, $\Delta y = \dfrac{2\sin\left(\dfrac{x+x+\Delta x}{2}\right)\sin\left(\dfrac{x+x-\Delta x}{2}\right)}{\cos(x+\Delta x)\cos x\left(\sqrt{\sec(x+\Delta x)} - \sqrt{\sec x}\right)}$

or, $\Delta y = \dfrac{2\sin\left(x+\dfrac{\Delta x}{2}\right)\sin\left(\dfrac{\Delta x}{2}\right)}{\Delta x\cdot\cos(x+\Delta x)\cos x\left(\sqrt{\sec(x+\Delta x)} - \sqrt{\sec x}\right)}$

Taking limit $\Delta x \to 0$ on both sides,

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{2\sin\left(x+\frac{\Delta x}{2}\right)\sin\left(\frac{\Delta x}{2}\right)}{\Delta x\cdot\cos(x+\Delta x)\cos x\left(\sqrt{\sec(x+\Delta x)} - \sqrt{\sec x}\right)}$$

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\left\{\frac{2\sin\left(x+\frac{\Delta x}{2}\right)}{\cos(x+\Delta x)\cos x\left(\sqrt{\sec(x+\Delta x)} - \sqrt{\sec x}\right)}\cdot\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}\cdot 2}\right\}$$

$$= \frac{2\sin x}{\cos x\cdot\cos x\left(\sqrt{\sec x}+\sqrt{\sec x}\right)}\cdot 1\cdot\frac{1}{2}$$

$$= \frac{\sin x}{\cos x\cdot\cos x\cdot 2\sqrt{\sec x}} = \frac{\tan x\sec x}{2\sqrt{\sec x}} = \frac{1}{2}\tan x\sqrt{\sec x}$$

2. Find the derivatives of the following functions.

(a) $\sin(px + q)$ (b) $\cos^3 x$ (c) $\tan^3(x^2)$

(d) $\sec^3\left(\frac{3ax+b}{c}\right)$ (e) $x^2 \tan x$ (f) $(1 + x^2)\sin x$

(g) $\frac{x^2}{\sin 3x}$ (h) $\frac{\sin\sqrt{x}}{\sqrt{x}}$ (i) $\tan^2 \sin(\sqrt{x})$

(j) $\sec^2(\tan\sqrt{x})$ (k) $\sin 3x \cos x$ (l) $\cos 6x \cos 2x$

(m) $\sin\frac{1+x^2}{1-x^2}$

Solution

(a) Let $y = \sin(px + q)$

Diff. both sides w.r to x

$$\frac{dy}{dx} = \frac{d}{dx}\{\sin(px+q)\}$$

$$= \frac{d(\sin(px+q))}{d(px+q)} \cdot \frac{d(px+q)}{dx} = \cos(px+q)\cdot p = p\cos(px+q).$$

(b) Let $y = \cos^3 x$

$$\frac{dy}{dx} = \frac{d}{dx}(\cos x)^3$$

$$= \frac{d(\cos x)^3}{d(\cos x)} \cdot \frac{d(\cos x)}{dx} = 3(\cos x)^2(-\sin x) = -3\cos^2 x \sin x.$$

(c) Let $y = \tan^3(x^2)$

$$\frac{dy}{dx} = \frac{d}{dx}(\tan^3(x^2))$$

$$= \frac{d(\tan x^2)^3}{d(\tan x^2)} \cdot \frac{d(\tan x^2)}{d(x^2)} \cdot \frac{d(x^2)}{dx}$$

$$= 3(\tan x^2)^2 \cdot \sec^2 x^2 \cdot 2x = 6x\tan^2 x^2 \sec^2 x^2.$$

(d) $y = \sec^3\left(\frac{3ax+b}{c}\right)$

$$\frac{dy}{dx} = \frac{d\left\{\sec\left(\frac{3ax+b}{c}\right)\right\}^3}{dx}$$

$$= \frac{d\left\{\sec\left(\frac{3ax+b}{c}\right)\right\}^3}{d\left(\sec\left(\frac{3ax+b}{c}\right)\right)} \cdot \frac{d\left(\sec\left(\frac{3ax+b}{c}\right)\right)}{d\left(\frac{3ax+b}{c}\right)} \cdot \frac{d\left(\frac{3ax+b}{c}\right)}{dx}$$

$$= 3\sec^2\left(\frac{3ax+b}{c}\right)\cdot\sec\left(\frac{3ax+b}{c}\right)\cdot\tan\left(\frac{3ax+b}{c}\right)\cdot\frac{3a}{c}$$

$$= \frac{9a}{c}\sec^3\left(\frac{3ax+b}{c}\right)\tan\left(\frac{3ax+b}{c}\right).$$

(e) Let $y = x^2 \tan x$

$$\frac{dy}{dx} = \frac{d}{dx}(x^2\tan x)$$

$$= x^2\frac{d}{dx}(\tan x) + \tan x\frac{d}{dx}(x^2)$$

$$= x^2\cdot\sec^2 x + \tan x \cdot 2x$$

$$= x^2\sec^2 x + 2x\tan x.$$

(f) Let $y = (1 + x^2)\sin x$

$$\frac{dy}{dx} = \frac{d}{dx}\{(1+x^2)\sin x\}$$

$$= (1+x^2)\frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(1+x^2)$$

$$= (1+x^2)\cdot\cos x + \sin x \cdot 2x$$

$$= (1+x^2)\cos x + 2x\sin x$$

(g) Let $y = \dfrac{x^2}{\sin 3x}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x^2}{\sin 3x}\right)$$

$$= \frac{\sin 3x \dfrac{d}{dx}(x^2) - x^2\dfrac{d}{dx}(\sin 3x)}{(\sin 3x)^2} = \frac{2x\sin 3x - 3x^2\cos 3x}{\sin^2 3x}$$

(h) Let $y = \dfrac{\sin\sqrt{x}}{\sqrt{x}}$

$$\frac{dy}{dx} = \frac{\sqrt{x}\dfrac{d}{dx}(\sin\sqrt{x}) - \sin\sqrt{x}\dfrac{d}{dx}\sqrt{x}}{(\sqrt{x})^2} = \frac{\sqrt{x}\cdot\cos\sqrt{x}\cdot\dfrac{1}{2\sqrt{x}} - \sin\sqrt{x}\cdot\dfrac{1}{2\sqrt{x}}}{x}$$

$$= \frac{\sqrt{x}\cos\sqrt{x} - \sin\sqrt{x}}{2x^{3/2}}$$

(i) Let $y = \tan^2(\sin\sqrt{x})$

$$\frac{dy}{dx} = \frac{d\{\tan\sin\sqrt{x}\}^2}{d(\tan\sin\sqrt{x})}\cdot\frac{d(\tan\sin\sqrt{x})}{d(\sin\sqrt{x})}\cdot\frac{d(\sin\sqrt{x})}{d(\sqrt{x})}\cdot\frac{d(\sqrt{x})}{dx}$$

$$= 2(\tan\sin\sqrt{x})\cdot\sec^2(\sin\sqrt{x})\cdot\cos\sqrt{x}\cdot\frac{1}{2\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}}\sec^2(\sin\sqrt{x})\cdot\tan(\sin\sqrt{x})\cdot\cos\sqrt{x}.$$

(j) Let $y = \sec^2(\tan\sqrt{x})$

$$\frac{dy}{dx} = \frac{d\{\sec\tan\sqrt{x}\}^2}{d(\sec\tan\sqrt{x})}\cdot\frac{d(\sec\tan\sqrt{x})}{d(\tan\sqrt{x})}\cdot\frac{d(\tan\sqrt{x})}{d(\sqrt{x})}\cdot\frac{d(\sqrt{x})}{dx}$$

$$= 2(\sec\tan\sqrt{x})\cdot\sec^2(\tan\sqrt{x})\cdot\tan(\tan\sqrt{x})\cdot\sec^2\sqrt{x}\cdot\frac{1}{2\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}}\sec^2(\tan\sqrt{x})\sec(\tan\sqrt{x})\sec^2\sqrt{x}.$$

(k) Let $y = \sin 3x\cos x$

$$= \frac{1}{2}(2\sin 3x\cos x) = \frac{1}{2}\{\sin(3x+x) + \sin(3x-x)\}$$

$$= \frac{1}{2}(\sin 4x + \sin 2x)$$

$$\frac{dy}{dx} = \frac{1}{2}\left\{\frac{d(\sin 4x)}{d(4x)}\cdot\frac{d(4x)}{d(x)} + \frac{d(\sin 2x)}{d(2x)}\cdot\frac{d(2x)}{d(x)}\right\}$$

$$= \frac{1}{2}(\cos 4x\cdot 4 + \cos 2x\cdot 2) = 2\cos 4x + \cos 2x.$$

(l) $y = \cos 6x \cos 2x$

$= \frac{1}{2}(2 \cos 6x \cos 2x) = \frac{1}{2}\{\cos(6x - 2x) + \cos(6x + 2x)\}$

$= \frac{1}{2}\{\cos 4x + \cos 8x\}$

$\frac{dy}{dx} = \frac{1}{2}(-4 \sin 4x - 8 \sin 8x) = -2 \sin 4x - 4 \sin 8x.$

(m) Let $y = \sin \frac{1 + x^2}{1 - x^2}$

$$\frac{dy}{dx} = \frac{d}{dx}\left\{\sin\left(\frac{1+x^2}{1-x^2}\right)\right\} = \frac{d\left\{\sin\left(\frac{1+x^2}{1-x^2}\right)\right\}}{d\left(\frac{1+x^2}{1-x^2}\right)} \cdot \frac{d\left(\frac{1+x^2}{1-x^2}\right)}{dx}$$

$$= \cos\left(\frac{1+x^2}{1-x^2}\right) \cdot \left\{\frac{(1-x^2)\cdot 2x - (1+x^2)\cdot(-2x)}{(1-x^2)^2}\right\}$$

$$= \cos\left(\frac{1+x^2}{1-x^2}\right) \cdot \left\{\frac{2x(1-x^2+1+x^2)}{(1-x^2)^2}\right\} = \frac{4x}{(1-x^2)^2} \cos\left(\frac{1+x^2}{1-x^2}\right)$$

3. Find $\frac{dy}{dx}$ if

(a) $x + y = \sin y$ (b) $x - y = \sin xy$ (c) $xy = \sin(x + y)$

(d) $x^2 + y^2 = \sin xy$ (e) $x^2y^2 = \tan xy$ (f) $x^2y = \sec xy^2$

Solution

(a) $x + y = \sin y$

$\frac{d}{dx}(x + y) = \frac{d}{dx}(\sin y)$

or, $\frac{d}{dx}(x) + \frac{d}{dx}(y) = \frac{d(\sin y)}{dy} \cdot \frac{dy}{dx}$

or, $1 + \frac{dy}{dx} = \cos y \frac{dy}{dx}$

or, $1 = (\cos y - 1)\frac{dy}{dx}$

$\therefore \frac{dy}{dx} = \frac{1}{\cos y - 1}$

(b) $x - y = \sin xy$

$\frac{d}{dx}(x - y) = \frac{d}{dx}(\sin xy)$

or, $\frac{d}{dx}(x) - \frac{d}{dx}(y) = \frac{d(\sin xy)}{d(xy)} \cdot \frac{d(xy)}{dx}$

or, $1 - \frac{dy}{dx} = \cos xy\left(x\frac{dy}{dx} + y\right)$

or, $1 - \frac{dy}{dx} = x \cos xy \frac{dy}{dx} + y \cos xy$

$\therefore \frac{dy}{dx} = \frac{1 - y \cos xy}{x \cos xy + 1} = \frac{1 - y \cos xy}{1 + x \cos xy}$

(c) $xy = \sin(x + y)$

$\frac{d}{dx}(xy) = \frac{d}{dx}(\sin(x + y))$

$x\frac{d}{dx}(y) + y\frac{d}{dx}(x) = \frac{d\{\sin(x + y)\}}{d(x + y)} \cdot \frac{d(x + y)}{dx}$

or, $x\frac{dy}{dx} + y = \cos(x + y)\left(1 + \frac{dy}{dx}\right)$

or, $\quad x\frac{dy}{dx} + y = \cos(x+y) + \cos(x+y)\frac{dy}{dx}$

or, $\quad \{x - \cos(x+y)\}\frac{dy}{dx} = \cos(x+y) - y$

$$\frac{dy}{dx} = \frac{\cos(x+y) - y}{x - \cos(x+y)}.$$

(d) Given, $x^2 + y^2 = \sin xy$

Differentiating both sides with respect to x, we get

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(\sin xy)$$

or, $\quad \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d(\sin xy)}{d(xy)} \cdot \frac{d(xy)}{dx}$

or, $\quad 2x + \frac{d(y^2)}{dy} \cdot \frac{dy}{dx} = \cos xy \cdot \left\{x\frac{d(y)}{dx} + y\frac{d}{dx}(x)\right\}$

or, $\quad 2x + 2y\frac{dy}{dx} = \cos xy\left(x\frac{dy}{dx} + y\right)$

or, $\quad 2x + 2y\frac{dy}{dx} = \cos xy\frac{dy}{dx} + y\cos xy$

or, $\quad 2x - y\cos xy = x\cos xy\frac{dy}{dx} - 2y\frac{dy}{dx}$

or, $\quad (x\cos xy - 2y)\frac{dy}{dx} = 2x - y\cos xy$

$\therefore \quad \frac{dy}{dx} = \frac{2x - y\cos xy}{x\cos xy - 2y}.$

(e) $x^2y^2 = \tan xy$

$$\frac{d}{dx}(x^2y^2) = \frac{d}{dx}(\tan xy)$$

or, $\quad x^2\frac{d(y^2)}{d(y)} \cdot \frac{dy}{dx} + y^2\frac{d}{dx}(x^2) = \frac{d(\tan xy)}{d(xy)} \cdot \frac{d(xy)}{dx}$

or, $\quad x^2 \cdot 2y \cdot \frac{dy}{dx} + y^2 \cdot 2x = \sec^2 xy\left(1 + x\frac{dy}{dx}\right) = \sec^2 xy\frac{dy}{dx} + y\sec^2 xy$

or, $\quad \frac{dy}{dx} = \frac{y\sec^2 xy - 2xy^2}{2x^2y - x\sec^2 xy} = \frac{y(\sec^2 xy - 2xy)}{-x(\sec^2 xy - 2xy)} = -\frac{y}{x}$

(f) $x^2y = \sec xy^2$

$$\frac{d}{dx}(x^2y) = \frac{d}{dx}(\sec xy^2)$$

or, $\quad x^2\frac{d}{dx}(y) + y\frac{d}{dx}(x^2) = \frac{d(\sec xy^2)}{d(xy^2)} \cdot \frac{d(xy^2)}{dx}$

or, $\quad x^2\frac{dy}{dx} + 2xy = \sec xy^2\tan xy^2\left\{x\frac{d}{dy}(y^2) \cdot \frac{dy}{dx} + y^2\frac{d}{dx}(x)\right\}$

or, $\quad x^2\frac{dy}{dx} + 2xy = \sec xy^2\tan xy^2 \cdot 2xy \cdot \frac{dy}{dx} + y^2\sec xy^2\tan xy^2$

or, $\quad 2xy - y^2\sec xy^2\tan xy^2 = (\sec xy^2 \cdot \tan xy^2 \cdot 2xy - x^2)\frac{dy}{dx}$

$\therefore \quad \frac{dy}{dx} = \frac{2xy - y^2\sec xy^2\tan xy^2}{2xy\sec xy^2\tan xy^2 - x^2}.$

4. Find $\frac{dy}{dx}$, when

(a) $x = a\cos^2\theta,\ y = b\sin^2\theta$

(b) $x = 2a\tan\theta,\ y = a\sec^2\theta$

(c) $x = a(t + \sin t),\ y = a(1 - \cos t)$

(d) $x = a(\cos t + t\sin t),\ y = a(\sin t - t\cos t)$

Solution

(a) Here, $x = a\cos^2\theta$

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(a\cos^2\theta)$$

$$= a\frac{d(\cos\theta)^2}{d(\cos\theta)} \cdot \frac{d(\cos\theta)}{d\theta}$$

$$= a \cdot 2\cos\theta \cdot (-\sin\theta)$$

$$= a(-\sin 2\theta) = -a\sin 2\theta$$

and $y = b\sin^2\theta$

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(b\sin^2\theta) = b\frac{d(\sin\theta)^2}{d(\sin\theta)} \cdot \frac{d(\sin\theta)}{d\theta} = b \cdot 2\sin\theta \cdot \cos\theta = b\sin 2\theta$$

Now, $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{b\sin 2\theta}{-a\sin 2\theta} = \dfrac{-b}{a}$

(b) We have,

$x = 2a\tan\theta$ ... (1)

$y = a\sec^2\theta$ ... (2)

From (1),

$$\frac{dx}{d\theta} = \frac{d}{d\theta}(2a\tan\theta) = 2a\sec^2\theta$$

From (2),

$$\frac{dy}{d\theta} = \frac{d}{d\theta}(a\sec^2\theta)$$

$$= a\frac{d}{d\theta}(\sec^2\theta) = a \times \frac{d(\sec^2\theta)}{d(\sec\theta)} \times \frac{d(\sec\theta)}{d\theta}$$

$$= a \times 2\sec\theta \times \sec\theta\tan\theta = 2a\sec^2\theta\tan\theta$$

Now, $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{2a\sec^2\theta\tan\theta}{2a\sec^2\theta} = \tan\theta$

$\therefore \dfrac{dy}{dx} = \tan\theta$

(c) Here, $x = a(t + \sin t)$

$$\frac{dx}{dt} = a\left\{\frac{d(t)}{dt} + \frac{d(\sin t)}{dt}\right\} = a(1 + \cos t)$$

And, $y = a(1 - \cos t)$

$$\frac{dy}{dt} = a\left\{\frac{d(1)}{dt} - \frac{d}{dt}(\cos t)\right\} = a(0 + \sin t) = a\sin t$$

Now, $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{a \cdot \sin t}{a(1 + \cos t)} = \dfrac{2\sin t/2\cos t/2}{2\cos^2 t/2} = \tan\dfrac{t}{2}$.

(d) Here,

$x = a(\cos t + t\sin t)$

$$\frac{dx}{dt} = \frac{d}{dt}\{a(\cos t + t\sin t)\} = a\left\{\frac{d}{dt}(\cos t) + \frac{d}{dt}(t\sin t)\right\}$$

$$= a\left\{-\sin t + t\frac{d}{dt}(\sin t) + \sin t\frac{d}{dt}(t)\right\} = a(-\sin t + t.\cos t + \sin t.1) = a(t\cos t) = at\cos t$$

And, $y = a(\sin t - t\cos t)$

$$\frac{dy}{dt} = \frac{d}{dt}\{a(\sin t - t\cos t)\}$$

$$= a\left\{\frac{d}{dt}(\sin t) - \frac{d}{dt}(t\cos t)\right\} = a\left[\cos t - \left\{t\frac{d}{dt}\cos t + \cos t\frac{d}{dt}(t)\right\}\right]$$

$$= a\left[\cos t - \{t.(-\sin t) + \cos t.1\}\right] = a(\cos t + t\sin t - \cos t) = a\,t\sin t$$

Now,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a\,t\sin t}{a\,t\cos t} = \tan t$$

5. Find the derivatives of

(a) $\cos^{-1} x^2$

(b) $\sin^{-1}(1-2x^2)$

(c) $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

(d) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

(e) $\tan^{-1}\frac{\sin 2x}{1+\cos 2x}$

Solution

(a) $y = \cos^{-1} x^2$

$$\frac{dy}{dx} = \frac{d}{dx}(\cos^{-1} x^2)$$

$$= \frac{d(\cos^{-1} x^2)}{d(x^2)} \cdot \frac{d(x^2)}{dx} = -\frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x = \frac{-2x}{\sqrt{1-x^4}}$$

(b) $y = \sin^{-1}(1-2x^2)$

$$\frac{dy}{dx} = \frac{d}{dx}[\sin^{-1}(1-2x^2)].$$

$$= \frac{\{\sin^{-1}(1-2x^2)\}}{d(1-2x^2)} \cdot \frac{d}{dx}(1-2x^2)$$

$$= \frac{1}{\sqrt{1-(1-2x^2)^2}} \cdot (-4x) = \frac{-4x}{\sqrt{1-(1-4x^2+4x^4)}} = \frac{-4x}{\sqrt{1-1+4x^2-4x^4}}$$

$$= \frac{-4x}{\sqrt{4x^2-4x^4}} = \frac{-4x}{2x\sqrt{1-x^2}} = \frac{-2}{\sqrt{1-x^2}}$$

(c) We have, $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Suppose, $x = \tan\theta$ ... (i)

$\therefore$ $y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$

or, $y = \tan^{-1}(\tan 2\theta)$

$\therefore$ $y = 2\theta$ ... (ii)

From (i)

$x = \tan\theta$

$\therefore$ $\frac{dx}{d\theta} = \frac{d}{d\theta}(\tan\theta) = \sec^2\theta$

and from (ii) $y = 2\theta$

$\therefore$ $\frac{dy}{d\theta} = \frac{d}{d\theta}(2\theta) = 2$

$$\text{Now, } \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2}{\sec^2\theta} = \frac{2}{1+\tan^2\theta} = \frac{2}{1+x^2}$$

Alternatively,

We have,

$$y = \tan^{-1}\left(\frac{2x}{1-x^2}\right) \quad \text{(i)}$$

Suppose, $x = \tan\theta$ ... (ii)

Now, (i) becomes

$$y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \tan^{-1}(\tan 2\theta)$$

or, $y = 2\theta$

or, $y = 2\tan^{-1}x$ [$\because$ from (ii)]

Differentiating both sides with respect to x,

$$\frac{dy}{dx} = \frac{d}{dx}(2\tan^{-1}x) = 2\frac{d}{dx}(\tan^{-1}x) = 2\left(\frac{1}{1+x^2}\right)$$

Hence, $\frac{dy}{dx} = \frac{2}{1+x^2}$

(d) Let $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

Put $x = \tan\theta$

Then, $y = \left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$$= \cos^{-1}(\cos 2\theta) \qquad \left[\because \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}\right]$$

$$= 2\theta = 2\tan^{-1}x$$

$$\frac{dy}{dx} = 2\frac{d}{dx}(\tan^{-1}x) = \frac{2}{1+x^2}.$$

(e) Let $y = \tan^{-1}\left(\frac{\sin 2x}{1+\cos 2x}\right)$

$$= \tan^{-1}\left(\frac{2\sin x\cos x}{1+2\cos^2 x-1}\right)$$

$$= \tan^{-1}\left(\frac{\sin x}{\cos x}\right) = \tan^{-1}(\tan x) = x$$

Differentiating both sides w.r. to 'x' $\frac{dy}{dx} = \frac{dx}{dx} = 1$

6. (a) **Find $\frac{dy}{dx}$ when $y = \sin\theta$ and $\theta = 5x^2 - 6x + 2$**

(b) **Find the derivative of $\sin x$ w.r. to $\cos x$.**

(c) **Find the derivative of $\tan x$ w.r. to $\cot x$.**

Solution

(a) $y = \sin\theta$ (i)

$\theta = 5x^2 - 6x + 2$ (ii)

Now, differentiating equation (i) w.r.t θ and differentiating equation (ii) w.r.t. x, we get

$$\frac{dy}{d\theta} = \frac{d}{d\theta}\sin\theta = \cos\theta = \cos(5x^2-6x+2) \qquad [\because \text{using (ii)}]$$

And $\frac{d\theta}{dx} = \frac{d}{dx}(5x^2-6x+2) = 10x-6$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta}\times\frac{d\theta}{dx} = \cos(5x^2-6x+2)\times(10x-6) = (10x-6)\cos(5x^2-6x+2).$$

Alternatively,

$$\frac{dy}{dx} = \frac{dy}{d\theta}\times\frac{d\theta}{dx} = \frac{d}{d\theta}\sin\theta\times\frac{d}{dx}(5x^2-6x+2)$$

$$= \cos\theta\times(10x-6) = (10x-6)\cos\theta$$

(b) Derivative of $\sin x$ w. r. to $\cos x$

$$= \frac{d(\sin x)}{d(\cos x)} = \frac{\frac{d(\sin x)}{dx}}{\frac{d(\cos x)}{dx}} = \frac{\cos x}{-\sin x} = -\cot x$$

(c) Derivative of $\tan x$ w. r. to $\cot x$

$$= \frac{d(\tan x)}{d(\cot x)} = \frac{\frac{d(\tan x)}{dx}}{\frac{d(\cot x)}{dx}} = \frac{\sec^2 x}{-\operatorname{cosec}^2 x} = -\frac{\sin^2 x}{\cos^2 x} = -\tan^2 x.$$

Objective Questions

1. $\dfrac{d}{dx}(\cos x) =$

(a) $-\sin x$ (b) $\sin x$

(c) $\tan x$ (d) $\sec x$

Ans: a

2. $\dfrac{d}{dx}(x^2 - \sin x) =$

(a) $x^2 + \cos x$ (b) $2x - \cos x$

(c) $2x + \sin x$ (d) $2x - \sin x$

Ans: b

$$\frac{d}{dx}(x^2 - \sin x) = \frac{d}{dx}(x^2) - \frac{d}{dx}(\sin x) = 2x - \cos x$$

3. If $y = \sin x \cos x$ then $\dfrac{dy}{dx} =$

(a) $\cos^2 x - \sin^2 x$ (b) $\sin^2 x - \cos^2 x$

(c) $\sin x + \cos x$ (d) $\cos x - \sin x$

Ans: a

$$\frac{dy}{dx} = \frac{d}{dx}(\sin x \cos x) = \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x)$$

$$= \sin x(-\sin x) + \cos x(\cos x) = \cos^2 x - \sin^2 x$$

4. If $y = \dfrac{\cos x}{1 - \sin x}$ then $\dfrac{dy}{dx} =$

(a) $\dfrac{1}{1 - \cos x}$ (b) $\dfrac{1}{1 + \cos x}$

(c) $\dfrac{1}{1 + \sin x}$ (d) $\dfrac{1}{1 - \sin x}$

Ans: d

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\cos x}{1 - \sin x}\right) = \frac{(1 - \sin x)\cdot(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}$$

$$= \frac{-\sin x + \sin^2 x + \cos^2 x}{(1 - \sin x)^2} = \frac{1 - \sin x}{(1 - \sin x)^2} = \frac{1}{1 - \sin x}$$

5. $\dfrac{d}{dx}(\sin x^{o}) =$

(a) $\cos x^{o}$ (b) $-\cos x^{o}$

(c) $\dfrac{\pi}{180}\cos x^{o}$ (d) $\dfrac{180}{\pi}\cos x^{o}$

Ans: c

$$\frac{d}{dx}(\sin x^{o}) = \frac{d}{dx}\left(\sin\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos x^{o}$$

6. Derivative of $\sin(x^2 + x)$ with respect to x is

(a) $(2x + 1)\cos(x^2 + x)$ (b) $-(2x + 1)\cos(x^2 + x)$

(c) $\cos(x^2 + x)$ (d) $-\cos(x^2 + x)$

Ans: a

$$\frac{d}{dx}\{\sin(x^2 + x)\} = \frac{d\{\sin(x^2 + x)\}}{d(x^2 + x)}\cdot\frac{d(x^2 + x)}{dx} = \cos(x^2 + x)\cdot(2x + 1)$$

$$= (2x + 1)\cos(x^2 + x)$$

7. If $y = \sin^5 x$ then $\dfrac{dy}{dx}$ at $x = \dfrac{\pi}{3}$ is

(a) $\dfrac{9}{8}$ (b) $\dfrac{8}{9}$

(c) $\dfrac{32}{45}$ (d) $\dfrac{45}{32}$

Ans: d

$$\frac{dy}{dx} = \frac{d(\sin x)^5}{d(\sin x)} \cdot \frac{d(\sin x)}{dx} = 5\sin^4 x \cdot \cos x$$

$$\frac{dy}{dx} \text{ at } x = \frac{\pi}{3} \text{ is } 5\sin^4\frac{\pi}{3}\cos\frac{\pi}{3} = 5\left(\frac{\sqrt{3}}{2}\right)^4\left(\frac{1}{2}\right) = \frac{45}{32}$$

8. If $y = \cos t$ and $x = \sin t$ then $\dfrac{dy}{dx} =$

(a) $\cot t$ (b) $-\cot t$

(c) $-\tan t$ (d) $\tan t$

Ans: d

$y = \cos t$ and $x = \sin t$

$$\frac{dy}{dt} = -\sin t, \quad \frac{dx}{dt} = \cos t$$

$$\text{Now, } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sin t}{\cos t} = -\tan t$$

9. If $y = \cos^{-1} x$ then $\dfrac{dy}{dx} =$

(a) $\dfrac{1}{\sqrt{1-x^2}}$ (b) $\dfrac{1}{\sqrt{x^2-1}}$

(c) $\dfrac{1}{\sqrt{x^2+1}}$ (d) $-\dfrac{1}{\sqrt{1-x^2}}$

Ans: d

10. The derivative of $\sin x$ with respect to $\cos x$ is

(a) $-\tan x$ (b) $\tan x$

(c) $-\cot x$ (d) $\cot x$

Ans: c

$$\frac{d(\sin x)}{d(\cos x)} = \frac{\frac{d(\sin x)}{dx}}{\frac{d(\cos x)}{dx}} = \frac{\cos x}{-\sin x} = -\cot x$$

EXERCISE 2 (C)

1. Find, from definition, the derivatives of :

(a) e^{ax} (b) e^{2x+3}

(c) $\ell n(3x+5)$ (d) $\log_a x$

Solution

(a) Let $y = e^{ax}$

Also let Δx and Δy be the small increments in x and y respectively. Then,

$y + \Delta y = e^{a(x+\Delta x)}$

or, $\Delta y = e^{ax + a\Delta x} - y$

or, $\Delta y = e^{ax + a\Delta x} - e^{ax}$

or, $\Delta y = e^{ax} \cdot e^{a\Delta x} - e^{ax}$

Dividing both sides by Δx

$$\frac{\Delta y}{\Delta x} = \frac{e^{ax}(e^{a\Delta x} - 1)}{\Delta x}$$

Taking limit $\Delta x \to 0$ on both sides,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{e^{ax}(e^{a\Delta x} - 1)}{a \cdot \Delta x} \cdot a$$

$$\frac{dy}{dx} = e^{ax} \cdot 1 \cdot a = ae^{ax}$$

(b) Let $y = e^{2x+3}$

Let Δx be an increment in x and Δy be the corresponding element in y

Then, $y + \Delta y = e^{2(x+\Delta x)+3}$

or, $\Delta y = e^{2x+2\Delta x+3} - e^{2x+3}$

or, $\Delta y = e^{2x+3}(e^{2\Delta x} - 1)$

Dividing both sides by Δx

$$\frac{\Delta y}{\Delta x} = \frac{e^{2x+3}(e^{2\Delta x} - 1)}{\Delta x}$$

Taking limit $\Delta x \to 0$ on both sides,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left\{ \frac{e^{2x+3}(e^{2\Delta x} - 1)}{2\Delta x} \cdot 2 \right\}$$

$$\frac{dy}{dx} = e^{2x+3} \cdot 1 \cdot 2 = 2e^{2x+3}$$

(c) Let $y = \ell n(3x + 5)$

Again, let Δx and Δy be the small increments in x and y respectively.

$y + \Delta y = \ell n\,\{3(x + \Delta x) + 5\}$

or, $\Delta y = \ell n\,(3x + 3\Delta x + 5) - \ell n(3x + 5)$

or, $\Delta y = \ell n \left(\dfrac{3x + 3\Delta x + 5}{3x + 5}\right)$

or, $\Delta y = \ell n \left(1 + \dfrac{3\Delta x}{3x + 5}\right)$

By definition of derivative, we have,

$$\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\ell n\left(1 + \dfrac{3\Delta x}{3x+5}\right)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\ell n\left(1 + \dfrac{3\Delta x}{3x+5}\right)}{\dfrac{3\Delta x}{3x+5}} \cdot \frac{3}{3x+5}$$

$$= 1 \cdot \frac{3}{3x+5} = \frac{3}{3x+5}$$

(d) Let $y = \log_a x = \log_a e \cdot \log_e x$

Again, let Δx and Δy be the small increments in x and y respectively. Then,

$y + \Delta y = \log_e (x + \Delta x) \cdot \log_a e$

or, $\Delta y = \log_e (x + \Delta x) \cdot \log_a e - \log_a e \cdot \log_e x$

or, $\Delta y = \log_a e\,\{\log_e (x + \Delta x) - \log_e x\}$

By definition of derivative, we have,

$$\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \log_a e \lim_{\Delta x \to 0} \frac{\log_e (x + \Delta x) - \log_e x}{\Delta x}$$

$$= \log_a e \cdot \lim_{\Delta x \to 0} \frac{\log_e \left(\dfrac{x + \Delta x}{x}\right)}{\Delta x} = \log_a e \cdot \lim_{\Delta x \to 0} \frac{\log_e \left(1 + \dfrac{\Delta x}{x}\right)}{\dfrac{\Delta x}{x} \cdot x}$$

$$= \log_a e \cdot 1 \cdot \frac{1}{x} = \frac{\log_a e}{x}$$

2. Find the derivatives of

(a) $\ell n\,(3x-2)$ (b) $\ell n\, x^5$ (c) $\ell n\,(ax^2+bx+c)$

(d) $\ell n\,(\sin x^2)$ (e) $\ell n\,(x+\tan x)$ (f) $\ell n\,(\ell n\, x)$

(g) $\ell n\,(e^{ax}+e^{-ax})$ (h) $x\,\ell n\, x$ (i) $\dfrac{\ell n\, x}{\cos x}$

Solution

(a) $y = \ell n(3x-2)$

Differentiating both sides w.r. to x

$$\frac{dy}{dx} = \frac{d}{dx}\{\ell n(3x-2)\}$$

$$= \frac{d\{\ell n(3x-2)\}}{d(3x-2)}\cdot\frac{d(3x-2)}{dx} = \frac{1}{3x-2}\cdot 3 = \frac{3}{3x-2}$$

(b) Let $y = \ell n\, x^5 = 5\,\ell n\, x$

$$\frac{dy}{dx} = \frac{d}{dx}(5\,\ell n\, x) = 5\cdot\frac{1}{x} = \frac{5}{x}$$

(c) Let $y = \ell n\,(ax^2+bx+c)$

$$\frac{dy}{dx} = \frac{d}{dx}\{\ell n\,(ax^2+bx+c)\}$$

$$= \frac{d\{\ell n\,(ax^2+bx+c)\}}{d(ax^2+bx+c)}\cdot\frac{d}{dx}(ax^2+bx+c)$$

$$= \frac{1}{ax^2+bx+c}\cdot(2ax+b) = \frac{2ax+b}{ax^2+bx+c}$$

(d) Let $y = \ell n\,(\sin x^2)$

$$\frac{dy}{dx} = \frac{d}{dx}\{\ell n\,(\sin x^2)\}$$

$$= \frac{d\{\ell n\,(\sin x^2)\}}{d(\sin x^2)}\cdot\frac{d(\sin x^2)}{d(x^2)}\cdot\frac{d(x^2)}{dx} = \frac{1}{\sin x^2}\cdot\cos x^2\cdot 2x = 2x\cot x^2.$$

(e) Let $y = \ell n\,(x+\tan x)$

$$\frac{dy}{dx} = \frac{d}{dx}\{\ell n\,(x+\tan x)\}$$

$$= \frac{d\{\ell n\,(x+\tan x)\}}{d\,(x+\tan x)}\cdot\frac{d\,(x+\tan x)}{dx} = \frac{1}{x+\tan x}\cdot(1+\sec^2 x) = \frac{1+\sec^2 x}{x+\tan x}$$

(f) Let $y = \ell n\,(\ell n\, x)$

$$\frac{dy}{dx} = \frac{d}{dx}\{\ell n\,(\ell n\, x)\} = \frac{d\{\ell n\,(\ell n\, x)\}}{d\,(\ell n\, x)}\cdot\frac{d\,(\ell n\, x)}{dx} = \frac{1}{\ell n\, x}\cdot\frac{1}{x} = \frac{1}{x\cdot\ell n\, x}$$

(g) Let $y = \ell n\,(e^{ax}+e^{-ax})$

$$\frac{dy}{dx} = \frac{d}{dx}\{\ell n\,(e^{ax}+e^{-ax})\}$$

$$= \frac{d\{\ell n\,(e^{ax}+e^{-ax})\}}{d\,(e^{ax}+e^{-ax})}\cdot\frac{d\,(e^{ax}+e^{-ax})}{dx} = \frac{1}{e^{ax}+e^{-ax}}\cdot\{ae^{ax}+(-a)\,e^{-ax}\} = \frac{a(e^{ax}-e^{-ax})}{e^{ax}+e^{-ax}}$$

(h) Let $y = x\,\ell n\, x$

$$\frac{dy}{dx} = \frac{d}{dx}(x\,\ell n\, x) = x\frac{d}{dx}(\ell n\, x) + \ell n\, x\frac{d}{dx}(x) = x\cdot\frac{1}{x} + \ell n\, x\cdot 1 = 1+\ell n\, x.$$

(i) Let $y = \dfrac{\ell n\, x}{\cos x}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\ell n\, x}{\cos x}\right) = \frac{\cos x\dfrac{d}{dx}(\ell n\, x) - \ell n\, x\dfrac{d}{dx}(\cos x)}{(\cos x)^2} = \frac{\cos x\cdot\dfrac{1}{x} - \ell n\, x\cdot(-\sin x)}{(\cos x)^2}$$

$$= \frac{\cos x + x\,\ell n\, x\cdot\sin x}{x\cos^2 x}$$

3. Find $\frac{dy}{dx}$, when

(a) $xy = \ell n(x + y)$ (b) $x^2 + y^2 = \ell n(x^2 + y^2)$

(c) $x = e^{\cos t}, y = e^{\sin t}$ (d) $x = \ell n\, t + \sin t, y = e^t + \cos t$

(e) $x^y = y^x$

Solution

(a) $xy = \ell n(x + y)$

Diff both sides w. r. to x

$$\frac{d}{dx}(xy) = \frac{d}{dx}\{\ell n(x + y)\}$$

or, $$x\frac{d}{dx}(y) + y\frac{d}{dx}(x) = \frac{d\{\ell n(x + y)\}}{d(x + y)} \cdot \frac{d(x + y)}{dx}$$

or, $$x\frac{dy}{dx} + y = \frac{1}{x + y}\left(1 + \frac{dy}{dx}\right)$$

or, $$(x + y)\left(x\frac{dy}{dx} + y\right) = 1 + \frac{dy}{dx}$$

or, $$x(x + y)\frac{dy}{dx} + y(x + y) = 1 + \frac{dy}{dx}$$

or, $$(x^2 + xy - 1)\frac{dy}{dx} = 1 - xy - y^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{1 - xy - y^2}{x^2 + xy - 1}.$$

(b) $x^2 + y^2 = \ell n(x^2 + y^2)$

Diff. both sides w. r. to x

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}\{\ell n(x^2 + y^2)\}$$

or, $$2x + 2y\frac{dy}{dx} = \frac{\{\ell n(x^2 + y^2)\}}{d(x^2 + y^2)} \cdot \frac{d(x^2 + y^2)}{dx}$$

or, $$2x + 2y\frac{dy}{dx} = \frac{1}{x^2 + y^2}\left(2x + 2y \cdot \frac{dy}{dx}\right)$$

or, $$\left(x + y \cdot \frac{dy}{dx}\right)(x^2 + y^2) = x + y\frac{dy}{dx}$$

or. $$x(x^2 + y^2) + y(x^2 + y^2)\frac{dy}{dx} = x + y\frac{dy}{dx}$$

or, $$\{y(x^2 + y^2) - y\}\frac{dy}{dx} = x - x(x^2 + y^2)$$

$$\therefore \quad \frac{dy}{dx} = \frac{x(1 - x^2 - y^2)}{y(x^2 + y^2 - 1)}$$

(c) $x = e^{\cos t}$

$$\frac{dx}{dt} = \frac{d(e^{\cos t})}{d(\cos t)} \cdot \frac{d(\cos t)}{dt}$$

$$= e^{\cos t} \cdot (-\sin t) = -\sin t\, e^{\cos t}.$$

and $y = e^{\sin t}$

$$\frac{dy}{dt} = \frac{d(e^{\sin t})}{dt} = \frac{d(e^{\sin t})}{d(\sin t)} \cdot \frac{d(\sin t)}{dt}$$

$$= e^{\sin t} \cdot \cos t.$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{e^{\sin t} \cdot \cos t}{-\sin t\, e^{\cos t}} = -e^{\sin t - \cos t} \cot t.$$

(d) Here, $x = \ell n\, t + \sin t$

$$\frac{dx}{dt} = \frac{1}{t} + \cos t = \frac{1 + t\cos t}{t}$$

and $y = e^t + \cos t$

$$\frac{dy}{dt} = e^t - \sin t$$

Now, $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{e^t - \sin t}{\frac{1 + t\cos t}{t}} = \frac{t(e^t - \sin t)}{1 + t\cos t}$$

(e) $x^y = y^x$

Taking 'ℓn' on both sides, we get,

$$y\, \ell n\, x = x\, \ell n\, y$$

Diff. both sides w.r. to x,

$$\frac{d}{dx}(y\, \ell n\, x) = \frac{d}{dx}(x\, \ell n\, y)$$

or, $$y\frac{d}{dx}(\ell n\, x) + \ell n\, x\frac{d}{dx}(y) = x\frac{d}{dx}(\ell n\, y) + \ell n\, y\frac{d}{dx}(x)$$

or, $$y \cdot \frac{1}{x} + \ell n\, x \cdot \frac{dy}{dx} = x \cdot \frac{1}{y} \cdot \frac{dy}{dx} + \ell n\, y$$

or, $$\left(\ell n\, x - \frac{x}{y}\right)\frac{dy}{dx} = \ell n\, y - \frac{y}{x}$$

or, $$\left(\frac{y\, \ell n\, x - x}{y}\right)\frac{dy}{dx} = \frac{x\, \ell n\, y - y}{x}$$

$$\therefore \quad \frac{dy}{dx} = \frac{y(x\, \ell n\, y - y)}{x(y\, \ell n\, x - x)}.$$

4. Find the derivatives of

(a) $e^{\sin x}$ **(b)** $e^{\sin(\ell n\, x)}$ **(c)** $(\sin x)^x$

Solution

(a) Let $y = e^{\sin x}$

$$\frac{dy}{dx} = \frac{d(e^{\sin x})}{d(\sin x)} \cdot \frac{d(\sin x)}{dx} = e^{\sin x} \cdot \cos x.$$

(b) Let $y = e^{\sin(\ell n\, x)}$

$$\frac{dy}{dx} = \frac{d(e^{\sin(\ell n\, x)})}{dx} = \frac{d(e^{\sin(\ell n\, x)})}{d(\sin(\ell n\, x))} \cdot \frac{d(\sin \ell n\, x)}{d(\ell n\, x)} \cdot \frac{d(\ell n\, x)}{dx}$$

$$= e^{\sin(\ell n\, x)} \cdot \cos \ell n\, x \cdot \frac{1}{x}$$

$$= \frac{1}{x} e^{\sin(\ell n\, x)} \cdot \cos(\ell n\, x).$$

(c) Let $y = (\sin x)^x$

Taking ℓn on both sides,

$$\ell n\, y = x\, \ell n \sin x$$

Diff. both sides w. r. to x

$$\frac{d}{dx}(\ell n\, y) = \frac{d}{dx}(x \cdot \ell n \sin x)$$

$$\frac{d(\ell n\, y)}{d(y)} \cdot \frac{dy}{dx} = x\frac{d}{dx}(\ell n \sin x) + \ell n \sin x\frac{d}{dx}(x)$$

or, $$\frac{1}{y} \cdot \frac{dy}{dx} = x \cdot \frac{1}{\sin x} \cdot \cos x + \ell n \sin x$$

or, $$\frac{dy}{dx} = y(x \cot x + \ell n \sin x)$$

$$\therefore \quad \frac{dy}{dx} = (\sin x)^x (x \cot x + \ell n \sin x)$$

5. Find all successive derivatives of

(a) $y = 2x^2 - x^3$ (b) $y = x^4 + 3x^2 + 5$

Solution

(a)
$y = 2x^2 - x^3$
$y_1 = 4x - 3x^2$
$y_2 = 4 - 6x$
$y_3 = -6$
$y_4 = 0$
$y_5 = y_6 = \ldots = 0$

(b)
$y = x^4 + 3x^2 + 5$
$y_1 = 4x^3 + 6x$
$y_2 = 12x^2 + 6$
$y_3 = 24x$
$y_4 = 24$
$y_5 = y_6 = y_7 = \ldots = 0$

Objective Questions

1. $\frac{d}{dx}(e^{2x}) =$

(a) $2e^{2x}$ (b) e^{2x}

(c) $\frac{e^{2x}}{2}$ (d) e^{x}

Ans: a

$\frac{d}{dx}(e^{2x}) = 2e^{2x}$ $\left[\because \frac{d}{dx}(e^{ax}) = ae^{ax}\right]$

2. $\frac{d}{dx}(5^x) =$

(a) 5^x (b) $x\,5^{x-1}$

(c) $\frac{5^x}{\log 5}$ (d) $5^x \log 5$

Ans: d

$\frac{d}{dx}(5^x) = 5^x \cdot \log 5$ $\left[\because \frac{d}{dx}(a^x) = a^x \log a\right]$

3. $\frac{d}{dx}(e^{mx+n}) =$

(a) e^{mx+n} (b) $\frac{e^{mx+n}}{m}$

(c) $n\,e^{mx+n}$ (d) $m\,e^{mx+n}$

Ans: d

$\frac{d}{dx}(e^{mx+n}) = \frac{d(e^{mx+n})}{d(mx+n)} \cdot \frac{d(mx+n)}{dx} = me^{mx+n}$

4. If $y = \log(2x+3)$ then $\frac{dy}{dx} =$

(a) $\frac{1}{2x+3}$ (b) $\frac{2}{2x+3}$

(c) $\frac{3}{2x+3}$ (d) $-\frac{1}{2x+3}$

Ans: b

$y = \log(2x+3)$

$\frac{dy}{dx} = \frac{d\{\log(2x+3)\}}{d(2x+3)} \cdot \frac{d(2x+3)}{dx} = \frac{1}{2x+3} \cdot 2 = \frac{2}{2x+3}$

5. If $y = \log t + e^t$ then $\frac{dy}{dx} =$

(a) $\log t + e^t$ (b) $e^t \log t$

(c) $\frac{1}{t} + e^t$ (d) $\frac{1}{t} - e^t$

Ans: c

$\frac{dy}{dt} = \frac{d}{dt}(\log t + e^t) = \frac{1}{t} + e^t$

6. If $y = e^{\sqrt{\sin x}}$ then $\frac{dy}{dx} =$

(a) $\dfrac{e^{\sqrt{\sin x}} \cdot \cos x}{2\sqrt{\sin x}}$ (b) $-\dfrac{e^{\sqrt{\sin x}} \cdot \cos x}{2\sqrt{\sin x}}$

(c) $\dfrac{e^{\sqrt{\sin x}} \cdot \cos x}{\sqrt{\sin x}}$ (d) $-\dfrac{e^{\sqrt{\sin x}} \cdot \cos x}{\sqrt{\sin x}}$

Ans: a

$$\frac{dy}{dx} = \frac{d}{dx}\left(e^{\sqrt{\sin x}}\right)$$

$$= \frac{d\left(e^{\sqrt{\sin x}}\right)}{d\left(\sqrt{\sin x}\right)} \cdot \frac{d\left(\sqrt{\sin x}\right)}{d(\sin x)} \cdot \frac{d(\sin x)}{dx} = e^{\sqrt{\sin x}} \cdot \frac{1}{2}(\sin x)^{\frac{1}{2}-1} \cdot \cos x = \frac{e^{\sqrt{\sin x}} \cdot \cos x}{2\sqrt{\sin x}}$$

7. The derivative of x^x w.r. to x is

(a) x^x (b) $x^x \cdot \log x$

(c) x^{x-1} (d) $x^x(1 + \log x)$

Ans: d

Let, $y = x^x$

$\log y = x \log x$

$$\frac{d}{dy}(\log y) \cdot \frac{dy}{dx} = x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(x)$$

$$\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \log x$$

$$\frac{dy}{dx} = y(1 + \log x) = x^x(1 + \log x)$$

EXERCISE 2 (D)

1. Find Δy, dy and $\Delta y - dy$ when $y = x^2 + 5x$ when $x = 2$ and $dx = 0.1$.

Solution

Here, $y = x^2 + 5x$, $x = 2$ and $dx = 0.1$

Again, $y + \Delta y = (x + \Delta x)^2 + 5(x + \Delta x)$

or, $\Delta y = x^2 + 2 \cdot x \cdot \Delta x + (\Delta x)^2 + 5x + 5\Delta x - x^2 - 5x$

or, $\Delta y = 2 \cdot x \cdot \Delta x + (\Delta x)^2 + 5\Delta x = 0.1\,(2 \times 2 + 0.1 + 5) = 0.91$

$dy = (2x + 5)\,dx = (2 \times 2 + 5) \times 0.1 = 0.9$

Now, $\Delta y - dy = 0.91 - 0.9 = 0.01$

2. What is the exact change in the value of $y = x^2$ when x changes from 10 to 10.1? What is the approximate change in y?

Solution

Here, $y = x^2$

$y + \Delta y = (x + \Delta x)^2$

or, $\Delta y = (x + \Delta x)^2 - x^2$ ... (i)

Here, $\Delta x = dx = 10.1 - 10 = 0.1$ and $x = 10$.

From (i), exact change $\Delta y = (10 + 0.1)^2 - 10^2 = (10.1)^2 - 10^2 = 2.01$

Again, $dy = 2x\,dx$

$\therefore$ Approximate change in y $(dy) = 2 \times 10 \times 0.1 = 2$

3. If the radius of sphere changes from 2 cm to 2.01 cm, find the approximate increase in its volume.

Solution

We have, volume of sphere $(V) = \frac{4}{3}\pi r^3$

Here, $r = 2$ cm, $r + \Delta r = 2.01$ cm

$\therefore$ $\Delta r = dr = 2.01 - 2 = 0.01$ cm

Approximate change in volume $(dV) = \frac{4}{3}\pi \cdot 3r^2 \cdot dr = \frac{4}{3}\pi \times 3 \times 2^2 \times 0.01 = 0.16\pi\ \text{cm}^3$

4. **If the radius of a circle is increased from 5 to 5.1 cm, find the approximate increase in area.**

Solution

Here, $r = 5$ cm, $r + \Delta r = 5.1$ cm

$\Delta r = dr = 5.1 - 5 = 0.1$ cm

We have, $A = \pi r^2$

$dA = \pi \cdot 2r \cdot dr$

Approximate increase in area $(dA) = \pi \times 2 \times 5 \times 0.1 = \pi \text{ cm}^2$

5. **The radius of a circle increases from 10 m to 10.1 m. Estimate the increase in the circle's area. Also find true change ΔA.**

Solution

Here, $r = 10$ m

$r + \Delta r = 10.1$ m

$\therefore \quad \Delta r = dr = 10.1 - 10 = 0.1$ m

We have, $A = \pi r^2$

$dA = \pi \cdot 2r\, dr$

Approximate change in area $(dA) = \pi \times 2 \times 10 \times 0.1 = 2\pi \text{ m}^2$

Again, $A + \Delta A = \pi(r + \Delta r)^2$

or, $\quad \Delta A = \pi(r + \Delta r)^2 - \pi r^2 = \pi r^2 + 2 \cdot r \cdot \Delta r \cdot \pi + \pi \cdot (\Delta r)^2 - \pi r^2 = \Delta r\,(2\pi r + \pi \cdot \Delta r)$

True change $= 0.1\,(2\pi \times 10 + \pi \times 0.1) = 2.01\pi \text{ m}^2$

6. **Find the approximate increase in the area of a cube if the edge increases from 10 cm to 10.2 cm. Calculate the percentage error in the use of differential approximation.**

Solution

Here, $x = 10$ cm, $x + \Delta x = 10.2$ cm

$\therefore \quad dx = \Delta x = 10.2 \text{ cm} - 10 \text{ cm} = 0.2$ cm

We have, area of cube $(A) = 6x^2$

$dA = 6 \cdot 2x \cdot dx = 6 \times 2 \times 10 \times 0.2 = 24 \text{ cm}^2$

Again, $A + \Delta A = 6(x + \Delta x)^2$

or, $\quad \Delta A = 6(x + \Delta x)^2 - 6x^2 = 6(10.2)^2 - 6 \times 10^2 = 24.24$

Error = Actual increase – Approximate increase $= 24.24 - 24 = 0.24$

Percentage error $= \dfrac{\text{error}}{A} \times 100 = \dfrac{0.24}{6 \times 10^2} \times 100 = 0.04\%$

Objective Questions

1. **Which of the following is not always true?**

   (a) If a function $f(x)$ is continuous at $x = a$ then $\lim\limits_{x \to a} f(x)$ exists.

   (b) If $f(x)$ is differentiable at $x = a$ then $f(x)$ is continuous at $x = a$.

   (c) If $f(x)$ is continuous at $x = a$ then $f(x)$ is differentiable at $x = a$.

   (d) If $f(x)$ is not continuous at $x = a$ then $f(x)$ is not differentiable at $x = a$.

Ans: c

2. **If $y = x^5 + 5x$ then $dy =$**

   (a) $5x^4 + 5$ (b) $(5x^4 + 5)\, dx$

   (c) $x^5 + 5x$ (d) $(x^5 + 5x)\, dx$

Ans: b

$y = x^5 + 5x$

$\dfrac{dy}{dx} = 5x^4 + 5$

$dy = (5x^4 + 5)\, dx$

3. **If $y = x^2 + 2x$; $x = 2$ and $dx = 0.1$ then $\Delta y =$**

   (a) 0.2 (b) 0.12

   (c) 0.47 (d) 0.61

Ans: d

$$y + \Delta y = (x + \Delta x)^2 + 2(x + \Delta x)$$

or. $$\Delta y = x^2 + 2 \cdot x \cdot \Delta x + (\Delta x)^2 + 2x + 2\Delta x - x^2 - 2x$$

$$\Delta y = 2 \cdot x \cdot \Delta x + (\Delta x)^2 + 2\Delta x = 2 \times 2 \times 0.1 + (0.1)^2 + 2 \times 0.1 = 0.61$$

4. If $y = f(x)$ be a given function and Δx, Δy be the small increments in x and y respectively, then the actual change in dependent variable y is

(a) $\Delta y = f(x + \Delta x)$

(b) $\Delta y = f(x + \Delta x) + f(x)$

(c) $\Delta y = f(x + \Delta x) - f(x)$

(d) $\Delta y = \dfrac{f(x + \Delta x)}{f(x)}$

Ans: c

5. The formula for calculating error is

(a) |Actual change|

(b) |Approximate change|

(c) |Actual change + Approximate change|

(d) |Actual change – Approximate change|

Ans: d

EXERCISE– 2 (E)

1. Find the slope and inclination with x–axis of the tangent of the following curves

(a) $3y = x^3 + 1$ at $x = 1$

(b) $y = -x^4 - 3x$ at $(-1, 2)$

(c) $x^2 + y^2 = 25$ at $(4, -3)$

Solution

(a) Here, $3y = x^3 + 1$

or. $$3\frac{dy}{dx} = 3x^2$$

or, $$\frac{dy}{dx} = x^2$$

At $x = 1$, $\dfrac{dy}{dx} = 1^2 = 1$

Slope at $x = 1$ is 1.

If θ is the inclination of tangent with x-axis then

$$\tan\theta = 1$$

or. $$\tan\theta = \tan\frac{\pi}{4}$$

$$\therefore \quad \theta = \frac{\pi}{4}$$

(b) Here, $y = -x^4 - 3x$ at $(-1, 2)$

$$\frac{dy}{dx} = -4x^3 - 3$$

At $x = -1$, $\dfrac{dy}{dx} = -4(-1)^3 - 3 = 4 - 3 = 1$.

Slope = 1

If θ is the inclination of tangent with x-axis then

$$\tan\theta = 1 = \tan\frac{\pi}{4}$$

$$\therefore \quad \theta = \frac{\pi}{4}$$

(c) Here, $x^2 + y^2 = 25$ at $(4, -3)$

or, $\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$

or, $2x + 2y\frac{dy}{dx} =$

$\therefore \frac{dy}{dx} = -\frac{x}{y}$

At $(x, y) = (4, -3)$, $\frac{dy}{dx} = \frac{-(4)}{-3} = \frac{4}{3}$

$\therefore$ Slope $= \frac{4}{3}$

If θ be the inclination of tangent with x-axis then

$\tan\theta = \frac{4}{3}$

$\theta = \tan^{-1}\left(\frac{4}{3}\right)$

2. At what angle does the curve $y(1 + x) = x$ cut x–axis?

Solution

Given, $y(1+x) = x$ ...(i)

The curve meets the x-axis where $y = 0$. So, putting $y = 0$ in (i), we get

From (i)

$$y = \frac{x}{1+x}$$

$$\frac{dy}{dx} = \frac{(1+x)\frac{dx}{dx} - x\frac{d}{dx}(1+x)}{(1+x)^2} = \frac{(1+x)\ 1 - x}{(1+x)^2} = \frac{1}{(1+x)^2}$$

At $x = 0$, $\frac{dy}{dx} = \frac{1}{(1+0)^2} = 1$

If θ be the angle made by tangent with x-axis then, $\tan\theta = 1 = \tan\frac{\pi}{4}$

$\therefore \theta = \frac{\pi}{4}$

3. Find the equations of the tangents and normals to the following curves.

(a) $y = 2x^3 - 5x^2 + 8$ at (2, 4)

(b) $x^2 + y^2 = 25$ at (3, 4)

Solution

(a) Given curve is $y = 2x^3 - 5x^2 + 8$

Differentiating both sides with respect to 'x'

$$\frac{dy}{dx} = 6x^2 - 10x$$

At (2, 4), $\frac{dy}{dx} = 6 \times 2^2 - 10 \times 2 = 4$

i.e. slope (m) = 4

The equation of tangent at (2, 4) and having slope 4 is

$y - y_1 = m(x - x_1)$

or, $y - 4 = 4(x - 2)$

or, $y - 4 = 4x - 8$

or, $4x - y - 4 = 0$

$\therefore 4x - y = 4$

Slope of normal $= -\frac{1}{\frac{dy}{dx}} = -\frac{1}{4}$

The equation of normal at $(x_1, y_1) = (2, 4)$ is

$y - y_1 = m(x - x_1)$

or, $y - 4 = -\frac{1}{4}(x - 2)$

or, $4y - 16 = -x + 2$

$\therefore$ $x + 4y = 18$

(b) Here, $x^2 + y^2 = 25$

Differentiating both sides w.r.t. x,

$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$

or, $2x + 2y\frac{dy}{dx} = 0$

or, $\frac{dy}{dx} = -\frac{x}{y}$

At $(x, y) = (3, 4)$; $\frac{dy}{dx} = -\frac{3}{4}$

$\therefore$ Slope of tangent $(m) = -\frac{3}{4}$

The equation of tangent at $(x_1, y_1) = (3, 4)$ is

$y - y_1 = m(x - x_1)$

or, $y - 4 = -\frac{3}{4}(x - 3)$

or, $4y - 16 = -3x + 9$

$\therefore$ $3x + 4y = 25$

Again, slope of normal $= -\dfrac{1}{\text{slope of tangent}} = -\dfrac{1}{\frac{-3}{4}} = \frac{4}{3}$

The equation of normal at $(x_1, y_1) = (3, 4)$ is

$y - y_1 = \frac{4}{3}(x - x_1)$

or, $y - 4 = \frac{4}{3}(x - 3)$

or, $3y - 12 = 4x - 12$

$\therefore$ $4x - 3y = 0$

4. **Find the points on the following curves where the tangents are parallel to the x–axis**

(a) $y = x^2 + 4x + 1$ (b) $y = x^3 - 2x^2 + 1$

Solution

(a) Here, $y = x^2 + 4x + 1$

$\frac{dy}{dx} = 2x + 4$

For tangent parallel to x-axis, we have,

$\frac{dy}{dx} = 0$

or, $2x + 4 = 0$

or, $x = -2$

When, $x = -2$, from (i) $y = (-2)^2 + 4(-2) + 1 = 4 - 8 + 1 = -3$

$\therefore$ Required point $= (-2, -3)$.

(b) Here, $y = x^3 - 2x^2 + 1$ ... (i)

$\frac{dy}{dx} = 3x^2 - 4x$

For tangent parallel to x-axis, $\frac{dy}{dx} = 0$

or, $3x^2 - 4x = 0$

or, $x = 0, \frac{4}{3}$

When $x = 0$, from (i) $y = 0^3 - 2 \times 0^2 + 1 = 1$

When $x = \frac{4}{3}$, from (i) $y = \left(\frac{4}{3}\right)^3 - 2\left(\frac{4}{3}\right)^2 + 1 = \frac{-5}{27}$

Required points are (0, 1) and $\left(\frac{4}{3}, \frac{-5}{27}\right)$

5. **Find the points on the curve $x^2 + y^2 = 25$ at which the tangents are parallel to the (a) x–axis (b) y–axis.**

Solution

Given curve is $x^2 + y^2 = 25$ ... (i)

Differentiating both sides w.r. to x,

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

or, $\frac{dy}{dx} = -\frac{x}{y}$

(a) For tangents parallel to x-axis, we have

$$\frac{dy}{dx} = 0$$

or, $-\frac{x}{y} = 0$

$\therefore$ $x = 0$

Putting the value of x in (i), we get

$0 + y^2 = 25$

or, $y = \pm 5$

$\therefore$ Required points are $(0, \pm 5)$

(b) For the tangent parallel to y-axis, we have

$$\frac{dx}{dy} = 0$$

or, $\frac{-y}{x} = 0$

or, $y = 0$

Then, from (i)

$x^2 = 25$

or, $x = \pm 5$

$\therefore$ Required points are $(\pm 5, 0)$.

6. (a) **Find the equation of the tangent to the curve $y = 2x^2 - 3x + 1$ which is parallel to the line $x - y + 5 = 0$.**

(b) **Find the equation of the tangent to the curve $y = 3x^2 - 3x + 5$ which is perpendicular to the line $x + 3y + 5 = 0$.**

Solution

(a) Given curve is $y = 2x^2 - 3x + 1$ ... (i)

$$\frac{dy}{dx} = 4x - 3$$

Slope of the line $x - y + 5 = 0$ is $-\frac{\text{coeff. of x}}{\text{coeff. of y}} = -\frac{1}{-1} = 1$.

Then, $4x - 3 = 1$

or, $4x = 4$

or, $x = 1$

Then, from (i), $y = 2 \times 1^2 - 3 \times 1 + 1 = 0$

The equation of tangent through the point $(x_1, y_1) = (1, 0)$ and having slope (m) = 1 is

$y - y_1 = m(x - x_1)$

or, $y - 0 = 1(x - 1)$

or, $y = x - 1$

$\therefore$ $x - y - 1 = 0$

(b) Given curve is $y = 3x^2 - 3x + 5$ ... (ii)

$$\frac{dy}{dx} = 6x - 3 = m_1 \text{ (say)}$$

Slope of the line $x + 3y + 5 = 0$ is $-\frac{1}{3}$.

But the tangent at the curve (i) is perpendicular to x + 3y − 5 = 0, so

$m_1 \cdot m_2 = -1$

or, $(6x - 3) \times \left(\frac{-1}{3}\right) = -1$

or, $2x - 1 = 1$

or, $2x = 2$

∴ $x = 1$

So, $m_1 = 6 \times 1 - 3 = 3$

Then, from (i), $y = 3 \times 1^2 - 3 \times 1 + 5 = 5$

Now, the equation of tangent at the point $(x_1, y_1) = (1, 5)$ and having the slope $(m_1) = 3$ is

$y - y_1 = m_1(x - x_1)$

or, $y - 5 = 3(x - 1)$

or, $y - 5 = 3x - 3$

or, $0 = 3x - y + 5 - 3$

∴ $3x - y + 2 = 0$

7. Find the angle of intersection of the following curves.

(a) $y^2 = x^3$ and $y = 2x$ at (0, 0)

(b) $xy = 6$ and $x^2y = 12$

Solution

(a)

Given curves are

$y = 2x$ ...(i)

$y^2 = x^3$

$y = x^{3/2}$ ...(ii)

Now, differentiating both sides of (i) with respect to 'x'

$$\frac{dy}{dx} = 2$$

At (0, 0); $\frac{dy}{dx} = 2$

i.e. $m_1 = 2$

Again, differentiating both sides of (ii) with respect to 'x'

$$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$$

At (0, 0); $\frac{dy}{dx} = 0$

i.e. $m_2 = 0$

If θ be the angle of intersection then,

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2} = \frac{2 - 0}{1 + 2 \times 0} = 2$$

∴ $\theta = \tan^{-1}(2)$

(b) Given, $xy = 6$ ... (i)

And $x^2y = 12$

or, $y = \frac{12}{x^2}$ ... (ii)

From (i) and (ii),

$x^2y = 12$

or, $x^2 \cdot \left(\frac{6}{x}\right) = 12$ [Using (i)]

or, $x = 2$

Putting $x = 2$ in (i), $y = 3$

The point of intersection = (2, 3).

Diff. (i) w.r.t. x,

$$y + x\frac{dy}{dx} = 0$$

$\Rightarrow$ $m_1 = \frac{dy}{dx} = -\frac{y}{x} = -\frac{3}{2}$ at the point (2, 3).

Again, diff (ii) w.r.t. to x,

$$2xy + x^2\frac{dy}{dx} = 0$$

$$\Rightarrow \quad m_2 = \frac{dy}{dx} = -\frac{2y}{x} = -2\cdot\frac{3}{2} = -3 \text{ at } (2, 3)$$

Let θ be the angle between the two curves. Then,

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2} = \frac{-\frac{3}{2}+3}{1+\frac{9}{2}} = \frac{\frac{6-3}{2}}{\frac{2+9}{2}} = \frac{3}{11}$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{3}{11}\right)$$

Objective Questions

1. The slope of tangent to the curve $y = f(x)$ at the point (x_1, y_1) is

(a) $\left(\frac{dy}{dx}\right)$ at (x_1, y_1) (b) $\left(\frac{dx}{dy}\right)$ at (x_1, y_1)

(c) $\left(\frac{dy}{dx}\right)$ at (y_1, x_1) (d) $\left(\frac{dx}{dy}\right)$ at (y_1, x_1)

Ans: a

2. The tangent to the curve $y = f(x)$ at a point P is horizontal if and only if

(a) $\frac{dy}{dx} = 0$ at P (b) $\frac{dy}{dx} = 1$ at P

(c) $\frac{dx}{dy} = 0$ at P (d) $\frac{dx}{dy} = 1$ at P

Ans: a

3. The equation of normal to the curve $y = f(x)$ at $P(x_1, y_1)$ is

(a) $y - y_1 = \left(\frac{dy}{dx}\right)_{(x,y)=(x_1,y_1)}(x - x_1)$ (b) $y - y_1 = -\left(\frac{dy}{dx}\right)_{(x,y)=(x_1,y_1)}(x - x_1)$

(c) $y - y_1 = \left(\frac{dx}{dy}\right)_{(x,y)=(x_1,y_1)}(x - x_1)$ (d) $y - y_1 = -\left(\frac{dx}{dy}\right)_{(x,y)=(x_1,y_1)}(x - x_1)$

Ans: d (Formula)

4. The slope of tangent to the curve $y = x^2 + 5x$ at (1, 2) is

(a) 2 (b) 5

(c) 7 (d) 10

Ans: c

$$y = x^2 + 5x$$

$$\frac{dy}{dx} = 2x + 5$$

Slope at (1, 2) = 2 × 1 + 5 = 7

5. The point on the curve $y = x^2 + 4x$ where the tangent is parallel to x–axis is

(a) (2, 4) (b) (4, 2)

(c) (–2, –4) (d) (–4, –2)

Ans: c

Here, $y = x^2 + 4x$ ... (i)

$$\frac{dy}{dx} = 2x + 4$$

For the tangent parallel to x-axis, $\frac{dy}{dx} = 0$

or, $2x + 4 = 0$

or, $x = -2$

Putting the value of x in (i)

$y = (-2)^2 + 4 \times (-2) = 4 - 8 = -4$

$\therefore$ Required point is $(-2, -4)$.

6. **The equation of normal to the curve $x^2 + y^2 = 25$ at $(3, 4)$ is**

(a) $4x + 3y = 5$ (b) $4x + 3y = 0$

(c) $4x - 3y = 5$ (d) $4x - 3y = 0$

Ans: d

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

or, $2x + 2y\frac{dy}{dx} = 0$

or, $\frac{dy}{dx} = -\frac{x}{y}$

or, $\frac{dy}{dx}$ at $(3, 4) = -\frac{3}{4}$

i.e., slope of tangent $= -\frac{3}{4}$

Hence, slope of normal $= -\frac{1}{\text{slope of tangent}} = \frac{4}{3}$

The equation of normal is $y - 4 = \frac{4}{3}(x - 3)$

or, $3y - 12 = 4x - 12$

$\therefore$ $4x - 3y = 0$

EXERCISE 2 (F)

1. **Evaluate the following using L Hospital's rule.**

(a) $\lim\limits_{x \to 4} \frac{x^2 - 16}{x - 4}$ (b) $\lim\limits_{x \to 3} \frac{x^3 - 27}{x^2 - 9}$ (c) $\lim\limits_{x \to -5} \frac{x^2 - 25}{x + 5}$

(d) $\lim\limits_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - x - 2}$ (e) $\lim\limits_{x \to 0} \frac{\sqrt{1 + x} - 1}{x}$ (f) $\lim\limits_{x \to 0} \frac{\sqrt{1 + x} - 1 - \frac{x}{2}}{x^2}$

(g) $\lim\limits_{x \to 0} \frac{1 - \cos x}{x + x^2}$ (h) $\lim\limits_{x \to 0} \frac{e^x - x - 1}{x^2}$ (i) $\lim\limits_{x \to 0} \frac{\tan x - x}{x - \sin x}$

(j) $\lim\limits_{x \to 0} \frac{x - \sin x \cos x}{x^3}$ (k) $\lim\limits_{x \to 0} \frac{(e^x - 1)\tan x}{x^2}$ (l) $\lim\limits_{t \to 0} \frac{t \sin t}{1 - \cos t}$

Solution

(a) $\lim\limits_{x \to 4} \frac{x^2 - 16}{x - 4}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim\limits_{x \to 4} \frac{2x}{1} = 2 \times 4 = 8$

(b) $\lim\limits_{x \to 3} \frac{x^3 - 27}{x^2 - 9}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim\limits_{x \to 3} \frac{3x^2}{2x} = \frac{3 \times 3^2}{2 \times 3} = \frac{9}{2}$

(c) $\lim\limits_{x \to -5} \frac{x^2 - 25}{x + 5}$ $\left[\frac{0}{0}\text{ form}\right]$

$= \lim\limits_{x \to -5} \frac{2x}{1} = 2 \times (-5) = -10$

(d) $\lim_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - x - 2}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x \to 2} \frac{2x - 5}{2x - 1} = \frac{2 \times 2 - 5}{2 \times 2 - 1} = -\frac{1}{3}$

(e) $\lim_{x \to 0} \frac{\sqrt{1 + x} - 1}{x}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1 + x}}}{1} = \frac{1}{2\sqrt{1 + 0}} = \frac{1}{2}$

(f) $\lim_{x \to 0} \frac{\sqrt{1 + x} - 1 - \frac{x}{2}}{x^2}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x \to 0} \frac{\frac{1}{2\sqrt{1 + x}} - \frac{1}{2}}{2x}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x \to 0} \frac{-\frac{1}{4(1 + x)^{\frac{3}{2}}}}{2}$

$= -\frac{1}{8(1 + 0)^{\frac{3}{2}}} = -\frac{1}{8}$

(g) $\lim_{x \to 0} \frac{1 - \cos x}{x + x^2}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x \to 0} \frac{\sin x}{1 + 2x}$

$= \frac{\sin 0}{1 + 2 \times 0} = \frac{0}{1} = 0.$

(h) $\lim_{x \to 0} \frac{e^x - x - 1}{x^2}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x \to 0} \frac{e^x - 1}{2x}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x \to 0} \frac{e^x}{2} = \frac{e^0}{2} = \frac{1}{2}$

(i) $\lim_{x \to 0} \frac{\tan x - x}{x - \sin x}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x \to 0} \frac{\sec^2 x - 1}{1 - \cos x}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x \to 0} \frac{2 \sec x \,.\, \sec x \tan x}{\sin x}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x \to 0} \frac{2 \sec^2 x \tan x}{\sin x}$

$= \lim_{x \to 0} \frac{2(\sec^2 x \,.\, \sec^2 x + \tan x \,.\, 2 \sec x \,.\, \sec x \,.\, \tan x)}{\cos x}$

$= \lim_{x \to 0} \frac{2(\sec^4 x + 2\sec^2 x \tan^2 x)}{\cos x} = \frac{2(1 + 0)}{1} = 2$

(j) $\lim_{x \to 0} \frac{x - \sin x \cos x}{x^3}$ $\left[\frac{0}{0} \text{ form}\right]$

$= \lim_{x \to 0} \frac{x - \frac{1}{2} \sin 2x}{x^3}$

$$= \lim_{x \to 0} \frac{1 - \cos 2x}{3x^2} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{2 \sin 2x}{6x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{\sin 2x}{3x}$$

$$= \lim_{x \to 0} \frac{2 \cos 2x}{3} = \frac{2 \times 1}{3} = \frac{2}{3}$$

(k) $\lim_{x \to 0} \dfrac{(e^x - 1) \tan x}{x^2} \qquad \left[\frac{0}{0} \text{ form}\right]$

$$= \lim_{x \to 0} \frac{(e^x - 1) \cdot \sec^2 x + e^x \tan x}{2x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{(e^x - 1) \cdot 2\sec^2 x \tan x + e^x \sec^2 x + e^x \sec^2 x + e^x \tan x}{2}$$

$$= \frac{0 + 1 + 1 + 0}{2} = \frac{2}{2} = 1.$$

(l) $\lim_{t \to 0} \dfrac{t \sin t}{1 - \cos t} \qquad \left[\frac{0}{0} \text{ form}\right]$

$$= \lim_{t \to 0} \frac{t \cos t + \sin t}{\sin t} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{t \to 0} \frac{t(-\sin t) + \cos t + \cos t}{\cos t}$$

$$= \frac{0 + 1 + 1}{1} = 2$$

2. **Evaluate the following limits using L Hospital's rule.**

(a) $\lim_{x \to \infty} \dfrac{5x^2 - 3x}{7x^2 + 1}$ (b) $\lim_{x \to \infty} \dfrac{2x^2 + 3x}{1 + 5x^2}$

(c) $\lim_{x \to \infty} \dfrac{x^5}{e^x}$ (d) $\lim_{x \to \frac{\pi}{2}} \dfrac{\tan x}{\tan 3x}$

Solution

(a) $\lim_{x \to \infty} \dfrac{5x^2 - 3x}{7x^2 + 1} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$

$$= \lim_{x \to \infty} \frac{10x - 3}{14x} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x \to \infty} \frac{10}{14}$$

$$= \frac{5}{7}$$

(b) $\lim_{x \to \infty} \dfrac{2x^2 + 3x}{1 + 5x^2} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$

$$= \lim_{x \to \infty} \frac{4x + 3}{10x} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x \to \infty} \frac{4}{10} = \frac{2}{5}$$

(c) $\lim_{x \to \infty} \dfrac{x^5}{e^x} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$

$$= \lim_{x \to \infty} \frac{5x^4}{e^x} \qquad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x \to \infty} \frac{20x^3}{e^x} \quad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x \to \infty} \frac{60x^2}{e^x} \quad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x \to \infty} \frac{120x}{e^x} \quad \left[\frac{\infty}{\infty} \text{ form}\right]$$

$$= \lim_{x \to \infty} \frac{120}{e^x} = 0$$

(d) $\lim_{x \to \frac{\pi}{2}} \frac{\tan x}{\tan 3x}$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\sec^2 x}{3 \sec^2 3x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{\cos^2 3x}{3 \cos^2 x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{1 + \cos 6x}{3(1 + \cos 2x)}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{-6 \sin 6x}{-6 \sin 2x} = \lim_{x \to \frac{\pi}{2}} \frac{\sin 6x}{\sin 2x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{6 \cos 6x}{2 \cos 2x}$$

$$= \frac{6 \cos 3\pi}{2 \cos \pi} = \frac{-6}{-2} = 3$$

Objective Questions

1. Which of the following is not an indeterminate form?

(a) $\frac{0}{0}$ (b) $\frac{\infty}{\infty}$

(c) $\infty - \infty$ (d) $0 - 0$

Ans: d

2. Which one of the following is wrong?

(a) $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{2x}{1} = 2 \times 2 = 4$ (b) $\lim_{x \to 2} \frac{x - 2}{x^2 - 2} = \frac{0}{2} = 0$

(c) $\lim_{x \to 2} \frac{x - 2}{x^2 - 2} = \lim_{x \to 2} \frac{1}{2x} = \frac{1}{4}$ (d) $\lim_{x \to 2} \frac{x - 2}{x^2 - 4} = \lim_{x \to 2} \frac{1}{2x} = \frac{1}{4}$

Ans: c

Option c is wrong since

$$\lim_{x \to 2} \frac{x - 2}{x^2 - 2} \quad \left[\text{Not in } \frac{0}{0} \text{ form}\right]$$

$$= \frac{2 - 2}{2^2 - 2} = \frac{0}{2} = 0$$

3. $\lim_{x \to 0} \frac{x}{\sin x} =$

(a) 0 (b) 1

(c) -1 (d) 2

Ans: b

$$\lim_{x \to 0} \frac{x}{\sin x} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{1}{\cos x} = \frac{1}{\cos 0} = \frac{1}{1} = 1$$

4. $\lim_{x \to 0} \frac{1 - \cos x}{x + x^2} =$

(a) 0 (b) 1

(c) 2 (d) 5

Ans: a

$$\lim_{x \to 0} \frac{1 - \cos x}{x + x^2} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 0} \frac{\sin x}{1 + 2x}$$

$$= \frac{\sin 0}{1 + 2 \times 0} = \frac{0}{1} = 0.$$

5. $\lim_{x \to 2} \frac{x^3 - 8}{x - 2} =$

(a) 3 (b) 4

(c) 6 (d) 12

Ans: d

$$\lim_{x \to 2} \frac{x^3 - 8}{x - 2} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 2} \frac{3x^2}{1} = 3 \times 2^2 = 12$$

6. $\lim_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - 4} =$

(a) $\frac{1}{4}$ (b) $-\frac{1}{4}$

(c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

Ans: b

$$\lim_{x \to 2} \frac{x^2 - 5x + 6}{x^2 - 4} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \to 2} \frac{2x - 5}{2x}$$

$$= \frac{2 \times 2 - 5}{2 \times 2} = -\frac{1}{4}$$

7. $\lim_{\theta \to \pi} \frac{\sin \theta}{\pi - \theta} =$

(a) 0 (b) 1

(c) −1 (d) 2

Ans: c

$$\lim_{\theta \to \pi} \frac{\sin \theta}{\pi - \theta} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{\theta \to \pi} \frac{\cos \theta}{-1}$$

$$= \frac{\cos \theta}{-1} = \frac{1}{-1} = -1$$

8. $\lim\limits_{x \to \pi/4} \dfrac{\sin x - \cos x}{x - \pi/4} =$

(a) 2 (b) $\dfrac{1}{2}$

(c) $\sqrt{2}$ (d) $\dfrac{1}{\sqrt{2}}$

Ans: c

$$\lim_{x \to \pi/4} \frac{\sin x - \cos x}{x - \pi/4} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x \to \pi/4} \frac{\cos x + \sin x}{1}$$

$$= \cos\frac{\pi}{4} + \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

9. $\lim\limits_{x \to \infty} \dfrac{3x^2 + 5x + 1}{4x^2 + 6x + 5} =$

(a) $\dfrac{3}{4}$ (b) $\dfrac{4}{3}$

(c) 4 (d) 3

Ans: a

$$\lim_{x \to \infty} \frac{3x^2 + 5x + 1}{4x^2 + 6x + 5} \qquad \left[\frac{\infty}{\infty}\text{ form}\right]$$

$$= \lim_{x \to \infty} \frac{6x + 5}{8x + 6}$$

$$= \lim_{x \to \infty} \frac{6}{8}$$

$$= \frac{3}{4}$$

10. $\lim\limits_{x \to \pi/2} \dfrac{\sec 3x}{\sec x} =$

(a) $\dfrac{1}{3}$ (b) $-\dfrac{1}{3}$

(c) 3 (d) -3

Ans: b

$$\lim_{x \to \pi/2} \frac{\sec 3x}{\sec x} \qquad \left[\frac{\infty}{\infty}\text{ form}\right]$$

$$= \lim_{x \to \pi/2} \frac{\cos x}{\cos 3x} \qquad \left[\frac{0}{0}\text{ form}\right]$$

$$= \lim_{x \to \pi/2} \frac{-\sin x}{-3\sin 3x}$$

$$= -\frac{1}{3}$$

EXERCISE 2 (G)

1. Verify Rolle's theorem for the following functions.

(a) $f(x) = x^2$ in $[-1, 1]$ (b) $f(x) = x^2 - 4$ in $-3 \le x \le 3$

(c) $f(x) = x(x-2)^2$ in $[0, 2]$ (d) $f(x) = \sin x$ in $[0, 2\pi]$

(e) $f(x) = \sqrt{16 - x^2}$ in $[-4, 4]$

Solution

(a) Given $f(x) = x^2$ in $[-1, 1]$

Since, f(x) is a polynomial function, so it is continuous in $[-1, 1]$

Again, $f'(x) = 2x$, which exists for all $x \in (-1, 1)$

So, f(x) is differentiable in $(-1, 1)$.

And, $f(-1) = (-1)^2 = 1$

$f(1) = 1^2 = 1$

$\therefore$ $f(-1) = f(1)$

So, all the conditions of Rolle's theorem are satisfied.

Hence, there exists at least a point $c \in (-1, 1)$ such that

$f'(c) = 0$

or, $2c = 0$

$\therefore$ $c = 0 \in (-1, 1)$

Hence, Rolle's theorem is verified.

(b) Given, $f(x) = x^2 - 4$ in $-3 \le x \le 3$

Since, f(x) is a polynomial function, so it is continuous in $[-3, 3]$.

Again, $f'(x) = 2x$, which exists for all $x \in (-3, 3)$.

So, f(x) is differentiable in $(-3, 3)$.

And, $f(-3) = (-3)^2 - 4 = 5$

$f(3) = 3^2 - 4 = 5$

$\therefore$ $f(-3) = f(3)$

Hence, all the conditions of Rolle's theorem are satisfied.

Hence, there exists at least a point $c \in (-3, 3)$ such that

$f'(c) = 0$

or, $2c = 0$

$\therefore$ $c = 0 \in (-3, 3)$

Hence, Rolle's theorem is verified.

(c) Here, $f(x) = x(x-2)^2 = x(x^2 - 4x + 4) = x^3 - 4x^2 + 4x$ in $[0, 2]$

Since, f(x) is a polynomial function, so it is continuous in $[0, 2]$.

Again, $f'(x) = 3x^2 - 8x + 7$ which exists for all x in $(0, 2)$.

Thus, f(x) is differentiable in $(0, 2)$.

Also, $f(0) = 0(0-2)^2 = 0$

$f(2) = 2(2-2)^2 = 0$

So, all the conditions of Rolle's theorem are satisfied. So, there exists at least one $c \in (0, 2)$ such that

$f'(c) = 0$

or, $3c^2 - 8c + 4 = 0$

or, $3c^2 - 6c - 2c + 4 = 0$

or, $3c(c-2) - 2(c-2) = 0$

or, $(c-2)(3c-2) = 0$

$\therefore$ $c = 2, \frac{2}{3}$.

Here, $c = 2 \notin (0, 2)$

but $c = \frac{2}{3} \in (0, 2)$

Hence, Rolle's theorem is verified.

(d) Here, $f(x) = \sin x$

For all $x \in [0, 2\pi]$, f(x) has a definite value, so f(x) is continuous in $[0, 2\pi]$

Again, $f'(x) = \cos x$ which exists for all $x \in (0, 2\pi)$.

$\therefore$ f(x) is differentiable in $(0, 2\pi)$.

Also, $f(0) = \sin 0 = 0$

$f(2\pi) = \sin 2\pi = 0$

$\therefore$ $f(0) = f(2\pi)$

$\therefore$ All the conditions of Rolle's theorem are satisfied.

Hence, there exists at least a point $c \in (0, 2\pi)$ such that $f'(c) = 0$

or, $\cos c = 0 = \cos\frac{\pi}{2} = \cos\frac{3\pi}{2}$

$\therefore$ $c = \frac{\pi}{2}, \frac{3\pi}{2} \in (0, 2\pi)$

Hence, Rolle's theorem is verified

(e) Given, $f(x) = \sqrt{16 - x^2}$ in $[-4, 4]$

For every value of $x \in [-4, 4]$, $f(x) = \sqrt{16 - x^2}$ has a definite value, so $f(x)$ is continuous for all x in $[-4, 4]$

Again, $f'(x) = \frac{1}{2}(16 - x^2)^{\frac{-1}{2}}(-2x) = \frac{-x}{\sqrt{16 - x^2}}$ which exists for all x in $(-4, 4)$

And, $f(-4) = \sqrt{16 - (-4)^2} = 0$

$f(4) = \sqrt{16 - 4^2} = 0$

$\therefore$ $f(-4) = f(4)$

Hence, all conditions of Rolle's theorem are satisfied. So there exists at least one point $c \in (-4, 4)$ such that

$f'(c) = 0$

or, $\frac{-c}{\sqrt{16 - c^2}} = 0$

or, $c = 0 \in (-4, 4)$

Hence, Rolle's theorem is verified.

2. **Using Rolle's theorem, find a point on the curve $f(x) = 3x - x^2$ in $[0, 3]$, where the tangent is parallel to x–axis.**

Solution

Given, $f(x) = 3x - x^2$ in $[0, 3]$

Since, $f(x)$ is a polynomial functions, so it is continuous in $[0, 3]$.

Also, $f'(x) = 3 - 2x$ which exists for all x in $(0, 3)$.

So, $f(x)$ is differentiable in $(0, 3)$.

And, $f(0) = 3 \times 0 - 0^2 = 0$

$f(3) = 3 \times 3 - 3^2 = 0$

$\therefore$ $f(0) = f(3)$

Hence, all the conditions of Rolle's theorem are satisfied. So, there exists at least one point $c \in (0, 3)$ such that

$f'(c) = 0$

or, $3 - 2c = 0$

or, $c = \frac{3}{2} \in (0, 3)$.

$\therefore$ $c = \frac{3}{2}$ is the x-coordinate of the point at which tangent is parallel to x-axis

Put, $x = \frac{3}{2}$ in $f(x) = 3x - x^2$, we have.

$$y = f(x) = 3 \times \frac{3}{2} - \left(\frac{3}{2}\right)^2 = \frac{9}{2} - \frac{9}{4} = \frac{9}{4}$$

$\therefore$ Required point is $\left(\frac{3}{2}, \frac{9}{4}\right)$

3. **Verify Lagrange's mean value theorem for the following functions.**

(a) $f(x) = x^2 - 1$ in $[1, 3]$ (b) $f(x) = x^2 + 2x - 1$ in $[0, 1]$

(c) $f(x) = x^{\frac{2}{3}}$ in $[0, 1]$ (d) $f(x) = \sqrt{x - 1}$, $x \in [1, 3]$

(e) $f(x) = e^x$; $x \in [0, 1]$.

Solution

(a) Here, $f(x) = x^2 - 1$ in $[1, 3]$

Since, f(x) is a polynomial function, so it is continuous in $[1, 3]$.

Again, $f'(x) = 2x$, which exists for all x in $(1, 3)$.

$\therefore$ f(x) is differentiable in $(1, 3)$.

Hence, all the conditions of MVT are satisfied. So, there exists at least $c \in (1, 3)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, $2c = \dfrac{f(3) - f(1)}{3 - 1}$

or, $2c = \dfrac{(9 - 1) - (1 - 1)}{2}$

or, $2c = 4$

or, $c = 2 \in (1, 3)$

Hence, MVT is verified.

(b) Here, $f(x) = x^2 + 2x - 1$ in $[0, 1]$

Since, f(x) is a polynomial function, so it is continuous in $[0, 1]$.

Again, $f'(x) = 2x + 2$ which exists for all x in $(0, 1)$.

$\therefore$ f(x) is differentiable in $(0, 1)$.

Hence, all the conditions of MVT are satisfied. So, there exists at least one point $c \in (0, 1)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, $2c + 2 = \dfrac{f(1) - f(0)}{1 - 0}$

or, $2c + 2 = \dfrac{2 - (-1)}{1}$

or, $2c = 1$

$\therefore$ $c = \dfrac{1}{2} \in (0, 1)$

Hence, MVT is verified.

(c) Here, $f(x) = x^{-\frac{2}{3}}$ in $[0, 1]$.

For all values of x in $[0, 1]$, $f(x) = x^{\frac{2}{3}}$ has a definite value, so f(x) is continuous in $[0, 1]$

Again, $f'(x) = \dfrac{2}{3} x^{\frac{-1}{3}} = \dfrac{2}{3x^{\frac{1}{3}}}$ which exists for all x in $(0, 1)$.

So, f(x) is differentiable in $(0, 1)$.

Hence, all the conditions of MVT are satisfied.

So, there exists at least one point $c \in (0, 1)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, $\dfrac{2}{3c^{\frac{1}{3}}} = \dfrac{f(1) - f(0)}{1 - 0}$

or, $\dfrac{2}{3c^{\frac{1}{3}}} = \dfrac{1 - 0}{1 - 0}$

or, $2 = 3c^{\frac{1}{3}}$

or, $\dfrac{2}{3} = c^{\frac{1}{3}}$

$\therefore$ $c = \dfrac{8}{27} \in (0, 1)$

Hence, MVT is verified

(d) Here, $f(x) = \sqrt{x-1}$, $x \in [1, 3]$

For all values of x in [1, 3], $f(x) = \sqrt{x-1}$, has a definite value so f(x) is continuous in [1, 3]

Again, $f'(x) = \frac{1}{2\sqrt{x-1}}$, which exists for all x in (1, 3)

So, f(x) is differentiable in (1, 3)

Thus, all the conditions of MVT are satisfied

By MVT there exists at least one point $c \in (1, 3)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, $$\frac{1}{2\sqrt{c-1}} = \frac{f(3) - f(1)}{3 - 1}$$

or, $$\frac{1}{2\sqrt{c-1}} = \frac{\sqrt{2}}{2}$$

or, $$1 = \sqrt{2}\sqrt{c-1}$$

or, $$c - 1 = \frac{1}{2}$$

or, $$c = \frac{3}{2} \in (1, 3)$$

Hence, MVT is verified

(e) Here, $f(x) = e^x$, $x \in [0, 1]$

For all values of x in [0, 1], f(x) has a definite value So, f(x) is continuous in [0, 1]

Again, $f'(x) = e^x$ which exists for all x in (0, 1).

So, f(x) is differentiable in (0, 1)

So, all the conditions of MVT are satisfied.

Hence, there exists at least one point $c \in (0, 1)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, $$e^c = \frac{e^1 - e^0}{1 - 0}$$

or, $$e^c = e - 1$$

∴ $$c = \log(e - 1) \in (0, 1)$$

Hence, MVT is verified.

4. **Using mean value theorem, find the point on the curve $f(x) = (x-1)(x-2)(x-3)$, the tangent at the point is parallel to the chord joining the points (1, 0) and (4, 6).**

Solution

Here, $f(x) = (x-1)(x-2)(x-3)$ in [1, 4]

$= x^3 - 6x^2 + 11x - 6$

Since f(x) is a polynomial function, so it is continuous in [1, 4]

Again, $f'(x) = 3x^2 - 12x + 11$ which exists for all $x \in (1, 4)$, so it is differentiable in (1, 4)

Hence, both the conditions of mean value theorem are satisfied. So there exists at least one $c \in (1, 4)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

But, $f(b) = f(4) = 4^3 - 6 \cdot 4^2 + 11 \cdot 4 - 6 = 6$

$f(a) = f(1) = 1^3 - 6 \cdot 1^2 + 11 \cdot 1 - 6 = 0$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow 3c^2 - 12c + 11 = \frac{6 - 0}{4 - 1}$$

$$\Rightarrow 3c^2 - 12c + 11 = 2$$

$$\Rightarrow 3c^2 - 12c + 9 = 0$$

$$\Rightarrow c^2 - 4c + 3 = 0$$

$\Rightarrow \quad (c-1)(c-3)=0$

$\Rightarrow \quad c = 1, 3$

But $c = 1 \notin (1, 4)$ and $c = 3 \in (1, 4)$

Hence mean value theorem is verified

$\therefore$ $c = 3$ is the x-coordinate of the point at which tangent drawn is parallel to the chord joining the points $(1, 0)$ and $(4, 6)$.

Put, $x = 3$ in $y = f(x) = (x-1)(x-2)(x-3)$, we get

$y = (3-1)(3-2)(3-3) = 0$

$\therefore$ Required point is $(3, 0)$.

Objective Questions

1. If a function $f(x)$ is continuous in $[a, b]$, differentiable in (a, b) and $f(a) = f(b)$ then there exists at least one point $c \in (a, b)$ such that

(a) $f'(c) = 1$ (b) $f'(c) = 0$

(c) $f'(c) = c$ (d) $f'(c)$ is undefined

Ans: b

2. If a function $f(x) = 1 - x^{\frac{2}{3}}$ is defined in the interval $[-1, 1]$ then $f'(0) =$

(a) 0 (b) 1

(c) -1 (d) does not exist

Ans: d

Here, $f(x) = 1 - x^{\frac{2}{3}}$

$$f'(x) = 0 - \frac{2}{3}x^{\frac{-1}{3}} = \frac{2}{3x^{\frac{1}{3}}}$$

$f'(0) = \frac{2}{0} = \infty$ i.e. $f'(0)$ is undefined.

3. The value of c prescribed by Rolle's theorem for $f(x) = x^2 - 4$ in $[-3, 3]$ is

(a) 0 (b) 3

(c) -3 (d) 1

Ans: a

Since, f(x) is polynomial function, so it is continuous in $[-3, 3]$.

Also, it is differentiable in $(-3, 3)$.

So, there exists $c \in (-3, 3)$ such that

$f'(c) = 0$

or, $2c = 0$

$\therefore$ $c = 0$

4. According to Lagrange's mean value theorem, if a function $f(x)$ is continuous in $[a, b]$ and differentiable in (a, b) then there exists at least one point $c \in (a, b)$ such that

(a) $f'(c) = b - a$ (b) $f'(c) = f(b) - f(a)$

(c) $f'(c) = \dfrac{b-a}{f(b)-f(a)}$ (d) $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Ans: d

☐

Application of Differentiation

EXERCISE – 3 (A)

1. (a) Show that the function $f(x) = 3x^3 - 24x + 1$ is increasing at $x = 4$ and decreasing at $x = \frac{1}{2}$.

(b) Examine whether the function $f(x) = 2x^3 - x^2 + 4$ at $x = 1$ and at $x = \frac{1}{4}$.

(c) Show that $f(x) = x - \frac{1}{x}$ is increasing for all $x \in \mathbb{R}$ except at $x = 0$.

(d) Show that the function $f(x) = x^2 - 6x + 3$ is decreasing on the interval (0, 2).

Solution

(a) Given function is $f(x) = 3x^3 - 24x + 1$

$$f'(x) = 9x^2 - 24$$

When $x = 4$, $f'(4) = 9 \times 4^2 - 24 = 120 > 4$

So, f(x) is increasing at $x = 4$

When $x = \frac{1}{2}$, $f'\left(\frac{1}{2}\right) = 9 \times \left(\frac{1}{2}\right)^2 - 24 = -21.72 < 0.$

So, f(x) is decreasing at $x = \frac{1}{2}$

(b) $f(x) = 2x^3 - x^2 + 4$

$f'(x) = 6x^2 - 2x$

When $x = 1$, $f'(1) = 6 \times 1^2 - 2 \times 1 = 4 > 0$

So, f(x) is increasing at $x = 1$

When $x = \frac{1}{4}$, $f'\left(\frac{1}{4}\right) = 6 \times \left(\frac{1}{4}\right)^2 - 2 \times \frac{1}{4} = -0.125 < 0.$

So, f(x) is decreasing at $x = \frac{1}{4}$

(c) Here, $f(x) = x - \frac{1}{x} = x - x^{-1}$

$$f'(x) = \frac{d}{dx}(x - x^{-1})$$

$$= 1 - (-1)x^{-1-1}$$

$= 1 + \frac{1}{x^2}$, which is always positive for any real x except $x = 0$

$\therefore$ f(x) is increasing for all $x \in R$ except at $x = 0$

(d) $f(x) = x^2 - 6x + 3$
$f'(x) = 2x - 6$
Here, $f'(x) = 2(x - 3) < 0$ for $x < 3$
i.e. $f(x)$ is decreasing on $(-\infty, 3)$. Since $(0, 2) \subset (-\infty, 3)$, so $f(x)$ is decreasing on $(0, 2)$.

2. **Find the intervals in which the following functions are increasing or decreasing.**
(a) $f(x) = x^2 - 2x$ (b) $f(x) = -x^2 - 3x + 3$
(c) $f(x) = x^3 - 3x^2 - 9x$ (d) $f(x) = -x^3 + 12x + 5, -3 \le x \le 3$

Solution

(a) $f(x) = x^2 - 2x$
$f'(x) = 2x - 2$
For critical points, $f'(x) = 0$
or, $2x - 2 = 0$
$\therefore$ $x = 1$

The point $x = 1$ divides the whole real line in two sub-intervals.
Now,

Intervals	Sign of f '(x)	Nature
$(-\infty, 1)$	–ve	decreasing
$(1, \infty)$	+ve	increasing

$\therefore$ f(x) is increasing on $(1, \infty)$ and decreasing on $(-\infty, 1)$.

(b) $f(x) = -x^2 - 3x + 3$
$f'(x) = -2x - 3$
For critical points, $f'(x) = 0$
or, $-2x - 3 = 0$
or, $x = -\frac{3}{2}$

Now,

Intervals	Sign of f '(x)	Nature
$\left(-\infty, -\frac{3}{2}\right)$	+ve	increasing
$\left(-\frac{3}{2}, \infty\right)$	–ve	decreasing

$\therefore$ f(x) is increasing on $\left(-\infty, -\frac{3}{2}\right)$ and decreasing on $\left(-\frac{3}{2}, \infty\right)$.

(c) $f(x) = x^3 - 3x^2 - 9x$
$f'(x) = 3x^2 - 6x - 9$
For critical points, $f'(x) = 0$
or, $3x^2 - 6x - 9 = 0$
or, $x^2 - 2x - 3 = 0$
or, $x^2 - 3x + x - 3 = 0$
or, $x(x - 3) + 1(x - 3) = 0$
or, $(x - 3)(x + 1) = 0$
$\therefore$ $x = 3, -1$

Now,

Intervals	Sign of f '(x)	Nature of f(x)
$(-\infty, -1)$	+ve	increasing
$(-1, 3)$	–ve	decreasing
$(3, \infty)$	+ve	increasing

$\therefore$ f(x) is increasing on $(-\infty, -1) \cup (3, \infty)$ and decreasing on $(-1, 3)$.

(d) $f(x) = -x^3 + 12x + 5, -3 \le x \le 3$
$f'(x) = -3x^2 + 12$
For critical points, $f'(x) = 0$
or, $-3x^2 + 12 = 0$
or, $x^2 - 4 = 0$
$\therefore$ $x = \pm 2$

Now,

Intervals	Sign of f '(x)	Nature of f(x)
$[-3, -2)$	$-$ve	decreasing
$(-2, 2)$	+ve	increasing
$(2, 3]$	$-$ve	decreasing

f(x) is increasing on $(-2, 2)$ and decreasing on $[-3, -2) \cup (2, 3]$

3. **Find the absolute maximum and minimum values of each function on the given interval.**

(a) $f(x) = \frac{2}{3}x - 5, -2 \le x \le 3$ (b) $f(x) = x^2 - 1, -1 \le x \le 2$

(c) $f(x) = x^3 - 3x^2 + 5$ on $[-2, 2]$ (d) $h(x) = x^{\frac{2}{3}}$ on $[-2, 3]$

Solution

(a) $f(x) = \frac{2}{3}x - 5$ in $[-2, 3]$

$f'(x) = \frac{2}{3}$

Since $f'(x) \ne 0$ for any values of x. So, there is no critical points. So we have to calculate f(x) at the end points only.

When $x = -2$, $f(-2) = \frac{2}{3}(-2) - 5 = -\frac{19}{3}$

When $x = 3$, $f(3) = \frac{2}{3} \times 3 - 5 = -3$

$\therefore$ Absolute max. value $= -3$ at $x = 3$.

Absolute min. value $= -\frac{19}{3}$ at $x = -2$.

(b) $f(x) = x^2 - 1$ in $[-2, 2]$

$f'(x) = 2x$

For critical points, $f'(x) = 0$

$2x = 0$

$x = 0$

When $x = -2$, $f(-2) = (-2)^2 - 1 = 3$

When $x = 0$, $f(0) = 0^2 - 1 = -1$

When $x = 2$, $f(2) = 2^2 - 1 = 3$

$\therefore$ Absolute max. value $= 3$ at $x = 2$

Absolute min. value $= -1$ at $x = 0$.

(c) $f(x) = x^3 - 3x^2 + 5$ on $[-2, 2]$

$f'(x) = 3x^2 - 6x$

For critical points, we have,

$f'(x) = 0$

$3x^2 - 6x = 0$

or, $x^2 - 2x = 0$

or, $x(x - 2) = 0$

$x = 0, 2$

When $x = -2$, $f(-2) = (-2)^3 - 3 \times (-2)^2 + 5 = -15$

When $x = 0$, $f(0) = 0^3 - 3 \times 0^2 + 5 = 5$

When $x = 2$, $f(2) = 2^3 - 3 \times 2^2 + 5 = 1$

Absolute max. value $= 5$ at $x = 0$

Absolute min. value $= -15$ at $x = -2$.

(d) $h(x) = x^{\frac{2}{3}}$

$h'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3x^{\frac{1}{3}}}$

Since h'(x) does not exist at x = 0.
So 0 is a critical point.

When x = –2, $h(-2) = (-2)^{\frac{2}{3}} = 4^{\frac{1}{3}} = \sqrt[3]{4}$

When x = 0, $h(0) = 0^{\frac{2}{3}} = 0$

When x = 3, $h(3) = 3^{\frac{2}{3}} = \sqrt[3]{9}$

∴ Absolute max. value = $\sqrt[3]{9}$ at x = 3
Absolute min. value = 0 at x = 0.

4. Find the local maximum and minimum values and point of inflection of following functions.

(a) $f(x) = 3x^2 - 6x + 4$ **(b)** $f(x) = x^3 - 3x + 1$

(c) $f(x) = x^3 - 3x^2 - 9x + 27$ **(d)** $f(x) = 2x^3 - 15x^2 + 36x + 5$

(e) $y = 4x^3 - 6x^2 - 9x + 1$ **on the interval (–1, 2).**

(f) $y = x + \frac{25}{x}$

Solution

Solution

Here, $f(x) = 3x^2 - 6x + 4$

$f'(x) = 6x - 6$

$f''(x) = 6$

For minimum and maximum values, we have,

$f'(x) = 0$

or, $6x - 6 = 0$

or, $x = 1$

When $x = 1$, $f''(1) = 6 > 0$

So, f(x) has minimum value at $x = 1$.

Minimum value = $f(1) = 3 \cdot 1^2 - 6 \cdot 1 + 4 = 1$

Since $f''(x) = 6 \neq 0$; so
f(x) has no point of inflection.

(b) Let $y = x^3 - 3x + 1$

$\frac{dy}{dx} = 3x^2 - 3$

$\frac{d^2y}{dx^2} = 6x$

For max. or min. $\frac{dy}{dx} = 0$

⇒ $3x^2 - 3 = 0$

∴ $x = \pm 1$

when x = 1

$\frac{d^2y}{dx^2} = 6 \times 1 = 6 > 0$ (minima) &

the minimum value at x = 1 is

$y = 1^3 - 3 \times 1 + 1 = -1$

when x = – 1

$\frac{d^2y}{dx^2} = 6 \times -1 = -6 < 0$ (maxima) &

the maximum value at x = – 1 is

$y = (-1)^3 - 3 \times -1 + 1 = 3$

For point of inflection, $f''(x) = 0$

or $6x = 0$

$\therefore x = 0$

(c) Let $f(x) = x^3 - 3x^2 - 9x + 27$ ... (i)

Differentiating both sides w.r. to 'x', we get

$$\frac{d}{dx} f(x) = \frac{d}{dx}(x^3 - 3x^2 - 9x + 27)$$

i.e. $f'(x) = 3x^2 - 6x - 9$ ... (ii)

Again, differentiating both sides w.r. to 'x' we get

$f''(x) = 6x - 6$ ... (iii)

For the maximum or minimum values of $f(x)$,

$f'(x) = 0$

$\therefore$ $3x^2 - 6x - 9 = 0$

or, $3x^2 - 9x + 3x - 9 = 0$

or, $3x(x - 3) + 3(x - 3) = 0$

or, $(x - 3)(3x + 3) = 0$

$\therefore$ $x = 3, -1$

When $x = 3$, $f''(x) = 6 \times 3 - 6 = 12 > 0$.

So $f(x)$ has a minimum value at $x = 3$ and the minimum value of $f(x)$ is

$f(3) = 3^3 - 3(3)^2 - 9 \times 3 + 27$

$= 27 - 27 - 27 + 27 = 0$

And when $x = -1$, $f''(x) = 6 \times (-1) - 6 = -12 < 0$.

So $f(x)$ has a maximum value at $x = -1$ and the maximum value of $f(x)$ is

$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 27$

$= -1 - 3 + 9 + 27 = 32$

Thus, the maximum value of $f(x) = 32$ at $x = -1$,

And, the minimum value of $f(x) = 0$ at $x = 3$

For point of inflection $f''(x) = 0$

or, $6x - 6 = 0$

$\therefore$ $x = 1$

(d) Here, $f(x) = 2x^3 - 15x^2 + 36x + 5$

$f'(x) = 6x^2 - 30x + 36$

$f''(x) = 12x - 30$

For stationary points, we have $f'(x) = 0$

or, $6x^2 - 30x + 36 = 0$

or, $x^2 - 5x + 6 = 0$

or, $(x - 2)(x - 3) = 0$

$\therefore$ $x = 2, 3$

At $x = 2$

$f''(2) = 12 \times 2 - 30 = -6 < 0$

So, $f(x)$ has maximum value at $x = 2$

Maximum value $= f(2)$

$= 2 \times 2^3 - 15 \times 2^2 + 36 \times 2 + 5 = 33$

At $x = 3$

$f''(3) = 12 \times 3 - 30 = 6 > 0$

So, $f(x)$ has minimum value at $x = 3$

Minimum value $= f(3)$

$= 2 \times 3^3 - 15 \times 3^2 + 36 \times 3 + 5 = 32$

For point of inflection, $f''(x) = 0$

or, $12x - 30 = 0$

$\therefore$ $x = \frac{5}{2}$

(e) Given curve is $y = 4x^3 - 6x^2 - 9x + 1$ ... (i)

Differentiating both sides with respect to 'x', we get

$$\frac{dy}{dx} = \frac{d}{dx}(4x^3 - 6x^2 - 9x + 1)$$

$$\frac{dy}{dx} = 12x^2 - 12x - 9 \quad \text{... (ii)}$$

Again, differentiating both sides with respect to 'x', we get

$$\frac{d^2y}{dx^2} = 24x - 12 \quad \text{(iii)}$$

For maxima or minima $\frac{dy}{dx} = 0$

i.e. $12x^2 - 12x - 9 = 0$

or, $4x^2 - 4x - 3 = 0$

or, $4x^2 - 6x + 2x - 3 = 0$

or, $2x(2x - 3) + 1(2x - 3) = 0$

or, $(2x - 3)(2x + 1) = 0$

$\therefore$ $x = \frac{3}{2}$ or $-\frac{1}{2}$

At $x = -\frac{1}{2}$

$$\frac{d^2y}{dx^2} = 24\left(-\frac{1}{2}\right) - 12 = -24 < 0$$

$\therefore$ The given function is maximum at $x = -\frac{1}{2}$.

Maximum value is

$$y_{max} = 4\left(-\frac{1}{2}\right)^3 - 6\left(-\frac{1}{2}\right)^2 - 9\left(-\frac{1}{2}\right) + 1$$

$$= 4\left(-\frac{1}{8}\right) - 6\left(\frac{1}{4}\right) + \frac{9}{2} + 1 = -\frac{1}{2} - \frac{3}{2} + \frac{9}{2} + 1 = \frac{-1 - 3 + 9 + 2}{2} = \frac{7}{2}$$

At $x = \frac{3}{2}$

$$\frac{d^2y}{dx^2} = 24 \times \frac{3}{2} - 12 = 24 > 0$$

So, the given function is minimum at $x = \frac{3}{2}$ and its minimum value is

$$y_{min} = 4\left(\frac{3}{2}\right)^3 - 6\left(\frac{3}{2}\right)^2 - 9\left(\frac{3}{2}\right) + 1$$

$$= 4 \times \frac{27}{8} - 6 \times \frac{9}{4} - \frac{27}{2} + 1 = \frac{27}{2} - \frac{27}{2} - \frac{27}{2} + 1 = -\frac{27}{2} + 1 = \frac{-27 + 2}{2} = \frac{-25}{2}$$

For the point of inflection,

$$\frac{d^2y}{dx^2} = 0$$

i.e. $24x - 12 = 0$

$\therefore$ $x = \frac{1}{2}$

(f) Let, $f(x) = x + \frac{25}{x}$

$f'(x) = 1 - 25x^{-2}$,

$f''(x) = 50x^{-3} = \frac{50}{x^3}$

For max. or min. $f'(x) = 0 \Rightarrow x = \pm 5$.

When $x = 5$

$f''(5) = \frac{50}{125} > 0$ (minima)

and the minimum value at $x = 5$ is $f(5) = 5 + \frac{25}{5} = 5 + 5 = 10$

Again, when $x = -5$, $f''(-5) = \frac{-50}{125} < 0$ (max.)

and the maximum value at $x = -5$ is $f(x) = -5 + \frac{25}{-5} = -5 + -5 = -10$

At last

It has no point of inflection since $\frac{d^2y}{dx^2} = f''(x) \neq 0$ for any real values of x

5. **Show that the function $x^3 + 6x + 30$ has neither a maximum value nor a minimum value.**

Solution

Let. $y = x^3 + 6x + 30$

$$\frac{dy}{dx} = 3x^2 + 6$$

$$\frac{d^2y}{dx^2} = 6x$$

and $\frac{d^3y}{dx^3} = 6 \neq 0$. This shows that it has neither maxima nor minima.

6. **Find the interval in which the given functions are concave upwards and downwards.**

(a) $f(x) = x^2 - 3x + 1$

(b) $f(x) = x^3 - 3x^2 + 5$

(c) $f(x) = x^4 - 8x^3 + 18x^2 - 24$

Solution

(a) $f(x) = x^2 - 3x + 1$

$f'(x) = 2x - 3$

$f''(x) = 2$ which is always positive for all x in $\mathbb{R}$.

So, f(x) is concave upwards on $(-\infty, \infty)$

(b) $f(x) = x^3 - 3x^2 + 5$

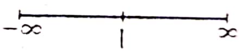
$f'(x) = 3x^2 - 6x$

$f''(x) = 6x - 6$

For point of inflection, $f''(x) = 0$

or, $6x - 6 = 0$

$\therefore$ $x = 1$.



Intervals	Sign of f ''(x)	Nature
$(-\infty, 1)$	−ve	concave downwards
$(1, \infty)$	+ve	concave upwards

$\therefore$ f(x) is concave downwards on $(-\infty, 1)$ and concave upwards on $(1, \infty)$.

(c) Given function is $f(x) = x^4 - 8x^3 + 18x^2 - 24$

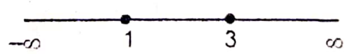
$\therefore$ $f'(x) = 4x^3 - 24x^2 + 36x$

And $f''(x) = 12x^2 - 48x + 36$

$= 12(x^2 - 4x + 3)$

$= 12(x^2 - 3x - x + 3)$

$= 12(x - 3)(x - 1)$



For the points of inflection, $f''(x) = 0$

$12(x - 3)(x - 1) = 0 \Rightarrow x = 1, 3$

$\therefore$ Required points of inflection are $x = 1$ and $x = 3$.

Intervals	Sign of f ''(x)	Remarks
$(-\infty, 1)$	+ ve	Concave upwards
$(1, 3)$	− ve	Concave downwards
$(3, \infty)$	+ ve	Concave upwards

$\therefore$ $f(x)$ is concave upwards on $(-\infty, 1) \cup (3, \infty)$ and downwards on $(1, 3)$.

7. **A man who has 144 m of fencing material wishes to enclose a rectangular garden. Find the maximum area he can enclose.**

Solution

Let, the sides of the rectangular garden be x and y.

Then, perimeter of the garden $= 2(x + y)$.

Given, perimeter = 144 m

$\therefore \quad 2(x+y) = 144$ m

$y = 72 - x \qquad \ldots(i)$

Let A be the area of the rectangular garden. Then,

$A = xy = x(72 - x) \qquad$ [Using (i)]

or, $\quad A = 72x - x^2$

Differentiating w.r.t. 'x', we get,

$$\frac{dA}{dx} = \frac{d(72x - x^2)}{dx}$$

or, $\quad \dfrac{dA}{dx} = 72 - 2x$

Again, differentiating w.r.t. 'x', we get

$$\frac{d^2A}{dx^2} = 0 - 2 = -2$$

For maximum or minimum area of the rectangular garden. $\dfrac{dA}{dx} = 0$

i.e. $\quad 72 - 2x = 0$

or, $\quad 2x = 72$

or, $\quad x = 36.$

When, $x = 36$, $\dfrac{d^2A}{dx^2} = -2 < 0.$

$\therefore$ Area A is maximum when $x = 36$ and when $x = 36$, $y = 72 - 36 = 36.$

Hence, the maximum area $= xy = 36 \times 36 = 1296$ m^2

8. **A gardener having 120 m of fencing wishes to enclose a rectangular plot of land and also to erect a fence across the land parallel to two of the sides. Find the maximum area he can enclose.**

Solution

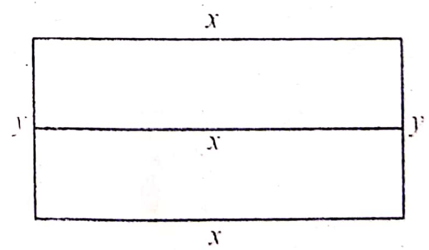
Let x be the length and y be the breadth of rectangular plot of land.

By question,

$3x + 2y = 120$

or, $\quad 2y = 120 - 3x$

$y = \dfrac{120 - 3x}{2} = 60 - \dfrac{3}{2}x \qquad \ldots(i)$



We have, area (A) = length × breadth

or, $\quad A = x \times \left(60 - \dfrac{3}{2}x\right) = 60x - \dfrac{3}{2}x^2$

$$\frac{dA}{dx} = 60 - \frac{3}{2} \cdot 2x = 60 - 3x$$

$$\frac{d^2A}{dx^2} = -3$$

For max. or min, we have, $\dfrac{dA}{dx} = 0$

or, $\quad 60 - 3x = 0$

or, $\quad 3x = 60$

$\therefore \quad x = 20$

When $x = 20$, $\dfrac{d^2A}{dx^2} = -3 < 0$

So, area is maximum when $x = 20$.

Max. area $= A_{max} = 20\left(60 - \dfrac{3}{2} \times 20\right) = 20 \times 30 = 600$ m^2

9. **Using derivatives, find two numbers whose sum is 20 and sum of whose squares is minimum.**

Solution

Let two numbers be x and y

By question, $x + y = 20$

$$y = 20 - x \qquad \text{(i)}$$

Let $S = x^2 + y^2$

or, $S = x^2 + (20 - x)^2$ [Using (i)]

or, $S = x^2 + 400 - 40x + x^2$

or, $S = 2x^2 - 40x + 400$

$$\frac{dS}{dx} = 4x - 40$$

$$\frac{d^2S}{dx^2} = 4$$

For max. or min. values, we have,

$$\frac{dS}{dx} = 0$$

or, $4x - 40 = 0$

or, $x = 10$

When $x = 10$, $\frac{d^2S}{dx^2} = 4 < 0$

So, S is minimum when $x = 10$.

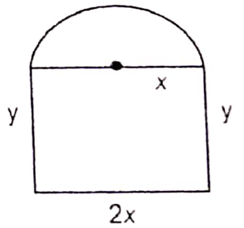
When $x = 10$, from (i) by $y = 20 - 10 = 10$

Required two numbers are 10 and 10.

10. **A window is in the form of a rectangle surmounted by a semi circle. If the total perimeter is 9 m, find the radius of the semicircle for the greatest window area.**

Solution

Let $2x$ be the length and y be the breadth of the rectangle of the window which is surmounted by a semi-circle as shown in the figure.



Total perimeter = 9m

or, $y + 2x + y + \pi x = 9$

or, $2y = 9 - 2x - \pi x$

or, $y = \dfrac{9 - 2x - \pi x}{2}$ ...(i)

Area of the window (A)

$= \text{area of rectangle} + \text{area of semi-circle}$

$= 2x \cdot y + \frac{1}{2} \cdot \pi \cdot x^2$

$= \dfrac{\pi x^2}{2} + 2x\left(\dfrac{9 - 2x - \pi x}{2}\right)$ [using (i)]

$= \dfrac{\pi x^2}{2} + 9x - 2x^2 - \pi x^2 = 9x - 2x^2 - \dfrac{\pi x^2}{2}$

$$\frac{dA}{dx} = 9 - 4x - \pi x$$

$$\frac{d^2A}{dx^2} = -4 - \pi$$

For maximum area, $\dfrac{dA}{dx} = 0$

or, $9 - 4x - \pi x = 0$

or, $(4 + \pi)x = 9$

$\therefore x = \dfrac{9}{4 + \pi}$

When $x = \frac{9}{4+\pi}$, $\frac{d^2A}{dx^2} = -4 - \pi < 0$

So, area is maximum when $x = \frac{9}{4+\pi}$

Hence, for the greatest window area, the radius of semi-circle $= \frac{9}{4+\pi}$ m

11. **A closed cylindrical can is to be made so that its volume is 26 cm³. Find its radius and height if the surface is to be a minimum.**

Solution

Let r be the radius and h be the height of cylindrical can

We have, $v = \pi r^2 h$

or, $26 = \pi r^2 h$

or, $h = \frac{26}{\pi r^2}$ ... (i)

Again, we have,

$S = 2\pi rh + 2\pi r^2$

or, $S = 2\pi r\left(\frac{26}{\pi r^2}\right) + 2\pi r^2$

or, $S = \frac{52}{r} + 2\pi r^2$

$\frac{dS}{dr} = -\frac{52}{r^2} + 4\pi r$

$\frac{d^2S}{dx^2} = \frac{104}{r^3} + 4\pi$

For max. or min. values, we have,

$\frac{dS}{dr} = 0$

or, $-\frac{52}{r^2} + 4\pi r = 0$

or, $-52 + 4\pi r^3 = 0$

or, $r^3 = \frac{52}{4\pi}$

or, $r^3 = \frac{13}{\pi}$

or, $r = \left(\frac{13}{\pi}\right)^{\frac{1}{3}}$

When $r = \left(\frac{13}{\pi}\right)^{\frac{1}{3}}$, $\frac{d^2S}{dr^2} = \frac{104}{\frac{13}{\pi}} + 4\pi$ which is positive

So, S is minimum when $r = \left(\frac{13}{\pi}\right)^{\frac{1}{3}}$

When $r = \left(\frac{13}{\pi}\right)^{\frac{1}{3}}$, from (i) $h = \frac{26}{\pi\left[\left(\frac{13}{\pi}\right)^{\frac{1}{3}}\right]^2}$

$= 2\left(\frac{13}{\pi}\right)\left(\frac{13}{\pi}\right)^{-\frac{2}{3}} = 2\left(\frac{13}{\pi}\right)^{\frac{1}{3}}$

Objective Questions

1. The rate of change of $y = x^2 - 3x + 5$ with respect to x is

(a) $2x - 3$ (b) $2x + 5$

(c) $x^2 + 5$ (d) $x^2 - 3$

Ans: a

$$\frac{dy}{dx} = 2x - 3$$

2. The critical points of the function $f(x) = 2x^3 - 6x^2 + 5$ are

(a) 1, 2 (b) 0, 2

(c) −1, 2 (d) 1, −2

Ans: b

$f'(x) = 6x^2 - 12x$

Now, $f'(x) = 0$, we get,

$6x(x - 2) = 0$

$x = 0, 2$

3. The function $f(x)$ is increasing on a domain D if

(a) $f'(x) > 0$ for all x in D (b) $f'(x) < 0$ for all x in D

(c) $f'(x) = 0$ for all x in D (d) $f'(x)$ does not exist for all x in D

Ans: a (definition)

4. The absolute maximum value of $f(x) = x^2$ on $[-2, 1]$ is

(a) −2 (b) 0

(c) 1 (d) 4

Ans: d

$f'(x) = 2x$

Now, $f'(x) = 0 \Rightarrow 2x = 0$

$\Rightarrow x = 0$

Now, $f(-2) = (-2)^2 = 4$

$f(0) = 0^2 = 0$

$f(1) = 1^2 = 1$

$\therefore$ Absolute maximum = 4

5. The conditions for the function $y = f(x)$ to have local minimum value are

(a) $\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} = 0$ (b) $\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} < 0$

(c) $\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} > 0$ (d) $\frac{dy}{dx} \neq 0$

Ans: c

6. The local maximum value of $-x^2 - 2x + 5$ is

(a) −1 (b) 0

(c) 2 (d) 6

Ans: d

Let $y = -x^2 - 2x + 5$

$$\frac{dy}{dx} = -2x - 2$$

$$\frac{d^2y}{dx^2} = -2$$

Now, $\frac{dy}{dx} = 0 \Rightarrow -2x - 2 = 0$

$\Rightarrow -2x = 2$

$\Rightarrow x = -1$

When $x = -1$, $\frac{d^2y}{dx^2} = -2 < 0$. (max. value)

$y_{max} = -(-1)^2 - 2 \times (-1) + 5 = 6$

7. The point of inflection of the function $y = x^4$ is

(a) 0 (b) 1

(c) 2 (d) no point of inflection

Ans: d

Since the function does not change its concavity, so there is no point of inflection.

8. The points of inflection of the function $y = x^4 - 4x^3 + 10$ are

(a)	0, 1	(b)	1, 2
(c)	−1, 2	(d)	0, 2

Ans: d

$$y = x^4 - 4x^3 + 10$$

$$\frac{dy}{dx} = 4x^3 - 12x^2$$

$$\frac{d^2y}{dx^2} = 12x^2 - 24x$$

$$\text{Now, } \frac{d^2y}{dx^2} = 0 \Rightarrow 12x^2 - 24x = 0$$

$$\Rightarrow x = 0, 2.$$

9. Let $y = f(x)$ be twice differentiable on an interval I. Then the graph of f over I is concave up if

(a)	$f''(x) = 0$ on I	(b)	$f''(x) > 0$ on I
(c)	$f''(x) < 0$ on I	(d)	$f''(x)$ does not exist on I

Ans: b

EXERCISE– 3 (B)

1. (a) **If the total cost is given by $C = 3Q^2 + 5Q + 50$, find the average cost and marginal cost and their values at the production level 6.**

   (b) **The demand function for a commodity is given by $P = 13 - 3x$. Find the average revenue and marginal revenue.**

   (c) **If the demand function, $P = 30 - 4x$ and total cost function, $C = 35x + 600$. Find the marginal profit.**

Solution

(a) Total cost (C) = $3Q^2 + 5Q + 50$

$$AC = \frac{C}{Q} = \frac{3Q^2 + 5Q + 50}{Q} = 3Q + 5 + \frac{50}{Q}$$

When Q = 6, $AC = 3 \times 6 + 5 + \frac{50}{6} = 31.33$

$$MC = \frac{d}{dQ}(C) = \frac{d}{dQ}(3Q^2 + 5Q + 50) = 6Q + 5$$

When Q = 6, M.C. = $6 \times 6 + 5 = 41$

(b) Demand function (P) = $13 - 3x$

Revenue function (R) = $P \cdot x = 13x - 3x^2$

$$\text{Average revenue (AR)} = \frac{13x - 3x^2}{x} = 13 - 3x$$

$$\text{Marginal revenue (MR)} = \frac{d}{dx}(R) = \frac{d}{dx}(13x - 3x^2) = 13 - 6x$$

(c) $R = P \cdot x = (30 - 4x) \cdot x = 30x - 4x^2$

$C = 35x + 600$

$$\begin{aligned}\text{Profit function } (\pi) &= R - C \\ &= 30x - 4x^2 - (35x + 600) = 30x - 4x^2 - 35x - 600 \\ &= -4x^2 - 5x - 600.\end{aligned}$$

$$\text{Marginal profit (MP)} = \frac{d}{dx}(\pi) = \frac{d}{dx}(-4x^2 - 5x - 600) = -8x - 5.$$

2. **A firm estimates that its daily total cost function is $C(x) = x^3 - 6x^2 + 13x + 15$ and its total revenue function is $R(x) = 28x$. Find the value of x that maximizes the daily profit.**

Solution

Given, total cost function $C(x) = x^3 - 6x^2 + 13x + 15$

Total revenue function $R(x) = 28x$

Profit function $\pi(x) = R(x) - C(x)$

$= 28x - (x^3 - 6x^2 + 13x + 15)$

$= 28x - x^3 + 6x^2 - 13x - 15$

$= -x^3 + 6x^2 + 15x - 15$

Differentiating both sides with respect to 'x'

$$\frac{d\pi(x)}{dx} = \frac{d}{dx}(-x^3 + 6x^2 + 15x - 15)$$

$$\pi'(x) = -3x^2 + 12x + 15$$

Again, differentiating both sides with respect to 'x'

$$\pi''(x) = \frac{d}{dx}(-3x^2 + 12x + 15)$$

$$= -6x + 12$$

For maximum or minimum,

$\pi'(x) = 0$

or, $-3x^2 + 12x + 15 = 0$

or, $x^2 - 4x - 5 = 0$

or, $(x + 1)(x - 5) = 0$

$\therefore$ $x = -1, 5$

When $x = -1$, $\pi''(x) = -6 \times (-1) + 12 = 18 > 0$

So, profit is minimum when $x = -1$ which is not required.

Again, when $x = 5$, $\pi''(x) = -6 \times 5 + 12 = -18 < 0$

Hence, profit is maximum at $x = 5$

3. **If the revenue function is $R = Q - 3Q^2$ and the cost function $C = Q^2 - 2Q$, find the value of the maximum profit.**

Solution

We have, profit $(\pi) = R - C = Q - 3Q^2 - Q^2 + 2Q$

$\Rightarrow$ $\pi = 3Q - 4Q^2$

$\Rightarrow$ $\frac{d\pi}{dQ} = 3 - 8Q$ and $\frac{d^2\pi}{dQ^2} = -8$

For maxima, $\frac{d\pi}{dQ} = 0 \Rightarrow 3 - 8Q = 0$

$\therefore$ $Q = \frac{3}{8}$,

when $Q = \frac{3}{8}$

$\frac{d^2\pi}{dQ^2} = -8 < 0$, (maxima)

and the maximum profit at $Q = \frac{3}{8}$ is $\pi_{max} = 3 \times \frac{3}{8} - 4 \times \frac{9}{64} = \frac{9}{8} - \frac{9}{16} = \frac{9}{16}$

4. **A firm has a demand function $P = 108 - 5Q$ and cost function $C = -12Q + Q^2$. Find the price at which the profit is maximum. Find the maximum profit.**

Solution

Demand function $(P) = 108 - 5Q$

Cost function $(C) = -12Q + Q^2$

Revenue function (R) = Demand function $\times$ Quantity

$= (108 - 5Q)\,Q$

$= 108\,Q - 5Q^2$

Profit $(\pi) = R - C$

$= 108\,Q - 5Q^2 - (-12Q + Q^2) = 108\,Q - 5Q^2 + 12Q - Q^2$

$= -6Q^2 + 120\,Q$

$$\frac{d\pi}{dQ} = -12Q + 120$$

$$\frac{d^2\pi}{dQ} = -12$$

For maxima and minima, $\frac{d\pi}{dQ} = 0$

or $\quad -12Q + 120 = 0$

or $\quad 12Q = 120$

$\therefore \quad Q = 10$

When $Q = 10$, $\frac{d^2\pi}{dQ} = -12 < 0$

Hence profit is maximum when $Q = 10$.

Price for max. profit $= 108 - 5 \times 10 = 58$

Max. profit $= -6 \times 10^2 + 120 \times 10 = -600 + 1200 = 600$.

5. **The demand equation for a certain commodity is $P = \frac{1}{3}Q^2 - \frac{15}{2}Q + 50$. Find the value of Q and the corresponding value of P that maximizes the revenue.**

Solution

Here, $P = \frac{1}{3}Q^2 - \frac{15Q}{2} + 50$

We have,

Revenue $(R) = P.Q. = \left(\frac{1}{3}Q^2 - \frac{15}{2}Q + 50\right)Q = \frac{Q^3}{3} - \frac{15}{2}Q^2 + 50Q$

$$\frac{dR}{dQ} = \frac{3Q^2}{3} - \frac{15}{2} \times 2Q + 50 = Q^2 - 15Q + 50$$

$$\therefore \quad \frac{d^2R}{dQ^2} = 2Q - 15,$$

For maxima, $\frac{dR}{dQ} = 0$

$\Rightarrow \quad Q^2 - 15Q + 50 = 0$

$\Rightarrow \quad Q^2 - 10Q - 5Q + 50$

$\Rightarrow \quad Q(Q - 10) - 5(Q - 10) = 0$

or, $\quad (Q - 10)(Q - 5) = 0$

$\therefore \quad Q = 5, 10,$

when $Q = 5$, $\frac{d^2R}{dQ^2} = 2 \times 5 - 15 = -5 < 0$

when $Q = 15$,

$$\frac{d^2R}{dQ^2} = 2 \times 15 - 20 = 10 > 0$$

So, at $Q = 15$, R is minimum. So we reject $Q = 15$.

The maximum revenue at $Q = 5$ is $P = \frac{Q^3}{3} - \frac{15}{2}Q^2 + 50Q = \frac{125}{3} - \frac{15}{2} \times 25 + 50 \times 5 = \frac{125}{6}$

6. **The demand function for a certain commodity is $P = \frac{Q^2}{3} - 10Q + 75$. Find the value of Q and the corresponding value of P that maximizes the revenue.**

Solution

$$P = \frac{Q^2}{3} - 10Q + 75$$

We have, $R = PQ$

or, $\quad R = \left(\frac{Q^2}{3} - 10Q + 75\right)Q$

or, $\quad R = \frac{Q^3}{3} - 10Q^2 + 75Q$

$$\frac{dR}{dQ} = \frac{3Q^2}{3} - 20Q + 75$$

$$\frac{d^2R}{dQ^2} = 2Q - 20$$

For maxima or minima $\frac{dR}{dQ} = 0$

or. $Q^2 - 20Q + 75 = 0$

or. $Q^2 - 15Q - 5Q + 75 = 0$

or. $Q(Q - 15) - 5(Q + 15) = 0$

or $Q = 5$ or 15

When $Q = 5$

$$\frac{d^2R}{dQ^2} = 2Q - 20 = 2 \times 5 - 20 = 10 - 20 = -10 < 0$$

Hence, the revenue is maxima and the maximum revenue at $Q = 5$ is

$$P = \frac{25}{3} - 50 + 75 = \frac{25 - 150 + 225}{3} = \frac{100}{3}$$

7. **Given the demand function $P = 20 - Q$ and the total cost function $C = Q^2 + 8Q + 2$, determine the optimal output Q, price P and total profit under profit maximization.**

Solution

Demand function $(P) = 20 - Q$

Revenue $(R) = P.Q$

$= (20 - Q).\ Q$

$= 20Q - Q^2$

Total cost function $(C) = Q^2 + 8Q + 2$

Profit function $(\pi) = R - C$

$= (20Q - Q^2) - (Q^2 + 8Q + 2)$

$= 20Q - Q^2 - Q^2 - 8Q - 2$

$= -2Q^2 + 12Q - 2$

$$\frac{d\pi}{dQ} = -4Q + 12$$

And $\frac{d^2\pi}{dQ^2} = -4$

For max. or min. $\frac{d\pi}{dQ} = 0$

$\Rightarrow \quad -4Q + 12 = 0$

$\Rightarrow \quad Q = 3$

When $Q = 3$, $\frac{d^2\pi}{dQ^2} = -4 < 0$.

So. profit (π) is maximum when $Q = 3$.

Maximum profit $= -2 \times 3^2 + 12 \times 3 - 2 = -18 + 36 - 2 = 16$.

When $Q = 3$, $P = 20 - 3 = 17$

$\therefore$ Under profit maximization $Q = 3$, $P = 17$, $\pi = 16$.

8. **Given the demand function $P = 20 - Q$ and total cost function $C = Q^2 + 8Q$, determine the price at which the profit is maximum. Also find the maximum profit.**

Solution

$P = 20 - Q$

$\therefore \quad R = P.Q. = (20 - Q)Q$

$C = Q^2 + 8Q$

Profit $(\pi) = R - C$

$= (20 - Q)\ Q - Q^2 - 8Q$

$= 20Q - Q^2 - Q^2 - 8Q$

$= 12Q - 2Q^2$

$$\frac{d\pi}{dQ} = 12 - 4Q$$

$$\frac{d^2\pi}{dQ} = -4 \text{ (maxima)}$$

For maxima or minima $\frac{d\pi}{dQ} = 0$

or, $12 - 4Q = 0$

$\therefore$ $Q = 3$

when $Q = 3$, $\frac{d^2\pi}{dQ^2} = -4$

So, when $Q = 3$, π is maximum

When $Q = 3$, $P = 20 - 3 = 17$

Max. Profit $(\pi) = 12 \times 3 - 2 \times 3^2 = 36 - 18 = 18$

9. A company produces Q units of output at a total cost of $C = \frac{1}{4}Q^2 + 3Q + 100$. Verify that the minimum average cost is equal to the marginal cost at the level minimizing the average cost.

Solution

Total cost function $(C) = \frac{1}{4}Q^2 + 3Q + 100$

Average cost $(AC) = \frac{C}{Q} = \frac{\frac{1}{4}Q^2 + 3Q + 100}{Q} = \frac{1}{4}Q + 3 + \frac{100}{Q}$

$$\frac{d(AC)}{dQ} = \frac{1}{4} - \frac{100}{Q^2}$$

$$\frac{d^2(AC)}{dQ^2} = \frac{200}{Q^3}$$

For maxima or minima, $\frac{d(AC)}{dQ} = 0$.

or, $\frac{1}{4} - \frac{100}{Q^2} = 0$

or, $\frac{1}{4} = \frac{100}{Q^2}$

or, $Q^2 = 400$

or, $Q^2 = (\pm 20)^2$

$\therefore$ $Q = \pm 20$

Since units of output cannot be negative, $Q = -20$ is rejected.

$\therefore$ $Q = 20$

When $Q = 20$, $\frac{d^2(AC)}{dQ^2} = \frac{200}{20^3} = \frac{1}{40} > 0$

So, AC is minimum at $Q = 20$.

Minimum average cost $= \frac{1}{4} \times 20 + 3 + \frac{100}{20} = 5 + 3 + 5 = 13$

Marginal cost $(MC) = \frac{d}{dQ}(C) = \frac{d}{dQ}\left(\frac{1}{4}Q^2 + 3Q + 100\right) = \frac{1}{4}.2Q + 3 = \frac{1}{2}Q + 3$.

When $Q = 20$, $MC = \frac{1}{2} \times 20 + 3 = 13$.

Hence minimum average cost is equal to the marginal cost at $Q = 20$.

10. A company produces Q units of output at a total cost of $\frac{1}{3}Q^3 - 18Q^2 + 160Q$. Find

(a) the output at which marginal cost is minimum,
(b) the output at which average cost is minimum, and
(c) the output at which average cost is equal to marginal cost.

Solution

Total cost function $(C) = \frac{1}{3}Q^3 - 18Q^2 + 160Q$

(a) Marginal cost (MC) $= \frac{d}{dQ}(C) = \frac{d}{dQ}\left(\frac{1}{3}Q^3 - 18Q^2 + 160Q\right) = Q^2 - 36Q + 160$

$\frac{d}{dQ}(MC) = 2Q - 36$

$\frac{d^2}{dQ^2}(MC) = 2$

For max. and min. values, we have,

$\frac{d}{dQ}(MC) = 0$

or, $2Q - 36 = 0$

$\therefore$ $Q = 18$

When $Q = 18$, $\frac{d^2}{dQ^2}(MC) = 2 > 0$.

So, MC is minimum at $Q = 18$.

(b) Average cost (AC) $= \frac{C}{Q} = \frac{\frac{1}{3}Q^3 - 18Q^2 + 160Q}{Q} = \frac{1}{3}Q^2 - 18Q + 160$

$\frac{d}{dQ}(AC) = \frac{1}{3}\cdot 2Q - 18$

$\frac{d^2}{dQ^2}(AC) = \frac{2}{3}$

For max. and min. values, we have,

$\frac{d}{dQ}(AC) = 0$

or, $\frac{2}{3}Q = 18$

$\therefore$ $Q = 27$.

When $Q = 27$, $\frac{d^2}{dQ^2}(AC) = \frac{2}{3} > 0$.

So, AC is minimum at $Q = 27$

(c) By given, AC = MC

or, $\frac{\frac{1}{3}Q^3 - 18Q^2 + 160Q}{Q} = \frac{d}{dQ}\left(\frac{1}{3}Q^3 - 18Q + 160\right)$

or, $\frac{1}{3}Q^2 - 18Q + 160 = Q^2 - 36Q + 160$

or, $Q^2 - 54Q + 480 = 3Q^2 - 108Q + 480$

or, $3Q^2 - Q^2 - 108Q + 54Q = 0$

or, $2Q^2 - 54Q = 0$

or, $2Q - 54 = 0$ [$\because Q = 0$ gives the trivial solution]

$\therefore$ $Q = 27$.

Objective Questions

1. **If the total cost function is $C = x^2 + 3$, then the average cost function is**

(a) $x^2 + 3$ (b) $2x$

(c) $x + \frac{3}{x}$ (d) $2x + 3$

Ans: c

$AC = \frac{C}{x} = \frac{x^2 + 3}{x} = x + \frac{3}{x}$

2. **The total cost function of a company is $C = x^2 + 30x + 500$ then marginal cost is**

(a) $2x + 30$ (b) $2x + 500$

(c) $x + 30 + \frac{500}{x}$ (d) $x^2 + 30x + 500$

Ans: a

$MC = \frac{d}{dx}(C) = \frac{d}{dx}(x^2 + 30x + 500) = 2x + 30.$

3. If total revenue function from the sales of x units of a product is R then average revenue function is given by

(a) $R \cdot x$ (b) $R + x$

(c) $R - x$ (d) $\frac{R}{x}$

Ans: d

$$MR = \frac{R}{x}$$

4. If x is the units of sales of a product, P is the price per unit and R is the revenue then marginal revenue (MR) is

(a) $P + x\frac{dP}{dx}$ (b) Px

(c) $\frac{dP}{dx} + 1$ (d) $\frac{dP}{dx}$

Ans: a

$$R = Px$$

Using product rule, we have

$$\frac{dR}{dx} = P\frac{dx}{dx} + x\frac{dP}{dx} = P + x\frac{dP}{dx}$$

5. The demand function of a product is $P = 200 - 5Q$ then marginal revenue is

(a) $200Q - 5Q^2$ (b) $200 - 10Q$

(c) $200 - 5Q$ (d) $100 - Q$

Ans: b

$$R = P \cdot Q = (200 - 5Q) \cdot Q = 200Q - 5Q^2$$

$$MR = \frac{d}{dQ}(R) = 200 - 10Q$$

6. A company produces Q units of output at a total cost of $\frac{1}{3}Q^3 - 18Q^2 + 160Q$. The output at which the marginal cost is minimum is

(a) 3 (b) 12

(c) 18 (d) 30

Ans: c

Here, $C = \frac{1}{3}Q^3 - 18Q^2 + 160Q$

$$MC = \frac{dC}{dQ} = Q^2 - 36Q + 160$$

$$\frac{d}{dQ}(MC) = 2Q - 36$$

$$\frac{d^2}{dQ^2}(MC) = 2$$

For max. or min. value, $\frac{d}{dQ}(MC) = 0$

$$2Q - 36 = 0$$

$$Q = 18.$$

When $Q = 18$, $\frac{d^2(MC)}{dQ^2} = 2 < 0$ (minimum)

$\therefore$ Required output $Q = 18$.

7. For profit maximization, the condition at the point of maximum profit is

(a) slope of the marginal revenue curve must be less than the slope of the marginal cost curve

(b) slope of the marginal revenue curve must be more than the slope of the marginal cost curve

(c) slope of the marginal revenue curve must be equal to the slope of the marginal cost curve

(d) none of the above

Ans: a

8. If the revenue function is $R = Q - 5Q^2$ and cost function is $C = Q^2 - 4Q$ then the value of Q that maximizes the profit is

(a) $\frac{12}{5}$ (b) 1

(c) 10 (d) $\frac{5}{12}$

Ans: d

Profit $(\pi) = R - C = -6Q^2 + 5Q$

$$\frac{d\pi}{dQ} = -12Q + 5$$

$$\frac{d^2\pi}{dQ^2} = -12$$

Set $\frac{d\pi}{dQ} = 0 \Rightarrow -12Q + 5 = 0$

$$\Rightarrow Q = \frac{5}{12}$$

When $Q = \frac{5}{12}$, $\frac{d^2\pi}{dQ^2} = -12 < 0$ (Maximum)

$\therefore$ Required value of $Q = \frac{5}{12}$.

EXERCISE 3 (C)

Graph the following functions:

(a) $y = x^2$ (b) $y = 6 - 2x - x^2$

(c) $y = x^3$ (d) $y = -2x^3 + 6x^2 - 3$

(e) $y = x^4 - 2x^2$ (f) $y = x^4 - 4x^3 + 10$

(g) $y = x^{\frac{5}{3}} - 5x^{\frac{2}{3}}$

Solution

(a) $y = x^2$

$y' = 2x$

$y'' = 2$

For critical points, we have,

$y' = 0$

$2x = 0$

$x = 0$

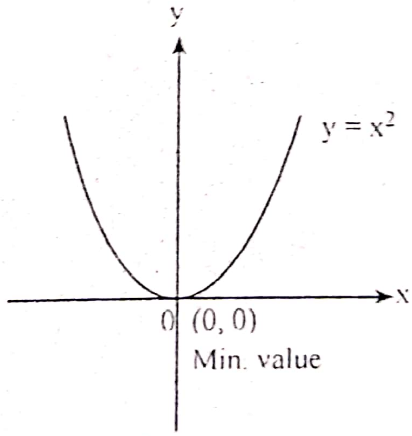
For $x > 0$, $y' = 2x > 0$ So, the curve rises on $(0, \infty)$

For $x < 0$, $y' = 2x < 0$ So, the curve falls on $(-\infty, 0)$

At $x = 0$, $y'' = 2 > 0$. So, y has minimum value at $x = 0$.

Min. value $= y_{min} = 0^2 = 0$

Since $y'' > 0$, the curve is concave up for all x.



(b) $y = 6 - 2x - x^2$

$y' = -2 - 2x$

$y'' = -2$

For critical points, we have,

$y' = 0$

$-2 - 2x = 0$

or, $x = -1$

For $x > 1$, $y' = -2 - 2x = -2(1 + x) < 0$

So the curve falls on $(1, \infty)$

For $x < 1$, $y' = -2(1 + x) > 0$ So, the curve rises on $(-\infty, 1)$

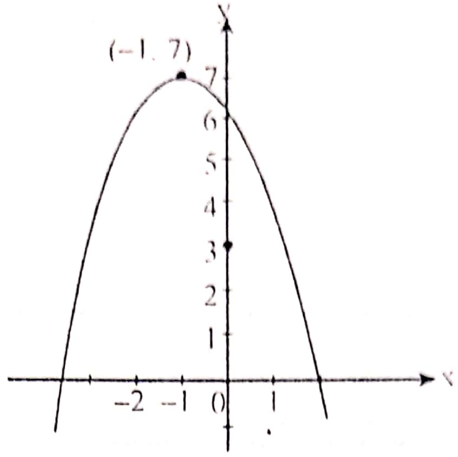
At $x = -1$, $y'' = -2 < 0$ So, y has max. value at $x = -1$

Max. value $= y_{max}$

$= 6 - 2(-1) - (-1)^2$

$= 6 + 2 - 1 = 7$

Since $y'' < 0$, so the curve is concave downward for all x.



(c) $y = x^3$

$y' = 3x^2$

$y'' = 6x$

For critical points, we have,

$y' = 0$

or, $3x^2 = 0$

$\therefore$ $x = 0$

For $x > 0$, $y' = 3x^2 > 0$. So, the curve rises on $(0, \infty)$.

For $x < 0$, $y' = 3x^2 < 0$. So, the curve falls on $(-\infty, 0)$

At $x = 0$, $y'' = 6 \times 0 = 0$ but $y''' = 6 \neq 0$.

So, y does not have any maximum and minimum values.

For point of inflection,

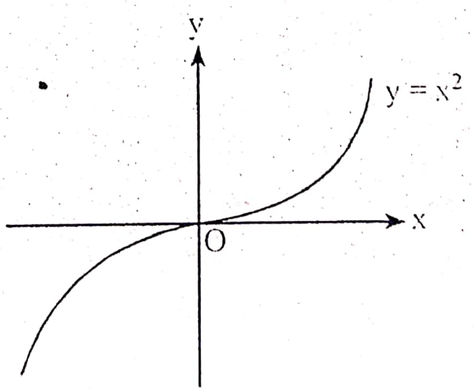
$y'' = 0$

or, $6x = 0$

$\therefore$ $x = 0$

For $x > 0$, $y'' > 0$, so the curve is concave upward for $x > 0$

For $x < 0$, $y'' < 0$, so the curve is concave downwards for $x < 0$.



(d) $y = -2x^3 + 6x^2 - 3$

$y' = -6x^2 + 12x = -6x(x - 2)$

$y'' = -12x + 12 = -12(x - 1)$

For critical points, we have,

$y' = 0$

or, $-6x(x - 2) = 0$

$\therefore$ $x = 0, 2$

Intervals	Sign of y'	Nature
$(-\infty, 0)$	− ve	decreasing
$(0, 2)$	+ ve	increasing
$(2, \infty)$	− ve	decreasing

$\therefore$ The curve rises on $(0, 2)$ and falls on $(-\infty, 0) \cup (2, \infty)$

At $x = 0$, $y'' = 12 > 0$.

So, y has local minimum at x = 0

$y_{min} = -2 \times 0^3 + 6 \times 0^2 - 3 = -3$

At x = 2, $y'' = -24 + 12 = -12 < 0$

So, y has local maximum at x = 2

$y_{max} = -2 \times 2^3 + 6 \times 2^2 - 3 = 5$

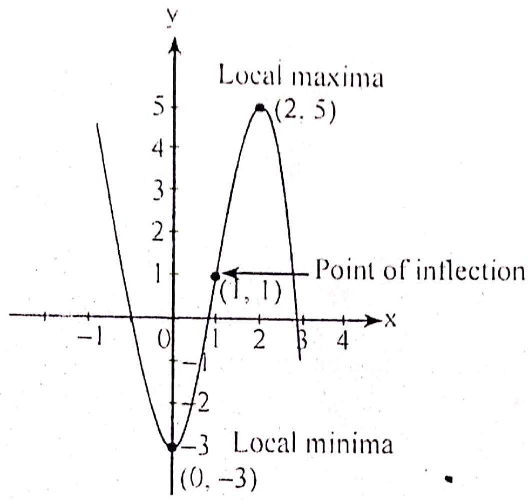
For point of inflection, we have,

$$y'' = 0$$

or, $-12(x - 1) = 0$

$\therefore$ $x = 1$

Intervals	**Sign of y''**	**Nature**
$(-\infty, 1)$	+ ve	Concave upwards
$(1, \infty)$	− ve	Concave downwards



(e) $y = x^4 - 2x^2$

Then, $y' = 4x^3 - 4x = 4x(x^2 - 1)$

And, $y'' = 12x^2 - 4$

For critical points, we have,

$$y' = 0$$

or, $4x(x^2 - 1) = 0$

or, $4x(x - 1)(x + 1) = 0$

$\therefore$ $x = 0, 1, -1.$

Intervals	**Sign of f '(x)**	**Nature**
$(-\infty, -1)$	− ve	decreasing
$(-1, 0)$	+ ve	increasing
$(0, 1)$	− ve	decreasing
$(1, \infty)$	+ ve	increasing

$\therefore$ The curve rises on $(-1, 0) \cup (1, \infty)$ and falls on $(-\infty, -1) \cup (0, 1)$.

At $x = \pm 1$, $y'' = 12(\pm 1)^2 - 4 = 8 > 0$

So, y has local minima at $x = \pm 1$

Local minima $= (\pm 1)^4 - 2(\pm 1)^2 = 1 - 2 = -1$

For points of inflection

$$y'' = 0$$

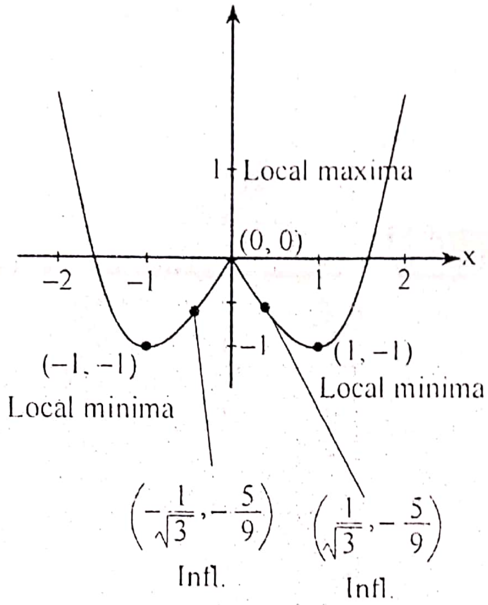
or, $12x^2 - 4 = 0$

or, $3x^2 - 1 = 0$

or, $x = \pm \dfrac{1}{\sqrt{3}}$

Intervals	Sign of f''(x)	Nature
$\left(-\infty, -\frac{1}{\sqrt{3}}\right)$	+ ve	Concave upwards
$\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$	− ve	Concave downwards
$\left(\frac{1}{\sqrt{3}}, \infty\right)$	+ ve	Concave upwards

$\therefore$ The function is concave upwards on $\left(-\infty, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, \infty\right)$ and concave downwards on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.



(f) Given $y = x^4 - 4x^3 + 10$

$y' = 4x^3 - 12x^2$

$y'' = 12x^2 - 24x$

For critical points, $y' = 0$

$4x^3 - 12x^2 = 0$

or, $4x^2(x - 3) = 0$

$\therefore$ $x = 0, 3$

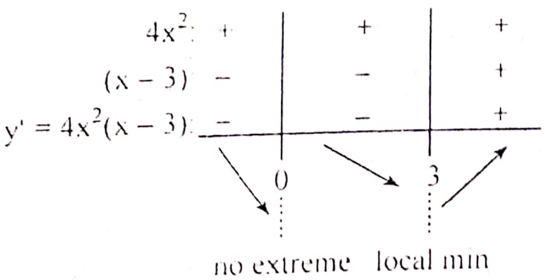
For possible point of inflection,

$y'' = 0$

$12x(x - 2) = 0$

$\therefore$ $x = 0, 2$.

Rise and fall:

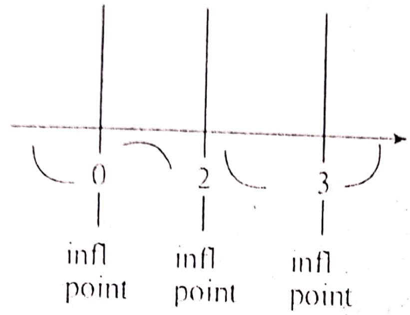


Concavity:

	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
$12x$	$-$	$+$	$+$
$(x-2)$	$-$	$-$	$+$
$y'' = 12x(x-2)$	$+$	$-$	$+$
	conc up	conc down	conc up

0: infl point; 2: infl point

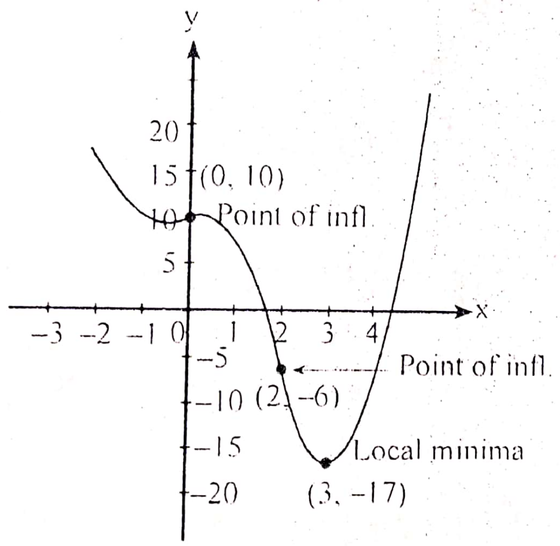
Summary:



Local min. value at 3 is

$$y_{min} = 3^4 - 4 \times 3^3 + 10 = -17$$

We take some points if we need.



(g) $y = x^{\frac{5}{3}} - 5x^{\frac{2}{3}}$

$$= x^{\frac{2}{3}}(x - 5)$$

$$y' = \frac{5}{3}x^{\frac{2}{3}} - \frac{10}{3}x^{-\frac{1}{3}}$$

$$= \frac{5}{3}x^{-\frac{1}{3}}(x - 2)$$

$$y'' = \frac{10}{9}x^{-\frac{1}{3}} + \frac{10}{9}x^{-\frac{4}{3}}$$

$$= 10x^{-\frac{4}{3}}(x + 1)$$

Here y' is zero at $x = 2$ and y' is undefined at $x = 0$

So, critical points are $x = 0$ and $x = 2$.

Also, y'' is zero at $x = -1$ and undefined at $x = 0$

So, possible points of inflection are $x = -1$ and $x = 0$

Rise and fall:

	$x<0$	$0<x<2$	$x>2$
$\frac{5}{3}x^{-\frac{1}{3}}$	$-$	$+$	$+$
$(x-2)$	$-$	$-$	$+$
$y'=\frac{5}{3}x^{-\frac{1}{3}}(x-2)$	$+$	$-$	$+$
	↗	0 ↘	2 ↗
		local max	local min

Concavity:

	$x<-1$	$-1<x<0$	$x>0$
$\frac{10}{9}x^{-\frac{4}{3}}$	$+$	$+$	$+$
$(x+1)$:	$-$	$+$	$+$
$y''=\frac{10}{9}x^{-\frac{4}{3}}(x+1)$:	$-$	$+$	$+$
	conc down	-1 conc up	0 conc up
		infl point	no point of infl

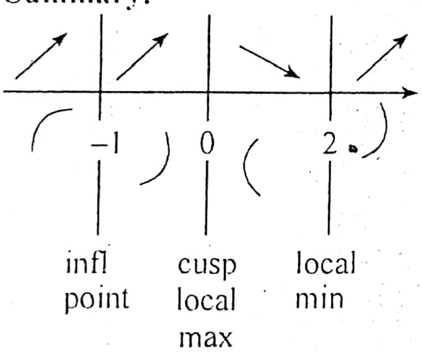
Here,

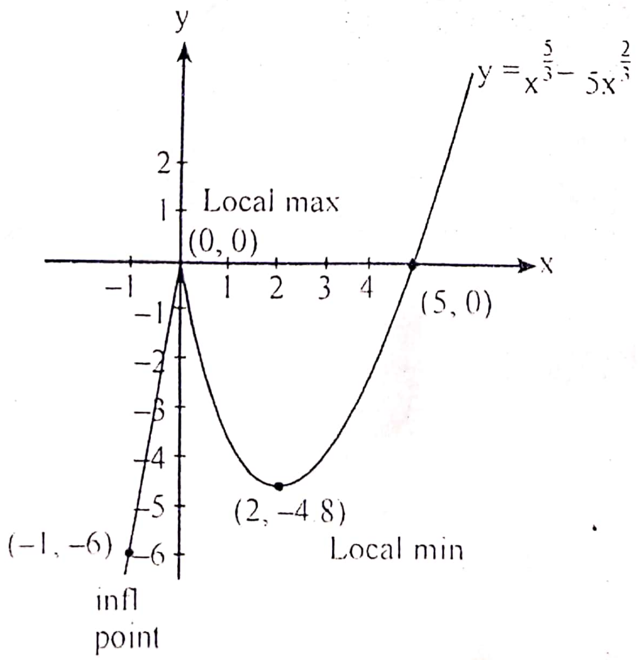
(i) the function y is continuous,

(ii) $y' \to \infty$ as $x \to 0^-$ and $y' \to -\infty$ as $x \to 0^+$,

(iii) the concavity does not change at $x = 0$. So, the graph has a cusp at $x = 0$.

Summary:





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4

UNIT

Integration and Its Application

EXERCISE 4 (A)

1. **Evaluate the integrals.**

(a) $\int(x + 1)\, dx$ (b) $\int(5 - 6x)dx$ (c) $\int(ax^2 + bx + c)dx$

(d) $\int\left(\frac{1}{x^2} - x^2 - \frac{1}{3x}\right) dx$ (e) $\int\left(\sqrt{x} + \sqrt[3]{x}\right) dx$ (f) $\int\left(x + \frac{1}{x}\right)^2 dx$

(g) $\int\frac{3x^3 + 4x + 5}{x^3}\, dx$

Solution

(a) $\int(x + 1)\, dx = \int x\, dx + \int dx = \frac{x^2}{2} + x + c$

(b) $\int(5 - 6x)dx = 5\int dx - 6\int x\, dx = 5x - \frac{6x^2}{2} + c = 5x - 3x^2 + c$

(c) $\int(ax^2 + bx + c)dx = a\int x^2\, dx + b\int x\, dx + c\int dx = \frac{ax^3}{3} + \frac{bx^2}{2} + cx + k$

(d) $\int\left(\frac{1}{x^2} - x^2 - \frac{1}{3x}\right) dx = \int x^{-2}\, dx - \int x^2\, dx - \frac{1}{3}\int\frac{1}{x}\, dx$

$= \frac{x^{-2+1}}{-2+1} - \frac{x^3}{3} - \frac{1}{3}\log x + c = -\frac{1}{x} - \frac{x^3}{3} - \frac{1}{3}\log x + c$

(e) $\int\left(\sqrt{x} + \sqrt[3]{x}\right) dx = \int x^{\frac{1}{2}}\, dx + \int x^{\frac{1}{3}}\, dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + c = \frac{2x^{\frac{3}{2}}}{3} + \frac{3x^{\frac{4}{3}}}{4} + c$

(f) $\int\left(x + \frac{1}{x}\right)^2 dx = \int\left(x^2 + 2\cdot x\cdot\frac{1}{x} + \frac{1}{x^2}\right) dx$

$= \int x^2\, dx + 2\int dx + \int x^{-2}\, dx = \frac{x^3}{3} + 2x + \frac{x^{-1}}{-1} + c = \frac{x^3}{3} + 2x - \frac{1}{x} + c$

(g) $\int\frac{3x^3 + 4x + 5}{x^3}\, dx = \int\left(\frac{3x^3}{x^3} + \frac{4x}{x^3} + \frac{5}{x^3}\right) dx = \int\left(3 + \frac{4}{x^2} + \frac{5}{x^3}\right) dx$

$= 3\int dx + 4\int x^{-2}\, dx + 5\int x^{-3}\, dx = 3x + \frac{4x^{-1}}{-1} + \frac{5x^{-2}}{-2} + c = 3x - \frac{4}{x} - \frac{5}{2x^2} + c$

2. Find the following integrals.

(a) $\int (7x-2)^3\, dx$ (b) $\int \sqrt{2-5x}\, dx$ (c) $\int \frac{1}{\sqrt{2x-9}}\, dx$

(d) $\int \frac{dx}{\sqrt{x+1}-\sqrt{x}}$ (e) $\int \frac{x^2-4}{x+2}\, dx$ (f) $\int \frac{2x+1}{x-1}\, dx$

(g) $\int \frac{3x+1}{x-2}\, dx.$

Solution

(a) $\int (7x-2)^3\, dx = \frac{(7x-2)^4}{7 \times 4} + c = \frac{1}{28}(7x-2)^4 + c$

(b) $\int (2-5x)^{1/2}\, dx = \frac{(2-5x)^{3/2}}{-5 \times \frac{3}{2}} + C = -\frac{2}{15}(2-5x)^{3/2} + c$

(c) $\int \frac{1}{\sqrt{(2x-9)}}\, dx = \int (2x-9)^{-1/2}\, dx = \frac{(2x-9)^{1/2}}{2 \times \frac{1}{2}} + c$

(d) $\int \frac{dx}{\sqrt{x+1}-\sqrt{x}} = \int \frac{dx}{\sqrt{x+1}-\sqrt{x}} \times \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}} = \int \frac{\sqrt{x+1}+\sqrt{x}}{x+1-x} dx$

$= \int (\sqrt{x+1}+\sqrt{x})\, dx = \frac{(x+1)^{3/2}}{3/2} + \frac{x^{3/2}}{3/2} + c = \frac{2}{3}[(x+1)^{3/2} + x^{3/2}] + c$

(e) $\int \frac{x^2-4}{x+2}\, dx$

$= \int \frac{(x+2)(x-2)}{x+2}\, dx = \int (x-2)\, dx = \int x\, dx - 2\int dx$

$= \frac{x^2}{2} - 2x + c$

(f) $\int \frac{2x+1}{x-1}\, dx = \int \frac{2x-2+3}{x-1} dx = \int \left\{\frac{2(x-1)}{x-1} + \frac{3}{x-1}\right\} dx$

$= \int \left(2 + \frac{3}{x-1}\right) dx = 2x + 3\log(x-1) + C$

(g) $\int \frac{3x+1}{x-2}\, dx = \int \frac{3(x-2)+7}{x-2}\, dx = \int \left\{\frac{3(x-2)}{x-2} + \frac{7}{x-2}\right\} dx$

$= 3\int dx + 7\int \frac{1}{x-2}\, dx$

$= 3x + 7\log(x-2) + c$

3. Evaluate the following integrals.

(a) $\int 3 \cos 5\theta \, d\theta$ (b) $\displaystyle\int \frac{\operatorname{cosec} \theta \cot \theta}{2} \, d\theta$

(c) $\int (2 \cos 2x - 3 \sin 3x) \, dx$ (d) $\int \cos^2 x \, dx$

(e) $\displaystyle\int \frac{1}{\cos^2 x \cdot \sin^2 x} \, dx$ (f) $\int \sqrt{1 + \cos x} \, dx$

(g) $\int \sin 4x \cos 2x \, dx$ (h) $\int (e^{ax} + e^{-bx}) dx$

(i) $\displaystyle\int \frac{e^{2x} + e^{x} + 3}{e^{x}} \, dx.$

Solution

(a) $\displaystyle\int 3 \cos 5\theta \, d\theta = \frac{3 \sin 5\theta}{5} + c = \frac{3}{5} \sin 5\theta + c$

(b) $\displaystyle\int \frac{\operatorname{cosec} \theta \cot \theta}{2} \, d\theta = \frac{1}{2} \int \operatorname{cosec} \theta \cot \theta \, d\theta = \frac{1}{2}(-\operatorname{cosec} \theta) + c = -\frac{1}{2} \operatorname{cosec} \theta + c$

(c) $\displaystyle\int (2 \cos 2x - 3 \sin 3x) \, dx = \int 2 \cos 2x \, dx - \int 3 \sin 3x \, dx = 2 \frac{\sin 2x}{2} + 3 \frac{\cos 3x}{3} + c$

$= \sin 2x + \cos 3x + c$

(d) $\int \cos^2 x \, dx$

$$= \int \frac{1 + \cos 2x}{2} \, dx = \int \frac{1}{2} dx + \int \frac{\cos 2x}{2} dx = \frac{1}{2} \int dx + \frac{1}{2} \int \cos 2x \, dx$$

$$= \frac{1}{2} x + \frac{1}{2} \cdot \frac{\sin 2x}{2} + c = \frac{1}{2} x + \frac{1}{4} \sin 2x + c$$

(e) $\displaystyle\int \frac{1}{\cos^2 x \sin^2 x} \, dx$

$$= \int \frac{\cos^2 x + \sin^2 x}{\cos^2 x \sin^2 x} \, dx = \int \left(\frac{\cos^2 x}{\cos^2 x \sin^2 x} + \frac{\sin^2 x}{\cos^2 x \sin^2 x} \right) dx = \int \left(\frac{1}{\sin^2 x} + \frac{1}{\cos^2 x} \right) dx$$

$$= \int \operatorname{cosec}^2 x \, dx + \int \sec^2 x \, dx = -\cot x + \tan x + c = \tan x - \cot x + c$$

(f) $\int \sqrt{1 + \cos x} \, dx$

$$= \int \sqrt{2 \cos^2 \frac{x}{2}} \, dx = \sqrt{2} \int \cos \frac{x}{2} \, dx = \sqrt{2} \cdot \frac{\sin \frac{x}{2}}{\frac{1}{2}} + c = 2\sqrt{2} \sin \frac{x}{2} + c$$

(g) $\int \sin 4x \cos 2x \, dx$

$$= \frac{1}{2} \int 2 \sin 4x \cos 2x \, dx = \frac{1}{2} \int \{(\sin (4x + 2x) + \sin (4x - 2x)\} \, dx$$

$$= \frac{1}{2} \int (\sin 6x + \sin 2x) \, dx = \frac{1}{2} \left\{ \left(-\frac{\cos 6x}{6} \right) + \left(-\frac{\cos 2x}{2} \right) \right\} + c$$

$$= -\frac{1}{12} \cos 6x - \frac{1}{4} \cos 2x + c$$

(h) $\int (e^{ax} + e^{-bx})dx = \int e^{ax}\,dx + \int e^{-bx}\,dx = \frac{e^{ax}}{a} + \frac{e^{-bx}}{-b} + c = \frac{e^{ax}}{a} - \frac{e^{-bx}}{b} + c$

(i) $\int \frac{e^{2x} + e^x + 3}{e^x}\,dx = \int (e^x + 1 + 3e^{-x})\,dx = e^x + x + \frac{3e^{-x}}{(-1)} + c = e^x + x - 3e^{-x} + c$

Objective Questions

1. $\int 2dx =$

(a) $x + c$ (b) $2x + c$

(c) $x^2 + c$ (d) $\frac{x^2}{2} + c$

Ans: a

Here, $\int 2\,dx = 2\int dx = 2x + c$

2. $\int \sqrt{x}\,dx =$

(a) $\sqrt{x} + c$ (b) $x^{\frac{3}{2}} + c$

(c) $\frac{2}{3}x^{\frac{3}{2}} + c$ (d) $\frac{3}{2}x^{\frac{2}{3}} + c$

Ans: c

$$\int x^{\frac{1}{2}}\,dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}x^{\frac{3}{2}} + c$$

3. $\int 3^x\,dx =$

(a) $3^x + c$ (b) $3^x \log 3 + c$

(c) $\frac{3^x}{\log 3} + c$ (d) $\frac{\log 3}{3^x} + c$

Ans: c

$$\int 3^x\,dx = \frac{3^x}{\log 3} + c$$

4. $\int \left(x^2 + \frac{1}{x} + e^x\right)dx =$

(a) $2x + \log x + e^x + c$ (b) $\frac{x^3}{3} + \log x + e^x + c$

(c) $x^3 + x + e^x + c$ (d) $3x^3 + \log x + e^x + c$

Ans: b

$$\int \left(x^2 + \frac{1}{x} + e^x\right)dx = \int x^2\,dx + \int \frac{1}{x}\,dx + \int e^x\,dx = \frac{x^3}{3} + \log x + e^x + c$$

5. $\int (3x + 2)^5\,dx =$

(a) $\frac{(3x + 2)^6}{6} + c$ (b) $5(3x + 2)^4 + c$

(c) $\frac{(3x + 2)^6}{18} + c$ (d) $\frac{(3x + 2)^5}{15} + c$

Ans: c

$$\int (3x + 2)^5\,dx = \frac{(3x + 2)^6}{3 \times 6} + c = \frac{(3x + 2)^6}{18} + c$$

6. $\int \csc x \cot x\,dx =$

(a) $-\csc x + c$ (b) $-\cot x + c$

(c) $\csc + c$ (d) $\cot x + c$

Ans: a

$\int \csc x \cot x\,dx = -\csc x + c$ (Formula)

7. $\int \sin ax \, dx =$

(a) $-\cos ax + c$ (b) $\cos ax + c$

(c) $\frac{\cos ax}{a} + c$ (d) $-\frac{\cos ax}{a} + c$

Ans: d

$$\int \sin ax \, dx = -\frac{\cos ax}{a} + c$$

8. $\int (1 + \tan^2\theta) \, d\theta =$

(a) $\cot\theta + c$ (b) $\tan\theta + c$

(c) $\operatorname{cosec}\theta + c$ (d) $\sec\theta + c$

Ans: b

$$\int (1 + \tan^2\theta) \, d\theta = \int \sec^2\theta \, d\theta = \tan\theta + c$$

9. $\int \sin^2 x \, dx =$

(a) $\frac{x}{2} - \frac{\sin 2x}{2} + c$ (b) $\frac{x}{2} - \frac{\sin 2x}{4} + c$

(c) $\frac{x}{2} - \frac{\cos 2x}{2} + c$ (d) $\frac{x}{2} - \frac{\cos 2x}{4} + c$

Ans: b

$$\int \sin^2 x \, dx = \int \frac{1 - \cos 2x}{2} \, dx = \frac{1}{2}\int (1 - \cos 2x) \, dx = \frac{1}{2} \left[\int dx - \int \cos 2x \, dx\right]$$

$$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + c = \frac{1}{2}x - \frac{\sin 2x}{4} + c$$

10. $\int e^{2x}(e^x + 1) \, dx =$

(a) $e^{3x} + e^{2x} + c$ (b) $3e^{3x} + 2e^{2x} + c$

(c) $\frac{e^{3x}}{3} + \frac{e^{2x}}{2} + c$ (d) $e^x + 3^{3x} + c$

Ans: c

$$\int e^{2x}(e^x + 1) \, dx = \int (e^{3x} + e^{2x}) \, dx = \int e^{3x} \, dx + \int e^{2x} \, dx = \frac{e^{3x}}{3} + \frac{e^{2x}}{2} + c$$

EXERCISE 4 (B)

1. **Evaluate the following integrals.**

(a) $\int_0^2 (x + 4) \, dx$ (b) $\int_0^1 (2x + 3) \, dx$

(c) $\int_1^3 (2x^2 + 5x) \, dx$ (d) $\int_1^4 (3x^2 + 2x + 5) \, dx$

(e) $\int_0^2 \left(x^3 + \frac{x^2}{2} + x\right) dx$ (f) $\int_{-2}^2 (x^3 - 2x + 3) \, dx$

Solution

(a) $\int_0^2 (x + 4) \, dx$

$$\text{Indefinite integral} = \int (x + 4) \, dx = \frac{x^2}{2} + 4x$$

$$\text{Definite integral} = \left[\frac{x^2}{2} + 4x\right]_0^2 = \left(\frac{4}{2} + 8\right) - \left(\frac{0}{2} + 0\right) = 2 + 8 = 10$$

(b) $\int_0^1 (2x + 3)\, dx$

Indefinite integral $= \int (2x + 3)\, dx = \frac{2x^2}{2} + 3x = x^2 + 3x$

Definite integral $= [x^2 + 3x]_0^1 = (1 + 3 \times 1) - (0 + 3 \times 0) = 4$

(c) $\int_1^3 (2x^2 + 5x)\, dx$

Indefinite integral $= \int (2x^2 + 5x)\, dx = \frac{2x^3}{3} + \frac{5x^2}{2} = \frac{2}{3}x^3 + \frac{5}{2}x^2$

Definite integral $= \left[\frac{2}{3}x^3 + \frac{5}{2}x^2\right]_1^3 = \left(\frac{2}{3} \times 27 + \frac{5}{2} \times 9\right) - \left(\frac{2}{3} \times 1 + \frac{5}{2} \times 1\right)$

$= \frac{36 + 45}{2} - \frac{4 + 15}{6} = \frac{81}{2} - \frac{19}{6} = \frac{293 - 19}{6} = \frac{224}{6} = \frac{112}{3}$

(d) $\int_1^4 (3x^2 + 2x + 5)\, dx$

Indefinite integral $= \int (3x^2 + 2x + 5)\, dx = 3\frac{x^3}{3} + \frac{2x^2}{2} + 5x = x^3 + x^2 + 5x$

Definite integral $= [x^3 + x^2 + 5x]_1^4 = (64 + 16 + 20) - (1 + 1 + 5) = 93$

(e) $\int_0^2 \left(x^3 + \frac{x^2}{2} + x\right) dx$

Indefinite integral $= \int \left(x^3 + \frac{x^2}{2} + x\right) dx = \frac{x^4}{4} + \frac{x^3}{2 \times 3} + \frac{x^2}{2} = \frac{1}{4}x^4 + \frac{1}{6}x^3 + \frac{1}{2}x^2$

Definite integral $= \left[\frac{1}{4}x^4 + \frac{1}{6}x^3 + \frac{1}{2}x^2\right]_0^2 = \left(\frac{1}{4} \times 16 + \frac{1}{6} \times 8 + \frac{1}{2} \times 4\right) - (0 + 0 + 0)$

$= 4 + \frac{4}{3} + 2 = \frac{22}{3}$

(f) $\int_{-2}^{2} (x^3 - 2x + 3)\, dx = \left[\frac{x^4}{4} - \frac{2x^2}{2} + 3x\right]_{-2}^{2} = \left[\frac{x^4}{4} - x^2 + 3x\right]_{-2}^{2}$

$= \left(\frac{2^4}{4} - 2^2 + 3 \times 2\right) - \left\{\frac{(-2)^4}{4} - (-2)^2 + 3(-2)\right\} = 6 + 6 = 12$

2. **Evaluate the following integrals.**

(a) $\int_0^\pi \sin x\, dx$ (b) $\int_0^\pi (1 + \cos x)\, dx$

(c) $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \csc\theta \cot\theta\, d\theta$ (d) $\int_{\frac{\pi}{2}}^{0} \frac{1 + \cos 2\theta}{2}\, d\theta$

(e) $\int_0^\pi \sin^2\left(1 + \frac{\theta}{2}\right) d\theta$

Solution

(a) $\int_0^{\pi} \sin x \, dx$

$= [-\cos x]_0^{\pi} = -\cos \pi - (-\cos(0)) = -(-1) - (-1) = 1 + 1 = 2$

(b) $\int_0^{\pi} (1 + \cos x) \, dx$

$= [x + \sin x]_0^{\pi} = (\pi + \sin \pi) - (0 + \sin 0) = \pi + 0 = \pi$

(c) $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \csc \theta \cot\theta \, d\theta$

$= [-\csc]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = -\csc \frac{3\pi}{4} - \left(-\csc \frac{\pi}{4}\right) = -\sqrt{2} - (-\sqrt{2}) = -\sqrt{2} + \sqrt{2} = 0$

(d) $\int_{\frac{\pi}{2}}^{0} \frac{1 + \cos 2\theta}{2} \, d\theta$

$= \frac{1}{2} \left[\theta + \frac{\sin 2\theta}{2}\right]_{\frac{\pi}{2}}^{0} = \frac{1}{2} \left\{\left(0 + \frac{\sin 0}{2}\right) - \left(\frac{\pi}{2} + \frac{\sin \pi}{2}\right)\right\} = \frac{1}{2}\left(0 + 0 - \frac{\pi}{2} - 0\right) = \frac{-\pi}{4}$

(e) $\int_0^{\pi} \sin^2\left(1 + \frac{\theta}{2}\right) d\theta$

$= \int_0^{\pi} \sin^2 \frac{1}{2}(\theta + 2) \, d\theta = \int_0^{\pi} \frac{1 - \cos(\theta + 2)}{2} \, d\theta = \frac{1}{2} \left[\theta - \frac{\sin(\theta + 2)}{1}\right]_0^{\pi}$

$= \frac{1}{2} [\theta - \sin(\theta + 2)]_0^{\pi} = \frac{1}{2} [\{\pi - \sin(\pi + 2)\} - \{0 - \sin(0 + 2)\}] = \frac{1}{2}(\pi + \sin 2 + \sin 2)$

$= \frac{1}{2}(\pi + 2\sin 2) = \frac{\pi}{2} + \sin 2$

3. Find $\frac{dy}{dx}$ if

(a) $y = \int_0^{\sqrt{x}} \cos t \, dt$ (b) $y = \int_0^{x} \sqrt{1 + t^2} \, dt$

Solution

(a) First method:

$\int_0^{\sqrt{x}} \cos t \, dt = [\sin t]_0^{\sqrt{x}} = \sin\sqrt{x} - \sin 0 = \sin\sqrt{x}$

Now, $\frac{dy}{dx} = \frac{d}{dx}\left(\int_0^{\sqrt{x}} \cos t \, dt\right) = \frac{d}{dx} \sin\sqrt{x} = \frac{d(\sin\sqrt{x})}{d(\sqrt{x})} \frac{d(\sqrt{x})}{dx} = \cos\sqrt{x} \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}} \cos\sqrt{x}$

Alternative Method

$$\frac{d}{dx}\left(\int_0^{\sqrt{x}} \cos t \, dt\right) = \cos \sqrt{x} \, \frac{d}{dx}(\sqrt{x}) = \cos\sqrt{x} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}} \cos\sqrt{x}$$

(b) $y = \int_0^x \sqrt{1 + t^2} \, dt$

$$\frac{dy}{dx} = \sqrt{1 + x^2}$$

Objective Questions

1. The set $P = \{0, 0.1, 0.2, 0.7, 0.9, 1\}$ is a partition of $[0, 1]$. Then $||P|| =$

(a) 0.1 (b) 0.3
(c) 0.4 (d) 0.5

Ans: d

$||P|| = 0.7 - 0.2 = 0.5$

2. The Riemann sums $\lim_{||P|| \to 0} \sum_{k=1}^{n} (c_k^2 + 2c_k - 1)\Delta x_k$ as an integral, if P denotes a partition of the interval $[1, 4]$, is

(a) $\int_1^4 (x^2 + 2x) \, dx$ (b) $\int_1^4 (x^2 - 1) \, dx$
(c) $\int_1^4 (x^2 + 2x - 1) \, dx$ (d) $\int_4^1 (x^2 + 2x - 1) \, dx$

Ans: c

3. If $f(x)$ has the constant value c on $[a, b]$ then $\int_a^b f(x) \, dx =$

(a) cb (b) ca
(c) 0 (d) $c(b - a)$

Ans: d

$$\int_a^b f(x) \, dx = \int_a^b c \, dx = c(b - a).$$

4. $\int_1^{\sqrt{5}} x \, dx =$

(a) 0 (b) 1
(c) 2 (d) 5

Ans: c

$$\int_1^{\sqrt{5}} x \, dx = \left[\frac{x^2}{2}\right]_0^{\sqrt{5}} = \frac{(\sqrt{5})^2}{2} - \frac{1^2}{2} = \frac{5 - 1}{2} = 2$$

5. $\int_0^{\pi} \sin x \, dx =$

(a) -1 (b) 0
(c) 1 (d) 2

Ans: d

$$\int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi} = -\cos \pi + \cos 0 = -(-1) + 1 = 2$$

6. $\int_{-\pi/4}^{0} \sec x \tan x \, dx =$

(a) $\sqrt{2}$ (b) 1

(c) $\sqrt{2} - 1$ (d) $1 - \sqrt{2}$

Ans: d

$$\int_{-\pi/4}^{0} \sec x \tan x \, dx = [\sec x]_{-\frac{\pi}{4}}^{0} = \sec 0 - \sec\left(\frac{-\pi}{4}\right) = \sec 0 + \sec\frac{\pi}{4} = 1 - \sqrt{2}$$

7. $\int_{0}^{2} (x^3 - x^2 - 2x) \, dx =$

(a) $-\frac{8}{3}$ (b) $-\frac{3}{8}$

(c) $\frac{8}{3}$ (d) $\frac{3}{8}$

Ans: d

$$\int_{0}^{2} (x^3 - x^2 - 2x) \, dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - \frac{2x^2}{2}\right]_0^2 = \left(\frac{2^4}{4} - \frac{2^3}{3} - 2^2\right) - 0 = -\frac{8}{3}$$

EXERCISE 4 (C)

1. **Evaluate the following integrals (substitution method).**

(a) $\int 2x(x^2 - 2)^3 \, dx$ (b) $\int 2x\sqrt{x^2 + 1} \, dx$

(c) $\int \frac{2x + 3}{\sqrt{x^2 + 3x}} \, dx$ (d) $\int (2x + 3) \, e^{x^2 + 3x + 5} \, dx$

(e) $\int \frac{(\log x)^5}{x} \, dx$ (f) $\int \frac{\sin(\log x)}{x} \, dx$

Solution

(a) Let $I = \int 2x(x^2 - 2)^3 \, dx$

Put, $y = x^2 - 2$. Then,

$$\frac{dy}{dx} = 2x$$

$\therefore$ $dy = 2x \, dx$

Now,

$$I = \int 2x\,(x^2 - 2)^3 \, dx = \int (x^2 - 2)^3 \, 2x \, dx = \int y^3 \, dy = \frac{y^4}{4} + c = \frac{(x^2 - 2)^4}{4} + c$$

(b) $\int 2x\sqrt{x^2 + 1} \, dx$

Let, $I = \int 2x\sqrt{x^2 + 1} \, dx$

Put, $y = x^2 + 1$

$dy = 2x \, dx$

$$\text{Then, } I = \int \sqrt{y} \, dy = \int y^{\frac{1}{2}} \, dy = \frac{y^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3} y^{\frac{3}{2}} + c = \frac{2}{3}(x^2 + 1)^{\frac{3}{2}} + c$$

(c) $\int \frac{2x+3}{\sqrt{x^2+3x}} dx$

Let $y = x^2 + 3x$ then

$\frac{dy}{dx} = 2x + 3$

$\therefore$ $dy = (2x + 3)\, dx$

Now,

$$\int \frac{2x+3}{\sqrt{x^2+3x}} dx = \int (x^2+3x)^{-1/2}(2x+3)\, dx = \int y^{-1/2} dy = \frac{y^{1/2}}{1/2} + c = 2\sqrt{x^2+3x} + c$$

(d) $\int (2x+3)\, e^{x^2+3x+5} dx$

Put, $y = x^2 + 3x + 5$

$dy = (2x + 3)\, dx$

Now,

$$I = \int e^y\, dy = e^y + c = e^{x^2+3x+5} + c$$

(e) Let $I = \int \frac{(\log x)^5}{x} dx$

Put, $y = \log x$

$dy = \frac{1}{x} dx$

Now,

$$I = \int y^5\, dy = \frac{y^6}{6} + c = \frac{(\log x)^6}{6} + c$$

(f) Let, $I = \int \frac{\sin(\log x)}{x} dx$

Put, $y = \log x$

$dy = \frac{1}{x} dx$

Then, $I = \int \sin y\, dy = -\cos y + c = -\cos(\log x) + c$

2. Evaluate the following integrals (Integration by parts)

(a) $\int x\, e^x\, dx$ **(b)** $\int x\, \ell n\, x\, dx$

(c) $\int x \cos x\, dx$ **(d)** $\int x^2 e^x\, dx$

(e) $\int e^x \cos x\, dx$ **(f)** $\int (x+1)\sqrt{x-1}\, dx$

Solution

(a) $\int x\, e^x\, dx$

$$= x\int e^x dx - \int\left[\frac{d(x)}{dx}\int e^x dx\right] dx = x\, e^x - \int e^x dx = x\, e^x - e^x + c = e^x (x-1) + c$$

(b) $\int x\, \ell n\, x\, dx$

$$= \int \ln x \;\; x\, dx = \ln x \int x\, dx - \int\left[\frac{d}{dx}(\ln x)\int x\, dx\right] dx = \ln x\, \frac{x^2}{2} - \int \frac{1}{x}\cdot\frac{x^2}{2} dx$$

$$= \ln x\, \frac{x^2}{2} - \frac{1}{2}\int x\, dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$$

(c) $\int x \cos x \, dx$

$$= x\int \cos x \, dx - \int \left\{\frac{d}{dx}(x)\int \cos x \, dx\right\} dx = x(\sin x) - \int 1 \cdot \sin x \, dx$$

$$= x \sin x + \cos x + c$$

(d) $\int x^2 e^x \, dx$

$$= x^2 \int e^x \, dx - \int \left[\frac{d}{dx}(x^2)\int e^x \, dx\right] dx = x^2 \cdot e^x - \int 2x \cdot e^x \, dx$$

$$= x^2 e^x - 2\left[x\int e^x \, dx - \int\left(\frac{d}{dx}(x)\int e^x \, dx\right) dx\right] = x^2 e^x - 2\,[xe^x - \int 1 \cdot e^x \, dx]$$

$$= x^2 e^x - 2[xe^x - e^x] + c = x^2e^x - 2xe^x + 2e^x + c$$

(e) $\int e^x \cos x \, dx$

Here, $I = \int e^x \cos x \, dx$

Integrating by parts, we get

$$I = \cos x \int e^x \, dx - \int\left[\frac{d}{dx}(\cos x)\int e^x \, dx\right] dx = e^x \cos x - \int (-\sin x)\, e^x \, dx$$

$$= e^x \cos x + \int \sin x \, e^x \, dx = e^x \cos x + \left\{\sin x \int e^x \, dx - \int\left[\frac{d}{dx}(\sin x)\int e^x \, dx\right] dx\right\}$$

$$= e^x \cos x + e^x \sin x - \int \cos x \, e^x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$$

or, $I = e^x \cos x + e^x \sin x - I \quad \left[\because I = \int e^x \cos x \, dx\right]$

or, $2I = e^x \cos x + e^x \sin x + c$

$\therefore \quad I = \dfrac{e^x}{2}(\cos x + \sin x) + c$

(f) $\int (x + 1)\sqrt{x - 1}\, dx$

$$= (x + 1)\int (x - 1)^{1/2} \, dx - \int\left[\frac{d(x + 1)}{dx}\int (x - 1)^{1/2} \, dx\right] dx$$

$$= (x + 1)\frac{(x - 1)^{3/2}}{3/2} - \int 1 \cdot \frac{(x - 1)^{3/2}}{3/2} \, dx = \frac{2}{3}(x + 1)(x - 1)^{3/2} - \frac{2}{3}\int (x - 1)^{3/2} \, dx$$

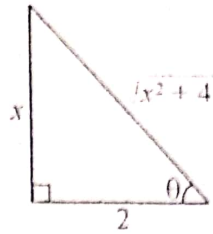
$$= \frac{2}{3}(x + 1)(x - 1)^{3/2} - \frac{2}{3}\frac{(x - 1)^{5/2}}{5/2} + c = \frac{2}{3}(x + 1)(x - 1)^{3/2} - \frac{4}{15}(x - 1)^{5/2} + c$$

3. Evaluate (trigonometric substitution).

(a) $\displaystyle\int \frac{dx}{\sqrt{x^2 + 4}}$ (b) $\displaystyle\int \frac{x^2 dx}{\sqrt{9 - x^2}}$ (c) $\displaystyle\int \frac{dx}{(x^2 - 1)^{\frac{3}{2}}}$

Solution

(a) Let. $I = \int \frac{dx}{\sqrt{x^2+4}}$



Put, $x = 2\tan\theta$

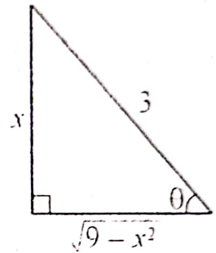
$dx = 2\sec^2\theta\, d\theta$

$$\text{Then, } I = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\tan^2\theta+4}} = \int \frac{2\sec^2\theta\, d\theta}{2\sec\theta} = \int \sec\theta\, d\theta = \log(\sec\theta + \tan\theta) + c$$

$$= \log\left(\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right) + c = \log(x + \sqrt{x^2+4}) - \log 2 + c_1$$

$$= \log(x + \sqrt{x^2+4}) + c \qquad \text{where } c = c_1 - \log 2$$

(b) Let. $I = \int \frac{x^2 dx}{\sqrt{9-x^2}}$



Put, $x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

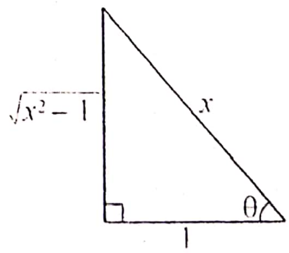
$$\text{Then, } I = \int \frac{(3\sin\theta)^2 \cdot 3\cos\theta\, d\theta}{\sqrt{9-9\sin^2\theta}} = \int \frac{9\sin^2\theta \cdot 3\cos\theta\, d\theta}{3\cos\theta} = \int 9\sin^2\theta\, d\theta$$

$$= 9\int \frac{1-\cos 2\theta\, d\theta}{2} = \int \frac{9}{2}(1-\cos 2\theta)\, d\theta = \frac{9}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] + c$$

$$= \frac{9}{2}[\theta - \sin\theta\cos\theta] + c = \frac{9}{2}\left[\sin^{-1}\frac{x}{3} - \frac{x}{3}\sqrt{1-\left(\frac{x}{3}\right)^2}\right] + c$$

$$= \frac{9}{2}\left[\sin^{-1}\frac{x}{3} - \frac{x}{9}\sqrt{9-x^2}\right] + c = \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + c$$

(c) Let. $I = \int \frac{dx}{(x^2-1)^{\frac{3}{2}}}$



Put, $x = \sec\theta$

$dx = \sec\theta\tan\theta\, d\theta$

$$\text{Then, } I = \int \frac{\sec\theta\tan\theta\, d\theta}{(\sec^2\theta-1)^{\frac{3}{2}}} = \int \frac{\sec\theta\tan\theta\, d\theta}{(\tan^2\theta)^{\frac{3}{2}}} = \int \frac{\sec\theta\tan\theta}{\tan^3\theta}\, d\theta = \int \frac{\sec\theta}{\tan^2\theta}\, d\theta$$

$$= \int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\, d\theta = \int \cot\theta\,\mathrm{cosec}\,\theta\, d\theta = -\,\mathrm{cosec}\,\theta + c = -\frac{x}{\sqrt{x^2-1}} + c$$

4. **Evaluate (partial fractions).**

(a) $\int \frac{5x-3}{(x+1)(x-3)}\, dx$ (b) $\int \frac{x}{(x-1)(2x+1)}\, dx$

Solution

(a) Let, $\frac{5x-3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$ ... (i)

$= \frac{A(x-3)+B(x+1)}{(x+1)(x-3)}$

or, $5x - 3 = A(x-3) + B(x+1)$ ... (ii)

When $x = 3$, from (ii),

$15 - 3 = B(3+1)$

$B = \frac{12}{4} = 3$

When $x = -1$, from (ii),

$-5 - 3 = A(-1-3)$

$A = 2$

Then from (i), $\frac{5x-3}{(x+1)(x-3)} = \frac{2}{x+1} + \frac{3}{x-3}$

Now, $\int \frac{5x-3}{(x+1)(x-3)}dx = \int \left(\frac{2}{x+1} + \frac{3}{x-3}\right)dx = \int \frac{2}{x+1}dx + \int \frac{3}{x-3}dx$

$= 2\log(x+1) + 3\log(x-3) + c$

(b) Let, $\frac{x}{(x-1)(2x+1)} = \frac{A}{x-1} + \frac{B}{2x+1}$ ... (i)

$= \frac{A(2x+1)+B(x-1)}{(x-1)(2x+1)}$

or, $x = A(2x+1) + B(x-1)$ ... (ii)

When $x = 1$, from (ii),

$1 = A(2\cdot 1 + 1)$

$A = \frac{1}{3}$

Again, when $x = 0$, from (ii)

$0 = A - B$

or, $B = A = \frac{1}{3}$

Now, from (i), $\frac{x}{(x-1)(2x+1)} = \frac{\frac{1}{3}}{x-1} + \frac{\frac{1}{3}}{2x+1} = \frac{1}{3(x-1)} + \frac{1}{3(2x+1)}$

$\therefore \int \frac{x}{(x-1)(2x+1)}dx = \frac{1}{3}\int \frac{1}{x-1}dx + \frac{1}{3}\int \frac{1}{2x+1}dx$

$= \frac{1}{3}\log(x-1) + \frac{1}{3}\cdot\frac{1}{2}\log(2x+1) + c$

$= \frac{1}{3}\log(x-1) + \frac{1}{6}\log(2x+1) + c$

5. Evaluate.

(a) $\int \tan x\, dx$ **(b)** $\int \operatorname{cosec} x\, dx$

(c) $\int \frac{dx}{x^2+10x+26}$ **(d)** $\int \frac{dx}{\sqrt{x^2-9}}$ **(e)** $\int \frac{dx}{\sqrt{x^2+4x+5}}$

Solution

(a) Let $I = \int \tan x\, dx$

Let, $I = \int \frac{\sin x}{\cos x}dx$

Put, $y = \cos x$

$dy = -\sin x\, dx$

Then, $I = \int \frac{-dy}{y} = -\log y + c = -\log(\cos x) + c = \log(\cos x)^{-1} + c = \log\left(\frac{1}{\cos x}\right) + c$

$= \log \sec x + c$

(b) We have, $I = \int \operatorname{cosec} x\, dx$

$$= \int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)}{(\operatorname{cosec} x - \cot x)} dx = \int \frac{\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x}{(\operatorname{cosec} x - \cot x)} dx$$

$$= \int \frac{\frac{d}{dx}(\operatorname{cosec} x - \cot x)}{(\operatorname{cosec} x - \cot x)} dx = \log(\operatorname{cosec} x - \cot x) + c$$

$$\left[\because \int \frac{f'(x)}{f(x)} dx = \log f(x) + c\right]$$

(c) $\int \frac{dx}{x^2 + 10x + 26}$

$$= \int \frac{dx}{x^2 + 2\cdot 5\cdot x + 5^2 + 1} = \int \frac{dx}{(x+5)^2 + 1^2} = \frac{1}{1}\tan^{-1}\frac{x+5}{1} + c = \tan^{-1}(x+5) + c$$

(d) $\int \frac{dx}{\sqrt{x^2 - 9}} = \int \frac{dx}{\sqrt{x^2 - 3^2}} = \log(x + \sqrt{x^2 - 3^2}) + c = \log(x + \sqrt{x^2 - 9}) + c$

(e) $\int \frac{dx}{\sqrt{x^2 + 4x + 5}}$

$$= \int \frac{dx}{\sqrt{x^2 + 2\cdot x\cdot 2 + 4 + 1}} = \int \frac{dx}{\sqrt{(x+2)^2 + 1^2}} = \log(x + 2 + \sqrt{(x+2)^2 + 1^2}) + c$$

$= \log(x + 2 + \sqrt{x^2 + 4x + 5}) + c$

Objective Questions

1. $\int(2x + 3)(x^2 + 3x)^3\, dx =$

(a) $\frac{(x^2+3x)^3}{3} + c$ (b) $\frac{(x^2+3x)^4}{4} + c$

(c) $(x^2 + 3x)^3 + c$ (d) $(x^2 + 3x)^4 + c$

Ans: b

Let, $y = x^2 + 3x$

$dy = (2x + 3)\, dx$

Then, $I = \int y^3\, dy = \frac{y^4}{4} + c = \frac{(x^2+3x)^4}{4} + c$

2. $\int(2ax + b)(ax^2 + bx + c)\, dx =$

(a) $\frac{1}{2}(ax^2 + bx + c) + k$ (b) $\frac{1}{2}(ax^2 + bx + c)^2 + k$

(c) $\frac{1}{2}(ax^2 + bx + c)^3 + k$ (d) $\frac{1}{2}\sqrt{ax^2 + bx + c} + k$

Ans: b

Let $y = ax^2 + bx + c$, then

$$\frac{dy}{dx} = 2ax + b$$

$\therefore$ $dy = (2ax + b)\,dx$

Now,

$$\int (2ax + b)(ax^2 + bx + c)\,dx$$

$$= \int y\,dy = \frac{y^2}{2} + k = \frac{(ax^2 + bx + c)^2}{2} + k$$

3. $\int x^2 e^{x^3}\,dx =$

(a) $\frac{1}{3} e^{x^3} + c$ (b) $\frac{1}{2} e^{x^2} + c$

(c) $e^{x^3} + c$ (d) $e^{x^2} + c$

Ans: a

Put, $y = x^3$

$dy = 3x^2\,dx$

Now,

$$I = \int e^y \frac{dy}{3} = \frac{1}{3} e^y + c = \frac{1}{3} e^{x^3} + c$$

4. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx =$

(a) $2 \sin\sqrt{x} + c$ (b) $-2 \sin\sqrt{x} + c$

(c) $2 \cos\sqrt{x} + c$ (d) $-2 \cos\sqrt{x} + c$

Ans: d

Put $y = \sqrt{x}$

$$dy = \frac{1}{2\sqrt{x}}\,dx$$

$$2dy = \frac{dx}{\sqrt{x}}$$

Then, $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int \sin y \cdot 2\,dy = -2\cos y + c = -2\cos\sqrt{x} + c$

5. $\int u \cdot v\,dx =$

(a) $\int u\,dx \cdot \int v\,dx$ (b) $\int u\,dx + \int v\,dx$

(c) $\int u\,dx - \int v\,dx$ (d) $u \int v\,dx - \int \left[\frac{d}{dx}(u) \int v\,dx\right] dx$

Ans: d (Formula)

6. $\int \frac{dx}{\sqrt{1 - x^2}} =$

(a) $\sin^{-1} x + c$ (b) $\cos^{-1} x + c$

(c) $\tan^{-1} x + c$ (d) $\cot^{-1} x + c$

Ans: a

Let, $I = \int \frac{dx}{\sqrt{1 - x^2}}$

Put $x = \sin\theta$

$dx = \cos\theta\,d\theta$

Then, $I = \int \frac{\cos\theta\,d\theta}{\sqrt{1 - \sin^2\theta}} = \int d\theta = \theta + c = \sin^{-1}x + c.$

EXERCISE 4 (D)

Evaluate the following definite integrals.

1. $\displaystyle\int_0^1 \frac{2x\,dx}{x^2+1}$ 2. $\displaystyle\int_0^1 \frac{16x\,dx}{8x^2+2}$ 3. $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\,dx$

4. $\displaystyle\int_0^{\frac{\pi}{2}} x\sin x\,dx$ 5. $\displaystyle\int_2^1 \frac{\sin(\log t)}{t}\,dt$ 6. $\displaystyle\int_2^1 \frac{8dx}{x^2-2x+2}$

7. $\displaystyle\int_0^4 \sqrt{16-x^2}\,dx$ 8. $\displaystyle\int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}}$

Solution

1. Let, $\displaystyle I = \int_0^1 \frac{2x\,dx}{x^2+1}$

Put, $y = x^2 + 1$

$dy = 2x\,dx$

When $x = 0$, $y = 0^2 + 1 = 1$

When $x = 1$, $y = 1^2 + 1 = 2$

Now, $\displaystyle I = \int_1^2 \frac{dy}{y} = [\log y]_1^2 = \log 2 - \log 1 = \log 2.$

2. Let, $\displaystyle I = \int_0^1 \frac{16x\,dx}{8x^2+2}$

Put, $y = 8x^2 + 2$

$dy = 16x\,dx$

When $x = 0$, $y = 2$

When $x = 1$, $y = 8 \times 1^2 + 2 = 10$

Now, $\displaystyle I = \int_2^{10} \frac{dy}{y} = [\log y]_2^{10} = \log 10 - \log 2 = \log\left(\frac{10}{2}\right) = \log 5.$

3. $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1+\cos 4x}\,dx$

$$= \int_0^{\frac{\pi}{4}} \sqrt{2\cos^2 2x}\,dx = \sqrt{2}\int_0^{\frac{\pi}{4}} \cos 2x\,dx = \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{\sqrt{2}}[\sin 2x]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{\sqrt{2}}\left[\sin\left(\frac{2\pi}{4}\right) - \sin 0\right] = \frac{1}{\sqrt{2}}(1-0) = \frac{1}{\sqrt{2}}$$

4. $\int_0^{\frac{\pi}{2}} x \sin x \, dx$

Let, $I = \int x \sin x \, dx$

$= x \int \sin x \, dx - \int \left\{ \frac{d}{dx}(x) \int \sin x \, dx \right\} dx = x \cdot (-\cos x) - \int 1 \, (-\cos x) \, dx$

$= -x \cos x + \int \cos x \, dx = -x \cos x + \sin x$

Now, $\int_0^{\frac{\pi}{2}} x \sin x \, dx = [-x \cos x + \sin x]_0^{\frac{\pi}{2}} = \left(\frac{-\pi}{2} \cos \frac{\pi}{2} + \sin \frac{\pi}{2} \right) - (-0 \cos 0 + \sin 0) = 1$

5. Let, $I = \int_2^1 \frac{\sin (\log t)}{t} \, dt$

Put, $y = \log t$

$dy = \frac{1}{t} \, dt$

When $t = 1$, $y = \log 1 = 0$

When $t = 2$, $y = \log 2$

Now, $I = \int_{\log 2}^{0} \sin (y) \, dy = [-\cos y]_{\log 2}^{0} = -\cos 0 - (-\cos \log 2) = \cos \log 2 - 1$

6. Let, $I = \int_2^1 \frac{8dx}{x^2 - 2x + 2}$

$= \int_2^1 \frac{8dx}{(x^2 - 2x + 1) + 1} = \int_2^1 \frac{8dx}{(x-1)^2 + 1^2} = 8 \left[\frac{1}{1} \tan^{-1} \frac{(x-1)}{1} \right]_2^1$

$= 8 \, [\tan^{-1} (x - 1)]_2^1 = 8 \, [\tan^{-1}(1 - 1) - \tan^{-1} (2 - 1)] = 8[\tan^{-1} 0 - \tan^{-1}(1)]$

$= 8 \left[0 - \frac{\pi}{4} \right] = -2\pi$

7. Let, $I = \int_0^4 \sqrt{16 - x^2} \, dx$

Put $x = 4 \sin \theta$, then $dx = 4 \cos \theta \, d\theta$

When $x = 0$, then $0 = 4 \sin \theta$

or, $\sin \theta = 0 = \sin 0 \Rightarrow \theta = 0$

When $x = 4$, then $4 = 4 \sin \theta \Rightarrow \sin \theta = 1$

or, $\sin \theta = \sin \pi/2 \Rightarrow \theta = \pi/2$

Now, $I = \int_0^{\pi/2} \sqrt{16 - 16 \sin^2 \theta} \times 4 \cos \theta d\theta$

$= \int_0^{\pi/2} 4 \sqrt{16 \cos^2 \theta} \times \cos \theta \, d\theta = 16 \int_0^{\pi/2} \cos \theta \times \cos \theta \, d\theta$

$$= \frac{16}{2}\int_0^{\pi/2} 2\cos^2\theta\, d\theta = 8\int_0^{\pi/2}(1+\cos 2\theta)\, d\theta = 8\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= 8\left[\left(\frac{\pi}{2}+\frac{\sin\pi}{2}\right)-(0+0)\right] = 8\left(\frac{\pi}{2}+\frac{0}{2}\right) = 4\pi$$

8 Let, $I = \int_0^{\frac{3}{2}} \frac{dx}{\sqrt{9-x^2}}$

Put $x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

When $x = 0, \theta = 0$

When $x = \frac{3}{2}$, $3\sin\theta = \frac{3}{2}$

$\therefore\quad \theta = \frac{\pi}{6}$

$$\text{Now, } I = \int_0^{\frac{\pi}{6}} \frac{3\cos\theta\, d\theta}{\sqrt{9-9\sin^2\theta}} = \int_0^{\frac{\pi}{6}} \frac{3\cos\theta\, d\theta}{3\cos\theta} = \int_0^{\frac{\pi}{6}} d\theta = [\theta]_0^{\frac{\pi}{6}} = \frac{\pi}{6} - 0 = \frac{\pi}{6}$$

Objective Questions

1. $\int_0^1 x\, e^x\, dx =$

(a) 0 (b) 1
(c) 2 (d) 3

Ans: b

Indefinite integral $= \int xe^x\, dx$

$$= x\int e^x\, dx - \int\left(\frac{dx}{dx}\int e^x\, dx\right)dx = xe^x - \int 1\cdot e^x dx = xe^x - e^x$$

Definite integral $= [xe^x - e^x]_0^1 = (1\cdot e^1 - e^1) - (0\cdot e^0 - e^0) = e - e - 0 + 1 = 1$

2. $\int_1^e \ell n\, x\, dx =$

(a) 1 (b) 2
(c) 3 (d) e

Ans: a

Indifinte integral $= \int \ln x\, dx$

$$= \ln x\int 1dx - \int\left[\frac{d}{dx}(\ln x)\int 1dx\right]dx = x\ln x - \int\frac{1}{x}\cdot x\cdot dx$$

$$= x\ln x - \int dx = x\ln x - x$$

Definite integral $= [x\ln x - x]_1^e = (e\ln e - e) - (1\cdot \ln 1 - 1) = (e-e) - 0 + 1 = 1$

3. $\int_0^1 x\sqrt{1-x^2}\,dx =$

(a) 2 (b) $\frac{1}{3}$

(c) 3 (d) $\frac{1}{2}$

Ans: b

Indefinite integral $= \int x\sqrt{1-x^2}\,dx$

Let $y = 1 - x^2$

$\frac{dy}{dx} = -2x \Rightarrow -\frac{1}{2}\,dy = x\,dx$

Given, integral reduces to

$$\int -\frac{1}{2}\sqrt{y}\,dy = -\frac{1}{2}\int y^{1/2}\,dy = -\frac{1}{2}\frac{y^{3/2}}{3/2} = -\frac{1}{3}(1-x^2)^{3/2}$$

$$\text{Definite integral} = \left[-\frac{1}{3}(1-x^2)^{3/2}\right]_0^1 = -\frac{1}{3}(1-1)^{3/2} + \frac{1}{3}(1-0)^{3/2} = \frac{1}{3}$$

4. $\int_0^1 \frac{dx}{\sqrt{x+1}-\sqrt{x}} =$

(a) 4 (b) $\frac{\sqrt{3}}{2}$

(c) $\frac{2}{\sqrt{3}}$ (d) $\frac{4\sqrt{2}}{3}$

Ans: d

$$\text{Indefinite integral} = \int \frac{dx}{\sqrt{x+1}-\sqrt{x}} \times \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}} = \int \frac{\sqrt{x+1}+\sqrt{x}}{x+1-x}$$

$$= \int \{(x+1)^{1/2} + x^{1/2}\} = \frac{(x+1)^{3/2}}{3/2} + \frac{x^{3/2}}{3/2} = \frac{2}{3}\,[(x+1)^{3/2} + x^{3/2}]$$

$$\text{Definite integral} = \frac{2}{3}\,[(x+1)^{3/2} + x^{3/2}]_0^1 = \frac{2}{3}\,[2^{3/2} + 1^{3/2} - 1^{3/2} - 0^{3/2}] = \frac{2}{3}\,[2^{3/2}] = \frac{4\sqrt{2}}{3}$$

5. $\int_0^3 \sqrt{9-x^2}\,dx =$

(a) 3 (b) $\frac{4}{9}$

(c) $\frac{9}{4}$ (d) 9

Ans: c

EXERCISE 4 (E)

1. **Evaluate the following improper integrals**

(a) $\int_0^\infty \frac{dx}{x^2+1}$ (b) $\int_0^1 \frac{dx}{\sqrt{x}}$ (c) $\int_{-1}^1 \frac{dx}{x^{\frac{2}{3}}}$

(d) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ (e) $\int_{0}^{1} \frac{dx}{\sqrt{1-x^2}}$

Solution

(a) $\int_0^{\infty} \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_0^b \frac{dx}{x^2+1} = \lim_{b\to\infty} [\tan^{-1} x]_0^b = \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

(b) $\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{b\to 0^+} \int_b^1 x^{\frac{-1}{2}} dx = \lim_{b\to 0^+} \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_b^1 = \lim_{b\to 0^+} \left[2x^{\frac{1}{2}}\right]_b^1 = \lim_{b\to 0^+} (2 - 2\sqrt{b})$

$= 2 - 0 = 2$

(c) $\int_{-1}^{1} \frac{dx}{x^{\frac{2}{3}}} = \int_{-1}^{1} x^{\frac{-2}{3}} dx = \int_{-1}^{0} x^{\frac{-2}{3}} dx + \int_0^1 x^{\frac{-2}{3}} dx = \lim_{b\to 0^-} \int_b^0 x^{\frac{-2}{3}} dx + \lim_{c\to 0^+} \int_c^1 x^{\frac{-2}{3}} dx$

$= \lim_{b\to 0^-} \left[3x^{\frac{1}{3}}\right]_b^0 + \lim_{c\to 0^+} \left[3x^{\frac{1}{3}}\right]_c^1 = \lim_{b\to 0^-} \left[3b^{\frac{1}{3}} - 3(-1)^{\frac{1}{3}}\right] + \lim_{c\to 0^+} \left[3(1)^{\frac{1}{3}} - 3c^{\frac{1}{3}}\right]$

$= (0 + 3) + (3 - 0) = 6$

(d) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_0^{\infty} \frac{dx}{1+x^2}$

$= \lim_{b\to -\infty} \int_b^0 \frac{dx}{1+x^2} + \lim_{c\to\infty} \int_0^c \frac{dx}{1+x^2} = \lim_{b\to -\infty} [\tan^{-1} x]_b^0 + \lim_{c\to\infty} [\tan^{-1} x]_0^c$

$= \lim_{b\to -\infty} [\tan^{-1} 0 - \tan^{-1} b] + \lim_{c\to\infty} [\tan^{-1} c - \tan^{-1} 0] = 0 - \left(\frac{-\pi}{2}\right) + \frac{\pi}{2} - 0 = \pi$

(e) $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to 1^-} \int_0^b \frac{dx}{\sqrt{1-x^2}}$

$= \lim_{b\to 1^-} [\sin^{-1} x]_0^b = \lim_{b\to 1^-} [\sin^{-1} b - \sin^{-1} 0] = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

2. **Test for convergence and divergence for the following integrals.**

(a) $\int_1^{\infty} \frac{dx}{x}$ (b) $\int_0^{\frac{\pi}{2}} \tan\theta \, d\theta$

Solution

(a) $\int_1^{\infty} \frac{dx}{x} = \lim_{b\to\infty} \int_1^b \frac{dx}{x} = \lim_{b\to\infty} [\ln x]_1^b = \lim_{b\to\infty} [\ln b - \ln 1] = \infty$

So, the integral diverges.

(b) $\int_0^{\frac{\pi}{2}} \tan\theta \, d\theta = \lim_{b\to \pi/2^-} \int_0^b \tan\theta \, d\theta = \lim_{b\to \pi/2^-} [-\ln|\sin\theta|]_0^b$

$= \lim_{b\to \pi/2^-} [-\ln|\cos b|] + \ln 1 = \infty$

So, the integral diverges.

Objective Questions

1. $\int_1^\infty \frac{dx}{x^2} =$

(a) 0 (b) 1

(c) 2 (d) $\sqrt{2}$

Ans: b

$$\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^2} = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty}\left[-\frac{1}{b}+\frac{1}{1}\right] = \lim_{b\to\infty}\left[1-\frac{1}{b}\right] = 1-0 = 1$$

2. $\int_0^\infty \frac{dx}{1+x^2} =$

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$ (d) π

Ans: c

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2} = \lim_{b\to\infty}[\tan^{-1}x]_0^b = \lim_{b\to\infty}[\tan^{-1}b - \tan^{-1}0]$$

$$= \lim_{b\to\infty}\tan^{-1}b = \frac{\pi}{2}$$

3. $\int_2^5 \frac{1}{\sqrt{x-2}}\,dx =$

(a) 2 (b) $2\sqrt{3}$

(c) 3 (d) $3\sqrt{3}$

Ans: b

$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t\to 2^+}\int_t^5 \frac{dx}{\sqrt{x-2}}$$ [Since, the integral becomes improper at $x = 2$]

$$= \lim_{t\to 2^+}[2\sqrt{x-2}]_t^5 = \lim_{t\to 2^+}2[\sqrt{3}-\sqrt{t-2}] = 2\sqrt{3}$$

4. The integral $\int_1^\infty \frac{dx}{x^3} =$

(a) 1 (b) $\frac{1}{3}$

(c) $\frac{1}{4}$ (d) $\frac{1}{2}$

Ans: d

$$\int_1^\infty \frac{dx}{x^3} = \lim_{b\to\infty}\int_1^b \frac{1}{x^3}\,dx = \lim_{b\to\infty}\left[-\frac{1}{2x^2}\right]_1^b = \lim_{b\to\infty}\left[-\frac{1}{2b^2}+\frac{1}{2}\right] = \frac{1}{2}$$

5. The integral $\int_1^\infty \frac{dx}{x^p}$ converges for

(a) $p > 1$ (b) $p < 1$

(c) $p = 1$ (d) $p = 0$

Ans: a

EXERCISE 4 (F)

1. **Find the area enclosed by the curve $y = 3x$, the x-axis and ordinates at $x = 0$ and $x = 4$.**

Solution

$$\text{Area} = \int_0^4 y\,dx = \int_0^4 3x\,dx = \left[\frac{3x^2}{2}\right]_0^4 = \frac{3\times 4^2}{2} - 0 = 24$$

2. **Find the area bounded by the curve $y = \sin x$, $x = 0$, $x = \pi$.**

Solution

$$\text{Area} = \int_0^\pi \sin x\,dx = [-\cos x]_0^\pi = -\cos\pi - (-\cos 0) = -(-1) - (-1) = 1 + 1 = 2$$

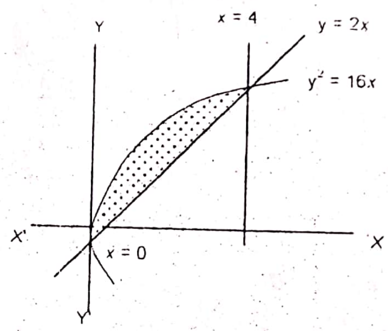
3. **Find the area of the region between the curves**
   (a) $y^2 = 16x$ and the line $y = 2x$ (b) $y = x^2 - 2$ and $y = 2$.

Solution

(a) The given equations of the curve and the line are respectively.

$y^2 = 16x$ ... (i)

$y = 2x$ ... (ii)



Firstly, we find the point of intersections of the curve (i) and line (ii). For this using equation (ii) in equation (i), we get

$(2x)^2 = 16x$

or, $4x^2 - 16x = 0$

or, $4x(x-4) = 0$

or, $x = 0;\ x = 4$

When $x = 0$; $y = 0$

When $x = 4$; $y = 8$

So, the curve (i) and line (ii) intersect at the points (0, 0) and (4, 8)

$\therefore$ Required area $= \int_0^4 (y_2 - y_1)\,dx$, where $y_2 = \sqrt{16x} = 4\sqrt{x}$ and $y_1 = 2x$

$$= \int_0^4 (4\sqrt{x} - 2x)\,dx = \left[\frac{4x^{3/2}}{3/2} - \frac{2x^2}{2}\right]_0^4 = \frac{8}{3}\left[x^{3/2}\right]_0^4 - \left[x^2\right]_0^4$$

$$= \frac{8}{3}\left[4^{3/2} - 0\right] - (4^2 - 0) = \frac{8}{3}(2^3) - 16 = \frac{8}{3}\times 8 - 16 = \frac{64-48}{3} = \frac{16}{3}\ \text{sq. units}$$

(b) Solving $y = x^2 - 2$ and $y = 2$, we get,

$x^2 - 2 = 2$

$x^2 = 4$

$x = \pm 2$

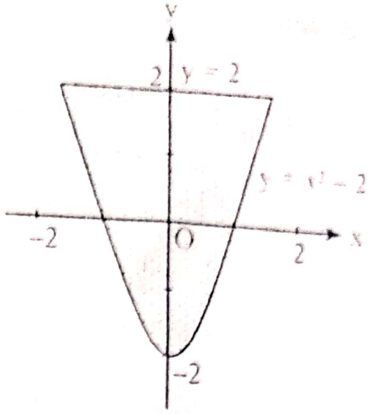
$$\text{Required area} = \int_{-2}^{2} (y_1 - y_2)\,dx = \int_{-2}^{2} [2 - (x^2 - 2)]\,dx$$

$$= \int_{-2}^{2} (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_{-2}^{2}$$

$$= \left(4 \times 2 - \frac{2^3}{3}\right) - \left\{4 \times (-2) - \frac{(-2)^3}{3}\right\}$$

$$= \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)$$

$$= 16 - \frac{8}{3} - \frac{8}{3} = \frac{48 - 16}{3} = \frac{32}{3}$$



4. Find the area bounded by the axis of coordinates, the curve $x^2 = 4a(y - 2a)$ and the ordinate of the point (h, k).

Solution

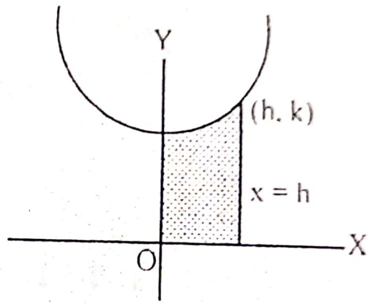
Given curve is $x^2 = 4a(y - 2a)$ ... (i)

The equation (i) can be written as

$$y = \frac{x^2}{4a} + 2a$$

Thus, we have to find the area bounded by the curve

$y = \dfrac{x^2}{4a} + 2a$, the x-axis and the ordinates $x = 0$ and $x = h$.



$$\text{Required area} = \int_0^h y\,dx = \int_0^h \left(\frac{x^2}{4a} + 2a\right) dx = \left[\frac{x^3}{4a \times 3} + 2ax\right]_0^h = \left(\frac{h^3}{12a} + 2ah\right) - 0$$

$$= \frac{h^3 + 24a^2h}{12a} = \frac{h}{12a}(h^2 + 24a^2)$$

5. Find the area bounded by the parabola $y^2 = 4x$ and the y–axis between the points $y = 0$ to $y = 2$.

Solution

$$\text{Area} = \int_0^2 x\,dy = \int_0^2 \frac{y^2}{4}\,dy = \left[\frac{y^3}{12}\right]_0^2$$

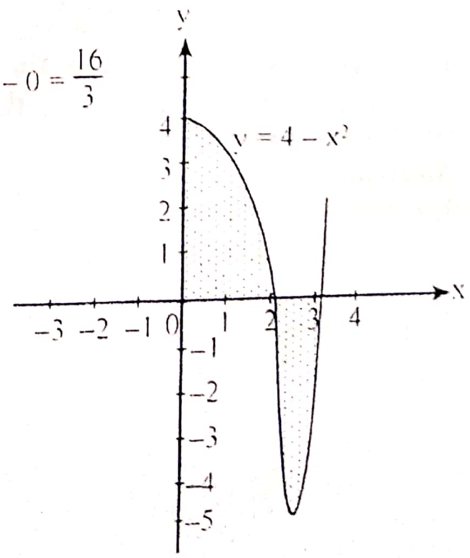
$$= \frac{2^3}{12} - \frac{0^3}{12} = \frac{2}{3}$$

6. Find the area of the region between the curve $y = 4 - x^2$ and $0 \le x \le 3$ and the x – axis.

Solution

Integral over [0, 2]

$$A_1 = \int_0^2 (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_0^2 = \left[4 \times 2 - \frac{2^3}{3}\right] - 0 = \frac{16}{3}$$



Integral over [2, 3]

$$A_2 = \int_2^3 (4 - x^2)\,dx = \left[4x - \frac{x^3}{3}\right]_2^3$$

$$= \left(4 \times 3 - \frac{3^3}{3}\right) - \left(4 \times 2 - \frac{2^3}{3}\right)$$

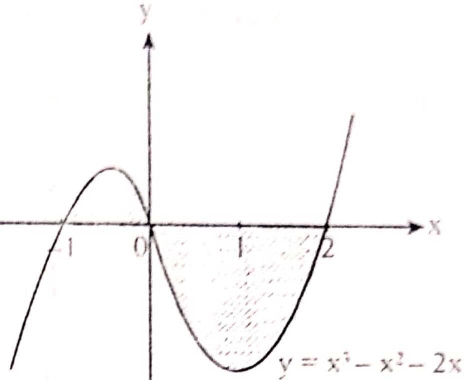
$$= (12 - 9) - \left(8 - \frac{8}{3}\right) = 3 - \frac{16}{3} = -\frac{7}{3}$$

$\therefore \quad A_2 = \left|\frac{-7}{3}\right| = \frac{7}{3}$

$\therefore \quad$ Total area $= A_1 + A_2 = \frac{16}{3} + \frac{7}{3} = \frac{23}{3}$

7. **Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x,\ -1 \le x \le 2$.**

Solution



Here, $f(x) = x^3 - x^2 - 2x$

Set $f(x) = 0$, we get

$x = 0, -1, 2$

The zeros partition $[-1, 2]$ into two subintervals $[-1, 0]$ on which $f \ge 0$ and $[0, 2]$ on which $f \le 0$

We integrate f over these two intervals and add the absolute values of areas of both portions.

Integral over [–1, 0];

$$\int_{-1}^{0} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - \frac{2x^2}{2}\right]_{-1}^{0}$$

$$= 0 - \left(\frac{1}{4} + \frac{1}{3} - 1\right) = \frac{5}{12}$$

Integral over [0, 2];

$$\int_{0}^{2} (x^3 - x^2 - 2x)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - \frac{2x^2}{2}\right]_{0}^{2} = \left(\frac{2^4}{4} - \frac{2^3}{3} - 2^2\right) - 0 = \frac{-8}{3}$$

$$\text{Total area} = \frac{5}{12} + \left|\frac{-8}{3}\right| = \frac{5}{12} + \frac{8}{3} = \frac{37}{12}$$

8. **Using integration, find the area of**

(a) $x^2 + y^2 = 36$. **(b)** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution

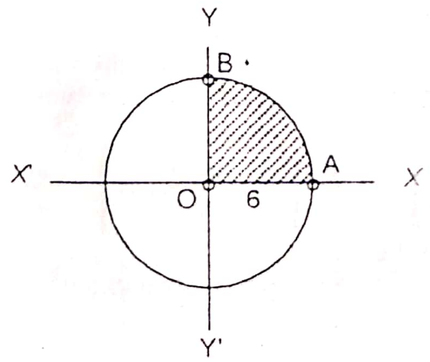
(a) Equation of given circle is $x^2 + y^2 = 36$

$\therefore \quad y = \sqrt{36 - x^2}$ ... (i)

The centre of circle (i) is at (0, 0) and radius OA = 6 units

since, circle (i) is symmetrical about both the axes, *x-axis* and y-axis divide it into four equal parts

OAB is the portion of the circle lying in the first quadrant, which is bounded by the axes and the curve (i).



$$\text{Now, Area of OAB} = \int_0^6 y\,dx = \int_0^6 \sqrt{36 - x^2}\,dx$$

Put $x = 6\sin\theta$, then $dx = 6\cos\theta\, d\theta$

When $x = 0$, then $0 = 6 \sin\theta$

or, $\sin\theta = 0 = \sin 0 \Rightarrow \theta = 0$

When $x = 6$, then $6 = 6\sin\theta \Rightarrow \sin\theta = 1$

or, $\sin\theta = \sin \pi/2 \Rightarrow \theta = \pi/2$

Now, Area of OAB $= \int_0^{\pi/2} \sqrt{36 - 36\sin^2\theta} \times 6\cos\theta\, d\theta = \int_0^{\pi/2} 6\sqrt{36\cos^2\theta} \times \cos\theta\, d\theta$

$$= 36\int_0^{\pi/2} \cos\theta \times \cos\theta\, d\theta = \frac{36}{2}\int_0^{\pi/2} 2\cos^2\theta\, d\theta = 18\int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta$$

$$= 18\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = 18\left[\left(\frac{\pi}{2} + \frac{\sin\pi}{2}\right) - (0 + 0)\right]$$

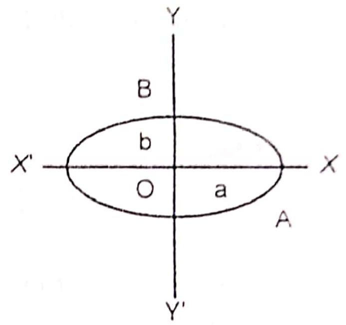
$$= 18\left(\frac{\pi}{2} + \frac{0}{2}\right) = 9\pi$$

If A be the area of circle, then

A = 4 × Area of OAB = 4 × 9π = 36π sq. units

Hence, the area of given circle = 36π sq. units.

(b) Given curve is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



The curve is symmetrical about *x-axis* and y-axis. So, to find the area of the whole ellipse, we first find the area of the portion lying in the first quadrant and then multiply it by 4. The area of the portion lying in the first quadrant is bounded by the curve, *x-axis* and the ordinates $x = 0$ and $x = a$. So its area is

$$A = \int_0^a y\, dx = \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx \quad \ldots(i)$$

Put $x = a\sin\theta$

Then, $dx = a\cos\theta\, d\theta$

$\therefore \quad \sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2\theta} = a\cos\theta$

When $x = 0$, $\theta = 0$

When $x = a$, $\theta = \frac{\pi}{2}$

Then, from (i)

$$A = \frac{b}{a}\int_0^{\pi/2} a\cos\theta \cdot a\cos\theta\, d\theta = ab\int_0^{\pi/2} \cos^2\theta\, d\theta = ab\int_0^{\pi/2} \frac{1 + \cos 2\theta}{2}\, d\theta$$

$$= \frac{ab}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{ab}{2}\left[\frac{\pi}{2} + \frac{\sin\pi}{2} - 0\right] = \frac{ab\pi}{4}$$

$\therefore$ The whole area of the ellipse

= 4(area of the portion lying in the first quadrant)

$= 4 \cdot \frac{ab\pi}{4} = \pi ab$ sq. units

9. **Find the length of the curve**

(a) $y = x^{\frac{3}{2}}$ **from** $x = 0$ **to** $x = 4$

(b) $y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$ **from** $x = 0$ **to** $x = 2$.

Solution

(a) Here, $y = x^{\frac{3}{2}}$; a = 0, b = 4

$$y = x^{\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{3}{2} x^{\frac{1}{2}}$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{9}{4} x$$

The length of the curve from x = 0 to x = 4 is

$$L = \int_0^4 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^4 \sqrt{1 + \frac{9}{4}x}\, dx = \frac{1}{2}\int_0^4 \sqrt{9x + 4}\, dx$$

$$= \frac{1}{2}\left[\frac{(9x+4)^{\frac{3}{2}}}{\frac{3}{2}\cdot 9}\right]_0^4 = \frac{1}{27}\left[40^{\frac{3}{2}} - 4^{\frac{3}{2}}\right] = \frac{1}{27}\,[80\sqrt{10} - 8\sqrt{2}]$$

(b) $y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{\frac{-1}{3}}\left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{2}{x}\right)^{\frac{1}{3}}$$

which is not defined at x = 0.

Therefore, we rewrite the equation to express x in terms of y.

$$y = \left(\frac{x}{2}\right)^{\frac{2}{3}}$$

or, $y^{\frac{3}{2}} = \frac{x}{2}$

or, $x = 2y^{\frac{3}{2}}$

$\therefore \frac{dx}{dy} = 2\left(\frac{3}{2}\right) y^{\frac{1}{2}} = 3y^{\frac{1}{2}}$

The derivative $\frac{dx}{dy}$ is continuous on [0, 1].

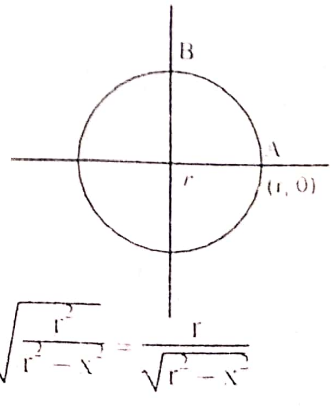
When x = 0, y = 0 and when x = 2, y = 1

$$\text{Length} = \int_0^1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + 9y}\, dy = \left[\frac{(1+9y)^{\frac{3}{2}}}{\frac{3}{2}\cdot 9}\right]_0^1 = \frac{2}{27}\,(10\sqrt{10} - 1)$$

10. (a) Using integration, find the circumference of the circle $x^2 + y^2 = r^2$.

(b) Find the length of the arc of the parabola $y^2 = 4ax$ cut off by the line $3y = 8x$.

Solution



(a) Given circle is $x^2 + y^2 = r^2$ ... (i)

Diff. both sides of (i) w.r. to x, we get,

$$2x + 2y\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{x}{y}$$

Now, $\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(-\frac{x}{y}\right)^2} = \sqrt{\frac{x^2 + y^2}{y^2}} = \sqrt{\frac{r^2}{r^2 - x^2}} = \frac{r}{\sqrt{r^2 - x^2}}$

Required circumference = 4 × length of arc AB

$$= 4\int_0^r \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = 4\int_0^r \frac{r}{\sqrt{r^2-x^2}}\,dx$$

$$= 4r\left[\sin^{-1}\frac{x}{r}\right]_0^r = 4r[\sin^{-1}1 - \sin^{-1}0] = 4r\left(\frac{\pi}{2}-0\right) = 2\pi r$$

(b) Given parabola is $y^2 = 4ax$ ... (i)
and the line is $3y = 8x$ ... (ii)
Eliminating x from (i) and (ii), we get,

$$y^2 = 4a \cdot \frac{3y}{8} = \frac{3ay}{2}$$

or, $$y^2 - \frac{3ay}{2} = 0$$

or, $$y\left(y - \frac{3a}{2}\right) = 0$$

or, $$y = 0, \frac{3a}{2}$$

Differentiating (i) w.r. to y,

$$2y = 4a\frac{dx}{dy}$$

or, $$\frac{dx}{dy} = \frac{y}{2a}$$

$$\text{Length of the arc} = \int_0^{3a/2}\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \int_0^{3a/2}\sqrt{1+\frac{y^2}{4a^2}}\,dy = \int_0^{3a/2}\sqrt{\frac{4a^2+y^2}{4a^2}}\,dy$$

$$= \frac{1}{2a}\int_0^{3a/2}\sqrt{y^2+(2a)^2}\,dy = \frac{1}{2a}\left[\frac{y\sqrt{y^2+(2a)^2}}{2} + \frac{(2a)^2}{2}\log\left(y+\sqrt{y^2+(2a)^2}\right)\right]_0^{\frac{3a}{2}}$$

$$= \frac{1}{2a}\left[\frac{\frac{3a}{2}\sqrt{\left(\frac{3a}{2}\right)^2+4a^2}}{2} + 2a^2\log\left\{\frac{3a}{2}+\sqrt{\left(\frac{3a}{2}\right)^2+(2a)^2}\right\}\right] - \frac{1}{2a}[2a^2\log 2a]$$

$$= \frac{3}{8}\left(\sqrt{\frac{9a^2+16a^2}{4}}\right) + a\log\left(\frac{3a}{2}+\sqrt{\frac{9a^2+16a^2}{4}}\right) - \frac{1}{2} - a\log 2a$$

$$= \frac{3}{16}\cdot 5a + a\log\left(\frac{3a}{2}+\frac{5a}{2}\right) - a\log 2a = \frac{15a}{16} + a\{\log(4a) - \log(2a)\}$$

$$= \frac{15a}{16} + a\log\left(\frac{4a}{2a}\right) = \frac{15a}{16} + a\log 2 = a\left(\log 2 + \frac{15}{16}\right)$$

11. (a) The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$ and the x–axis is revolved about the x–axis to generate a solid. Find its volume.

(b) Find the volume of the solid generated by revolving the region between the y–axis and the curve $x = \frac{2}{y}$, $1 \le y \le 4$ about the y–axis.

(c) Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.

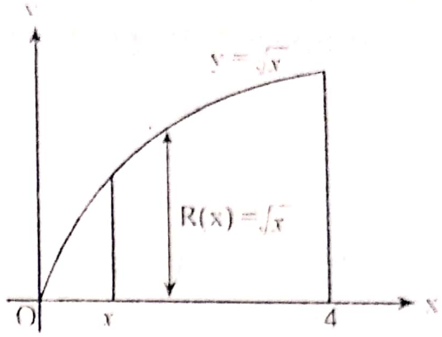
Solution

(a) The volume is

$$V = \int_a^b \pi [R(x)]^2\, dx = \int_0^4 \pi [\sqrt{x}\,]^2\, dx$$

$$= \int_0^4 \pi x\, dx = \pi \int_0^4 x\, dx$$

$$= \pi \left[\frac{x^2}{2}\right]_0^4 = \pi \left[\frac{4^2}{2} - \frac{0^2}{2}\right] = 8\pi$$

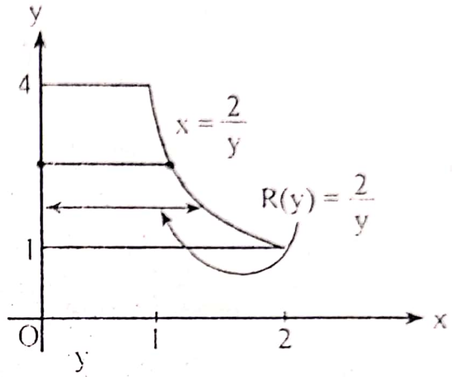


(b) The volume is

$$V = \int_c^d \pi [R(y)]^2\, dy = \int_1^4 \pi \left(\frac{2}{y}\right)^2 dy = \int_1^4 4\, y^{-2}\, dy$$

$$= 4\pi \left[\frac{y^{-2+1}}{-2+1}\right]_1^4 = 4\pi \left[-\frac{1}{y}\right]_1^4$$

$$= 4\pi \left[-\frac{1}{4} + \frac{1}{1}\right] = 4\pi \times \frac{3}{4} = 3\pi.$$



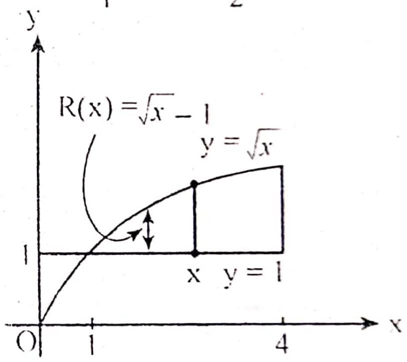
(c) The volume is

$$V = \int_1^4 \pi [R(x)]^2\, dx = \int_1^4 \pi [\sqrt{x} - 1]^2\, dx$$

$$= \pi \int_1^4 (x - 2\sqrt{x} + 1)\, dx$$

$$= \pi \left[\frac{x^2}{2} - 2 \cdot \frac{2}{3} \cdot x^{3/2} + x\right]_1^4$$

$$= \pi \left[\frac{4^2}{2} - \frac{4}{3} \cdot 4^{3/2} + 4\right] - \pi \left[\frac{1^2}{2} - 2 \cdot \frac{2}{3} \cdot 1^{3/2} + 1\right] = \frac{7\pi}{6}$$



12. (a) **Find the lateral surface area of the cone generated by revolving the line segment $y = \frac{x}{2}$, $0 \le x \le 4$, about the x–axis.**

(b) **Find the lateral surface area of the cone generated by revolving the line segment $y = \frac{x}{2}$, $0 \le x \le 4$ about the y–axis.**

Solution

(a) Here, a = 0, b = 4, $y = \frac{x}{2}$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}$$

$$\text{Area (S)} = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^4 2\pi \cdot \frac{x}{2} \sqrt{1 + \left(\frac{1}{2}\right)^2}\, dx$$

$$= \int_0^4 \pi x \cdot \frac{\sqrt{5}}{2}\, dx = \frac{\sqrt{5}\,\pi}{2} \left[\frac{x^2}{2}\right]_0^4 = \frac{\sqrt{5}\,\pi}{4} [4^2 - 0^2] = 4\sqrt{5}\,\pi.$$

(b) Here, $y = \frac{x}{2}$, $0 \le x \le 4$

When x = 0, $y = \frac{0}{2} = 0$

When $x = 4$, $y = \frac{4}{2} = 2$

$\therefore \quad c = 0,\ d = 2,\ x = 2y$

$\therefore \quad \frac{dx}{dy} = 2$

And, $\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + 2^2} = \sqrt{5}$

We have,

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^2 2\pi \cdot 2y\sqrt{5}\, dy$$

$$= 4\sqrt{5}\,\pi \int_0^2 y\, dy = 4\sqrt{5}\,\pi \left[\frac{y^2}{2}\right]_0^2 = 4\sqrt{5}\,\pi \left[\frac{2^2}{2} - 0\right] = 8\sqrt{5}\,\pi.$$

Objective Questions

1. The area bounded by the curve $y = 2x$, the x-axis and the ordinates at $x = 0$ and $x = 3$ is

(a) 1 (b) 2

(c) 3 (d) 9

Ans: d

$$A = \int_0^3 y dx = \int_0^3 2x\, dx = [x^2]_0^3 = 3^2 - 0 = 9$$

2. The area bounded by the curve $x = f(y)$, the y-axis and the lines at $y = a$ and $y = b$ is

(a) $\int_0^a f(y)\, dy$ (b) $\int_0^b f(y)\, dy$

(c) $\int_a^b f(y)\, dy$ (d) $\int_a^b [f(y)]^2\, dy$

Ans: c

3. The area of the region enclosed by $y = x^2$ and $y = \sqrt{x}$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$

(c) $\frac{1}{4}$ (d) $\frac{1}{5}$

Ans: b

$$A = \int_a^b \{\text{upper function} - \text{lower function}\}\, dx$$

$$= \int_0^1 (\sqrt{x} - x^2)\, dx = \left[\frac{2x^{\frac{3}{2}}}{3} - \frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

4. The area between two curves $y = f(x)$ and $y = g(x)$ and two ordinates at $x = a$ and $x = b$ is

(a) $\int_a^b f(x)\, dx$ (b) $\int_a^b g(x)\, dx$

(c) $\int_a^b [f(x) + g(x)]\, dx$ (d) $\int_a^b [f(x) - g(x)]\, dx$

Ans: d (Formula)

5. If f is smooth on $[a, b]$ then the length of the curve $y = f(x)$ form a to b is

(a) $\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ (b) $\int_a^b \sqrt{1-\left(\frac{dy}{dx}\right)^2}\,dx$

(c) $\int_a^b \sqrt{\left(\frac{dy}{dx}\right)^2 - 1}\,dx$ (d) $\int_a^b \sqrt{1+\frac{dy}{dx}}\,dx$

Ans: a

6. The perimeter of $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is
(a) a (b) $2a$
(c) $4a$ (d) $6a$

Ans: d

7. The volume of solid obtained by rotating about the y-axis the region between $y = x$ and $y = x^2$ is
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{6}$ (d) π

Ans: c

$$V = \int_0^1 2\pi x\,(x - x^2)\,dx = 2\pi \int_0^1 (x^2 - x^3)\,dx = 2\pi\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = \frac{\pi}{6}$$

EXERCISE 4 (G)

1. Use composite trapezoidal rule to evaluate
(a) $\int_0^3 2x\,dx,\ n = 3$ (b) $\int_1^2 x^{-2}\,dx,\ h = \frac{1}{2}$
(c) $\int_0^1 \frac{dx}{1+x^2},\ n = 2$ (d) $\int_0^{\pi} \sin x\,dx,\ n = 4$

Solution

(a) Here, $a = 0$, $b = 3$, $y = f(x) = 2x$, $n = 3$

We have, $h = \frac{b-a}{n} = \frac{3-0}{3} = 1$

The four points to be considered are $x_0 = 0$, $x_1 = 1$, $x_2 = 2$, $x_3 = 3$. The value of the integrand at these points are tabulated below:

x	0	1	2	3
f(x)	0	2	4	6

By trapezoidal rule, we have

$$\int_0^3 f(x)\,dx \approx \frac{h}{2}[y_0 + 2y_1 + 2y_2 + y_3] = \frac{1}{2}[0 + 2\times 2 + 2\times 4 + 6] = 9.$$

(b) Here, $y = f(x) = x^{-2}$

$a = 1, b = 2, h = \frac{1}{2}$

We have,

$h = \frac{b-a}{n}$

or, $\frac{1}{2} = \frac{2-1}{n}$

or, $n = 2$

The three points to be considered are $x_0 = 1$, $x_1 = 1.5$, $x_2 = 2$ respectively. The values of the function at these points are tabulated below.

End point	$x_0 = 1$	$x_1 = 1.5$	$x_2 = 2$
$y = f(x) = x^{-2} = \frac{1}{x^2}$	1	0.4444	0.25

Hence, by using trapezoidal rule, we have

$$\int_1^2 x^{-2}\,dx \approx \frac{h}{2}[y_0 + 2y_1 + y_2] = \frac{1/2}{2}[1 + 2 \times 0.4444 + 0.25] = 0.5347$$

(c) Here, a = 0, b = 1, n = 2.

$\therefore \quad h = \frac{b-a}{n} = \frac{1-0}{2} = 0.5$

Three points to be considered are $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$

The values of function at these points are tabulated below.

End point	$x_0 = 0$	$x_1 = 0.5$	$x_2 = 1$
$y = f(x) = (1 + x^2)^{-1} = \frac{1}{1+x^2}$	1	0.8	0.5

Hence, by using Trapezoidal rule, we have

$$\int_0^1 (1 + x^2)^{-2}\,dx \approx \frac{h}{2}[f(x_0) + 2f(x_1) + f(x_2)] = \frac{0.5}{2}[1 + 2 \times 0.8 + 0.5] = 0.775$$

(d) Here, a = 0, b = π, n = 4

$\therefore \quad h = \frac{b-a}{n} = \frac{\pi - 0}{4} = \frac{\pi}{4}$

The five points to be considered are $x_0 = 0$, $x_1 = \frac{\pi}{4}$, $x_2 = \frac{\pi}{2}$, $x_3 = \frac{3\pi}{4}$, $x_4 = \pi$ respectively.

The values of function at these points are tabulated below.

End point	$x_0 = 0$	$x_1 = \frac{\pi}{4}$	$x_2 = \frac{\pi}{2}$	$x_3 = \frac{3\pi}{4}$	$x_4 = \pi$
$y = \sin x$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0

Using trapezoidal rule, we have

$$\int_0^\pi \sin x\,dx \approx \frac{h}{2}[y_0 + 2y_1 + 2y_2 + 2y_3 + y_4]$$

$$= \frac{\pi}{8}\left[0 + 2\cdot\frac{1}{\sqrt{2}} + 2\cdot 1 + 2\cdot\frac{1}{\sqrt{2}} + 0\right] = \frac{\pi}{8}(2\sqrt{2} + 2) = 1.896.$$

2. Use the trapezoidal rule with $n = 4$ to estimate $\int_1^2 x^2\,dx$. Compare the estimate with the exact value of the integral.

Solution

Here, a = 1, b = 2, f(x) = x^2, n = 4

We have, $h = \frac{b-a}{n} = \frac{2-1}{4} = 0.25$

The five points to be considered are $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$

x	0	0.25	0.5	0.75	1
$y = f(x) = x^2$	0	0.0625	0.25	0.5625	1

Using trapezoidal rule, we have

$$\int_0^1 f(x)\,dx \approx \frac{h}{2}[y_0 + 2y_1 + 2y_2 + 2y_3 + y_4]$$

$$= \frac{0.25}{2}[0 + 2 \times 0.0625 + 2 \times 0.25 + 2 \times 0.5625 + 1] = 0.34375$$

$\therefore$ Approximate value = 0.34375

$$\text{Exact value} = \int_1^2 x^2\,dx = \left[\frac{x^3}{3}\right]_1^2 = \left(\frac{2^3}{3} - \frac{1^3}{3}\right) = \frac{8-1}{3} = \frac{7}{3} = 2.33333$$

3. **Using composite trapezoidal rule, evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ with three points of the intervals. Find the error of approximation. How many points are to be considered to make the approximated value within 0.0001?**

Solution

There are three points of intervals. So, no. of intervals n = 2.

Here, a = 0, b = 1, n = 2.

$\therefore \quad h = \frac{b-a}{n} = \frac{1-0}{2} = 0.5$

Three points to be considered are $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$.

The values of function at these points are tabulated below

End point	$x_0 = 0$	$x_1 = 0.5$	$x_2 = 1$
$y = f(x) = (1 + x^2)^{-1} = \frac{1}{1+x^2}$	1	0.8	0.5

Hence, by using Trapezoidal rule, we have

$$\int_0^1 (1 + x^2)^{-2}\,dx \approx \frac{h}{2}[f(x_0) + 2f(x_1) + f(x_2)] = \frac{0.5}{2}[1 + 2 \times 0.8 + 0.5] = 0.775$$

Approximate value = 0.775

$$\text{Exact value} = \int_0^1 \frac{dx}{1+x^2} = [\tan^{-1} x]_1^2 = \tan^{-1}(1) = \frac{\pi}{4} = 0.7854$$

Error = Actual value – approximate value
= 0.7854 – 0.775 = 0.0104.

Here, $f(x) = \frac{1}{1+x^2}$

$$f'(x) = -\frac{2x}{(1+x^2)^2}$$

$$f''(x) = \frac{2(3x^2-1)}{(1+x^2)^3}$$

The max. value of $|f''(x)|$ occurs at x = 1. It is 2.

$\therefore \quad M = 2$

$$\text{Error bounds} = \frac{(b-a)^3 M}{12n^2} = \frac{(1-0)^3 \cdot 2}{12 \times n^2} = \frac{1}{6n^2}.$$

By given, $\frac{1}{6n^2} \leq 0.0001$

or, $\frac{1}{6n^2} \leq 10^{-4}$

or, $n^2 \geq \frac{10^4}{6}$

or, $n \geq 40.8 \approx 41$

$\therefore$ Required no. of points to be considered are 41 + 1 = 42 or more.

[$\because$ no. of points = n + 1]

4. Evaluate using Simpson's $\frac{1}{3}$ rule.

(a) $\int_0^1 5x^4\,dx,\ n = 4$ (b) $\int_0^{0.2} \sqrt{1-2x^2}\,dx$

(c) $\int_0^{\pi} \sin x\,dx,\ n = 6$

Solution

(a) Here, a = 0, b = 1, n = 4

We have, $h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$

The five points to be considered are $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$

The values of the integrand at these points are tabulated below:

x	0	0.25	0.5	0.75	1
f(x)	0	0.0195	0.3125	1.5820	5

Using Simpson's $\frac{1}{3}$ rule, we have

$$\int_0^1 5x^4\,dx \approx \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + y_4]$$

$$= \frac{0.25}{3}[0 + 4 \times 0.0195 + 2 \times 0.3125 + 4 \times 1.5820 + 5] = 1.0026.$$

(b) Here, $y = f(x) = \sqrt{1-2x^2}$

a = 0, b = 0.2, n = 2

We have, $h = \frac{b-a}{n} = \frac{0.2-0}{2} = 0.1$

The three points to be considered at $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$ respectively. The values of the functions at these points are tabulated below.

End point	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$
$y = f(x) = \sqrt{1-2x^2}$	1	0.9899	0.9592

Using Simpson's $\frac{1}{3}$ rule, we have,

$$\int_0^1 \sqrt{1-2x^2}\,dx \approx \frac{h}{3}[y_0 + 4y_1 + y_2] = \frac{0.1}{3}[1 + 4 \times 0.9899 + 0.9592] = 0.1973$$

(c) Here, $y = f(x) = \sin x$

a = 0, b = π, n = 6

We have, $h = \frac{b-a}{n} = \frac{\pi-0}{6} = \frac{\pi}{6}$

The seven points to be considered are $x_0 = 0$, $x_1 = \frac{\pi}{6}$, $x_2 = \frac{\pi}{3}$, $x_3 = \frac{\pi}{2}$, $x_4 = \frac{2\pi}{3}$, $x_4 = \frac{5\pi}{6}$ and $x_6 = \pi$. The values of the function at these points are tabulated below:

End point	$x_0 = 0$	$x_1 = \frac{\pi}{6}$	$x_2 = \frac{\pi}{3}$	$x_3 = \frac{\pi}{2}$	$x_4 = \frac{2\pi}{3}$	$x_5 = \frac{5\pi}{6}$	$x_6 = \pi$.
$y = f(x) = \sin x$	0	0.5	0.8660	1	0.8660	0.5	0

Using Simpson's $\frac{1}{3}$ rule, we have

$$\int_0^{\pi} \sin x\, dx \approx \frac{h}{3}[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6]$$

$$= \frac{\pi/6}{3}[0 + 4 \times 0.5 + 2 \times 0.8660 + 4 \times 1 + 2 \times 0.8660 + 4 \times 0.5 + 0]$$

$$= \frac{\pi}{18}(11.464) = 2.0008$$

5. **Using Simpson's $\frac{1}{3}$ rule, evaluate $\int_0^1 \frac{1}{1+x}dx$ with three points of the intervals. Find the error of approximation. How many points are to be considered to make the approximated value within 10^{-4}?**

Solution

Here, a = 0, b = 1, n = 2

We have, $h = \frac{b-a}{n} = \frac{1-0}{2} = 0.5$

The three points to be considered are $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$

The values of the integrand at these points are tabulated below:

x	0	0.5	1
f(x)	1	0.6667	0.5

Using Simpson's $\frac{1}{3}$ rule, we have

$$\int_0^1 \frac{1}{1+x}dx \approx \frac{h}{3}[y_0 + 4y_1 + y_2] = \frac{0.5}{3}[1 + 4 \times 0.6666 + 0.5] = 0.6944$$

$$\text{Exact value} = \int_0^1 \frac{1}{1+x}dx = [\log(1+x)]_0^1 = [\log 2 - \log 0] = 0.6931$$

| Error | = | Actual value – approximate value |

= |0.6944 – 0.6931| = 0.0013

Here, $f(x) = \frac{1}{1+x}$ $\qquad f'(x) = -\frac{1}{(1+x)^2}$

$f''(x) = \frac{2}{(1+x)^3}$ $\qquad f'''(x) = -\frac{6}{(1+x)^4}$

$f^{iv}(x) = \frac{24}{(1+x)^5}$

In [0, 1], the largest value of $\left|\frac{24}{(1+x)^5}\right|$ occurs at x = 0.

The value is 24. Thus, M = 24.

Now,

$$\text{Error} = \frac{(b-a)^5 M}{180n^4} = \frac{(1-0)^5 \cdot 24}{180n^4} = \frac{2}{15n^4}$$

By given, $\frac{2}{15n^4} \le 10^{-4}$

or, $\frac{2}{15n^4} \le \frac{1}{10^4}$

or, $n^4 \ge \frac{2 \times 10^4}{15}$

or, $n \ge 6.04 \approx 7$

∴ Required no. of points to be considered are 7 + 1 = 8 or more.

6. Using Simpson's $\frac{3}{8}$ rule, evaluate $\int_1^4 e^x \ln x \, dx$, $n = 3$.

Solution

Here,

$a = 1, b = 4, n = 3$ $\therefore$ $h = \frac{b-a}{n} = \frac{4-1}{3} = 1$

The 4 points to be considered are $x_0 = a = 1$. $x_1 = x_0 + h = 1 + 1 = 2$. $x_2 = x_1 + h = 2 + 1 = 3$ and $x_3 = b = 4$

The values of the integrand at each of these points are tabulated below.

x	$x_0 = 1$	$x_1 = 2$	$x_2 = 3$	$x_3 = 4$
$y = f(x) = e^x \ln x$	0	5.12170	22.06621	75.68911

Since n = 3 which is odd, so we have to apply Simpson's $\frac{3}{8}$ rule. Using Simpson's $\frac{3}{8}$ rule, we have

$$\int_1^4 e^x \ln x \, dx \approx \frac{3h}{8} [y_0 + 3 xy_1 + 3y_2 + y_3]$$

$$= \frac{3 \times 1}{8} [0 + 3 \times 5.12170 + 3 \times 22.06621 + 75.68911] = 58.969815$$

Objective Questions

1. To approximate $\int_a^b f(x) \, dx$, the interval $[a, b]$ has divided into equal n subintervals then length of each interval (h)

(a) $b - a$ (b) $n(b - a)$

(c) $\frac{b+a}{n}$ (d) $\frac{b-a}{n}$

Ans: d

2. The approximate value of $\int_0^4 x^2 \, dx$ using mid-point rule with one interval is

(a) 2 (b) 4

(c) 8 (d) 16

Ans: d

$$\int_0^4 x^2 \, dx \approx (4, 0) f\left(\frac{0+4}{2}\right) = 4 f(2) = 4 \times 2^2 = 16$$

3. Using Trapezoidal rule, the value of $\int_0^2 (2x^2 - 1) \, dx$, $n = 4$, is

(a) 1.5 (b) 2.5

(c) 3.5 (d) 4.5

Ans: c

Here, a = 0, b = 2, n = 4 $\therefore$ $h = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$

The five points to be considered are $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$, $x_4 = 2$

The values of function at these points are tabulated below

End point	$x_0 = 0$	$x_1 = 0.5$	$x_2 = 1$	$x_3 = 1.5$	$x_4 = 2$
$y = 2x^2 - 1$	-1	-0.5	1	3.5	7

Hence, by using trapezoidal rule, we have

$$\int_0^2 (2x^2 - 1) \, dx \approx \frac{h}{2} [y_0 + 2y_1 + 2y_2 + 2y_3 + y_4]$$

$$= \frac{0.5}{2} [-1 + 2 \times (-0.5) + 2 \times 1 + 2 \times 3.5 + 7] = 0.25 [-1 - 1 + 2 + 7 + 7] = 3.5$$

4. Using Trapezoidal rule, the value of integral $\int_0^1 \frac{dx}{1+x^2}$, $n = 2$ is

(a) 0.125 (b) 0.775
(c) 0.992 (d) 1.257

Ans: b

Here, a = 0, b = 1, n = 2.

$\therefore \quad h = \frac{b-a}{n} = \frac{1-0}{2} = 0.5$

Three points to be considered are $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$

The values of function at these points are tabulated below

End point	$x_0 = 0$	$x_1 = 0.5$	$x_2 = 1$
$y = f(x) = (1 + x^2)^{-1} = \frac{1}{1+x^2}$	1	0.8	0.5

Hence, by using Trapezoidal rule, we have

$$\int_0^1 (1 + x^2)^{-2}\, dx \approx \frac{h}{2}\,[f(x_0) + 2f(x_1) + f(x_2)]$$

$$= \frac{0.5}{2}\,[1 + 2 \times 0.8 + 0.5] = 0.775$$

5. Simpson's $\frac{1}{3}$ rule can't be applied when number of subintervals (n) =

(a) 2 (b) 4
(c) 6 (d) 3

Ans: d

6. Simpson's error bounds with usual notations of symbols is

(a) $\frac{(b-a)^3 M}{12n^4}$ (b) $\frac{(b-a)^4 M}{12n^2}$

(c) $\frac{(b-a)^5 M}{180n^4}$ (d) $\frac{(b-a)^6 M}{12n^5}$

Ans: c

7. The approximate value of $\int_0^{0.2} \sqrt{1-2x^2}\, dx$ with $n = 2$ using Simpson's $\frac{1}{3}$ rule is

(a) 0.1973 (b) 0.9899
(c) 0.9592 (d) 1

Ans: d

Here, $y = f(x) = \sqrt{1-2x^2}$

a = 0, b = 0.2, n = 2

We have, $h = \frac{b-a}{n} = \frac{0.2-0}{2} = 0.1$

The three points to be considered at $x_0 = 0$, $x_1 = 0.1$, $x_2 = 0.2$ respectively. The values of the functions at these points are tabulated below:

End point	$x_0 = 0$	$x_1 = 0.1$	$x_2 = 0.2$
$y = f(x) = \sqrt{1-2x^2}$	1	0.9899	0.9592

Using Simpson's $\frac{1}{3}$ rule,

we have $\int_0^1 \sqrt{1-2x^2}\, dx \approx \frac{h}{3}\,[y_0 + 4y_1 + y_2] = \frac{0.1}{3}\,[1 + 4 \times 0.9899 + 0.9592] = 0.1973$

5

UNIT

Differential Equations

EXERCISE – 5(A)

1. Find the order and degree of the following differential equations.

(a) $\frac{dy}{dx} = 2$ (b) $\frac{d^2y}{dx^2} = \sin x$

(c) $x\frac{d^3y}{dx^3} + y + \left(\frac{dy}{dx}\right)^4 = 0$ (d) $(y'')^3 + 4y' = e^x$

(e) $\frac{d^2y}{dx^2} = \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3}$

Solution

(a) order = 1, degree = 1 (b) order = 2, degree = 1 (c) order = 3, degree = 1

(d) order = 2, degree = 3 (e) order = 2, degree = 2

2. Solve the following differential equations using separation of variables.

(a) $x\,dx - y\,dy = 0$ (b) $\frac{dy}{dx} = \frac{y}{x}$

(c) $\frac{dy}{dx} = \frac{x^3 + 1}{y^3 + 1}$ (d) $(1 + x^2)y' = 1$

(e) $y\,dx - x\,dy = xy\,dx$ (f) $(xy^2 + x)dx + (yx^2 + y)dy = 0$

(g) $\frac{dy}{dx} = \frac{e^x + 1}{y}$ (h) $\frac{dy}{dx} = e^{x-y} + e^{-y}$

(i) $e^{x-y}dx + e^{y-x}dy = 0$ (j) $\sec^2 x \tan y\,dx + \sec^2 y \tan x\,dy = 0$

(k) $\frac{dy}{dx} + \frac{1 + \cos 2y}{1 - \cos 2x} = 0$

Solution

(a) $xdx - ydy = 0$

or, $xdx = ydy$

Integrating both sides, we get,

$$\int xdx = \int y\,dy + \frac{c}{2}$$

or, $\frac{x^2}{2} = \frac{y^2}{2} + \frac{c}{2}$

$\therefore$ $x^2 - y^2 = c$

(b) $\frac{dy}{dx} = \frac{y}{x}$

or, $\frac{dy}{y} = \frac{dx}{x}$

Integrating, we have,

$$\int\frac{dx}{x} = \int\frac{dy}{y} + \log c$$

or, $\log x = \log y + \log c$

or, $\log x = \log (cy)$

$\therefore$ $\log x = \log (cy)$_

$\therefore$ $x = cy$

(c) Given, $\frac{dy}{dx} = \frac{x^3 + 1}{y^3 + 1}$

or, $(y^3 + 1)\,dy = (x^3 + 1)\,dx$

Integrating, we get

$$\frac{y^4}{4} + y = \frac{x^4}{4} + x + c$$

(d) Given, $(1 + x^2)\frac{dy}{dx} = 1$

or, $\frac{dy}{dx} = \frac{1}{1 + x^2}$

or, $dy = \frac{1}{1 + x^2}\,dx$

Integrating, we have

$$\int dy = \int \frac{1}{1 + x^2}\,dx + c$$

$\therefore$ $y = \tan^{-1} x + c$

(e) $y\,dx - x\,dy = xy\,dx$

Diving both sides by xy

$$\frac{dx}{x} - \frac{dy}{y} = dx$$

Integrating, we have,

$$\int \frac{dx}{x} - \int \frac{dy}{y} = \int dx + c$$

or, $\log x - \log y = x + c$

or, $\log\left(\frac{x}{y}\right) = x + c$

(f) Here, $\frac{dy}{dx} = \frac{e^x + 1}{y}$

or, $y\,dy = (e^x + 1)\,dx$

Integrating, we have

$$\frac{y^2}{2} = e^x + x + c$$

or, $y^2 = 2e^x + 2x + 2c_1$

$\therefore$ $y^2 = 2e^x + 2x + c$, where $C = 2c_1$

(g) Here, $(xy^2 + x)\,dx + (yx^2 + y)\,dy = 0$

or, $x(y^2 + 1)\,dx + y(x^2 + 1)\,dy = 0$

Dividing both sides by $(x^2 + 1)(y^2 + 1)$, we get

or, $\frac{x}{x^2 + 1}\,dx + \frac{y}{y^2 + 1}\,dy = 0$

or, $\frac{2x\,dx}{x^2 + 1} + \frac{2y\,dy}{y^2 + 1} = 0$

Integrating, we get

$\log(x^2 + 1) + \log(y^2 + 1) = \log c$

or, $\log(x^2 + 1)(y^2 + 1) = \log c$

$\therefore$ $(x^2 + 1)(y^2 + 1) = c$

(h) $\frac{dy}{dx} = e^{x-y} + e^{-y}$

or, $\frac{dy}{dx} = e^x . e^{-y} + e^{-y}$

or, $\frac{dy}{dx} = e^{-y}(e^x + 1)$

or, $\frac{dy}{e^{-y}} = (e^x + 1)dx$

or, $e^y\,dy = (e^x + 1)\,dx$

Integrating, we have,

$$\int e^y\,dy = \int (e^x + 1)\,dx + c$$

$e^y = e^x + x + c$

(i) Here, $e^{x-y}\,dx + e^{y-x}\,dy = 0$

or, $\frac{e^x}{e^y}\,dx + \frac{e^y}{e^x}\,dy = 0$

or, $e^{2x}\,dx + e^{2y}\,dy = 0$

Integrating we have

$$\frac{e^{2x}}{2} + \frac{e^{2y}}{2} = \frac{c}{2}$$

$e^{2x} + e^{2y} = c$

(j) Given, $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

Diving both sides by tanx tany, we get

$$\frac{\sec^2 x}{\tan x} dx + \frac{\sec^2 y}{\tan y} dy = 0$$

Integrating, we have

$$\int \frac{\sec^2 x}{\tan x} dx + \int \frac{\sec^2 y}{\tan y} dy = \log c$$

or, $\log(\tan x) + \log(\tan y) = \log c \quad \left[\because \int \frac{f'(x)}{f(x)} dx = \log f(x) + c \right]$

or, $\log(\tan x \tan y) = \log c$

$\therefore \quad \tan x \tan y = c$

(k) $\frac{dy}{dx} + \frac{1 + \cos 2y}{1 - \cos 2x} = 0$

or, $\frac{dy}{dx} = -\frac{1 + \cos 2y}{1 - \cos 2x}$

or, $\frac{dy}{dx} = -\frac{2\cos^2 y}{2\sin^2 x}$

or, $\frac{dy}{\cos^2 y} = -\frac{dx}{\sin^2 x}$

or, $\sec^2 y \, dy = -\operatorname{cosec}^2 x \, dx$

Integrating, we get,

$$\int (-\operatorname{cosec}^2 x) \, dx = \int \sec^2 y \, dy + c$$

$\therefore \quad \cot x = \tan y + c$

3. Solve the initial value problems

(a) $\frac{dy}{dx} = 2x - 7, y(2) = 0$ (b) $\frac{dy}{dx} = 10 - x, y(0) = -1$

(c) $\frac{dy}{dx} = 9x^2 - 4x + 5, y(-1) = 0$

Solution

(a) $\frac{dy}{dx} = 2x - 7$

or, $dy = (2x - 7)dx$

Integrating, we have,

$$\int dy = \int (2x - 7) \, dx + c$$

$$y = \frac{2x^2}{2} - 7x + c$$

or, $y = x^2 - 7x + c$ (i)

Given, $y(2) = 0$

i.e. when $x = 2, y = 0$

Then, From (i)

$0 = 2^2 - 7 \times 2 + c$

or, $c = 10$

Putting the value of c in (i)

$y = x^2 - 7x + 10$

(b) Given, $\frac{dy}{dx} = 10 - x$

$dy = (10 - x) \, dx$

Integrating, we have,

$$\int dy = \int (10 - x)dx + c$$

$$y = 10x - \frac{x^2}{2} + c \qquad (i)$$

Given $y(0) = -1$

i.e. when $x = 0, y = -1$

Then, from (i)

$-1 = 10 \times 0 - \frac{0^2}{2} + c$

$\therefore \quad c = -1$

Putting the value of c in (i)

$y = 10x - \frac{x^2}{2} - 1$

(c) $\frac{dy}{dx} = 9x^2 - 4x + 5$

$dy = (9x^2 - 4x + 5)\,dx$

Integrating,

$$y = \frac{9x^3}{3} - \frac{4x^2}{2} + 5x + c$$

or, $y = 3x^3 - 2x^2 + 5x + c$ ... (i)

By given, $y(-1) = 0$

i.e. when $x = -1$, $y = 0$

Then, from (i),

$$0 = 3(-1)^3 - 2(-1)^2 + 5(-1) + c$$

or, $0 = -3 - 2 - 5 + c$

$\therefore$ $c = 10$

Putting the value of c in (i),

$$y = 3x^3 - 2x^2 + 5x + 10.$$

Objective Questions

1. The order of the differential equation $x\frac{d^4y}{dx^4} + y + \left(\frac{dy}{dx}\right)^3 = 0$ is

(a) 1 (b) 2

(c) 3 (d) 4

Ans: d

2. The degree of the differential equation $(y''')^2 + 3y' = e^x$ is

(a) 4 (b) 3

(c) 2 (d) 1

Ans: c

3. Which of the following is not an ordinary differential equation?

(a) $\frac{dy}{dx} = \sin x$ (b) $\frac{d^2y}{dx^2} + 3y = 0$

(c) $2xy\frac{dy}{dx} = y^2 - x^2$ (d) $\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial y^2} = 0$

Ans: d option d is PDE

4. The solution of $\frac{dy}{dx} = x^2$ is

(a) $y = x^2 + c$ (b) $y^2 = x + c$

(c) $y = \frac{x^2}{2} + c$ (d) $y = \frac{x^3}{3} + c$

Ans: d

$\frac{dy}{dx} = x^2$

$dy = x^2dx$

Integrating, we have,

$$\int dy = \int x^2\,dx + c$$

$$\therefore \quad y = \frac{x^3}{3} + c.$$

5. The solution of the differential equation $\frac{dy}{dx} = \frac{x^2}{y^2}$ is

(a) $\frac{x^2}{2} + \frac{y^2}{2} = c$ (b) $\frac{x^2}{2} - \frac{y^2}{2} = c$

(c) $\frac{x^3}{3} - \frac{y^3}{3} = c$ (d) $\frac{x^3}{3} + \frac{y^3}{3} = c$

Ans: c

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

or, $y^2 dy = x^2 dx$

Integrating, we have

$$\int y^2\, dy + c = \int x^2\, dx$$

or, $\frac{y^3}{3} + c = \frac{x^3}{3}$

$\therefore$ $\frac{x^3}{3} - \frac{y^3}{3} = c$

6. **The solution of differential equation $\frac{dy}{dx} = \frac{y^2}{x^2}$ is**

(a) $x + y = c\,xy$ (b) $x - y = c\,xy$
(c) $x^2 + y^2 = c\,xy$ (d) $x^2 - y^2 = c\,xy$

Ans: b

$$\frac{dy}{y^2} = \frac{dx}{x^2}$$

or, $x^{-2}dx = y^{-2}dy$

Integrating, we have,

$$-\frac{1}{x} = -\frac{1}{y} + c$$

or, $\frac{1}{y} - \frac{1}{x} = c$

or, $\frac{x - y}{xy} = c$

$\therefore$ $x - y = cxy$

7. **The solution of differential equation $\frac{dy}{dx} + 4x = 2e^{2x}$ is**

(a) $y = e^{2x} - x^2 + c$ (b) $y = e^{2x} - 2x^2 + c$
(c) $y = 2e^{2x} + 2x^2 + c$ (d) $y = 2e^{2x} + x^2 + c$

Ans: b

$$\frac{dy}{dx} + 4x = 2\,e^{2x}$$

or, $\frac{dy}{dx} = 2e^{2x} - 4x$

or, $dy = (2e^{2x} - 4x)\,dx$

Integrating, we have

$$y = \frac{2e^{2x}}{2} - \frac{4x^2}{2} + c$$

$\therefore$ $y = e^{2x} - 2x^2 + c$

8. **The solution of differential equation $y' = \frac{1 + y^2}{1 + x^2}$ is**

(a) $x - y = c(1 - xy)$ (b) $x - y = c(1 + xy)$
(c) $x + y = c(1 - xy)$ (d) $x + y = c(1 + xy)$

Ans: b

$$\frac{dy}{dx} = \frac{1 + y^2}{1 + x^2}$$

or, $(1 + x^2)\,dy = (1 + y^2)dx$

or, $\frac{dx}{1 + x^2} = \frac{dy}{1 + y^2}$

or, $\frac{dx}{1 + x^2} - \frac{dy}{1 + y^2} = 0$

Integrating,

$$\tan^{-1}x - \tan^{-1}y = \tan^{-1}c$$

or, $$\tan^{-1}\frac{x-y}{1+xy} = \tan^{-1}c$$

or, $$\frac{x-y}{1+xy} = c$$

$\therefore$ $$x - y = c(1+xy)$$

EXERCISE – 5(B)

Solve the following differential equations:

1. $\dfrac{dy}{dx} = \dfrac{x+y}{x}$

Solution

$$\frac{dy}{dx} = \frac{x+y}{x}$$

$$\frac{dy}{dx} = 1 + \frac{y}{x}$$

The given equation is of the form $\dfrac{dv}{dx}\varphi\left(\dfrac{y}{x}\right)$

Put $y = vx$. Then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Now, equation (i) can be written as

$$v + x\frac{dv}{dx} = 1 + \frac{vx}{x}$$

or, $$v + \frac{xdv}{dx} = 1 + v$$

or, $$x\frac{dv}{dx} = 1$$

or, $$dv = \frac{dx}{x}$$

Integrating, we get,

$$\int dv = \int\frac{dx}{x} + \log c$$

or, $$v = \log x + \log c$$

or, $$v = \log cx$$

or, $$\frac{y}{x} = \log cx$$

$\therefore$ $$y = x\log cx.$$

2. $\dfrac{dy}{dx} = \dfrac{2x+y}{x}$

Solution

Given, $$\frac{dy}{dx} = \frac{2x+y}{x} \quad \ldots(i)$$

So, put $y = vx$. then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Now, equation (i) can be written as

$$v + x\frac{dv}{dx} = \frac{2x}{x} + \frac{vx}{x}$$

or, $$v + x\frac{dv}{dx} = 2 + v$$

or, $$x\frac{dv}{dx} = 2$$

or, $$dv = 2\frac{dx}{x}$$

Integrating, we have,

$$\int dv = 2\int\frac{dx}{x} + \log c$$

or, $$v = 2\log x + \log c$$

or, $$v = \log x^2 + \log c$$

or, $$\frac{y}{x} = \log cx^2$$

$\therefore$ $$y = x\log cx^2$$

3. $\dfrac{dy}{dx} = \dfrac{2y-x}{x}$

Solution

$$\frac{dy}{dx} = \frac{2y-x}{x} \quad \ldots(i)$$

This is a homogeneous equation

So, put $y = vx$. Then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Now, equation (i) can be written as

$$v + x\frac{dv}{dx} = \frac{2vx}{x} - \frac{x}{x}$$

or $$x\frac{dv}{dx} = 2v - v - 1$$

or $\quad x\dfrac{dv}{dx} = v - 1$

or $\quad \dfrac{dv}{v-1} = \dfrac{dx}{x}$

Integrating, we have

$$\log (v - 1) = \log x + \log c$$

or, $\quad \log (v - 1) = \log cx$

or, $\quad v - 1 = cx$

or, $\quad \dfrac{y}{x} - 1 = cx$

or, $\quad \dfrac{y - x}{x} = cx$

$\therefore \quad y - x = cx^2$

4. $\dfrac{dy}{dx} = \dfrac{xy}{x^2 + y^2}$

Solution

Here, $(x^2 + y^2)\, dy = xy\, dx$

or, $\quad \dfrac{dy}{dx} = \dfrac{xy}{x^2 + y^2}$ ...(i)

Put $y = vx$

Then, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Now, equation (i) can be written as

$$v + x\frac{dv}{dx} = \frac{x.\,vx}{x^2 + v^2x^2}$$

or, $\quad x\dfrac{dv}{dx} = \dfrac{v}{1 + v^2} - v$

or, $\quad x\dfrac{dv}{dx} = \dfrac{v - v - v^3}{1 + v^2}$

or, $\quad x\dfrac{dv}{dx} = \dfrac{-v^3}{1 + v^2}$

or, $\quad \dfrac{1 + v^2}{v^3}\, dv = \dfrac{-dx}{x}$

or, $\quad \dfrac{1}{v^3}\, dv + \dfrac{1}{v}\, dv + \dfrac{dx}{x} = 0$

or, $\quad \dfrac{1}{v}\, dv + \dfrac{dx}{x} = -\dfrac{1}{v^3}\, dv$

Integrating, we get

$$\log v + \log x + \log c = \frac{-v^{-3+1}}{(-3 + 1)}$$

or, $\quad \log (v.\, x.\, c) = \dfrac{1}{2v^2}$

or, $\quad \log\left(\dfrac{y}{x}.x.\,c\right) = \dfrac{1}{2\left(\dfrac{y}{x}\right)^2}$

or, $\quad \log (cy) = \dfrac{x^2}{2y^2}$

$$x^2 = 2y^2 \log cy$$

5. $2xy\dfrac{dy}{dx} = x^2 + y^2$

Solution

Given, $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{2xy}$ ... (i)

This is a homogeneous equation

Put $y = vx$. Then, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Now, equation (i) can be written as

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{2x\,vx}$$

or, $v + x\dfrac{dv}{dx} = \dfrac{1 + v^2}{2v}$

or, $x\dfrac{dv}{dx} = \dfrac{1 + v^2}{2v} - v$

or, $x\dfrac{dv}{dx} = \dfrac{1 + v^2 - 2v^2}{2v}$

or, $x\dfrac{dv}{dx} = \dfrac{1 - v^2}{2v}$

or, $\dfrac{2v}{v^2 - 1}dv = -\dfrac{dx}{x}$

or, $\dfrac{2v}{v^2 - 1}dv + \dfrac{dx}{x} = 0$

Integrating,

$\log(v^2 - 1) + \log x = \log c$

or, $\log((v^2 - 1).x) = \log c$

or, $x(v^2 - 1) = c$

or, $x\left(\dfrac{y^2}{x^2} - 1\right) = c$

or, $\dfrac{x(y^2 - x^2)}{x^2} = c$

∴ $y^2 - x^2 = cx$

6. $xy\dfrac{dy}{dx} = x^2 - y^2$

Solution

$\dfrac{dy}{dx} = \dfrac{x^2 - y^2}{xy}$ ... (i)

This is a homogeneous equation

Put $y = vx$. Then, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Now, equation (i) can be written as

$$v + x\frac{dv}{dx} = \frac{x^2 - v^2x^2}{x\,vx}$$

or, $v + x\dfrac{dv}{dx} = \dfrac{1 - v^2}{v}$

or, $x\dfrac{dv}{dx} = \dfrac{1 - v^2}{v} - v$

or, $x\dfrac{dv}{dx} = \dfrac{1 - v^2 - v^2}{v}$

or, $x\dfrac{dv}{dx} = \dfrac{1 - 2v^2}{v}$

or, $x\dfrac{dv}{dx} = -\dfrac{(2v^2 - 1)}{v}$

or, $\dfrac{vdv}{2v^2 - 1} = -\dfrac{dx}{x}$

or, $\dfrac{4vdv}{2v^2 - 1} = -4\dfrac{dx}{x}$

Integrating, we have,

$\log(2v^2 - 1) = -4\log x + \log c$

or, $\log(2v^2 - 1) + 4\log x = \log c$

or, $\log(2v^2 - 1) + \log x^4 = \log c$

or, $\log x^4(2v^2 - 1) = \log c$

or, $x^4(2v^2 - 1) = c$

or, $x^4\left\{2\left(\dfrac{y}{x}\right)^2 - 1\right\} = c$

or, $x^4\dfrac{(2y^2 - x^2)}{x^2} = c$

or, $x^2(2y^2 - x^2) = c$

∴ $x^2(x^2 - 2y^2) + c = 0$

7. $\dfrac{dy}{dx} = \dfrac{x + y}{x - y}$

Solution

Given, equation is $\dfrac{dy}{dx} = \dfrac{x + y}{x - y}$ ... (i)

This is a homogeneous equation

So, put $y = vx$. Then, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

Now, equation (i) becomes,

$$v + x\frac{dv}{dx} = \frac{x + vx}{x - vx}$$

or, $v + x\dfrac{dv}{dx} = \dfrac{1 + v}{1 - v}$

or, $x\dfrac{dv}{dx} = \dfrac{1 + v}{1 - v} - v$

or, $x\dfrac{dv}{dx} = \dfrac{1+v-v(1-v)}{1-v}$

or, $x\dfrac{dv}{dx} = \dfrac{1+v-v+v^2}{1-v}$

or, $x\dfrac{dv}{dx} = \dfrac{1+v^2}{1-v}$

or, $\dfrac{1-v}{1+v^2}\,dv = \dfrac{dx}{x}$

or, $\left(\dfrac{1}{1+v^2} - \dfrac{v}{1+v^2}\right) dv = \dfrac{dx}{x}$

Integrating

$$\int \frac{1}{1+v^2}\,dv - \frac{1}{2}\int \frac{2v}{1+v^2}\,dv = \int \frac{dx}{x} + \log c$$

or, $\tan^{-1} v - \dfrac{1}{2}\log(1+v^2) = \log x + \log c$

or, $\tan^{-1} v - \log(1+v^2)^{\frac{1}{2}} = \log cx$

or, $\tan^{-1} v = \log\sqrt{1+v^2} + \log cx$

or, $\tan^{-1} v = \log cx\sqrt{1+v^2}$

or, $\tan^{-1}\left(\dfrac{y}{x}\right) = \log cx\sqrt{1+\dfrac{y^2}{x^2}}$

or, $\tan^{-1}\left(\dfrac{y}{x}\right) = \log\sqrt{x^2+y^2} + c$

or, $\tan^{-1}\left(\dfrac{y}{x}\right) = \log(x^2+y^2)^{\frac{1}{2}} + c$

or, $\tan^{-1}\left(\dfrac{y}{x}\right) = \dfrac{1}{2}\log(x^2+y^2) + c$

8. $\dfrac{dy}{dx} = \dfrac{y}{x} - \sin^2\dfrac{y}{x}$

Solution

Given equation is, $\dfrac{dy}{dx} = \dfrac{y}{x} - \sin^2\dfrac{y}{x}$

Put $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

Now, the given equation can be written as

$v + x\dfrac{dv}{dx} = v - \sin^2 v$

or, $x\dfrac{dv}{dx} = -\sin^2 v$

or, $\dfrac{-dv}{\sin^2 v} = \dfrac{dx}{x}$

Integrating, we have

$$\int -\operatorname{cosec}^2 v\,dv = \int \frac{dx}{x} + c$$

or $\cot v = \log x + c$

$\therefore$ $\cot\left(\dfrac{y}{x}\right) = \log x + c$

Objective Questions

1. The first order and first degree homogeneous differential equation can be written as

(a) $\frac{dy}{dx} = \phi\left(\frac{1}{x}\right)$ (b) $\frac{dy}{dx} = \phi\left(\frac{1}{y}\right)$

(c) $\frac{dy}{dx} = \phi(y)$ (d) $\frac{dy}{dx} = \phi\left(\frac{y}{x}\right)$

Ans: d

2. Which of the following differential equation is not homogeneous?

(a) $\frac{dy}{dx} = \frac{y}{x}$ (b) $\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$

(c) $\frac{dy}{dx} = \frac{y + 1}{x + y + 1}$ (d) $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$

Ans: c

3. Which of the following is a homogeneous differential equation?

(a) $x(x + y)\, dy = y(x - y)\, dx$ (b) $\cos^2 x \frac{dy}{dx} + y = 1$

(c) $\frac{dy}{dx} + \frac{y}{x} = y^2$ (d) $\tan x\, dy + y\, dx = \sec x\, dx$

Ans: a

4. The variable separated form of the homogeneous differential equation $\frac{dy}{dx} = \phi\left(\frac{y}{x}\right)$ by putting $y = vx$ is

(a) $\frac{dv}{\phi(v) - v} = \frac{dx}{x}$ (b) $\frac{dv}{\phi(v) + v} = \frac{dx}{x}$

(c) $\frac{dv}{v} = \frac{dx}{x}$ (d) $[\phi(v) - v]\, dv = x dx$

Ans: a

Given, $\frac{dy}{dx} = \phi\left(\frac{y}{x}\right)$ ....(i)

Put y = vx. Then,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Then (i) can be written as

$$v + x\frac{dv}{dx} = \phi(v)$$

$$\Rightarrow \quad \frac{dv}{\phi(v) - v} = \frac{dx}{x}$$

5. The solution of differential equation $\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$ is

(a) $x^2 = 2y^2 (\ell n\, x + c)$ (b) $2x^2 = y^2 (\ell n\, x + c)$

(c) $y^2 = 2x^2 (\ell n\, x + c)$ (d) $y^2 = x^2 (\ell n\, x + c)$

Ans: c

Solution

Given differential equation is

$$xy\frac{dy}{dx} = x^2 + y^2$$

or, $\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$ ....(i)

Put $y = vx$ then

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Then, from (i)

$$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{x \cdot vx}$$

or, $$v + x\frac{dv}{dx} = \frac{1 + v^2}{v}$$

or, $$x\frac{dv}{dx} = \frac{1 + v^2}{v} - v$$

or, $$x\frac{dv}{dx} = \frac{1 + v^2 - v^2}{v}$$

or, $$x\frac{dv}{dx} = \frac{1}{v}$$

or, $$vdv = \frac{dx}{x}$$

Integrating, we have

$$\frac{v^2}{2} = \ln x + c$$

or, $$\left(\frac{y}{x}\right)^2 = 2(\ln x + c)$$

$\therefore$ $$y^2 = 2x^2(\ln x + c)$$

EXERCISE – 5 (C)

Solve the following differential equations by reducing to exact form.

1. $x\,dy + y\,dx = 0$

Solution

$$xdy + ydx = 0$$

or, $$d(xy) = 0$$

Integrating, we have.

$$xy = c$$

2. $2xy\,dy + y^2\,dx = 0$

Solution

$$ydx - xdy = 0$$

Dividing both sides by y^2, we get,

$$\frac{ydx - xdy}{y^2} = 0$$

or, $$d\left(\frac{x}{y}\right) = 0$$

Integrating, we get,

$$\frac{x}{y} = c$$

$\therefore$ $$x = cy$$

3. $y\,dx - x\,dy = 0$

Solution

$$2xydy + y^2dx = 0$$

or, $$x.2ydy + y^2\,dx = 0$$

or, $$xd(y^2) + y^2\,d(x) = 0$$

or, $$d(xy^2) = 0$$

Integrating, we get

$\therefore$ $$xy^2 = c$$

4. $2xy\,dx - x^2\,dy = 0$

Solution

$$2xydx - x^2dy = 0$$

or, $$y.d(x^2) - x^2d(y) = 0$$

Dividing both sides by y^2

$$\frac{yd(x) - x^2d(y)}{y^2} = 0$$

or, $$d\left(\frac{x^2}{y}\right) = 0$$

Integrating, we get $\frac{x^2}{y} = c$

$\therefore$ $$x^2 = cy$$

5. $y\,dx + (x + y)\,dy = 0$

Solution

$$y\,dx + (x + y)\,dy = 0$$

or, $y\,dx + x\,dy + y\,dy = 0$

or $(y\,dx + x\,dy) + y\,dy = 0$

or $d(xy) + d\left(\frac{y^2}{2}\right) = 0$

or, $d\left(xy + \frac{y^2}{2}\right) = 0$

Integrating, we get, $xy + \frac{y^2}{2} = \frac{c}{2}$

$\therefore\quad 2xy + y^2 = c$

6. $(2xy + y^2)\,dy + (y^2 + x)\,dx = 0$

Solution

$$2xy\,dy + y^2\,dy + y^2\,dx + x\,dx = 0$$

or, $2xy\,dy + y^2\,dx + y^2\,dy + x\,dx = 0$

or, $x\,d(y^2) + y^2\,d(x) + d\left(\frac{y^3}{3}\right) + d\left(\frac{x^2}{2}\right) = 0$

or, $d(xy^2) + d\left(\frac{y^3}{3}\right) + d\left(\frac{x^2}{2}\right) = 0$

or, $d\left(xy^2 + \frac{y^3}{3} + \frac{x^2}{2}\right) = 0$

Integrating, we get,

$$xy^2 + \frac{y^3}{3} + \frac{x^2}{2} = \frac{c}{6}$$

or, $6xy^2 + 2y^3 + 3x^2 = c$

$\therefore\quad 3x^2 + 6xy^2 + 2y^3 = c$

7. $\frac{dy}{dx} = \frac{y - x + 1}{y - x + 5}$

Solution

Given,

$$\frac{dy}{dx} = \frac{y - x + 1}{y - x + 5}$$

or, $ydy - xdy + 5dy = ydx - xdx + dx$

or, $xdx + ydy - xdy - ydx - dx + 5dy = 0$

or, $xdx + ydy - (xdy + ydx) - dx + 5dy = 0$

or, $d\left(\frac{x^2}{2}\right) + d\left(\frac{y^2}{2}\right) - d(xy) - d(x) + d(5y) = 0$

or, $d\left(\frac{x^2}{2} + \frac{y^2}{2} - xy - x + 5y\right) = 0$

Integrating, we get,

$$\frac{x^2}{2} + \frac{y^2}{2} - xy - x + 5y = \frac{c}{2}$$

$\therefore\quad x^2 + y^2 - 2xy - 2x + 10y = c$

8. $(x^2 + 5xy^2)dx + (5x^2y + y^2)dy = 0$

Solution

$$(x^2 + 5xy^2)dx + (5x^2y + y^2)dy = 0$$

or, $x^2\,dx + 5xy^2\,dx + 5x^2\,y\,dy + y^2\,dy = 0$

or, $x^2\,dx + 5(xy^2\,dx + x^2\,y\,dy) + y^2\,dy = 0$

or, $d\left(\frac{x^3}{3}\right) + d\left(\frac{y^3}{3}\right) + 5.\frac{1}{2}\,d(x^2\,y^2) = 0$

or, $d\left(\frac{x^3}{3} + \frac{y^3}{3} + \frac{5}{2}x^2y^2\right) = 0$

Integrating, we get,

$$\frac{x^3}{3} + \frac{y^3}{3} + \frac{5x^2\,y^2}{2} = \frac{c}{6}$$

$\therefore\quad 2x^3 + 2y^3 + 15x^2y^2 = c$

9. $\sin x \cos x\,dx + \sin y \cos y\,dy = 0$

Solution

$$\sin x \cos x\,dx + \sin y \cos y\,dy = 0$$

or, $2 \sin x \cos x\,dx + 2 \sin y \cos y\,dy = 0$

or, $d(\sin^2 x) + d(\sin^2 y) = 0$

or, $d(\sin^2 x + \sin^2 y) = 0$

Integrating, we get,

$$\sin^2 x + \sin^2 y = c$$

Objective Questions

1. **Which of the following differential equation is not exact?**

(a) $x\,dy + y\,dx = 0$ (b) $x\,dx + y\,dy = 0$

(c) $x\,dx - y\,dy = 0$ (d) $x\,dy - y\,dx = 0$

Ans: d

Option (d) should multiply both sides by $\frac{1}{x^2}$ to make it exact

2. **Which of the following equation has the solution $y = cx$?**

(a) $y\,dx - x\,dy = 0$ (b) $x\,dy - y\,dx = 0$

(c) $x\,dy + y\,dx = 0$ (d) $x\,dx + y\,dy = 0$

Ans: b

$$y = cx$$

$$\frac{y}{x} = c$$

Differentiating both sides w.r. to x,

$$\frac{x\frac{dy}{dx} - y\frac{d}{dx}(x)}{x^2} = 0$$

$$x\frac{dy}{dx} - y = 0$$

$$xdy - ydx = 0$$

3. **The solution of differential equation $y\,dx + x\,dy = 0$ is**

(a) $x + y = c$ (b) $x - y = c$

(c) $xy = c$ (d) $\frac{x}{y} = c$

Ans: c

Given, $y.\,dx + x\,dy = 0$

or, $d(xy) = 0$

Integrating, we get, $xy = c$

4. **The solution of differential equation $2xy\,dy - y^2\,dx = 0$ is**

(a) $y^2 = cx$ (b) $x = cy$

(c) $y = cx$ (d) $x^2 = cy$

Ans: a

Given, $2xy\,dy - y^2\,dx = 0$

$$x\,d(y^2) - y^2\,dx = 0$$

Diving both sides by x^2, we get,

$$\frac{x\,d(y^2) - y^2\,d(x)}{x^2} = 0$$

or, $d\left(\frac{y^2}{x}\right) = 0$

Integrating, we have,

$$\frac{y^2}{x} = c$$

$\therefore$ $y^2 = cx$

5. **The solution of differential equation $(x^2 - ay)dx + (y^2 - ax)dy = 0$ is**

(a) $x^3 + y^3 = c$ (b) $x^3 + y^3 + 3axy = c$

(c) $x^3 + y^3 - 3axy = c$ (d) $x^3 + y^3 + a^3 = c$

Ans: c

$$x^2\,dx - ay\,dx + y^2\,dy - ax\,dy = 0$$

$$d\left(\frac{x^3}{3}\right) + d\left(\frac{y^3}{3}\right) - a(y\,dx + x\,dy) = 0$$

or, $d\left(\frac{x^3}{3}\right) + d\left(\frac{y^3}{3}\right) - a\,d(xy) = 0$

or, $d\left(\frac{x^3}{3}\right) + d\left(\frac{y^3}{3}\right) - d(axy) = 0$

or, $d\left(\frac{x^3}{3} + \frac{y^3}{3} - axy\right) = 0$

Integrating, we get,

$$\frac{x^3}{3} + \frac{y^3}{3} - axy = \frac{c}{3}$$

$\therefore$ $x^3 + y^3 - 3axy = c$

EXERCISE – 5 (D)

Solve the following linear differential equations.

1. $\frac{dy}{dx} + y = 1$

Solution

$$\frac{dy}{dx} + y = 1 \qquad \ldots (i)$$

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get,

$P = 1, Q = 1$

$I.F.\ e^{\int Pdx} = e^{\int 1\, dx} = e^x.$

Multiplying both sides of (i) by e^x, we have

$$e^x . \frac{dy}{dx} + e^x . y = e^x$$

or, $d(y . e^x) = e^x dx$

Integrating, we get,

$$ye^x = \int e^x dx + c$$

or, $ye^x = e^x + c$

$\therefore$ $y = 1 + ce^{-x}$

2. $\frac{dy}{dx} - y = e^x$

Solution

$$\frac{dy}{dx} - y = e^x \qquad \ldots (i)$$

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get,

$P = -1, Q = e^x$

$I.F.\ e^{\int Pdx}\ e^{\int -dx} = e^{-x}$

Multiplying both sides of (i0 by e^{-x}, we have,

$$e^{-x} \frac{dy}{dx} - e^{-x} . y = e^{-x} . e^x$$

or, $\frac{d}{dx}(y . e^{-x}) = 1$

or, $d(y . e^{-x}) = dx$

Integrating, we have,

$y . e^{-x} = x + c$

$\therefore$ $y = e^x (x + c)$

3. $\frac{dy}{dx} + \frac{y}{x} = x$

Solution

$$\frac{dy}{dx} + \frac{y}{x} = x \quad \text{...(i)}$$

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get,

$$P = \frac{1}{x}, Q = x$$

$$\text{I.F.} = e^{\int P\,dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

Multiplying both sides of (i) by x, we get

$$x.\frac{dy}{dx} + x.\frac{y}{x} = x.x$$

or, $\frac{d}{dx}(x.y) = x^2$

or, $d(xy) = x^2\,dx$

Integrating,

$$xy = \int x^2\,dx + c$$

or, $xy = \frac{x^3}{3} + c$

4. $x\frac{dy}{dx} + y = x^4$

Solution

$$x\frac{dy}{dx} + y = x^4$$

or, $\frac{dy}{dx} + \frac{y}{x} = x^3 \quad \text{...(i)}$

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get,

$$P = \frac{1}{x}, Q = x^3$$

$$\text{I.F.} = e^{\int P\,dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

Multiplying both sides of (i) by x,

$$x.\frac{dy}{dx} + x.\frac{y}{x} = x.x^3$$

or, $\frac{d}{dx}(x.y) = x^4$

or, $d(xy) = x^4\,dx$

Integrating, $xy = \frac{x^5}{5} + c$

5. $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$

Solution

Here, $(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2$

or $\frac{dy}{dx} + \frac{2x}{1 + x^2}y = \frac{4x^2}{1 + x^2} \quad \text{...(i)}$

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get

$$P = \frac{2x}{1+x^2} \text{ and } Q = \frac{4x^2}{1+x^2}$$

Now, $I.F = e^{\int Pdx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1 + x^2$

Multiplying both sides of (i) by $1 + x^2$, we get,

$$(1+x^2)\frac{dy}{dx} + 2xy = 4x^2$$

or, $\frac{d}{dx}\{y(1+x^2)\} = 4x^2$

or, $d\{y(1+x^2)\} = 4x^2\,dx$

Integrating, we get,

$$y(1+x^2) = \int 4x^2\,dx + c$$

or, $y(1+x^2) = 4 \cdot \frac{x^3}{3} + c$

$\therefore$ $y(1+x^2) = \frac{4x^3}{3} + c$

6. $\frac{dy}{dx} + 2y\tan x = \sin x$

Solution

Given equation is $\frac{dy}{dx} + 2y\tan x = \sin x$ ... (i)

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get,

$P = 2\tan x$, $Q = \sin x$.

Now, $I.F = e^{\int Pdx} = e^{\int 2\tan x\,dx}$

$= e^{2\int \tan x\,dx}$

$= e^{2\log\sec x}$

$= e^{\log\sec^2 x}$

$= \sec^2 x$

Multiplying both sides of (i) by $\sec^2 x$

$$\sec^2 x.\frac{dy}{dx} + \sec^2 x.\ 2y.\ \tan x = \sec^2 x.\sin x$$

or. $d(y.\sec^2 x) = \sec x\tan x\,dx$

Integrating, we get

$$y\sec^2 x = \int \sec x\tan x\,dx + c$$

$\therefore$ $y\sec^2 x = \sec x + c$.

7. $\sin x\frac{dy}{dx} + y\cos x = x\sin x$

Solution

Given, $\sin x\frac{dy}{dx} + \cos x.y = x\sin x$

Dividing both sides by $\sin x$,

$$\frac{dy}{dx} + \frac{\cos x}{\sin x}y = x \quad \text{... (i)}$$

Comparing (i) with $\frac{dy}{dx} + Py = Q$, we get

$$P = \frac{\cos x}{\sin x},\ Q = x$$

$$I.F = e^{\int Pdx} = e^{\int \frac{\cos x}{\sin x}dx} = e^{\log\sin x} = \sin x$$

Multiplying both sides of equation (i) by $\sin x$

$$\sin x \frac{dy}{dx} + \cos x \cdot y = x \sin x$$

or, $d(y \cdot \sin x) = x \sin x \, dx$

Integrating,

$$y \sin x = \int x \sin x \, dx + c$$

or, $$y \sin x = x \int \sin x \, dx - \int \left[\frac{dx}{dx} \int \sin x \, dx\right] dx + c$$

or, $$y \sin x = -x \cos x - \int 1 \, (-\cos x) \, dx + c$$

or, $y \sin x = -x \cos x + \sin x + c$

$\therefore$ $y \sin x = \sin x - x \cos x + c$

8. $\cos^2 x \dfrac{dy}{dx} + y = 1$

Solution

Here, $\cos^2 x \dfrac{dy}{dx} + y = 1$

Dividing both sides by $\cos^2 x$, we have,

$$\frac{dy}{dx} + \sec^2 x \cdot y = \sec^2 x \qquad \ldots (i)$$

Comparing equation (i) with $\dfrac{dy}{dx} + Py = Q$, we get,

$P = \sec^2 x$, $Q = \sec^2 x$

$I.F = e^{\int P dx} = e^{\int \sec^2 x \, dx} = e^{\tan x}$

Multiplying equation (i) both sides by $e^{\tan x}$, we get

$$e^{\tan x} \frac{dy}{dx} + e^{\tan x} \sec^2 x \cdot y = e^{\tan x} \sec^2 x$$

$$d(y \cdot e^{\tan x}) = e^{\tan x} \sec^2 x \, dx$$

Integrating, we have,

$$y \cdot e^{\tan x} = \int d(e^{\tan x}) + c$$

or, $y \cdot e^{\tan x} = e^{\tan x} + c$

$\therefore$ $y = 1 + c \, e^{-\tan x}$

9. $(1 + x^2) \dfrac{dy}{dx} + y = e^{\tan^{-1} x}$

Solution

Given equation is: $(1 + x^2) \dfrac{dy}{dx} + y = e^{\tan^{-1} x}$

or, $$\frac{dy}{dx} + \frac{1}{1 + x^2} y = \frac{e^{\tan^{-1} x}}{1 + x^2} \qquad \ldots (i)$$

Comparing equation (i) with $\dfrac{dy}{dx} + Py = Q$, we get

$$P = \frac{1}{1 + x^2}, \; Q = \frac{e^{\tan^{-1} x}}{1 + x^2}$$

$$I.F = e^{\int P dx} = e^{\int \frac{1}{1+x^2} dx} = e^{\tan^{-1} x}$$

Multiplying both sides of (i) by $e^{\tan^{-1}x}$, we have

$$e^{\tan^{-1}x}\frac{dy}{dx} + e^{\tan^{-1}x}\cdot\frac{1}{1+x^2}\cdot y = \frac{(e^{\tan^{-1}x})^2}{1+x^2}$$

or, $\quad d(y\,e^{\tan^{-1}x}) = \frac{(e^{\tan^{-1}x})^2}{1+x^2}\,dx$

Integrating, we get

$$y\,e^{\tan^{-1}x} = \int\frac{(e^{\tan^{-1}x})^2}{1+x^2}\,dx$$

Put $\tan^{-1}x = z$

Then, $\frac{1}{1+x^2}\,dx = dz$

$$\int\frac{(e^{\tan^{-1}x})^2}{1+x^2}\,dx = \int e^{2z}\,dz = \frac{1}{2}e^{2z} + c = \frac{1}{2}(e^{\tan^{-1}x})^2 + c$$

$$\therefore \quad y\,e^{\tan^{-1}x} = \frac{1}{2}(e^{\tan^{-1}x})^2 + c$$

$$y = \frac{1}{2}e^{\tan^{-1}x} + c\,e^{-\tan^{-1}x}$$

Objective Questions

1. **Which of the following is not linear differential equation of first order and first degree?**

(a) $y' + y = x$ (b) $y' + \frac{y}{x} = 2x$

(c) $y' - y = xy^2$ (d) $y' + 2y = e^x$

Ans: c

It is not in the form $\frac{dy}{dx} + Py = Q$ where P and Q are function of x only or constant. But, here $Q = xy^2$ which is a function of x and y both.
So, it is not linear.

2. **The integrating factor of the linear differential equation $\frac{dy}{dx} + P\cdot y = Q$, where P and Q are constants or functions of x only, is**

(a) $e^{\int Q\,dx}$ (b) $e^{\int PQ\,dx}$

(c) $e^{\int (P+Q)dx}$ (d) $e^{\int P\,dx}$

Ans: d

(formula)

3. **The differential equation $\frac{dy}{dx} + P\cdot y = Q$ is said to be linear differential equation if**

(a) P and Q both are functions of x and y both.
(b) Q must be function of y.
(c) P must be function of y.
(d) P and Q are constants or functions of x only.

Ans: d

4. **The integrating factor of differential equation $\frac{dy}{dx} + \frac{y}{x} = e^x$ is**

(a) $\frac{1}{x}$ (b) $-\frac{1}{x}$

(c) x (d) $-x$

Ans: c

Comparing $\frac{dy}{dx} + \frac{y}{x} = e^x$ with $\frac{dy}{dx} + Py = Q$, we get

$$P = \frac{1}{x}$$

$$I.F. = e^{\int P\,dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

5. The integrating factor of differential equation $\frac{dy}{dx} + y \tan x = \sec x$ is

(a) $\tan x$ (b) $\sec x$
(c) $\sin x$ (d) $\cos x$

Ans: b

Here, $p = \tan x$

$I.F. = e^{\int \tan x\,dx} = e^{\log \sec x} = \sec x.$

6. The solution of differential equation $\frac{dy}{dx} + y = e^x$

(a) $y = \frac{1}{2}e^x + ce^{-x}$ (b) $y = e^x + ce^{-x}$

(c) $y = \frac{1}{2}e^x + c$ (d) $y = \frac{1}{2}e^{-x} + ce^x$

Ans: a

Given, $\frac{dy}{dx} + y = e^x$ ... (i)

Here, $P = 1$

$I.F. = e^{\int P\,dx} = e^{\int dx} = e^x$

Multiplying equation (i) by e^x, we get,

$$d(y.\,e^x) = e^x.\,e^x\,dx$$

or, $$d(y.\,e^x) = e^{2x}\,dx$$

Integrating,

$$ye^x = \frac{e^{2x}}{2} + c$$

$$y = \frac{1}{2}e^x + ce^{-x}$$

❑

6
UNIT

Computational Method

EXERCISE – 6 (A)

1. **Draw the graph of the following inequalities.**

   (a) $2x + 1 > x + 3$ (b) $2x + 1 \le 3$

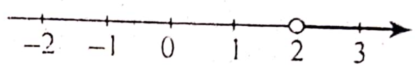
   (c) $3x + y < 6$ (d) $4x + y \ge 8$

Solution

(a) $2x + 1 > x + 3$

or, $2x - x > 3 - 1$

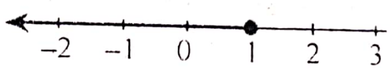
or, $x > 2$.



(b) $2x + 1 \le 3$

or, $2x \le 2$

or, $x \le 1$



(c) $3x + y < 6$

The corresponding equation of boundary line is $3x + y = 6$.

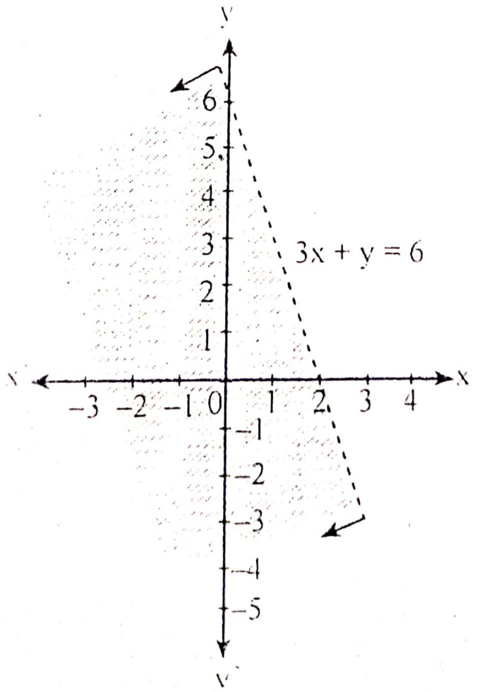
x	0	2
y	6	0

Taking testing point (0, 0) in $3x + y < 6$, we get,

$3 \times 0 + 0 < 6$

i.e $0 < 6$ (true)

So, the solution of $3x + y < 6$ is the plane region containing the origin but not including the boundary line.



d. $4x + y \ge 8$

The corresponding equation of boundary line is

$4x + y = 8$

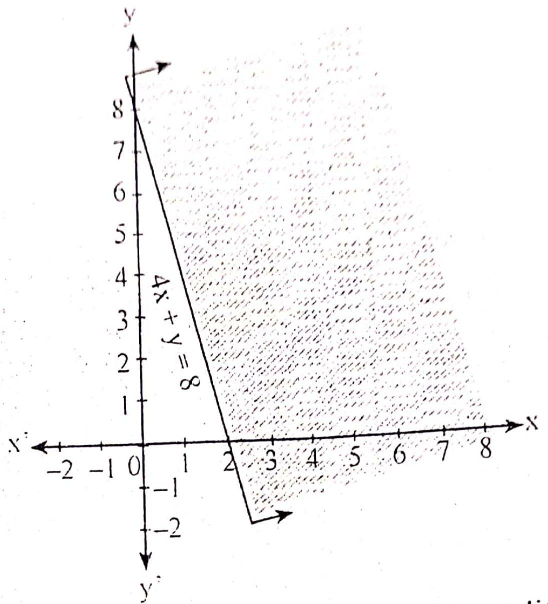
x	0	2
y	8	0

Taking testing point (0, 0) in $4x + y \geq 8$, we get.

$4 \times 0 + 0 \geq 8$

i.e. $0 \geq$ (false)

So, the solution set of $4x + y \geq 8$ is the closed plane region not containing the origin.



2. **Draw the graph of the following system of linear inequalities.**

(a) $x \geq 5, y \geq 2$ **(b)** $x - y \leq 2, x + y \leq 4$

Solution

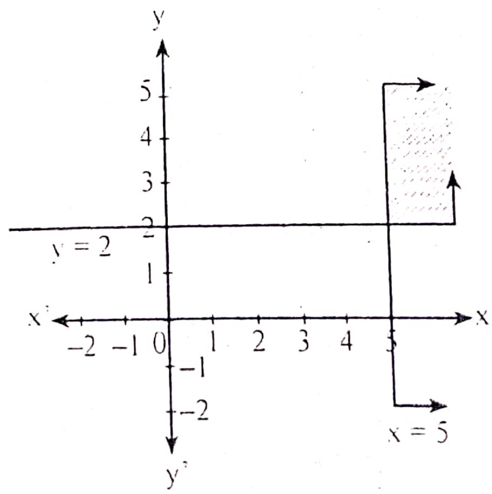
a. The corresponding equations of boundary lines are

$x = 5$ . . . (i)

$y = 2$ . . . (ii)

From (i) $x = 5$ is a line parallel to y – axis and through the point (5, 0). Taking testing point (0, 0) in $x \geq 5$, we get $0 \geq 5$ (false)

From (ii) $y = 2$ is a line parallel to x – axis and through the point (0, 2). Taking testing point (0, 0) in $y \geq 2$, we get, $0 \geq 2$ (false)



b. The corresponding equations of boundary lines are

$x - y = 2$ . . . (i)

$x + y = 4$ . . . (ii)

From (i), $x - y = 2$

x	0	2
y	– 2	0

Taking testing point (0, 0) in $x - y \le 2$, we get,

$0 - 0 \le 2$

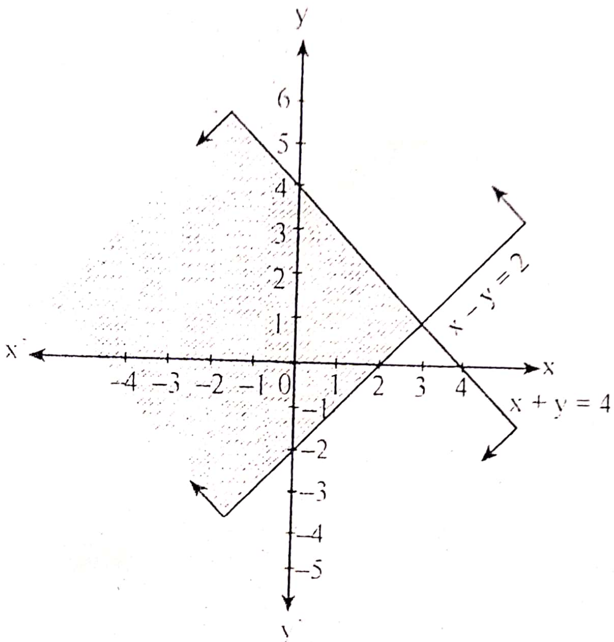
i.e. $0 \le 2$ (true)

From (ii), $x + y = 4$

x	0	4
y	4	0

Taking testing point (0, 0) in $x + y \le 4$, we get.

$0 + 0 \le 4$ (true)



3. **Graph the following system of inequalities and find the vertices if they exist.**

(a) $x + y \le 6, x \ge 0, y \ge 0$

(b) $3x + 4y \le 24, x \ge 2, y \ge 1$

(c) $x + y \le 6, x - y \ge -2, x \ge 0, y \ge 0$

Solution

(a) The corresponding equations of boundary lines are

$x + y = 6$ ... (i)

$x = 0$ ... (ii)

$y = 0$ ... (iii)

From (i), $x + y = 6$

x	0	6
y	6	0

Taking testing point (0, 0) in $x + y \le 6$, we get

$0 + 0 \le 6$ (true)

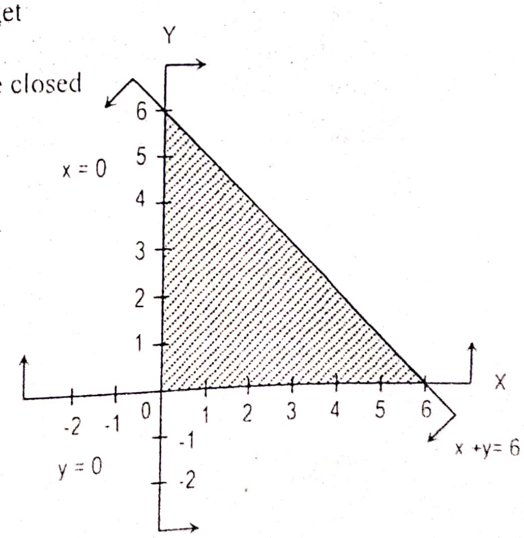
So, the region determined by $x + y \le 6$ is the closed half plane containing the origin

From (ii) $x = 0$ which is y–axis

And, $x \ge 0$ gives the closed right half plane.

From (iii) $y = 0$ which is x-axis.

And $y \ge 0$ gives the closed upper half plane.



(b) The corresponding equations of boundary lines are

$3x + 4y = 24$ ...(i)

$x = 2$ ...(ii)

$y = 1$ ...(iii)

From (i) $3x + 4y = 24$

x	0	8
y	6	0

Taking testing point (0, 0) in $3x + 4y \le 24$, we get

$3 \times 0 + 4 \times 0 \le 24$ (true)

From (ii) $x = 2$

x	2	2
y	0	1

Taking testing point (0, 0) in $x \ge 2$, we get

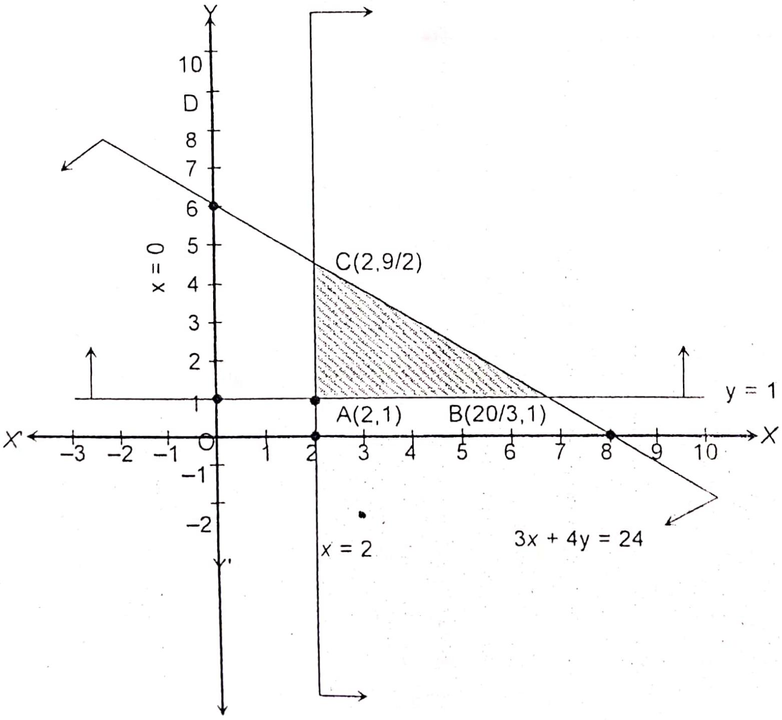
$0 \ge 2$ (false)

From (iii) $y = 1$

x	0	3
y	1	1

Taking testing point (0, 0) in $y \ge 1$, we get

$0 \ge 1$ (false)



(c) The corresponding equations of given inequalities are

$x + y = 6$ ... (i)

$x - y = -2$ ... (ii)

$x = 0$ ... (iii)

$y = 0$ ... (iv)

From (i), $x + y - 6$

x	0	6
y	6	0

Taking testing point (0, 0) in $x + y \le 6$, we get

$0 + 0 \le 6$ (true)

So, it contains origin.

From (ii), $x - y = -2$

x	0	-2
y	2	0

Taking testing point (0, 0) in $x - y \geq -2$, we get

$0 - 0 \geq -2$

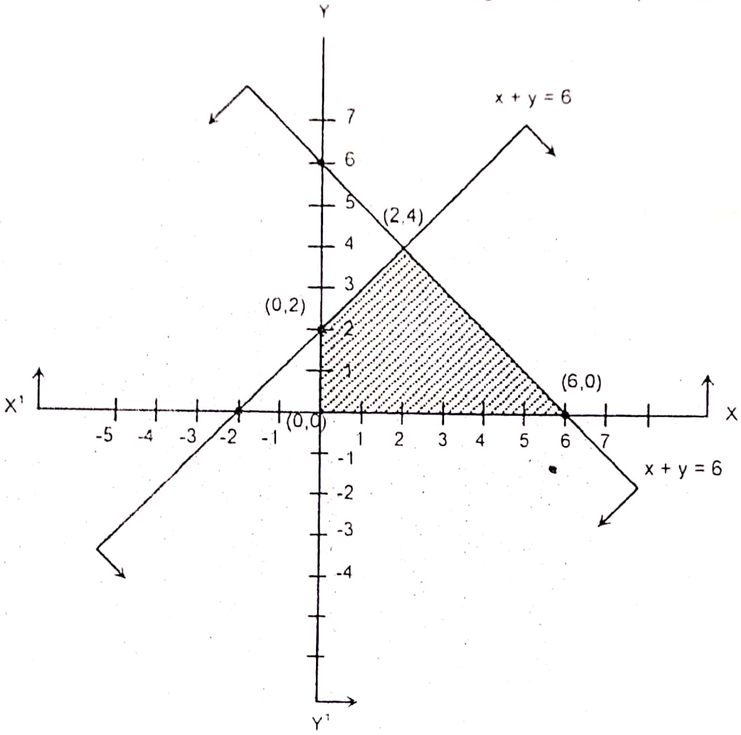
$0 \geq -2$ (true)

So, it contains origin

From (iii) $x = 0$ which is y – axis

From (iv) $y = 0$ which is x – axis

And, $x \geq 0$, $y \geq 0$ means we have to consider the region in first quadrant only.



4. **Solve the following linear programming problem using graphic method.**

(a) **Maximize $Z = x + y$ subject to the constraints**

$2x + y \leq 14$; $x + 2y \leq 10$, $x, y \geq 0$

(b) **Minimize $F = 7x + 6y$ subject to the constraints**

$4x - 9y \leq 36$, $x \geq 9$, $y \leq 4$

(c) **Maximize and minimize $Z = 45x + 80y$ subject to**

$x + 2y \leq 7$, $x - y \leq 4$, $x \geq 0$, $y \geq 0$

(d) **Find the extreme values of the objective function $F = 16x - 2y + 40$ subject to constraints**

$3x + 4y \leq 24$, $0 \leq y \leq 4$ and $0 \leq x \leq 7$

(e) **Optimize $F = x + 2y$ subject to**

$2x + y \leq 7$, $x \leq y$, $x \geq 1$

Solution

(a) Given inequalities are

$2x + y \leq 14$

$x + 2y \leq 10$

$x, y \geq 0$

The corresponding equations of the given inequalities are

$2x + y = 14$ ... (i)

$x + 2y = 10$ ... (ii)

$x = 0$ ... (iii)

$y = 0$ ... (iv)

From (i), $2x + y = 14$

When $x = 0$, $y = 14$

When $y = 0$, $x = 7$

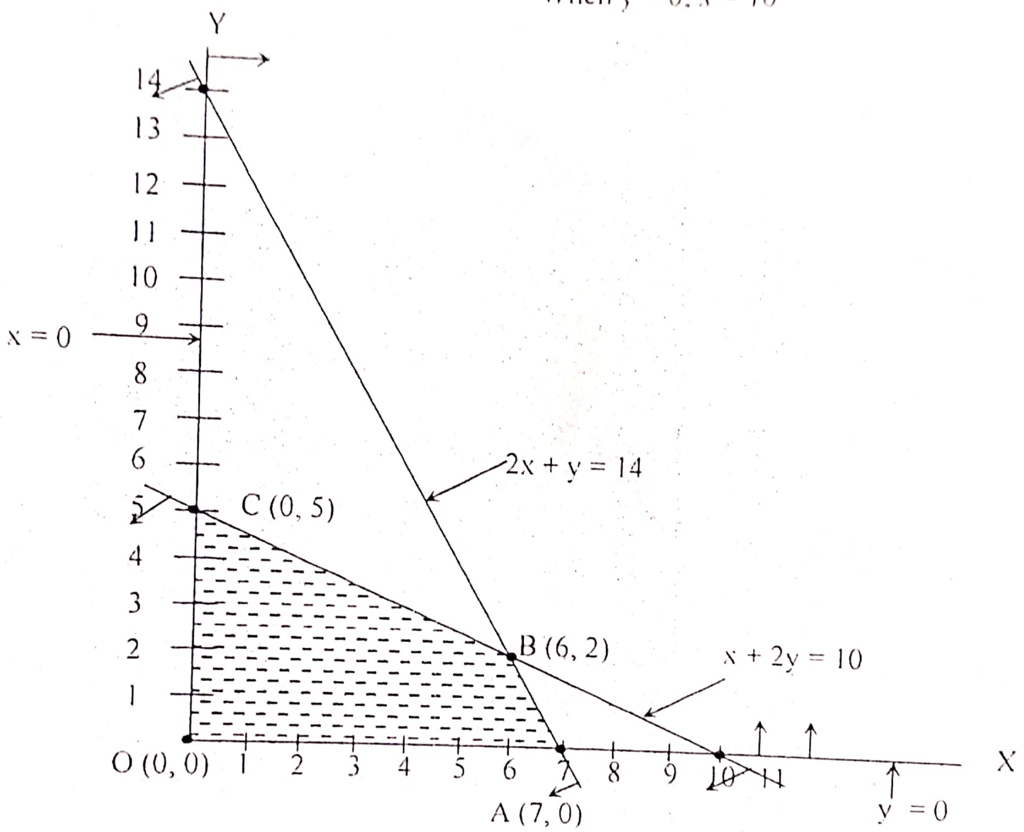
$\therefore$ The boundary line (i) passes through (0, 14) and (7, 0)

Taking testing point (0, 0) i.e. put $x = 0$, $y = 0$ in $2x + y \leq 14$ we get $0 + 0 \leq 14$ i.e. $0 \leq 14$ which is true.

$\therefore$ The graph of $2x + y \leq 14$ is the plane region containing the origin.

From (ii), $x + 2y = 10$

When $x = 0$; $y = 5$ When $y = 0$; $x = 10$



$\therefore$ The boundary line (ii) passes through (0, 5) and (10, 0)

Taking testing point (0, 0) i.e. put $x = 0$; $y = 0$ in $x + 2y \leq 10$

We have, $0 + 0 \leq 10$

$0 + 0 \leq 10$ which is true.

Hence its graph is the plane region containing the origin.

$x = 0$ is the y–axis and $y = 0$ is the x–axis

$x \geq 0$ is the right half plane containing y–axis and

$y \geq 0$ is the upper half plane containing x–axis

The feasible region OABC is shown in the graph. The vertices of the feasible region OABC are O (0, 0), A (7, 0), B (6, 2), and C (0, 5)

Vertices	x	y	$Z = x + y$
O (0, 0)	0	0	$z = 0 + 0 = 0$
A (7, 0)	7	0	$z = 7 + 0 = 7$
B (6, 2)	6	2	$z = 6 + 2 = 8$
C (0, 5)	0	5	$z = 0 + 5 = 5$

$\therefore$ Max. $Z = 8$ at (6, 2)

(b) The corresponding equations of boundary lines are

$4x - 9y = 36$ ...(i)

$x = 9$ ...(ii)

$y = 4$ ...(iii)

From (i) $4x - 9y = 36$

x	0	9
y	−4	0

Taking testing point (0, 0) in $4x - 9y \le 36$, we get

$4 \times 0 - 9 \times 0 \le 36$ (true)

Hence, the solution set of $4x - 9y \le 36$ is the plane region containing the origin

From (ii) $x = 9$

x	9	9
y	0	4

Taking testing point (0, 0) in $x \ge 9$, we get $0 \ge 9$ (false)

So, the solution set of $x \ge 9$ is the plane region without containing the origin

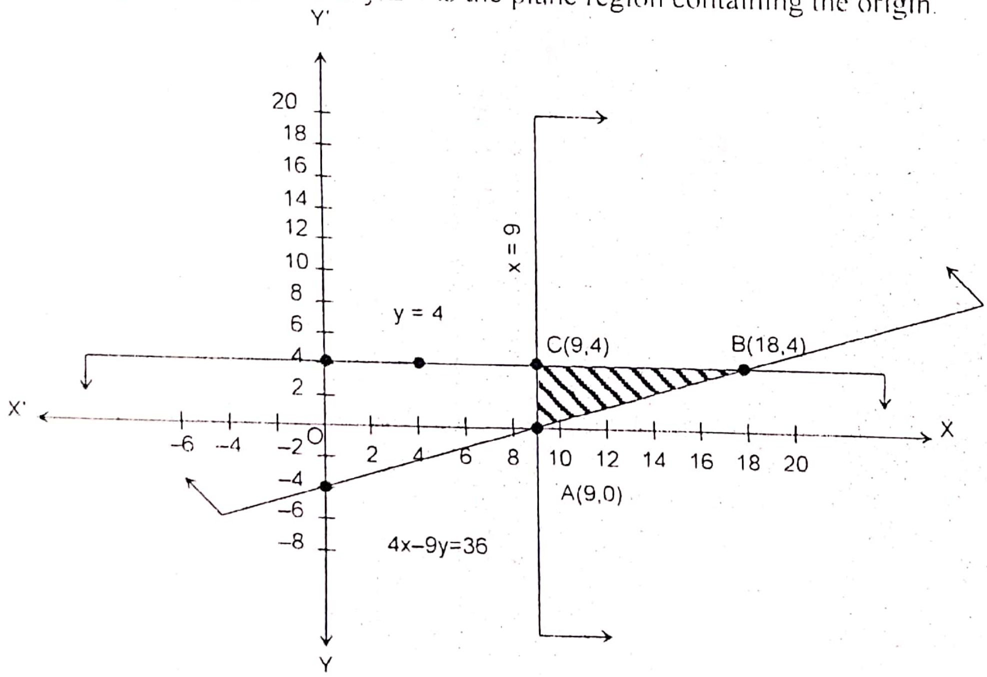
From (iii) $y = 4$

x	0	4
y	4	4

Taking testing point (0, 0) in $y \le 4$, we get

$0 \le 4$ (true)

Hence, the solution set of $y \le 4$ is the plane region containing the origin.



Here, ABC is the feasible region where the coordinates of A, B and C are (9,0), (18,4) and (9, 4) respectively.

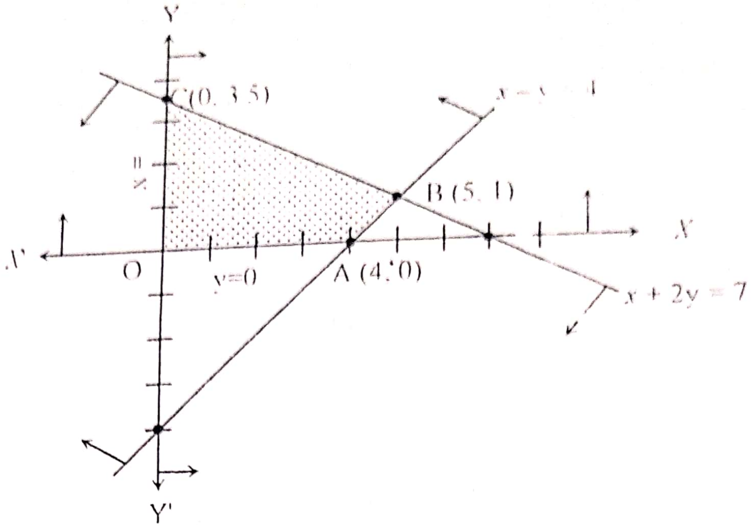
Vertices	Value of F = 7x + 6y
A (9, 0)	$F = 7 \times 9 + 6 \times 0 = 63$
B(18, 4)	$F = 7 \times 18 + 6 \times 4 = 150$
C(9, 4)	$F = 7 \times 9 + 6 \times 4 = 87$

Minimum value of F = 63 at (9, 0)

(c)

Linear inequalities	Boundary lines	Points	Testing points	Result
$x + 2y \le 7$	$x + 2y = 7$	(0, 3.5) (7, 0)	(0, 0)	$0 \le 7$ (True)
$x - y \le 4$	$x - y = 4$	(0, − 4) (4, 0)	(0, 0)	$0 \le 4$ (True)

The inequalities $x \ge 0$, $y \ge 0$ indicate that the common region lies in the first quadrant



Hence the common region is quadrilateral AOBC, which is shaded on the graph.

Vertices	$F(x, y) = 45x + 80y$
A(0, 3.5)	$F = 45 \times 0 + 80 \times 3.5 = 280$
B(4, 0)	$F = 45 \times 4 + 80 \times 0 = 180$
C(5, 1)	$F = 45 \times 5 + 80 \times 1 = 305$
O(0, 0)	$F = 45 \times 0 + 80 \times 0 = 0$

$\therefore$ Max. F = 305 at (5, 1)

Min F = 0 at (0, 0)

(d) The boundary equations of given inequalities are

$3x + 4y = 24$ ...(i)

$y = 0$ ...(ii)

$y = 4$ ...(iii)

$x = 0$ ... (iv)

$x = 7$ ...(v)

From (i) $3x + 4y = 24$

x	0	8
y	6	0

Taking testing point (0,0) in $3x + 4y \le 24$, we get

$3 \times 0 + 4 \times 0 \le 24$ (true)

The graph of $3x + 4y \le 24$ contains origin.

From (ii) $y = 0$ which is x-axis.

$y \ge 0$ gives the upper half plane containing x-axis.

From (iii) $y = 4$ is the line parallel to x-axis and through the point (0, 4)

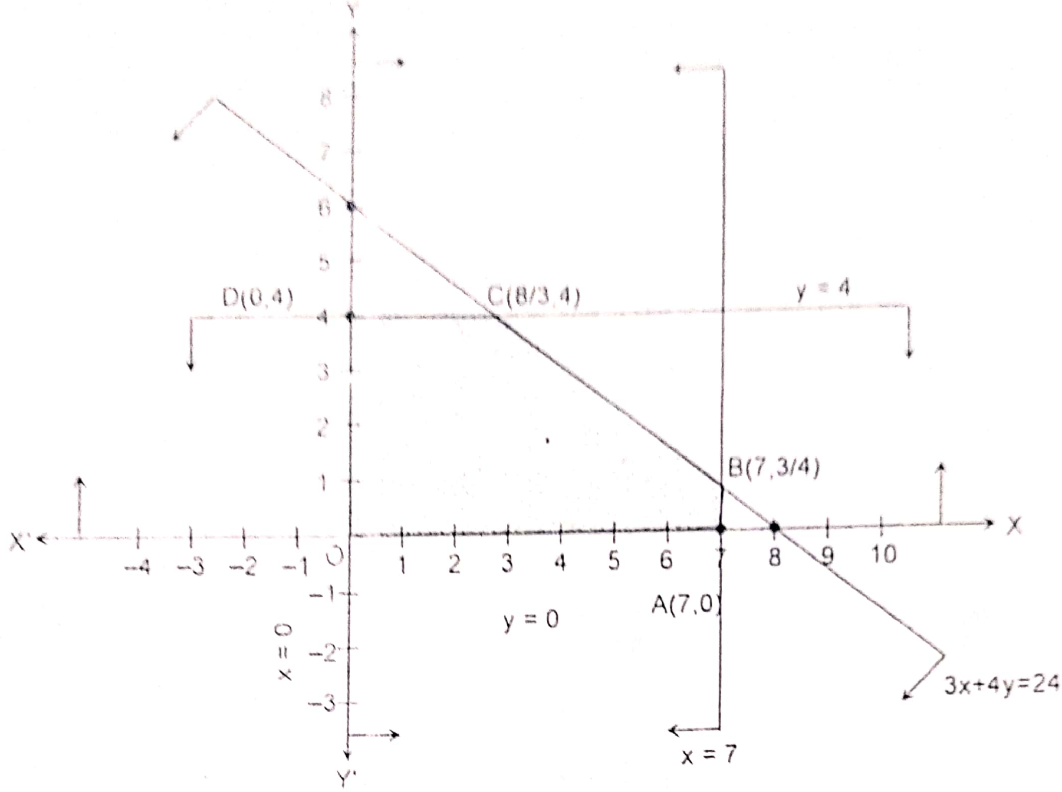
Taking testing point (0,0) in $y \le 4$, we get

$0 \le 4$ (true)

From (iv) $x = 0$ which is y-axis.

$x \ge 0$ gives the right half plane containing the y-axis

From (v) $x = 7$ is the line parallel to y-axis and through the point (7, 0)



The required solution is shaded in the figure.

Vertices	Value of F = 16x – 2y + 40
O(0,0)	$F = 16 \times 0 - 2 \times 0 + 40 = 40$
A(7,0)	$F = 16 \times 7 - 2 \times 0 + 40 = 152$
$B\left(7, \frac{3}{4}\right)$	$F = 16 \times 7 - 2 \times \frac{3}{4} + 40 = 150.5$
$C\left(\frac{8}{3}, 4\right)$	$F = 16 \times \frac{8}{3} - 2 \times 4 + 40 = 74.67$
D(0,4)	$F = 16 \times 0 - 2 \times 4 + 40 = 32$

$\therefore$ Max. value of F = 152 at (7, 0)

Min. value of F = 32 at (0, 4)

(e) The corresponding equation of boundary lines are

$2x + y = 7$ ... (i)

$x = y$ ... (ii)

$x = 1$ ... (iii)

From (i) $2x + y = 7$

x	0	3
y	7	1

Taking testing point (0, 0) in $2x + y \le 7$, we get,

$2 \times 0 + 0 \le 7$ (True)

From (ii) $x = y$

x	0	1
y	0	1

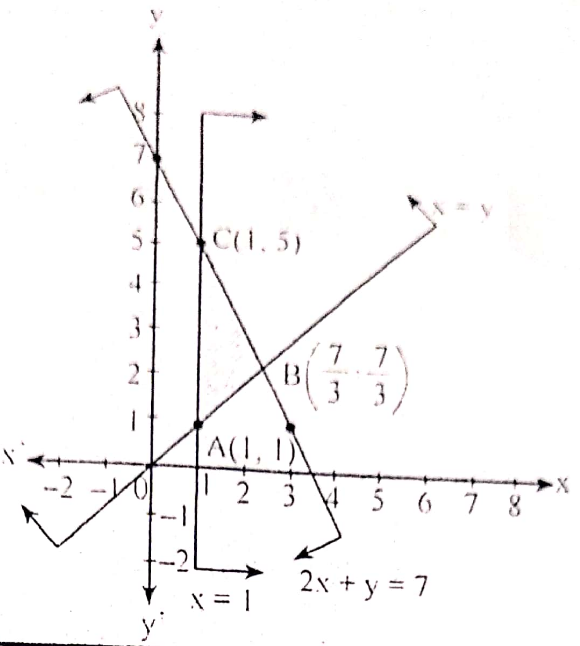
Taking testing point (1, 0) in $x \le y$, we get,

$1 \le 0$ (false)

From (iii) $x = 1$ which is a line parallel to y – axis and through the point (1, 0)

Taking testing point (0, 0) in $x \ge 1$, we get

$0 \ge 1$ (false)



Vertices	Value of F = x + 2y
A(1, 1)	$F = 1 + 2 \times 1 = 3$
B$\left(\frac{7}{3}, \frac{7}{3}\right)$	$F = \frac{7}{3} + 2 \times \frac{7}{3} = 7$
C(1, 5)	$F = 1 + 2 \times 5 = 11$

Max. F = 11 at (1, 7)

Min F = 3 at (1; 1)

5. An electric company produces two products A and B that are produced and sold on a weekly basis. Product A requires 2 men and B requires only one man as a labour. The company employees a total of 60 workers. The weekly production cannot exceed 25 for product A and 35 for product B because of limited available facilities. Profit margin on A is Rs. 60 and on B is Rs. 40. Formulate it as a linear programming problem and solve for maximum profit using graphic method.

Solution

Expressing the given information in the following tabular form.

	Product A	Product B	Total Workers
Men	2	1	60
Profit (Rs.)	60	40	
Maximum production unit	25	35	

Let x and y be the units produced of type A and type B respectively

Formulation of problem

For objective function:

Profit of 1 unit of A is Rs. 60

Profit of x units of A is Rs. 60x

Profit of 1 units of B is Rs. 40

Profit of y units of B is Rs. 40y

$\therefore$ Total profit (P) = Rs. 60x + Rs. 40y is to be maximized

For constraints.

The product of A cannot exceeds 25 units

$\therefore$ $x \le 25$

Similarly, the product of B cannot exceeds 35 units

$\therefore$ $y \le 35$

For men

1 unit of product A requires 2 men

x units of product A requires 2x men

Similarly,

1 unit of product B requires 1 man

y units of product B required y men

∴ The total required men = 2x + y

but, the company ha a 60 workers

∴ $2x + y \leq 60$

Non– negativity constraints:

x and y are the quantity of production so they cannot be negative $x \geq 0$, $y \geq 0$

Hence, the following is the mathematical formulation of the given problem

Maximize profit, P = Rs. 60x + Rs. 40y

Subject to constraints:

$x \leq 25$

$y \leq 35$

$2x + y \leq 60$

$x \geq 0, y \geq 0$

Graphical Solution

The corresponding equation of the given constraints are

$x = 25$ .... (i)

$y = 35$ .... (ii)

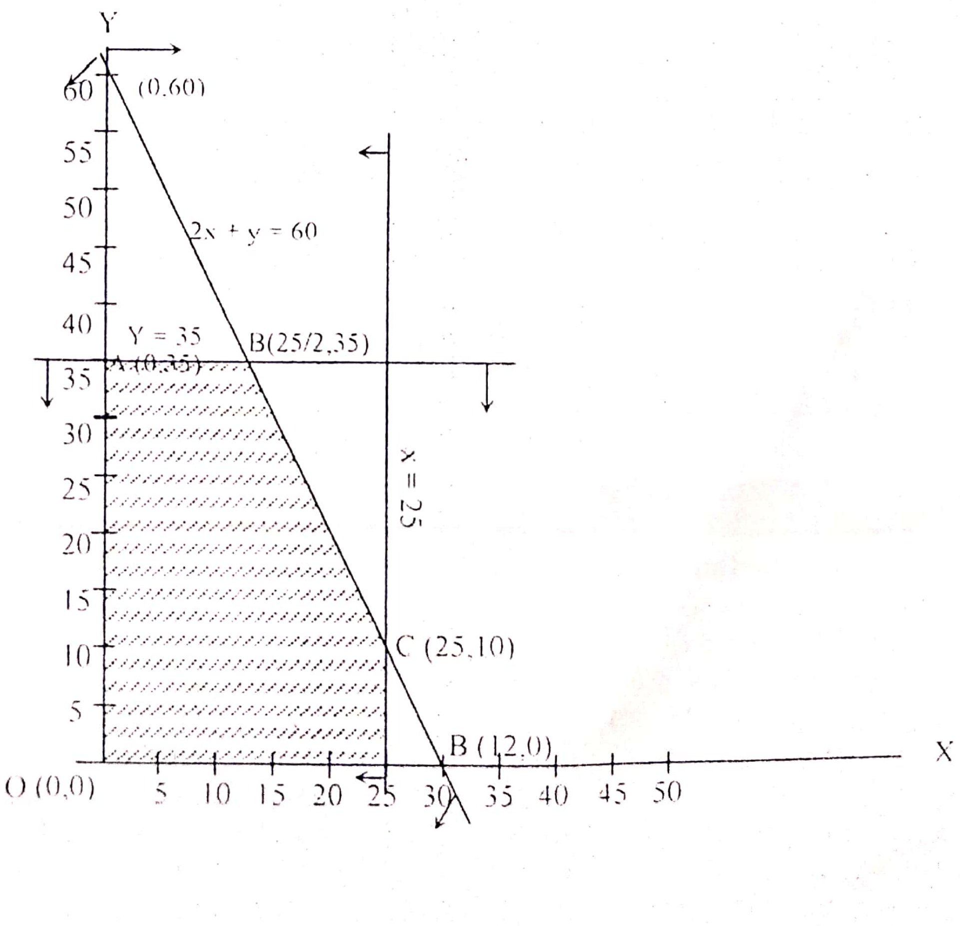
$2x + y = 60$ ... (iii)

x = 25 is a line parallel to y–axis at a distance of 25 from the y–axis

y = 35 is a line parallel to x–axis at a distance of 35 from the x–axis.

From equation (iii) when x = 0, y = 60 and when y = 0, x = 30

$x \geq 0$, $y \geq 0$ implies that all the points lie only in the first quadrant.



To determine the feasible region we use the origin (0, 0) as a test point

Put x = 0 and y = 0 in the given constraints then we have

$0 \le 25$, $0 \le 35$, $0 \le 60$ which are true

Hence, the solution set of the given constraints contains the origin

The shaded portion OABCD is the feasible region. The co-ordinates of the point B and C are obtained as B (25/2, 35) and C = (25, 10)

Calculation of value of the objective function

Corner points	Co-ordinates (x, y)	Objective function, P = 60x + 40y
O	(0, 0)	0
A	(35, 0)	2100
B	(25/2, 35)	$\boxed{2150}$
C	(25, 10)	1900
D	(25, 0)	1500

∴ The maximum value of P is 2150 at vertex B (25/2, 35)

The maximum profit is Rs. 2150 which is obtained by producing 25/2 = 12.5 units of product A and 35 units of product B.

6. **Two spare parts X and Y are to be produced in a batch. Each one has to go through two processes P and Q. The time required in hours per unit and total time available are given below:**

	X	Y	Total Hours Available
Process P	**3**	**4**	**24**
Process Q	**9**	**4**	**36**

Profit per unit of X and Y are Rs. 5 and Rs. 6 respectively. Find how many number of spare parts of X and Y are to be produced in this batch to maximize the profit.

Solution

	X	Y	Total hrs available
Process P	3	4	24
Process Q	9	4	36
Profit (Rs.)	5	6	

Let x and y be the number of spare parts of X and Y respectively, that are to be produced in the fetch.

The above problem in the LPP form be stated as

Total profit, Z = Rs. (5x + 6y) is to be maximized subject to constraints

$3x + 4y \le 24$

$9x + 4y \le 36$

$x \ge 0, y \ge 0$

Graphical Solution

The corresponding equation of the given constraints are

$3x + 4y = 24$ .... (i)

$9x + 4y = 36$ .... (ii)

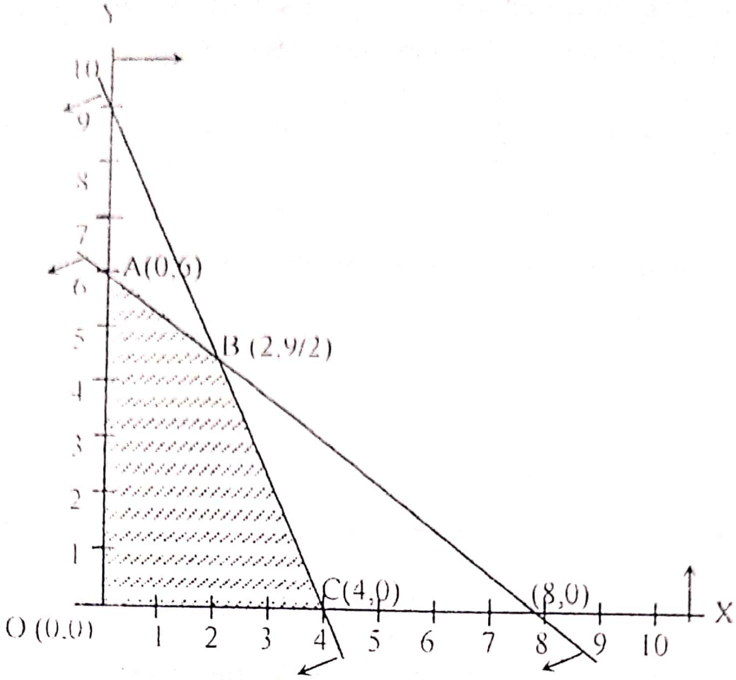
From equation (i), when x = 0, y = 6 and when y = 0, x = 8

∴ The line (i) passes through the points (0, 6) and (8, 0)

From equation (ii), when x = 0, y = 9 and y = 0, x = 4

∴ The line (ii) passes through the points (0, 9) and (4, 0)

$x \ge 0, y \ge 0$ implies that all the points of the solution set lies only on the first quadrant.



To determine the feasible region, we use origin (0, 0) as a test point. Put x = 0 and y = 0, in the given constraints then

$0 \le 24$, $0 \le 36$, which are true. Hence all the inequalities enclose the planes with origin

The shaded portion OABC is the feasible region. On solving equation (i) and (ii), we get the co-ordinates of B. The co-ordinates of point B is (2, 9/2)

Calculation of value of the objective function

Corner points	Co-ordinates (x, y)	Objective function, $Z = 5x + 6y$
O	(0, 0)	0
A	(0,6)	36
B	(2, 9/2)	37
C	(4, 0)	20

The maximum value of Z is 37 which is obtained at B (2, 9/2).

∴ The maximum profit is Rs. 37, which is obtained by producing 2 spare parts of type X and 4.5 spare parts of type Y.

Objective Questions

1. **In maximization problem, optimal solution at the corner point yields**
   (a) mean value of objective function (b) highest value of objective function
   (c) mid value of objective function (d) lowest value of objective function

Ans: b

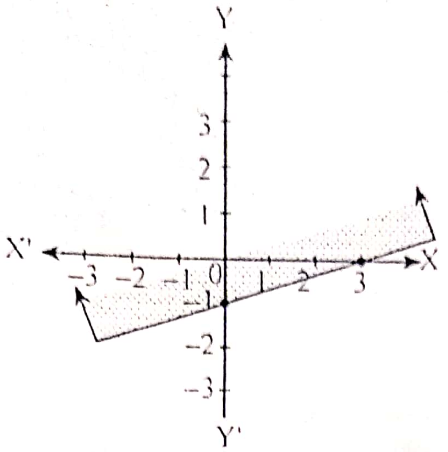
2. **The inequality $x \ge 0$ represents**
   (a) closed upper half plane (b) open upper half plane
   (c) closed right half plane (d) open right half plane

Ans: c

3. **The inequality $y \ge 0$ represents**
   (a) closed upper half plane (b) open upper half plane
   (c) closed right half plane (d) open right half plane

Ans: a

4.

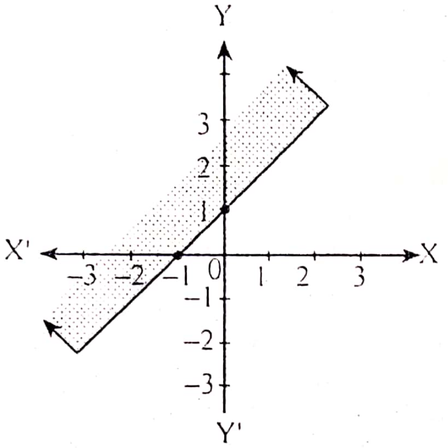


The inequality representing the above graph is

(a) $x - 3y \leq 3$ (b) $x - 3y < 3$
(c) $x - 3y \geq 3$ (d) $x - 3y > 3$

Ans: a

5.



The inequality representing the above graph is

(a) $y - x \leq 1$ (b) $y - x < 1$
(c) $y - x \geq 1$ (d) $y - x > 1$

Ans: c

6. The solution set of $4x + 3 \geq 2x - 1$ is

(a) $\{x : x > -2\}$ (b) $\{x : x \leq -2\}$
(c) $\{x : x < -2\}$ (d) $\{x : x \geq -2\}$

Ans: c

Here, $4x + 3 \geq 2x - 1$

$4x - 2x \geq -1 - 3$

or, $2x \geq -4$

or, $x \geq -2$

The solution set is $\{x : x \geq -2\}$

EXERCISE – 6 (B)

1. Using simplex method, find the optimal solutions of the following linear programming problems.

(a) Max. $Z = 7x_1 + 5x_2$
subject to $x_1 + 2x_2 \leq 6$
$4x_1 + 3x_2 \leq 12$
$x_1, x_2 \geq 0$

(b) Max. $Z = 9x + y$
subject to $2x + y \leq 8$
$4x + 3y \leq 18$
$x \geq 0, y \geq 0$

(c) Max. $P = x + 3y$
subject to $x + y \leq 4$
$x - y \leq 1$
$x \geq 0, y \geq 0$

(d) Max. $P = 50x_1 + 80x_2$
subject to $x_1 + 2x_2 \leq 32$
$3x_1 + 4x_2 \leq 84$
$x_1, x_2 \geq 0$

(e) Max. $F = x_1 + 7x_2$
subject to $x_1 - 2x_2 \geq -8$
$x_1 + 2x_2 \leq 4$
$x_1, x_2 \geq 0$

(f) Max. $Z = 4x_1 + 2x_2$
subject to $x_1 + x_2 \leq 50$
$x_1 \leq 40$
$x_1 \geq 0, x_2 \geq 0$

(g) Max. $Z = 2x_1 + 12x_2 + 8x_3$
subject to $2x_1 - 2x_2 + x_3 \leq 100$
$x_1 - 2x_2 + 5x_3 \leq 80$
$10x_1 + 5x_2 + 4x_3 \leq 300$
$x_1, x_2, x_3 \geq 0$

Solution

(a) Let x_3 and x_4 be non-negative slack variables. Adding the slack variables, we can write the given LPP in the following form

$x_1 + 2x_2 + x_3 = 6$
$4x_1 + 3x_2 + x_4 = 12$
$Z = 7x_1 + 5x_2$

$\Rightarrow$ $x_1 + 2x_2 + x_3 + 0 . x_4 + 0 . Z = 6$
$4x_1 + 3x_2 + 0 . x_3 + x_4 + 0 . Z = 12$
$-7x_1 - 5x_2 + 0 . x_3 + 0 . x_4 + Z = 0$

Simplex tableau

Basic Variables	x_1	x_2	x_3	x_4	Z	RHS(b)
x_3	1	2	1	0	0	6
x_4	$\boxed{4}$	3	0	1	0	12
	−7 ↑	−5	0	0	1	0

Here, −7 is the most negative entry in the last row. So, first column is the pivot column.
Since $\frac{6}{1} = 6, \frac{12}{4} = 3$ and $3 < 6$, so 4 is the pivot element.

Applying $R_2 \rightarrow \frac{1}{4} R_2$

Basic Variables	x_1	x_2	x_3	x_4	Z	RHS(b)
x_3	1	2	1	0	0	6
x_1	1	3/4	0	1/4	0	3
	−7	−5	0	0	1	0

Applying $R_1 \rightarrow R_1 - R_2$ and $R_3 \rightarrow R_3 + 7R_2$

Basic Variables	x_1	x_2	x_3	x_4	Z	RHS(b)
x_3	0	5/4	1	−1/4	0	3
x_1	1	3/4	0	1/4	0	3
	0	1/4	0	7/4	1	21

Since all the entries in the last row are non-negative, so the solution is optimal.
Max. value of $Z = 21$ when $x_1 = 3$ and $x_2 = 0$

(b) Let r and s be non - negative sack variables. Then given LPP can be written as

2x + y + r = 8
4x + 3y + s = 18
z = 9x + y

$\Rightarrow$ 2x + y + r + 0 s + 0 Z = 8
4x + 3y + 0 r + s + 0 Z = 18
− 9x − y + 0 r + 0 s + Z = 0

Simple tableau

Basic variables	x	y	r	s	Z	RHS (b)
r	$\boxed{2}$	1	1	0	0	8
s	4	3	0	1	0	18
	−9 ↑	−1	0	0	1	0

Here, – 9 is the most negative entry in last row. So, first column is the pivot column.

Since $\frac{8}{2} = 4$, $\frac{18}{4} = 4.5$, $4 < 4.5$. So 2 is the pivot element.

Applying $R_1 \to \frac{1}{2} R_1$

Basic variables	x	y	r	s	Z	RHS (b)
x	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	4
s	4	3	0	1	0	18
	–9	–1	0	0	1	0

Applying $R_2 \to R_2 - 4R_1$, $R_3 \to R_3 + 9R_1$

Basic variables	x	y	r	s	Z	RHS (b)
x	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	4
s	0	1	– 2	1	0	2
	0	$\frac{7}{2}$	$\frac{9}{2}$	0	1	36

Since the last row contains all non-negative entries, so optimal solution is obtained.

∴ Max. $z = 36$ at $x = 4$, $y = 0$

(c) Let r and s be the non-negative slack variables. Adding the slack variables, we can express the given LPP in the following form:

$x + y + r = 4$

$x - y + s = 1$

$x + 3y = p$

⇒ $x + y + r + 0.s + 0.p = 4$

$x - y + 0.r + s + 0.p = 1$

$-x - 3y + 0.r + 0.s + p = 0$

Simplex Tableau

x	y	r	s	P	RHS
1	$\boxed{1}$	1	0	0	4
1	-1	0	1	0	1
–1	–3	0	0	1	0
	↑				

Here –3 is the most negative entry. So, column second is the pivot column.

Now, $\frac{4}{1} = 4$ and $\frac{1}{-1} = -1$ (We shouldn't take negative ratio). So, 1 is the pivot element.

Applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + 3R_1$

x	y	r	s	P	RHS
1	1	1	0	0	4
2	0	1	1	0	5
2	0	3	0	1	12

Since all the entries in the last row are non-negative, the optimal solution is obtained.

The optimal solution is

Max. $P = 12$ at $x = 0$ and $y = 4$

(d) Let r and s be the non-negative slack variables. Adding the slack variables, we can express the given LPP in the following form:

$x + y + r = 4$

$x - y + s = 1$

$x + 3y = P$

⇒ $x + y + r + 0.s + 0.P = 4$

$x - y + 0.r + s + 0.p = 1$

$-x - 3y + 0.r + 0.s + p = 0$

Simplex Tableau

x	y	r	s	P	RHS
1	$\boxed{1}$	1	0	0	4
1	-1	0	1	0	1
-1	-3	0	0	1	0
	↑				

Here −3 is the most negative entry. So, column second is the pivot column.

Now, $\frac{4}{1} = 4$ and $\frac{1}{-1} = -1$ (We shouldn't take negative ratio). So, 1 is the pivot element

Applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + 3R_1$

x	y	r	s	P	RHS
1	1	1	0	0	4
2	0	1	1	0	5
2	0	3	0	1	12

Since all the entries in the last row are non-negative, the optimal solution is obtained.
The optimal solution is

Max. P = 12 at x = 0 and y = 4

(e) The inequalities are

$$-x_1 + 2x_2 \le 8$$
$$x_1 + 2x_2 \le 4$$
$$x_1 \ge 0,\ x_2 \ge 0$$

Let x_3 and x_4 be the non-negative slack variables. Then given LPP can be written as

$\Rightarrow \quad -x_1 + 2x_2 + x_3 = 8$
$$x_1 + 2x_2 + x_4 = 4$$
$$F = x_1 + 7x_2$$

i.e. $-x_1 + 2x_2 + x_3 + 0.x_4 + 0.F = 8$
$$x_1 + 2x_2 + 0.x_3 + x_4 + 0.F = 4$$
$$-x_1 - 7x_2 + 0.x_3 + 0.x_4 + F = 0$$

Simplex Tableau

Basic variables	x_1	x_2	x_3	x_4	F	RHS
x_3	−1	$\boxed{2}$	1	0	0	8
x_4	1	2	0	1	0	4
	−1	−7	0	0	1	0

Here, − 7 is the most negative entry in the last row, so second column is the pivot column.

Since $\frac{8}{2} = 4$, $\frac{4}{2} = 2$. 2 < 4 so the element 2 in the second row is the pivot element.

Applying $R_2 \to \frac{1}{2} R_2$

Basic variables	x_1	x_2	x_3	x_4	F	RHS (b)
x_3	−1	2	1	0	0	8
x_2	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	2
	−1	−7	0	0	1	0

Applying $R_1 \to R_1 - 2R_2$ and $R_3 \to R_3 + 7R_2$

Basic variables	x_1	x_2	x_3	x_4	F	RHS
x_3	−2	0	1	−1	0	4
x_2	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	2
	$\frac{5}{2}$	0	0	$\frac{7}{2}$	1	14

This is the optimal solution as last row contains all non – negative entries.
So, Max F = 14 at $x_1 = 0$, $x_2 = 2$

(f) Let x_3 and x_4 be the non-negative slack variables. Then given LPP can be written as

$x_1 + x_2 + x_3 = 50$
$x_1 + x_4 = 40$
$z = 4x_1 + 2x_2$

$\Rightarrow$ $x_1 + x_2 + x_3 + 0.x_4 + 0\,Z = 50$
$x_1 + 0\cdot x_2 + 0.x_3 + x_4 + 0\,Z = 40$
$-4x_1 - 2x_2 + 0\cdot x_3 + 0\,x_4 + Z = 0$

Simplex tableau

Basic variables	x_1	x_2	x_3	x_4	Z	RHS
x_3	1	1	1	0	0	50
x_4	$\boxed{1}$	0	0	1	0	40
	–4	–2	0	0	1	0

Here, – 4 is the most negative entry in the last row. So, first column is the pivot column.

Since $\frac{50}{1} = 50$, $\frac{40}{1} = 40$; $40 < 50$; so 1 in the second row is the pivot element.

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 + 4R_2$

Basic variables	x_1	x_2	x_3	x_4	Z	RHS
x_3	0	$\boxed{1}$	1	– 1	0	10
x_1	1	0	0	1	0	40
	0	– 2	0	4	1	160

Again, second column is the pivot column, and 1 is the pivot element

Applying $R_3 \to R_3 + 2R_1$

Basic variables	x_1	x_2	x_3	x_4	Z	RHS
x_2	0	1	1	– 1	0	10
x_1	1	0	0	1	0	40
	0	0	2	2	1	180

All the entries in the last row are non-negative. So the optimal solution is obtained

$\therefore$ Max $z = 180$ at $x_1 = 40$, $x_2 = 10$

(g) Let x_4, x_5 and x_6 be the non-negative slack variables. Then given LPP can be written as

$2x_1 - 2x_2 + x_3 + x_4 = 100$
$x_1 - 2x_2 + 5x_3 - x_5 = 80$
$10x_1 + 5x_2 + 4x_3 + x_6 = 80$
$z = 2x_1 + 12x_2 + 8x_3.$

$\Rightarrow$ $2x_1 - 2x_2 + x_3 + x_4 + 0.x_5 + 0.x_6 + 0.z = 100$
$x_1 - 2x_2 + 5x_3 + 0.x_4 + x_5 + 0.x_6 + 0.z = 80$
$10x_1 + 5x_2 + 4x_3 + 0.x_4 + 0.x_5 + x_6 + 0.z = 80$
$-2x_1 - 12x_2 - 8x_3 + 0.x_4 + 0.x_5 + 0.x_6 + z = 0$

Simplex Tableau

Basic variables	x_1	x_2	x_3	x_4	x_5	x_6	Z	RHS
x_4	2	–2	1	1	0	0	0	100
x_5	1	– 2	5	0	1	0	0	80
x_6	10	$\boxed{5}$	4	0	0	1	0	80
	–2	–12	–8	0	0	0	1	0

Since – 12 is the most negative entry in the last row, so second column is the pivot column

Since $\frac{80}{5} = 16$ and other ratios are negative, so 5 is the pivot element.

Applying $R_3 \to \frac{1}{5} R_3$.

Basic variables	x_1	x_2	x_3	x_4	x_5	x_6	Z	RHS
x_4	2	−2	1	1	0	0	0	100
x_5	1	−2	5	0	1	0	0	80
x_2	2	1	$\frac{4}{5}$	0	0	$\frac{1}{5}$	0	16
	−2	−12	−8	0	0	0	1	0

Applying $R_1 \to R_1 + 2R_3$, $R_2 \to R_2 + 2R_3$, $R_4 \to R_4 + 12R_3$

Basic variables	x_1	x_2	x_3	x_4	x_5	x_6	Z	RHS
x_4	6	0	$\frac{13}{5}$	1	0	$\frac{2}{5}$	0	132
x_5	5	0	$\frac{33}{5}$	0	1	$\frac{2}{5}$	0	112
x_2	2	1	$\frac{4}{5}$	0	0	$\frac{1}{5}$	0	16
	22	0	$\frac{8}{5}$	0	0	$\frac{12}{5}$	1	192

Since the last row has all non - negative entries, so optimal solution is obtained.

Max. $Z = 192$ at $x_1 = 0$, $x_2 = 16$, $x_3 = 0$

2. **Using simplex method, find the optimal solutions of the following linear programming problems.**

(a) **Min.** $Z = 10x + 15y$
subject to $x + y \geq 8$
$5x + 3y \geq 30$
$x \geq 0, y \geq 0$

(b) **Min.** $F = 3x_1 + x_2$
subject to $4x_1 + 3x_2 \geq 12$
$x_1 + x_2 \geq 5$
$x_1, x_2 \geq 0$

(c) **Min.** $C = 3x_1 + 2x_2$
subject to $-x_1 + x_2 \leq -3$
$2x_1 + 5x_2 \geq 10$
$x_1, x_2 \geq 0$

(d) **Min.** $C = 4x + 2y$
subject to $x + y \geq 21$
$3x + y \geq 27$
$x, y \geq 0$

Solution

(a) Let A = augmented matrix formed from the constraints and the objective function

$$= \left(\begin{array}{cc|c} 1 & 1 & 8 \\ 5 & 3 & 30 \\ \hline 10 & 15 & 0 \end{array}\right)$$

$$A^T = \left(\begin{array}{cc|c} 1 & 5 & 10 \\ 1 & 3 & 15 \\ \hline 8 & 30 & 0 \end{array}\right)$$

The corresponding dual problem of given LPP is

Max. $W = 8r + 30s$

subject to

$r + 5s \leq 10$

$r + 3s \leq 15$

$r, s \geq 0$

Let us introduce the non-negative slack variables x and y. Then above LPP can be written as

$r + 5s + x = 10$

$r + 3s + y = 15$

$w = 8r + 30s$

$\Rightarrow$ $r + 5s + x + 0.y + 0.w = 10$

$r + 3s + 0.x + y + 0.w = 15$

$-8r - 30s + 0.x + 0.y + w = 0$

Basic variables	r	s	x	y	w	RHS
x	1	$\boxed{5}$	1	0	0	10
y	1	3	0	1	0	15
	−8	−30 ↑	0	0	1	0

Here, – 30 is the most negative entry in last row. So, second column is the pivot column.

Since $\frac{10}{5} = 2$, $\frac{15}{3} = 5$ and $2 < 5$, so 5 is the pivot element.

Applying $R_1 \to \frac{1}{5} R_1$

Basic variables	r	s	x	y	w	RHS
s	$\frac{1}{5}$	1	$\frac{1}{5}$	0	0	2
y	1	3	0	1	0	15
	– 8	– 30	0	0	1	0

Applying $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 + 30 R_1$

Basic variables	r	s	x	y	w	RHS
s	$\boxed{\frac{1}{5}}$	1	$\frac{1}{5}$	0	0	2
y	$\frac{2}{5}$	0	$-\frac{3}{5}$	1	0	9
	– 2 ↑	0	6	0	1	60

Since $\frac{2}{\frac{1}{5}} = 10$, $\frac{9}{\frac{2}{5}} = \frac{45}{2}$ and $10 < \frac{45}{2}$, so

$\frac{1}{5}$ is the pivot element.

Applying $R_1 \to 5R_1$

Basic variables	r	s	x	y	w	RHS
r	1	5	1	0	0	10
y	$\frac{2}{5}$	0	$-\frac{3}{5}$	1	0	9
	– 2	0	6	0	1	60

Applying $R_2 \to R_2 - \frac{2}{5} R_1$ and $R_3 \to R_3 + 2R_1$

Basic variables	r	s	x	y	w	RHS
r	1	5	1	0	0	10
y	0	–2	–1	1	0	5
	0	10	8	0	1	80

Here, all the entries in the last row are non-negative, so the solution is optimal.

Thus, max. w = 80 at r = 10, s = 0

∴ Min z = 80 at x = 8, y = 0

(b) Let A = $= \left(\begin{array}{cc|c} 4 & 3 & 12 \\ 1 & 1 & 5 \\ \hline 3 & 1 & 0 \end{array}\right)$

$A^T = \left(\begin{array}{cc|c} 4 & 1 & 3 \\ 3 & 1 & 1 \\ \hline 12 & 5 & 0 \end{array}\right)$

The corresponding dual problem of given LPP is

Max. $z = 12y_1 + 5y_2$ subject to

$4y_1 + y_2 \le 3$

$3y_1 + y_2 \le 1$

$y_1 \ge 0,\ y_2 \ge 0$

Introducing the non-negative slack variables x_1 and x_2 then the given LPP can be written as

$$4y_1 + y_2 + x_1 = 3$$
$$3y_1 + y_2 + x_2 = 1$$
$$z = 12y_1 + 5y_2$$
$$\Rightarrow \quad 4y_1 + y_2 + x_1 + 0x_2 + 0z = 3$$
$$3y_1 + y_2 + 0x_1 + x_2 + 0z = 1$$
$$-12y_1 - 5y_2 + 0x_1 + 0x_2 + z = 0$$

Simplex Tableau

Basic variables	y_1	y_2	x_1	x_2	z	RHS
x_1	4	1	1	0	0	3
x_2	$\boxed{3}$	1	0	1	0	1
	−12 ↑	−5	0	0	1	0

Here, first column is the pivot column.

Since $\frac{3}{4} = \frac{3}{4}, \frac{1}{3} = \frac{1}{3}$ and $\frac{1}{3} < \frac{3}{4}$, so 3 is the pivot element.

Applying $R_2 \to \frac{1}{3} R_2$

Basic variables	y_1	y_2	x_1	x_2	z	RHS
x_1	4	1	1	0	0	3
y_1	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{3}$
	−12	−5	0	0	1	0

Applying $R_1 \to R_1 - 4R_2$ and $R_3 \to R_3 + 12R_2$

Basic variables	y_1	y_2	x_1	x_2	z	RHS
x_1	0	$-\frac{1}{3}$	1	$-\frac{4}{3}$	0	$\frac{5}{3}$
y_1	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{3}$
	0	−1 ↑	0	4	1	4

Again, second column is the pivot column

And $\frac{1}{3}$ is the pivot element.

Applying $R_2 \to 3R_2$

Basic variables	y_1	y_2	x_1	x_2	z	RHS
x_1	0	$-\frac{1}{3}$	1	$-\frac{4}{3}$	0	$\frac{5}{3}$
y_2	3	1	0	1	0	1
	0	−1	0	4	1	4

Applying $R_1 \to R_1 + \frac{1}{3} R_2$, $R_3 \to R_3 + R_2$

Basic variables	y_1	y_2	x_1	x_2	z	RHS
x_1	1	0	1	−1	0	2
y_2	3	1	0	1	0	1
	3	0	0	5	1	5

This is the optimal solution

Max $z = 5$ at $y_1 = 0, y_2 = 1$

Min $F = 5$ at $x_1 = 0, x_2 = 5$

(c) Given inequalities are

$x_1 - x_2 \geq 3$

$2x_1 + 5x_2 \geq 10$

$x_1, x_2 \geq 0$

$$\text{Let } A = \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 2 & 5 & 10 \\ \hline 3 & 2 & 0 \end{array}\right)$$

$$A^T = \left(\begin{array}{cc|c} 1 & 2 & 3 \\ -1 & 5 & 2 \\ \hline 3 & 10 & 0 \end{array}\right)$$

The corresponding dual problem of given LPP is

Max: $Z = 3y_1 + 10y_2$ subject to

$y_1 + 2y_2 \leq 3$

$-y_1 + 5y_2 \leq 2$

$y_1, y_2 \geq 0.$

Introducing the non-negative slack variables $x_1, x_2 \geq 0$, the above LPP can be written as

$y_1 + 2y_2 + x_1 = 3$

$-y_1 + 5y_2 + x_2 = 2$

$z = 3y_1 + 10y_2$

$\Rightarrow$ $y_1 + 2y_2 + x_1 + 0.x_2 + 0.z = 3$

$-y_1 + 5y_2 + 0.x_1 + x_2 + 0.z = 2$

$-3y_1 - 10y_2 + 0.x_1 + 0.x_2 + z = 0$

Simplex Tableau

Basic variables	y_1	y_2	x_1	x_2	z	RHS
x_1	1	2	1	0	0	3
x_2	-1	$\boxed{5}$	0	1	0	2
	-3	-10 ↑	0	0	1	0

Here, second column is the pivot column.

Since $\frac{3}{2} = 1.5, \frac{2}{5} = 0.4$ and $0.4 < 1.5$, so

5 is the pivot element.

Applying $R_2 \to \frac{1}{5} R_2$

Basic variables	y_1	y_2	x_1	x_2	z	RHS
x_1	1	2	1	0	0	3
y_2	$-\frac{1}{5}$	1	0	$\frac{1}{5}$	0	$\frac{2}{5}$
	-3	-10	0	0	1	0

Applying $R_1 \to R_1 - 2R_2$ and

$R_3 \to R_3 + 10R_2$

Basic variables	y_1	y_2	x_1	x_2	z	RHS
x_1	$\boxed{\frac{7}{5}}$	0	1	$-\frac{2}{5}$	0	$\frac{11}{5}$
y_2	$-\frac{1}{5}$	1	0	$\frac{1}{5}$	0	$\frac{2}{5}$
	-5 ↑	0	0	2	1	4

Again, $\frac{7}{5}$ is the pivot

Applying $R_1 \to \frac{5}{7} R_1$

Basic variables	y_1	y_2	x_1	x_2	z	RHS
y_1	1	0	$\frac{5}{7}$	$-\frac{2}{7}$	0	$\frac{11}{7}$
y_2	$-\frac{1}{5}$	1	0	$\frac{1}{5}$	0	$\frac{2}{5}$
	-5	0	0	2	1	4

Applying $R_2 \to R_2 + \frac{1}{5} R_1$ and $R_3 \to R_3 + 5 R_1$

Basic variables	y_1	y_2	x_1	x_2	z	RHS
y_1	1	0	$\frac{5}{7}$	$-\frac{2}{7}$	0	$\frac{11}{7}$
y_2	0	1	$\frac{1}{7}$	$\frac{1}{7}$	0	$\frac{5}{7}$
	0	0	$\frac{25}{7}$	$\frac{4}{7}$	1	$\frac{83}{7}$

This is the optimal solution.

$$\text{Max } z = \frac{83}{7} \text{ at } y_1 = \frac{11}{7}, y_2 = \frac{5}{7}$$

$$\text{Min } C = \frac{83}{7} \text{ at } x_1 = \frac{25}{7}, x_2 = \frac{4}{7}$$

(d) Let $A = \left(\begin{array}{cc|c} 1 & 1 & 21 \\ 3 & 1 & 27 \\ \hline 4 & 2 & 0 \end{array}\right)$

$$A' = \left(\begin{array}{cc|c} 1 & 3 & 4 \\ 1 & 1 & 2 \\ \hline 21 & 27 & 0 \end{array}\right)$$

The corresponding dual problem of given LPP is

Max $Z = 21r + 27s$ subject to

$$r + 3s \le 4$$
$$r + s \le 2$$
$$r \ge 0, s \ge 0$$

Introducing the non-negative slack variables x, y, then above LPP can be written as

$$r + 3s + x = 4$$
$$r + s + y = 2$$
$$Z = 21r + 27s$$

$$\Rightarrow \quad r + 3s + x + 0.y + 0.Z = 4$$
$$r + s + 0.x + y + 0.Z = 2$$
$$-21r - 27s + 0.x + 0.y + Z = 0$$

Simplex tableau

Basic variables	r	s	x	y	Z	RHS
x	1	$\boxed{3}$	1	0	0	4
y	1	1	0	1	0	2
	-21	-27	0	0	1	0

Second column is the pivot column. Since $\frac{4}{3} = \frac{4}{3}, \frac{2}{1} = 2$ and $\frac{4}{3} < 2$, so 3 is the pivot element.

Applying $R_1 \to \frac{1}{3} R_1$

Basic variables	r	s	x	y	Z	RHS
s	$\frac{1}{3}$	1	$\frac{1}{3}$	0	0	$\frac{4}{3}$
y	1	1	0	1	0	2
	-21	-27	0	0	1	0

Applying $R_2 \to R_2 - R_1$
$R_3 \to R_3 + 27R_1$

Basic variables	r	s	x	y	Z	RHS
s	$\frac{1}{3}$	1	$\frac{1}{3}$	0	0	$\frac{4}{3}$
y	$\boxed{\frac{2}{3}}$	0	$-\frac{1}{3}$	1	0	$\frac{2}{3}$
	-12 ↑	0	9	0	1	36

Since $\frac{\frac{4}{3}}{\frac{1}{3}} = 4, \frac{\frac{2}{3}}{\frac{2}{3}} = 1$ and $1 < 4$, so $\frac{2}{3}$ is the pivot element.

Applying $R_2 \to \frac{3}{2} R_2$

Basic variables	r	s	x	y	Z	RHS
s	$\frac{1}{3}$	1	$\frac{1}{3}$	0	0	$\frac{4}{3}$
r	1	0	$-\frac{1}{2}$	$\frac{3}{2}$	0	1
	-12	0	9	0	1	36

Applying $R_1 \to R_1 - \frac{1}{3} R_2$ and $R_3 \to R_3 + 12 R_2$

Basic variables	r	s	x	y	Z	RHS
s	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	1
r	1	0	$-\frac{1}{2}$	$\frac{3}{2}$	0	1
	0	0	3	18	1	48

This is the optimal solution as the last row contains non-negative entries.
Max $Z = 48$ at $r = 1$ and $s = 1$
Min $C = 48$ at $x = 3$ and $y = 185$

Objective Questions

1. **In less than or equal to constraints, the non-negative variable that is used to balance both side is**
   (a) condition variable (b) surplus variable
   (c) slack variable (d) solving variable

Ans: c

2. **In simplex method, the basic feasible solution must satisfy**
   (a) negativity constraint (b) non-negativity constraint
   (c) basic constraint (d) non-basic constraint

Ans: b

3. **Dual problem is formulated with the help of information available in another is called**
   (a) primal problem (b) dual problem
   (c) prime problem (d) optimal problem

Ans: a

4. **Every LPP is associated with another LPP is**
   (a) slack (b) surplus
   (c) primal (d) dual

Ans: d

5. The entry in the simplex tableau in the entering variable column and the departing variable row is

(a) departing element
(b) entering element
(c) pivot element
(d) basic element

Ans: c

6. The dual problem of the linear programming problem

Min. $Z = 2x_1 + 3x_2$,

subject to, $2x_1 + x_2 \geq 1$

$-x_1 + x_2 \geq 5$

$x_1, x_2 \geq 0$ is

(a) Max. $Z^* = 2x_1 + 3x_2$
subject to
$2x_1 + x_2 \geq 1$
$-x_1 + x_2 \geq 5$
$x_1, x_2 \geq 0$

(b) Max. $Z^* = y_1 + 5y_2$
subject to
$2y_1 - y_2 \leq 2$
$y_1 + y_2 \leq 3$
$y_1 \geq 0, y_2 \geq 0$

(c) Min. $Z^* = y_1 + 5y_2$
subject to
$2y_1 - y_2 \leq 2$
$y_1 + y_2 \leq 3$
$y_1 \geq 0, y_2 \geq 0$

(d) none of the above

Ans: b

EXERCISE 6 (C)

Solve the following system of equations by Gauss elimination method.

(a) $3x_1 - 2x_2 = 1$
$x_1 + 5x_2 = 6$

(b) $2x + 3y = 4$
$3x + 2y = -4$

(c) $5x_1 - 8x_2 - 28 = 0$
$3x_1 - 7x_2 - 5 = 0$

(d) $x - 2y + 3z = 2$
$2x - 3y + z = 1$
$3x - y + 2z = 9$

(e) $3x_1 + x_2 + x_3 = 5$
$x_1 - 4x_2 + x_3 = -2$
$x_1 + x_2 - 3x_3 = -1$

(f) $2x_2 + 3x_3 = 7$
$3x_1 - 2x_2 + 2x_3 = 1$
$2x_1 + 3x_2 - 3x_3 = 5$

Solution

(a) Given equations are

$3x_1 - 2x_2 = 1$ ... (i)

$x_1 + 5x_2 = 6$ ... (ii)

Multiplying equation (i) by $\frac{1}{3}$ and subtracting it from equation (ii),

$x_1 + 5x_2 = 6$

$x_1 - \frac{2}{3}x_2 = \frac{1}{3}$

$-\quad +\quad =$

$\frac{17}{3}x_2 = \frac{17}{3}$ ... (iii)

Now, we have the following system of equation

$3x_1 - 2x_2 = 1$ ... (i)

$\frac{17}{3}x_2 = \frac{17}{3}$ ... (iii)

From (iii), we get $x_2 = 1$

Using $x_2 = 1$ in (i), we get,

$3x_1 - 2 \times 1 = 1$

$x_1 = 1$

$x_1 = 1, x_2 = 1$

(b) Given equations are:

$2x + 3y = 4$ ... (i)

$3x + 2y = -4$ ... (ii)

Multiplying equation (i) by $\frac{3}{2}$ and then subtracting from (ii),

$$\begin{array}{rrr} 3x & +2y & = -4 \\ 3x & +\frac{9}{2}y & = 6 \\ - & - & - \\ \hline & -\frac{5}{2}y & = -10 \end{array} \quad \text{...(iii)}$$

Now, we have the following equations

$2x + 3y = 4$ ...(i)

$-\frac{5}{2}y = -10$ ... (iii)

From equation (iii), we have $y = 4$

Using $y = 4$ in equation (i), we have

$2x + 3 \times 4 = 4$

or, $x = -4$

$\therefore$ The required solution is $x = -4$, $y = 4$.

(c) Given equation are

$5x_1 - 8x_2 = 28$ ... (i)

$3x_1 - 7x_2 = 5$ ... (ii)

Multiplying equation (i) by $\frac{3}{5}$ and subtracting from equation (ii), we get,

$$\begin{array}{rrr} 3x_1 & -7x_2 & = 5 \\ 3x_1 & -\frac{24}{5}x_2 & = \frac{84}{5} \\ - & + & - \\ \hline & -\frac{11}{5}x_2 & = -\frac{59}{5} \end{array} \quad \text{... (iii)}$$

Now, we have the following system of equation,

$5x_1 - 8x_2 = 28$ ... (i)

$-\frac{11}{5}x_2 = -\frac{59}{5}$ ... (ii)

From (iii), we have $x_2 = \frac{59}{11}$

Using $x_2 = \frac{59}{11}$ in equation (i), we get

$$3x_1 - 7 \times \frac{59}{11} = 5 \Rightarrow x_1 = \frac{156}{11}$$

$\therefore$ $x_1 = \frac{156}{11}$, $x_2 = \frac{59}{11}$

d. Given equations are:

$x - 2y + 3z = 2$ ...(i)

$2x - 3y + z = 1$ ...(ii)

$3x - y + 2z = 9$ ... (iii)

Multiplying equation (i) by 2 and then subtracting from equation (ii)

$$\begin{array}{rrrr} 2x & -3y & +z & = 1 \\ 2x & -4y & +6z & = 4 \\ - & + & - & - \\ \hline & y & -5z & = -3 \end{array} \quad \text{...(iv)}$$

Again, multiplying equation (i) by 3 then subtracting from equation (iii)

$$\begin{array}{rrrl} 3x & -\ y & +\ 2z & =\ 9 \\ 3x & -\ 6y & +\ 9z & =\ 6 \\ - & + & - & - \\ \hline & 5y & -\ 7z & =\ 3 \end{array} \qquad \ldots(v)$$

Multiplying equation (iv) by 5 then subtracting from (v)

$$\begin{array}{rrl} 5y & -\ 7z & =\ 3 \\ 5y & -\ 25z & =\ -15 \\ - & + & + \\ \hline & 18z & =\ 18 \end{array} \qquad \ldots(vi)$$

Now, we have the following three equations

$$x - 2y + 3z = 2 \qquad \ldots(i)$$

$$y - 5z = -3 \qquad \ldots(iv)$$

$$18z = 18 \qquad \ldots(vi)$$

From equation (vi), we have $z = 1$

Using $z = 1$ in equation (iv), we have

$y - 5 \times 1 = -3$

$\therefore$ $y = 2$

Again, using $y = 2$ and $z = 1$ in equation (i), we have

$x - 2 \times 2 + 3 \times 1 = 2$

$\therefore$ $x = 3$

$\therefore$ The required solution is $x = 3$, $y = 2$, $z = 1$

e. Given equations are

$$3x_1 + x_2 + x_3 = 5 \qquad \ldots(i)$$

$$x_1 - 4x_2 + x_3 = -2 \qquad \ldots(ii)$$

$$x_1 + x_2 - 3x_3 = -1 \qquad \ldots(iii)$$

Multiplying equation (i) by $\frac{1}{3}$ and subtracting from (ii)

$$\begin{array}{rrrrl} x_1 & - & 4x_2 & +\ x_3 & =\ -2 \\ x_1 & + & \frac{1}{3}x_2 & +\ \frac{1}{3}x_3 & =\ \frac{5}{3} \\ - & - & & - & - \\ \hline & - & \frac{13}{3}x_2 & -\ \frac{2}{3}x_3 & =\ -\frac{11}{3} \end{array} \qquad \ldots(iv)$$

Again, multiplying equation (i) by $\frac{1}{3}$ and subtracting from (iii),

$$\begin{array}{rrrrl} x_1 & + & x_2 & -\ 3x_3 & =\ -1 \\ x_1 & + & \frac{1}{3}x_2 & +\ \frac{1}{3}x_3 & =\ \frac{5}{3} \\ - & - & & - & - \\ \hline & & \frac{2}{3}x_2 & -\ \frac{10}{3}x_3 & =\ -\frac{8}{3} \end{array} \qquad \ldots(v)$$

Again, multiplying equation (iv) by $\frac{2}{13}$ and adding with equation (v),

$$\begin{array}{rrl} -\frac{2}{3}x_2 & +\ \frac{4}{39}x_3 & =\ -\frac{22}{39} \\ \frac{2}{3}x_2 & -\ \frac{10}{3}x_3 & =\ -\frac{8}{3} \\ \hline & -\ \frac{126}{39}x_3 & =\ -\frac{126}{39} \end{array} \qquad \ldots(vi)$$

Now, we have the following system of equations

$$3x_1 + x_2 + x_3 = 5 \qquad \ldots(i)$$

$$-\frac{13}{3}x_2 + \frac{2}{3}x_3 = -\frac{11}{3} \qquad \ldots(iv)$$

$$-\frac{126}{39}x_3 = -\frac{126}{39} \qquad \ldots(vi)$$

From equation (vi), we have, $x_3 = 1$

Using $x_3 = 1$ in (iv)

$$-\frac{13}{3}x_2 = -\frac{11}{3} - \frac{2}{3}$$

$\therefore x_2 = 1$

Again, using $x_3 = 1$ and $x_2 = 1$ in (i), we get

$3x_1 + 1 + 1 = 5$

$x_1 = 1$

$\therefore$ $x_1 = 1, x_2 = 1, x_3 = 1$

f. The given equations are

$2x_2 + 3x_3 = 7$

$3x_1 - 2x_2 + 2x_3 = 1$

$2x_1 + 3x_2 - 3x_3 = 5$

The coefficient of the first variable x_1 is zero in the first equation. So, interchanging the first two equations, we have

$3x_1 - 2x_2 + 2x_3 = 1$ ...(i)

$2x_2 + 3x_3 = 7$ ...(ii)

$2x_1 + 3x_2 - 3x_3 = 5$ ...(iii)

Multiplying equation (i) by $\frac{2}{3}$ and the subtracting from equation (iii)

$$\begin{array}{rrrl} 2x_1 & +3x_2 & -3x_3 & = 5 \\ 2x_1 & -\frac{4}{3}x_2 & +\frac{4}{3}x_3 & = \frac{2}{3} \\ - & + & - & - \\ \hline & \frac{13}{3}x_2 & -\frac{13}{3}x_3 & = \frac{13}{3} \end{array}$$

or, $x_2 - x_3 = 1$ ... (iv)

Again, multiplying equation (ii) by $\frac{1}{2}$ and subtracting from equation (iv), we have

$$\begin{array}{rrl} x_2 & -x_3 & = 1 \\ x_2 & +\frac{3}{2}x_3 & = \frac{7}{2} \\ - & - & - \\ \hline & -\frac{5}{2}x_3 & = -\frac{5}{2} \end{array}$$ ... (v)

Now, we have the following three equations

$3x_1 - 2x_2 + 2x_3 = 1$ ...(i)

$x_2 - x_3 = 1$ ... (ii)

$-\frac{5}{2}x_3 = -\frac{5}{2}$ ...(v)

From equation (v), $x_3 = 1$

Using $x_3 = 1$ in equation (iv), we have $x_2 = 2$

Again, using $x_2 = 2$ and $x_3 = 1$ in equation (i), we have

$x_1 = 1$

$\therefore$ The required solution is $x_1 = 1, x_2 = 2, x_3 = 1$.

2. Test the consistency of the following system. Solve if possible.

(a) $x - y - 2z = -1$
$2x + y + z = 2$
$3x + 2y + 9z = 4$

(b) $x_1 + x_2 - x_3 = 1$
$2x_1 + 3x_2 + 3x_3 = 3$
$x_1 - 3x_2 + 3x_3 = 2$

(c) $x_1 + 2x_2 + 3x_3 = 1$
$4x_1 + 5x_2 + 6x_3 = 3$
$7x_1 + 8x_2 + 9x_3 = 5$

(d) $x_1 + 2x_2 + 5x_3 = 4$
$-2x_1 + x_2 + 3x_3 = 12$
$6x_1 - 3x_2 - 9x_3 = 24$

(e) $x_2 - 4x_3 = 8$
$2x_1 - 3x_2 + 2x_3 = 1$
$5x_1 - 8x_2 + 7x_3 = 1$

Solution

(a) Given equations are

$x - y - 2z = -1$ ... (i)

$2x + y + z = 2$ ... (ii)

$3x + 2y + 9z = 4$ ... (iii)

Multiplying equation (i) by 2, then subtracting from equation (ii) we have

$$\begin{array}{rrrr} 2x & +y & +z & =2 \\ 2x & -2y & -4z & =-2 \\ - & + & + & + \\ \hline & 3y & +5z & =4 \end{array}$$

... (iv)

Again, multiplying equation (i) by 3 and then subtracting from equation (iii), we have

$$\begin{array}{rrrr} 3x & +2y & +9z & =4 \\ 3x & -3y & -6z & =-3 \\ - & + & + & + \\ \hline & 5y & +15z & =7 \end{array}$$

... (v)

Multiplying equation (iv) by $\frac{5}{3}$ and subtracting from equation (v)

$$\begin{array}{rrr} 5y & +15z & =7 \\ 5y & +\frac{25}{3}z & =\frac{20}{3} \\ - & - & - \\ \hline & \frac{20}{3}z & =\frac{1}{3} \end{array}$$

... (vi)

Now, we have the following three equations

$x - y - 2z = -1$ ... (i)

$3y + 5z = 4$ ...(iv)

$\frac{20}{3}z = \frac{1}{3}$ ... (vi)

From equation (vi), we can find a finite value of z and then finite values of x and y. So, the system of equation is consistent.

From (vi), $z = \frac{1}{20}$

Using $z = \frac{1}{20}$ in equation (iv)

$$3y + 5 \times \frac{1}{20} = 4$$

or, $y = \frac{5}{4}$

Again, using $y = \frac{5}{4}$ and $z = \frac{1}{20}$ in equation (i), we have,

$$x = \frac{7}{20}$$

$\therefore x = \frac{7}{20}$, $y = \frac{5}{4}$ and $z = \frac{1}{20}$

(b) Given equations are

$x_1 + x_2 - x_3 = 1$ (i)

$2x_1 + 3x_2 + 3x_3 = 3$ (ii)

$x_1 - 3x_2 + 3x_3 = 2$ (iii)

Multiplying equation (i) by 2 and subtracting it form equation (ii)

$$\begin{array}{l} 2x_1 + 3x_2 + 3x_3 = 3 \\ 2x_1 + 2x_2 - 2x_3 = 2 \\ \quad - \quad - \quad + \quad - \\ \hline x_2 + 5x_3 = 1 \end{array}$$

(iv)

Again, subtracting equation (i) from equation (iii), we get,

$$\begin{array}{l} x_1 - 3x_2 + 3x_3 = 2 \\ x_1 + x_2 - x_3 = 1 \\ - \quad - \quad + \quad - \\ \hline - 4x_2 + 4x_3 = 1 \end{array} \quad \text{(v)}$$

Multiplying equation (iv) by 4 and adding with equation (v)

$$\begin{array}{l} -4x_2 + 4x_3 = 1 \\ 4x_2 + 20x_3 = 4 \\ \hline 24x_3 = 5 \end{array} \quad \text{(vi)}$$

Now, we have the following system of equations.

$x_1 + x_2 - x_3 = 1$ ... (i)

$x_2 + 5x_3 = 1$ ... (ii)

$24x_3 = 5$ ... (iii)

From (vi), $x_3 = \frac{5}{24}$

Using $x_3 = \frac{5}{24}$ in equation (iv), $x_2 = -\frac{1}{24}$

Using x_2 and x_3 in (i), $x_1 = \frac{5}{4}$

$\therefore x_1 = \frac{5}{4}, x_2 = -\frac{1}{24}, x_3 = \frac{5}{24}$

Hence the system is consistent and the solution is $x_1 = \frac{5}{4}, x_2 = -\frac{1}{24}, x_3 = \frac{5}{24}$

(c) Given equation are

$x_1 + 2x_2 + 3x_3 = 1$ ... (i)

$4x_1 + 5x_2 + 6x_3 = 3$ ... (ii)

$7x_1 + 8x_2 + 9x_3 = 5$ ... (iii)

Multiplying equation (i) by 4 and subtracting it from (ii),

$$\begin{array}{l} 4x_1 + 5x_2 + 6x_3 = 3 \\ 4x_1 + 8x_2 + 12x_3 = 4 \\ - \quad - \quad - \quad - \\ \hline - 3x_2 - 6x_3 = -1 \end{array} \quad \text{... (iv)}$$

Again, multiplying equation (i) by 7 and subtracting it from equation (iii)

$$\begin{array}{l} 7x_1 + 8x_2 + 9x_3 = 5 \\ 7x_1 + 14x_2 + 21x_3 = 4 \\ - \quad - \quad - \quad - \\ \hline - 6x_2 - 12x_3 = -2 \end{array} \quad \text{... (v)}$$

Multiplying equation (iv) by 2 and subtracting from equation (v),

$$\begin{array}{l} - 6x_2 - 12x_3 = -2 \\ - 6x_2 + 12x_3 = -2 \\ + \quad + \quad + \\ \hline 0 = 0 \end{array} \quad \text{... (vi)}$$

Now, we have following system of equation

$x_1 + 2x_2 + 3x_3 = 1$ ... (i)

$-3x_2 - 6x_3 = -1$ (ii)

$0\, x_3 = 0$ ... (vi)

Equation (vi) is true for all values of x_3

Hence, we get infinitely many solutions.

So, the system is consistent

Thus, if $x_3 = k$ then from (iv),

$-3x_2 - 6k = -1$

or, $x_2 = \frac{1 - 6k}{3}$

Again using x_2 and x_3 in equation (i), we get

$$x_1 + 2\left(\frac{1-6k}{3}\right) + 3k = 1$$

or $$x_1 = 1 - 3k - 2\left(\frac{1-6k}{3}\right)$$

$$= \frac{3 - 9k - 2 + 12k}{3}$$

$$= \frac{1+3k}{3}$$

$$\therefore \quad x_1 = \frac{1+3k}{3},\ x_2 = \frac{1-6k}{3},\ x_3 = k$$

(d) Given equations are

$x_1 + 2x_2 + 5x_3 = 4$ ... (i)

$-2x_1 + x_2 + 3x_3 = 12$ ... (ii)

$6x_1 - 3x_2 - 9x_3 = 24$ ... (iii)

Multiplying equation (i) by 2 and adding with equation (ii)

$$\begin{array}{r} -2x_1 + x_2 + 3x_3 = 12 \\ 2x_1 + 4x_2 + 10x_3 = 8 \\ \hline 5x_2 + 13x_3 = 20 \end{array} \quad \text{... (iv)}$$

Multiplying equation (i) by 6 and subtracting it from equation (iii)

$$\begin{array}{r} 6x_1 - 3x_2 - 9x_3 = 24 \\ 6x_1 + 12x_2 + 30x_3 = 24 \\ - \quad - \quad - \quad - \\ \hline -15x_2 - 39x_3 = 0 \end{array} \quad \text{... (v)}$$

Multiplying equation (iv) by 3 and adding with equation (v)

$$\begin{array}{r} -15x_2 - 39x_3 = 0 \\ 15x_2 + 39x_3 = 20 \\ \hline 0 = 20 \end{array} \quad \text{... (vi)}$$

Now, we have the following system of equations

$x_1 + 2x_2 + 5x_3 = 4$ ... (i)

$5x_2 + 13x_3 = 20$ ... (iv)

$0.x_3 = 20$ ... (vi)

Here, no value of x_3 satisfies the equation (vi) and therefore the system has no solution. Hence the system is inconsistent.

(e) $x_2 - 4x_3 = 8$

$2x_1 - 3x_2 + 2x_3 = 1$

$5x_1 - 8x_2 + 7x_3 = 1$

Interchanging the equations, we have

$2x_1 - 3x_2 + 2x_3 = 1$ ... (i)

$5x_1 - 8x_2 + 7x_3 = 1$ ... (ii)

$x_2 - 4x_3 = 8$ ... (iii)

Multiplying equation (i) by $\frac{5}{2}$ and subtracting it from (ii)

$$\begin{array}{r} 5x_1 - 8x_2 + 7x_3 = 1 \\ 5x_1 - \frac{15}{2}x_2 + 5x_3 = \frac{5}{2} \\ - \quad + \quad - \quad - \\ \hline -\frac{1}{2}x_2 + 2x_3 = -\frac{3}{2} \end{array} \quad \text{... (iv)}$$

Multiplying (iii) by $\frac{1}{2}$ and adding with equation (iv).

$$\begin{array}{r} -\frac{1}{2}x_2 + 2x_3 = -\frac{3}{2} \\ \frac{1}{2}x_2 - 2x_3 = 4 \\ \hline 0 = \frac{5}{2} \end{array} \quad \text{(v)}$$

Now, we have the following system of equation.

$2x_1 - 3x_2 + 2x_3 = 1$ (i)

$-\frac{1}{2}x_2 + 2x_3 = -\frac{3}{2}$ (iv)

$0\,x_3 = \frac{5}{2}$ (v)

Here, no value of x_3 satisfies the third equation and therefore the system has no solution. Hence the system is inconsistent.

3. Using the Gauss elimination with partial pivoting, solve.

$x_1 + x_2 + x_3 = 6$

$2x_1 - x_2 = 4$

$4x_1 - 3x_2 - x_3 = 5$

Solution

$x_1 + x_2 + x_3 = 6$ ... (i)

$2x_1 - x_2 = 4$ ... (ii)

$4x_1 - 3x_2 - x_3 = 5$ ... (iii)

Multiplying equation (i) by 2 and subtracting from (ii),

$$\begin{array}{rrrl} 2x_1 & - x_2 & & = 4 \\ 2x_1 & + 2x_2 & + 2x_3 & = 12 \\ - & - & - & - \\ \hline & -3x_2 & - 2x_3 & = -8 \end{array} \quad \ldots \text{(iv)}$$

Multiplying equation (i) by 4 and subtracting from equation (iii)

$$\begin{array}{rrrl} 4x_1 & - 3x_2 & - x_3 & = 5 \\ 4x_1 & + 4x_2 & + 4x_3 & = 24 \\ - & - & - & - \\ \hline & -7x_2 & - 5x_3 & = -19 \end{array} \quad \ldots \text{(v)}$$

Since $|-7| > |-3|$, so we interchange equation (iv) and equation (v).

$-7x_2 - 5x_3 = -19$ ... (vi)

$-3x_2 - 2x_3 = -8$ ... (vii)

Multiplying equation (vi) by $\frac{3}{7}$ and subtracting from (vii)

$$\begin{array}{rrl} -3x_2 & - 2x_3 & = -8 \\ -3x_2 & - \frac{15}{7}x_3 & = -\frac{57}{7} \\ + & + & + \\ \hline & \frac{1}{7}x_3 & = \frac{1}{7} \end{array} \quad \ldots \text{(viii)}$$

Now, we have the following system of equations,

$x_1 + x_2 + x_3 = 6$ ... (i)

$-7x_2 - 5x_3 = -19$ ... (v)

$\frac{1}{7}x_3 = \frac{1}{7}$ ... (viii)

From equation (viii), $x_3 = 1$.

Using $x_3 = 1$ in equation (v), $-7x_2 - 5 = -19$

or, $x_2 = 2$

Using x_2 and x_3 in equation (i)

$x_1 + 2 + 1 = 6$

$x_1 = 3$

$\therefore$ $x_1 = 3, x_2 = 2, x_3 = 1$

Objective Questions

1. Which of the following equation is not linear?

(a) $3x + 4y = 5$ (b) $2x - y = 1$

(c) $x_1 + 2x_2 = 5$ (d) $x^2 + 3x + 2 = 0$

Ans: d

Option d is quadratic equation.

2. The system $x - y = 1$ and $x - y = 5$ has

(a) one solution
(b) infinitely many solutions
(c) no solution
(d) none of the above

Ans: c

Since the lines $x - y = 1$ and $x - y = 5$ are parallel, so the system has no solution.

3. **In Gauss elimination method, the coefficients of the variables of the equation a_{ij} where $i = j$ are known as**

(a) basic elements
(b) non-basic elements
(c) pivot elements
(d) common elements

Ans: c

4. **The system of equations $2x + y = 7$ and $x + 3y = 11$ is**

(a) consistent and independent
(b) inconsistent and independent
(c) consistent and dependent
(d) inconsistent and dependent

Ans: a

The system has unique solution $x = 2$ and $y = 3$. So, the system is consistent and independent

5. **In Gauss elimination method, the coefficient of the first variable in first equation must be**

(a) zero
(b) non-zero
(c) negative
(d) positive

Ans: b

EXERCISE – 6 (D)

1. **Examine whether the following system of equations are diagonally dominant?**

(a) $3x_1 + x_2 + x_3 = 5$
$-x_1 + 5x_2 + 2x_3 = 6$
$-2x_1 - x_2 + 5x_3 = 2$

(b) $x + 3y - z = 5$
$2x - 5y + 2z = 1$
$3x + 2y + 7z = 1$

Solution

(a) Here, $|3| > |1| + |1| = 2$
$|5| > |-1| + |2| = 3$
and $|5| > |-2| + |-1| = 3$
So, the system is diagonally dominant.

(b) Here, $|1| < |3| + |-1| = 4$
So, the system is not diagonally dominant.

2. **Solve the given system of equations by Gauss-Seidel method in second iteration.**

(a) $3x_1 + x_2 = 4$
$x_1 - 2x_2 = 3$

(b) $-4x_1 + x_2 - x_3 = -8$
$3x_1 + 6x_2 + 2x_3 = 1$
$x_1 - x_2 + 3x_3 = 2$

Solution

From given equations, we have,

$$x_1 = \frac{1}{3}(4 - x_2)$$

$$x_2 = \frac{1}{2}(x_1 - 3)$$

Initially, $x_1 = 0$, $x_2 = 0$

Iteration I

$$x_1 = \frac{1}{3}(4 - 0) = 1.333$$

$$x_2 = \frac{1}{2}(1.333 - 3) = -0.834$$

Iteration II

$$x_1 = \frac{1}{3}(4 + 0.834) = 1.611$$

$$x_2 = \frac{1}{2}(1.611 - 3) = -0.695$$

(b) From given equations, we can write,

$$x_1 = \frac{1}{4}(8 + x_2 - x_3)$$

$$x_2 = \frac{1}{6}(1 - 3x_1 - 2x_3)$$

$$x_3 = \frac{1}{3}(2 - x_1 + x_2)$$

Initially, $x_1 = 0, x_2 = 0, x_3 = 0$

Iteration I:

$$x_1 = \frac{1}{4}(8 + 0 - 0) = 2$$

$$x_2 = \frac{1}{6}(1 - 3 \times 2 - 2 \times 0) = -0.833$$

$$x_3 = \frac{1}{3}(2 - 2 - 0.833) = -0.278$$

Iteration II:

$$x_1 = \frac{1}{4}(8 - 0.833 + 0.278) = 1.861$$

$$x_2 = \frac{1}{6}(1 - 3 \times 1.861 + 2 \times 0.278) = -0.671$$

$$x_3 = \frac{1}{3}(2 - 1.861 - 0.671) = -0.177$$

3. **Solve the following system of equations using Gauss-Seidel method.**

(a) $3x_1 - 2x_2 = 1$
$-x_1 + 4x_2 = 3$

(b) $3x_1 + x_2 = 5$
$x_1 + 2x_2 = 5$

(c) $3x + y - z = 2$
$2x - 5y + z = 20$
$x - 3y - 8z = 3$

(d) $8x_1 + x_2 + x_3 = 13$
$2x_1 + 5x_2 + x_3 = 15$
$-x_1 + 4x_2 - 7x_3 = -14$

Solution

(a) $3x_1 - 2x_2 = 1$
$-x_1 + 4x_2 = 3$

From given equations, we have

$$x_1 = \frac{1}{3}(1 + 2x_2)$$

$$x_2 = \frac{1}{4}(x_1 + 3)$$

Initially, $x_1 = 0, x_2 = 0$

Iteration I

$$x_1 = \frac{1}{3}(1 + 2 \times 0) = 0.333$$

$$x_2 = \frac{1}{4}(0.333 + 3) = 0.833$$

Iteration II

$$x_1 = \frac{1}{3}(1 + 2 \times 0.833) = 0.889$$

$$x_2 = \frac{1}{4}(0.889 + 3) = 0.972$$

Iteration III

$$x_1 = \frac{1}{3}(1 + 2 \times 0.972) = 0.981$$

$$x_2 = \frac{1}{4}(0.981 + 3) = 0.995$$

Iteration IV

$$x_1 = \frac{1}{3}(1 + 2 \times 0.995) = 0.997$$

$$x_2 = \frac{1}{4}(0.997 + 3) = 0.999$$

Iteration V

$$x_1 = \frac{1}{3}(1 + 2 \times 0.999) = 0.999$$

$$x_2 = \frac{1}{4}(0.999 + 3) = 1.000$$

Iteration VI

$$x_1 = \frac{1}{3}(1 + 2 \times 1) = 1.000$$

$$x_2 = \frac{1}{4}(1 + 3) = 1.000$$

From 5th and 6th iterations, the values of x_1 and x_2 are equal so,

$x_1 = 1$ $x_2 = 1$

(b) Given equations can be written as:

$$x_1 = \frac{1}{3}(5 - x_2), \qquad x_2 = \frac{1}{2}(5 - x_1)$$

Initially, $x_1 = 0$, $x_2 = 0$

1st iteration:

$$x_1 = \frac{1}{3}(5 - 0) = 1.667, \qquad x_2 = \frac{1}{2}(5 - 1.667) = 1.667$$

2nd iteration:

$$x_1 = \frac{1}{3}(5 - 1.667) = 1.111, \quad x_2 = \frac{1}{2}(5 - 1.111) = 1.945$$

3rd iteration:

$$x_1 = \frac{1}{3}(5 - 1.945) = 1.018, \quad x_2 = \frac{1}{2}(5 - 1.018) = 1.991$$

4th iteration:

$$x_1 = \frac{1}{3}(5 - 1.991) = 1.003, \quad x_2 = \frac{1}{2}(5 - 1.003) = 1.998$$

5th iteration:

$$x_1 = \frac{1}{3}(5 - 1.998) = 1.001, \quad x_2 = \frac{1}{2}(5 - 1.001) = 2.000$$

6th iteration:

$$x_1 = \frac{1}{3}(5 - 2) = 1, \qquad x_2 = \frac{1}{2}(5 - 1) = 2$$

The values of x_1 and x_2 in 5th and 6th iteration are same

$\therefore$ $x_1 = 1$, $x_2 = 2$

(c) Given equations can be written as:

$$x = \frac{1}{3}(2 - y + z) \qquad \ldots(i)$$

$$y = \frac{1}{5}(2x + z - 20) \qquad \ldots(ii)$$

$$z = \frac{1}{8}(x - 3y - 3) \qquad \ldots(iii)$$

Initially, $x = 0$, $y = 0$, $z = 0$

1st iteration:

Put $x = 0$, $y = 0$, $z = 0$ in equation (i), we get

$$x = \frac{1}{3}(2 - 0 + 0) = 0.667$$

Again, put most recent values i.e. $x = 0.667$ and $z = 0$ in equation (ii).

$$y = \frac{1}{5}(2 \times 0.667 + 0 - 20) = -3.733$$

And, put $x = 0.667$ and $y = -3.733$ in equation (iii).

$$z = \frac{1}{8}(0.667 + 3 \times 3.733 - 3) = 1.108$$

2nd iteration

$$x = \frac{1}{3}(2 + 3.733 + 1.108) = 2.280$$

$$y = \frac{1}{5}(2 \times 2.280 + 1.108 - 20) = -2.866$$

$$z = \frac{1}{8}(2.280 + 3 \times 2.866 - 3) = 0.985$$

3rd iteration

$$x = \frac{1}{3}(2 + 2.866 + 0.985) = 1.950$$

$$y = \frac{1}{5}(2 \times 1.950 + 0.985 - 20) = -3.023$$

$$z = \frac{1}{8}(1.950 + 3 \times 3.023 - 3) = 1.002$$

4th iteration

$$x = \frac{1}{3}(2 + 3.023 + 1.002) = 2.008$$

$$y = \frac{1}{5}(2 \times 2.008 + 1.002 - 20) = -2.996$$

$$z = \frac{1}{8}(2.008 + 3 \times 2.996 - 3) = 1.000$$

5th iteration

$$x = \frac{1}{3}(2 + 2.996 + 1) = 1.999$$

$$y = \frac{1}{5}(2 \times 1.999 + 1 - 20) = -3.001$$

$$z = \frac{1}{8}(1.999 + 3 \times 3.001 - 3) = 1.000$$

6th iteration

$$x = \frac{1}{3}(2 + 3.001 + 1) = 2.000$$

$$y = \frac{1}{5}(2 \times 2 + 1 - 20) = -3$$

$$z = \frac{1}{8}(2 + 3 \times 3 - 3) = 1$$

7th iteration

$$x = \frac{1}{3}(2 + 3 + 1) = 2$$

$$y = \frac{1}{5}(2 \times 2 + 1 - 20) = -3$$

$$z = \frac{1}{8}(2 + 3 \times 3 - 3) = 1$$

From 6th and 7th iterations, x, y and z have equal values

So, $x = 2$, $y = -3$, $z = 1$

(d) From given equations, we can write,

$$x_1 = \frac{1}{8}(13 - x_2 - x_3)$$

$$x_2 = \frac{1}{5}(15 - 2x_1 - x_3)$$

$$x_3 = \frac{1}{7}(14 - x_1 + 4x_2)$$

Initially, $x_1 = 0, x_2 = 0, x_3 = 0$

Iteration I

$$x_1 = \frac{1}{8}(13 - 0 - 0) = 1.625$$

$$x_2 = \frac{1}{5}(15 - 2 \times 1.625 - 0) = 2.35$$

$$x_3 = \frac{1}{7}(14 - 1.625 + 4 \times 2.35) = 3.111$$

Iteration II

$$x_1 = \frac{1}{8}(13 - 2.35 - 3.111) = 0.942$$

$$x_2 = \frac{1}{5}(15 - 2 \times 0.942 - 3.111) = 2.001$$

$$x_3 = \frac{1}{7}(14 - 0.942 + 4 \times 2.001) = 3.009$$

Iteration III

$$x_1 = \frac{1}{8}(13 - 2.001 - 3.009) = 0.999$$

$$x_2 = \frac{1}{5}(15 - 2 \times 0.999 - 3.009) = 1.999$$

$$x_3 = \frac{1}{7}(14 - 0.999 + 4 \times 1.999) = 2.999$$

Iteration IV

$$x_1 = \frac{1}{8}(13 - 1.999 - 2.999) = 1.000$$

$$x_2 = \frac{1}{5}(15 - 2 \times 1.000 - 2.999) = 2.000$$

$$x_3 = \frac{1}{7}(14 - 1.000 + 4 \times 2.000) = 3.000$$

Iteration V

$$x_1 = \frac{1}{8}(13 - 2 - 3) = 1.000$$

$$x_2 = \frac{1}{5}(15 - 2 \times 1.000 - 3.000) = 2.000$$

$$x_3 = \frac{1}{7}(14 - 1.000 + 4 \times 2.000) = 3.000$$

Since 4^{th} and 5^{th} iterations the values of x_1, x_2 and x_3 are equal, so,
$x_1 = 1, x_2 = 2, x_3 = 3$

Objective Questions

1. A simple way for initial guess in Gauss Seidel method is to assume that
   - (a) they are all ones
   - (b) they are all negatives
   - (c) they are all positive
   - (d) they are all zeros

Ans: d

2. The convergence of the solution in Gauss Seidel method is assured when the system is
   - (a) diagonally dominant
   - (b) diagonally non-dominant
   - (c) both a and b
   - (d) none

Ans a

3. The solution of equations $3x + 2y + 9 = 0$ and $2x - 3y + 6 = 0$ in first iteration is

(a) (0, 3) (b) (0, –3)

(c) (–3, 0) (d) (3, 0)

Ans: c

4. If a small change in the coefficient of the variable in the system of equations shows a small deviation in the solution then the system is

(a) ill-conditioned (b) well-conditioned

(c) both a and b (d) none

Ans: b

5. The solution of the system of equation $3x_1 + x_2 = 5$ and $x_1 + 2x_2 = 5$ in second iteration using Gauss Seidel method is

(a) $x_1 = 0, x_2 = 0$ (b) $x_1 = 1.667, x_2 = 1.667$

(c) $x_1 = 1.111, x_2 = 1.945$ (d) $x_1 = 1.018, x_2 = 1.991$

Ans: c

Given equations can be written as:

$$x_1 = \frac{1}{3}(5 - x_2),\ x_2 = \frac{1}{2}(5 - x_1)$$

Initially, $x_1 = 0,\ x_2 = 0$

1st iteration:

$$x_1 = \frac{1}{3}(5 - 0) = 1.667,\ x_2 = \frac{1}{2}(5 - 1.667) = 1.667$$

2nd iteration:

$$x_1 = \frac{1}{3}(5 - 1.667) = 1.111,\ x_2 = \frac{1}{2}(5 - 1.111) = 1.945$$

EXERCISE – 6 (E)

1. Find the inverses of following matrices using Gauss–Jordan method.

(a) $\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$ (b) $\begin{pmatrix} 3 & 6 \\ 7 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

Solution

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$

We argument A with identity matrix I.

$$(A \mid I) = \left(\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & -1 & -3 & 1 \end{array}\right) R_2 \to R_2 - 3R_1$$

$$\sim \left(\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & -1 \end{array}\right) R_2 \to (-1)\ R_2$$

$$\sim \left(\begin{array}{cc:cc} 1 & 0 & -5 & 2 \\ 0 & 1 & 3 & -1 \end{array}\right) R_1 \to R_1 - 2R_2$$

$$A^{-1} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$$

(b) Let $A = \begin{pmatrix} 3 & 6 \\ 7 & 2 \end{pmatrix}$

We argument A with identity matrix I.

$$\text{i.e. } [A \mid I] = \left(\begin{array}{cc:cc} 3 & 6 & 1 & 0 \\ 7 & 2 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc:cc} 1 & 2 & \frac{1}{3} & 0 \\ 7 & 2 & 0 & 1 \end{array}\right) R_1 \to \frac{1}{3} R_1$$

$$\sim \left(\begin{array}{cc|cc} 1 & 2 & \frac{1}{3} & 0 \\ 0 & -12 & \frac{-7}{3} & 1 \end{array}\right) R_2 \rightarrow R_2 - 7R_1$$

$$\sim \left(\begin{array}{cc|cc} 1 & 2 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{7}{36} & \frac{-1}{12} \end{array}\right) R_2 \rightarrow \left(\frac{-1}{12}\right) R_2$$

$$\sim \left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{18} & \frac{1}{6} \\ 0 & 1 & \frac{7}{36} & -\frac{1}{12} \end{array}\right) R_1 \rightarrow R_1 - 2R_2$$

$$\therefore A^{-1} = \begin{pmatrix} -\frac{1}{18} & \frac{1}{6} \\ \frac{7}{36} & \frac{-1}{12} \end{pmatrix}$$

(c) Let $A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

We augment A with identity matrix I

$$\text{i.e. } (A \mid I) = \left(\begin{array}{ccc:ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc:ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \begin{array}{l} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc:ccc} 1 & 0 & 3 & 4 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) R_1 \rightarrow R_1 - 3R_2$$

$$\sim \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & 7 & -3 & -3 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) R_1 \rightarrow R_1 - 3R_3$$

$$\therefore \quad A^{-1} = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

2. **Solve the following system of equations by inverse matrix method.**

(a) $x + 2y = 4$
$3x - y = 5$

(b) $2x + 5y = 3$
$x + 3y = 2$

(c) $2x_1 + 3x_2 + 4 = 0$
$3x_1 = 5 + 2x_2$

(d) $x_1 - 2x_2 - x_3 = 1$
$x_1 - x_2 + 2x_3 = 9$
$2x_1 - 3x_2 - x_3 = 4$

(e) $x + y + z = 9$
$2x + 5y + 7z = 52$
$2x + y - z = 0$

(f) $9y - 5x = 3$
$x + z = 1$
$z + 2y = 2$

(g) $2x + 3y = 4$
$4x - z = 5$
$4y + 3z = -5$

Solution

(a) Writing the given equations in matrix form

$$\begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

i.e. $AX = B$

$\Rightarrow X = A^{-1}B$ (i)

Now, we augment A with identity matrix I

i.e. $(A \mid I) = \left(\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 3 & -1 & 0 & 1 \end{array}\right)$

$\sim \left(\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & -7 & -3 & 1 \end{array}\right) R_2 \to R_2 - 3R_1$

$\sim \left(\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{3}{7} & -\frac{1}{7} \end{array}\right) R_2 \to \left(-\frac{1}{7}\right) R_2$

$\sim \left(\begin{array}{cc:cc} 1 & 0 & \frac{1}{7} & \frac{2}{7} \\ 0 & 1 & \frac{3}{7} & -\frac{1}{7} \end{array}\right) R_1 \to R_1 - 2R_2$

$\therefore A^{-1} = \begin{pmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{3}{7} & -\frac{1}{7} \end{pmatrix}$

From (i)

$X = \begin{pmatrix} \frac{1}{7} & \frac{2}{7} \\ \frac{3}{7} & -\frac{1}{7} \end{pmatrix} \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

or, $X = \begin{pmatrix} \frac{4}{7} + \frac{10}{7} \\ \frac{12}{7} - \frac{5}{7} \end{pmatrix}$

or, $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\therefore x = 2, y = 1$

(b) Writing the given equation in matrix form

$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

which is in the form of

$AX = B$

$\Rightarrow X = A^{-1} B$ ... (i)

We augment A with identity matrix I

$(A \mid I) = \left(\begin{array}{cc:cc} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right)$

$\sim \left(\begin{array}{cc:cc} 1 & 3 & 0 & 1 \\ 2 & 5 & 1 & 0 \end{array}\right) R_1 \leftrightarrow R_2$

$\sim \left(\begin{array}{cc:cc} 1 & 3 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right) R_2 \to R_2 - 2R_1$

$\sim \left(\begin{array}{cc:cc} 1 & 3 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right) R_2 \to (-1) R_2$

$\sim \left(\begin{array}{cc:cc} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{array}\right) R_1 \to R_1 - 3R_2$

$\therefore A^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$

From (i)

$$X = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

or, $$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9-10 \\ -3+4 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\therefore$ $x = -1,\ y = 1$

(c) Given equation are

$2x_1 + 3x_2 = -4$

$3x_1 - 2x_2 = 5$

Writing the given equations in matrix form,

$$\begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$$

i.e. $AX = B$

$\Rightarrow$ $X = A^{-1} B$ ... (i)

We augment A with identity matrix I

$$\text{i.e. } (A \mid I) = \left(\begin{array}{cc:cc} 2 & 3 & 1 & 0 \\ 3 & -2 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cc:cc} 1 & \frac{3}{2} & \frac{1}{2} & 0 \\ 3 & -2 & 0 & 1 \end{array}\right) R_1 \to \frac{1}{2} R_1$$

$$\sim \left(\begin{array}{cccc} 1 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & \frac{-13}{2} & \frac{-3}{2} & 2 \end{array}\right) R_2 \to R_2 - 3R_1$$

$$\sim \left(\begin{array}{cc:cc} 1 & \frac{3}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{3}{13} & \frac{-2}{13} \end{array}\right) R_2 \to \left(-\frac{2}{13}\right) R_2$$

$$\sim \left(\begin{array}{cc:cc} 1 & 0 & \frac{2}{13} & \frac{3}{13} \\ 0 & 1 & \frac{3}{13} & \frac{-2}{13} \end{array}\right) R_1 \to R_1 - \frac{3}{2} R_2$$

$$\therefore A^{-1} = \begin{pmatrix} \frac{2}{13} & \frac{3}{13} \\ \frac{3}{13} & \frac{-2}{13} \end{pmatrix}$$

$$\text{From (i), } X = \begin{pmatrix} \frac{2}{13} & \frac{3}{13} \\ \frac{3}{13} & \frac{-2}{13} \end{pmatrix}\begin{pmatrix} -4 \\ 5 \end{pmatrix}$$

$$\text{or, } \quad X = \begin{pmatrix} \frac{-8}{13} + \frac{15}{13} \\ \frac{-12}{13} - \frac{10}{13} \end{pmatrix}$$

$$\text{or } \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{7}{13} \\ -\frac{22}{13} \end{pmatrix}$$

$$\therefore \quad x = \frac{7}{13},\ y = -\frac{12}{13}$$

(d) Writing the given equation in matrix form

$$\begin{bmatrix} 1 & -2 & -1 \\ 1 & -1 & 2 \\ 2 & -3 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 4 \end{bmatrix}$$

which is in the form of $AX=B$

$\Rightarrow \quad X = A^{-1}B \qquad \ldots(i)$

To find A^{-1}, we augment A with identity matrix I.

$$[A \mid I] = \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 1 & -1 & 2 & 0 & 1 & 0 \\ 2 & -3 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -1 & : & 1 & 0 & 0 \\ 0 & 1 & 3 & : & -1 & 1 & 0 \\ 0 & 1 & 1 & : & -2 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 5 & : & -1 & 2 & 0 \\ 0 & 1 & 3 & : & -1 & 1 & 0 \\ 0 & 0 & -2 & : & -1 & -1 & 1 \end{bmatrix} \begin{matrix} R_1 \to R_1 + 2R_2 \\ R_3 \to R_3 - R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 5 & : & -1 & 2 & 0 \\ 0 & 1 & 3 & : & -1 & 1 & 0 \\ 0 & 0 & 1 & : & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \end{bmatrix} R_3 \to \frac{1}{(-2)} R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & : & \frac{-7}{2} & \frac{-1}{2} & \frac{5}{2} \\ 0 & 1 & 0 & : & \frac{-5}{2} & \frac{-1}{2} & \frac{3}{2} \\ 0 & 0 & 1 & : & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \end{bmatrix} \begin{matrix} R_2 \to R_2 - 3R_3 \\ R_1 \to R_1 - 5R_3 \end{matrix}$$

$$\therefore \quad A^{-1} = \begin{bmatrix} \frac{-7}{2} & \frac{-1}{2} & \frac{5}{2} \\ \frac{-5}{2} & \frac{-1}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \end{bmatrix}$$

Now, from (i)

$$X = A^{-1}B$$

$$\text{or,} \quad X = \begin{bmatrix} \frac{-7}{2} & \frac{-1}{2} & \frac{5}{2} \\ \frac{-5}{2} & \frac{-1}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 9 \\ 4 \end{bmatrix}$$

$$\text{or,} \quad X = \begin{bmatrix} \frac{-7}{2} - \frac{9}{2} + \frac{20}{2} \\ \frac{-5}{2} - \frac{9}{2} + \frac{12}{2} \\ \frac{1}{2} + \frac{9}{2} - \frac{4}{2} \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$\therefore \quad x_1 = 2,\ x_2 = -1,\ x_3 = 3$

(e) Writing the given equation in matrix form

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 9 \\ 52 \\ 0 \end{pmatrix}$$

i.e. $AX = B$

$\Rightarrow X = A^{-1}B$ ... (i)

We augment A with identify matrix I.

$$\text{i.e. } (A \mid I) = \begin{pmatrix} 1 & 1 & 1 & : & 1 & 0 & 0 \\ 2 & 5 & 7 & : & 0 & 1 & 0 \\ 2 & 1 & -1 & : & 0 & 0 & 1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & : & 1 & 0 & 0 \\ 0 & 3 & 5 & : & -2 & 1 & 0 \\ 0 & -1 & -3 & : & -2 & 0 & 1 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 1 & : & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{3} & : & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & -1 & -3 & : & -2 & 0 & 1 \end{pmatrix} R_2 \to \frac{1}{3} R_2$$

$$\sim \begin{pmatrix} 1 & 0 & -\frac{2}{3} & : & \frac{5}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{5}{3} & : & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{-4}{3} & : & \frac{-8}{3} & \frac{1}{3} & 1 \end{pmatrix} \begin{matrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 + R_2 \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 0 & \frac{-2}{3} & : & \frac{5}{3} & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{5}{3} & : & -\frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 & : & 2 & \frac{-1}{4} & \frac{-3}{4} \end{pmatrix} R_3 \to \left(\frac{-3}{4}\right) R_3$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & : & 3 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & : & -4 & \frac{3}{4} & \frac{5}{4} \\ 0 & 0 & 1 & : & 2 & -\frac{1}{4} & -\frac{3}{4} \end{pmatrix} \begin{matrix} R_1 \to R_1 + \frac{2}{3} R_3 \\ R_2 \to R_2 - \frac{5}{3} R_3 \end{matrix}$$

$$\therefore A^{-1} = \begin{pmatrix} 3 & -\frac{1}{2} & -\frac{1}{2} \\ -4 & \frac{3}{4} & \frac{5}{4} \\ 2 & \frac{-1}{4} & \frac{-3}{4} \end{pmatrix}$$

$$\text{From (i), } X = \begin{pmatrix} 3 & -\frac{1}{2} & -\frac{1}{2} \\ -4 & \frac{3}{4} & \frac{5}{4} \\ 2 & -\frac{1}{4} & -\frac{3}{4} \end{pmatrix}\begin{pmatrix} 9 \\ 52 \\ 0 \end{pmatrix}$$

$$\text{or,}\quad X = \begin{pmatrix} 3\times 9 - \frac{1}{2}\times 52 - 0 \\ -4\times 9 + \frac{3}{4}\times 52 + 0 \\ 2\times 9 + \left(-\frac{1}{4}\right)\times 52 - 0 \end{pmatrix}$$

$$\text{or,}\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

$\therefore$ $x = 1$

$y = 3$

$z = 5$

(f) Writing the given equation in matrix form,

$$\begin{pmatrix} -5 & 9 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

i.e. $AX = B$

$\Rightarrow$ $X = A^{-1}B$ . . . (i)

We augment A with identity matrix I.

$$\text{i.e. } (A \mid I) = \left(\begin{array}{ccc:ccc} -5 & 9 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc:ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ -5 & 9 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) R_1 \leftrightarrow R_2$$

$$\sim \left(\begin{array}{ccc:ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 9 & 5 & 1 & 5 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) R_2 \rightarrow R_2 + 5R_1$$

$$\sim \left(\begin{array}{ccc:ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{5}{9} & \frac{1}{9} & \frac{5}{9} & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) R_2 \rightarrow \frac{1}{9}R_2$$

$$\sim \left(\begin{array}{ccc:ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{5}{9} & \frac{1}{9} & \frac{5}{9} & 0 \\ 0 & 0 & \frac{-1}{9} & \frac{-2}{9} & \frac{-10}{9} & 1 \end{array}\right) R_3 \rightarrow R_3 - 2R_2$$

$$\sim \left(\begin{array}{ccc:ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{5}{9} & \frac{1}{9} & \frac{5}{9} & 0 \\ 0 & 0 & 1 & 2 & 10 & -9 \end{array}\right) R_3 \rightarrow (-5)R_3$$

$$\sim \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & -2 & -9 & 9 \\ 0 & 1 & 0 & -1 & -5 & 5 \\ 0 & 0 & 1 & 2 & 10 & -9 \end{array}\right) \begin{array}{l} R_2 \rightarrow R_2 - \frac{5}{9}R_3 \\ R_1 \rightarrow R_1 - R_3 \end{array}$$

$$\therefore A^{-1} = \begin{pmatrix} -2 & -9 & 9 \\ -1 & -5 & 5 \\ 2 & 10 & -9 \end{pmatrix}$$

From (i)

$$X = \begin{pmatrix} -2 & -9 & 9 \\ -1 & -5 & 5 \\ 2 & 10 & -9 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6-9+18 \\ -3-5+10 \\ 6+10-18 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}$$

$\therefore \quad x = 3,\ y = 2,\ z = -2$

(g) Writing the given equations in matrix form

$$\begin{pmatrix} 2 & 3 & 0 \\ 4 & 0 & -1 \\ 0 & 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ -5 \end{pmatrix}$$

i.e. $AX = B$

$\Rightarrow X = A^{-1}B \qquad \ldots \text{(i)}$

We augment A with identity matrix I.

$$(A \mid I) = \left(\begin{array}{ccc:ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 4 & 0 & -1 & 0 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc:ccc} 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 4 & 0 & -1 & 0 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right) R_1 \to \frac{1}{2} R_1$$

$$\sim \left(\begin{array}{ccc:ccc} 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & -6 & -1 & -2 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right) R_2 \to R_2 - 4R_1$$

$$\sim \left(\begin{array}{ccc:ccc} 1 & \frac{3}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right) R_2 \to \left(-\frac{1}{6}\right) R_2$$

$$\sim \left(\begin{array}{ccc:ccc} 1 & 0 & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & \frac{7}{3} & \frac{-4}{3} & \frac{2}{3} & 1 \end{array}\right) R_1 \to R_1 - \frac{3}{2} R_2$$

$R_3 \to R_3 - 4R_2$

$$\sim \left(\begin{array}{ccc:ccc} 1 & 0 & -\frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 1 & \frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & 0 \\ 0 & 0 & 1 & -\frac{4}{7} & \frac{2}{7} & \frac{3}{7} \end{array}\right) R_3 \to \frac{3}{7} R_3$$

$$\sim \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & -\frac{1}{7} & \frac{9}{28} & \frac{3}{28} \\ 0 & 1 & 0 & \frac{3}{7} & -\frac{3}{14} & -\frac{1}{14} \\ 0 & 0 & 1 & -\frac{4}{7} & \frac{2}{7} & \frac{3}{7} \end{array}\right) \begin{array}{l} R_2 \to R_2 - \frac{1}{6} R_3 \\ R_1 \to R_1 + \frac{1}{4} R_3 \end{array}$$

$$\therefore A^{-1} = \begin{pmatrix} -\frac{1}{7} & \frac{9}{28} & \frac{3}{28} \\ \frac{3}{7} & -\frac{3}{14} & -\frac{1}{14} \\ \frac{-4}{7} & \frac{2}{7} & \frac{3}{7} \end{pmatrix}$$

$$\text{From (i) } X = \begin{pmatrix} -\frac{1}{7} & \frac{9}{28} & \frac{3}{28} \\ \frac{3}{7} & \frac{-3}{14} & -\frac{1}{14} \\ \frac{-4}{7} & \frac{2}{7} & \frac{3}{7} \end{pmatrix} \begin{pmatrix} 4 \\ 5 \\ -5 \end{pmatrix}$$

$$\text{or, } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{7} \times 4 + \frac{9}{28} \times 5 + \frac{3}{28} \times (-5) \\ \frac{3}{7} \times 4 - \frac{3}{14} \times 5 - \frac{1}{14} \times (-5) \\ \frac{-4}{7} \times 4 + \frac{2}{7} \times 5 + \frac{3}{7} \times (-5) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} \\ 1 \\ -3 \end{pmatrix}$$

Objective Questions

1. A square matrix A is singular if determinant of A is

(a) negative (b) zero

(c) positive (d) non-zero

Ans: b

2. If $A = \begin{bmatrix} 2 & 5 \\ a & 3 \end{bmatrix}$ does not have an inverse then $a =$

(a) $\frac{6}{5}$ (b) $\frac{5}{6}$

(c) 6 (d) 5

Ans: a

$$\begin{vmatrix} 2 & 5 \\ a & 3 \end{vmatrix} = 0$$

or, $6 - 5a = 0$

or, $6 = 5a$

$\therefore$ $a = \frac{6}{5}$

3. The inverse of $\begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix}$ is

(a) $\begin{bmatrix} -2 & -3 \\ -4 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & -4 \\ -3 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} -2 & 1 \\ -3 & -4 \end{bmatrix}$ (d) $\begin{bmatrix} -3 & -1 \\ 2 & 4 \end{bmatrix}$

Ans: b

4. The solution of the system $x - y = 2$ and $2x + 3y = 9$ is

(a) (−3, 1) (b) (3, −1)

(c) (−3, −1) (d) (3, 1)

Ans: d

EXERCISE – 6 (F)

1. Apply the method of bisection to find the
   (a) square root of 2 within two places of decimal in [1, 2].
   (b) the approximate value of $\sqrt[3]{7}$ within an error of 10^{-3}.
   (c) cube root of 50 within 2 places of decimal in [3, 4].

Solution

Let x be the square root of 2

Then, $x^2 = 2$

or, $x^2 - 2 = 0$

Let, $f(x) = x^2 - 2$

Here, a = 1, b = 2

a	b	$m = \frac{a+b}{2}$	f(a)	f(b)	f(m)
1	2	1.5	– 1	2	0.25
1	1.5	1.25	– 1	0.25	– 0.4375
1.25	1.5	1.375	– 0.4375	0.25	– 0.1094
1.375	1.5	1.4375	– 0.1094	0.25	0.0664
1.375	1.4375	1.4063	– 0.1094	0.0664	– 0.0225
1.4063	1.4375	1.4219	– 0.2246	0.0664	0.0217
1.4063	1.4219	1.4141	– 0.0225	0.0217	– 0.0004
1.4141	1.4219	1.418	– 0.0004	0.0217	0.0107
1.4141	1.418		– 0.0004	0.0107	

Since a and b are same upto 2 places of decimal, so the required roof is 1.41

(b) Let $x = \sqrt[3]{7}$

Then, $x^3 = 7$

or, $x^3 - 7 = 0$

Let $f(x) = x^3 - 7$

Here, $f(1) = 1^3 - 7 = -6$

$f(2) = 2^3 - 7 = 1$

Since f(1) and f(2) have opposite sign, so a root lies between 1 and 2.

a	b	$m = \frac{a+b}{2}$	f(a)	f(b)	f(m)
1	2	1.5	– 6	1	– 3.625
1.5	2	1.75	– 3.625	1	– 1.64063
1.75	2	1.875	–1.64063	1	– 0.40820
1.875	2	1.9375	– 0.40820	1	0.27319
1.875	1.9375	1.90625	– 0.40820	0.27319	– 0.07309
1.90625	1.9375	1.92186	– 0.07309	0.27319	0.09864
1.90625	1.92186	1.91406	– 0.07309	0.09864	0.01234
1.90625	1.91406	1.91016	– 0.07309	0.01234	– 0.03043
1.91016	1.91406	1.91211	– 0.03043	0.01234	– 0.00901
1.91211	1.91406	1.91309	– 0.00901	0.01234	0.00174
1.91211	1.91309	1.9126	– 0.00901	0.00174	– 0.00364
1.9126	1.91309	1.91285	– 0.00364	0.00174	– 0.00095

Here, $|f(m)| = |-0.00095| = 0.00095 < 10^{-3}$

So, the required root is 1.91285

(c) Let x be the cube root of 50.

Then, $x^3 = 50$

or, $x^3 - 50 = 0$

Let $f(x) = x^3 - 50$

Here, a = 3, b = 4

Now.

a	b	$m = \frac{a+b}{2}$	f(a)	f(b)	f(m)
3	4	3.5	−23	14	−7.125
3.5	4	3.75	−7.125	14	2.73438
3.5	3.75	3.625	−7.125	2.73438	−2.36523
3.625	3.75	3.6875	−2.36523	2.73438	0.14138
3.625	3.6875	3.65625	−2.36523	0.14138	−1.12265
3.65625	3.6875	3.67188	−1.12265	0.14138	−0.49334
3.67188	3.6875	3.67969	−0.49334	0.14138	−0.17656
3.67969	3.6875	3.68359	−0.17656	0.14138	−0.01797
3.68359	3.6875		−0.01797	0.14138	

Since a and b are same upto 2 places of decimal, so the required root is 3.68

2. **Using the bisection method to find the root of the equation.**

(a) $x^2 + x - 4 = 0$ **in [1, 2] correct to 2 places of decimals.**

(b) $x^3 - 4x + 1 = 0$ **in [1, 2] correct to 2 places of decimals.**

(c) $x^3 - 2x - 5 = 0$ **lying between 2 and 3 correct to 3 places of decimals with error less than 0.001.**

(d) $2x^3 - 5x + 2 = 0$ **lying between 1 and 2 with error less than** 10^{-2}.

(e) $\cos x = 3x - 1$ **correct to 4 places of decimals.**

(f) $e^{-x} = 10x$ **correct to 4 places of decimals.**

Solution

(a) Let $f(x) = x^2 + x - 4$

Here, a = 1, b = 2

$f(1) = 1^2 + 1 - 4 = -2$

$f(2) = 2^2 + 2 - 4 = 2$

Since f(1) and f(2) have opposite signs, so there is a root between 1 and 2.

a	b	$m = \frac{a+b}{2}$	f(a)	f(b)	f(m)
1	2	1.5	−2	2	−0.25
1.5	2	1.75	−0.25	2	0.8125
1.5	1.75	1.625	−0.25	0.8125	0.2656
1.5	1.625	1.5625	−0.25	0.2656	0.0039
1.5	1.5625	1.5313	−0.25	0.0039	−0.1238
1.5313	1.5625	1.5469	-0.1238	0.0039	−0.0602
1.5469	1.5625	1.5547	−0.0602	0.0039	−0.0282
1.5547	1.5625	1.5586	−0.0282	0.0039	−0.0122
1.5586	1.5625	1.5606	−0.0122	0.0039	−0.0039
1.5606	1.5625		−0.0039	0.0039	

The values of a and b are same to two places of decimal in last row. Hence, the approximate root to two places of decimal is 1.56.

(b) Let, $f(x) = x^3 - 4x + 1$

$f(1) = 1^3 - 4 \times 1 + 1 = -2$

$f(2) = 2^3 - 4 \times 2 + 1 = 1$

Since f(1) and f(2) have opposite signs, a real root lies between 1 and 2.

a	b	$m = \frac{a+b}{2}$	f(a)	f(b)	f(m)
1	2	1.5	−2	1	−1.624
1.5	2	1.75	−1.625	1	−0.6406
1.75	2	1.875	−0.6406	1	0.0918
1.75	1.875	1.8125	−0.6406	0.0918	−0.2957

1.8125	1.875	1.8437	-0.2957	0.0918	-0.1073
1.8437	1.875	1.8594	-0.1073	0.0918	-0.0092
1.8594	1.875	1.8672	-0.0092	0.0918	0.0411
1.8594	1.8672	1.8633	-0.0092	0.0411	0.0172
1.8594	1.8633	1.8614	-0.0092	0.0172	0.0038
1.8594	1.8614	1.8604	-0.0092	0.0038	-0.0026
1.8604	1.8614		-0.0026	0.0038	

Since the values of a and b are same for 2 places of decimal, so required root is 1.86

(c) Let $f(x) = x^3 - 2x - 5$

Here, a = 2, b = 3

$f(2) = 2^3 - 2 \times 2 - 5 = -1$

$f(3) = 3^3 - 2 \times 3 - 5 = 16$

Since f(2) and f(3) have opposite sign, so one root lies between 2 and 3

a	b	$m = \frac{a+b}{2}$	f(a)	f(b)	f(m)
2	3	2.5	-1	16	5.625
2	2.5	2.25	-1	5.625	1.8906
2	2.25	2.125	-1	1.8906	0.3457
2	2.125	2.0625	-1	0.3457	-0.3513
2.0625	2.125	2.0938	-0.3513	0.3457	-0.0084
2.0938	2.125	2.1094	-0.0084	0.3457	0.1671
2.0938	2.1094	2.1016	-0.0084	0.1671	0.0790
2.0938	2.1016	2.0977	-0.0084	0.0790	0.0352
2.0938	2.0977	2.0958	-0.0084	0.0352	0.0139
2.0938	2.0958	2.0948	-0.0084	0.0139	0.0028
2.0938	2.0948	2.0943	-0.0084	0.0028	-0.0028
2.0943	2.0948		-0.0028	0.0028	

The values of a and b are same to three places of decimal in last row. Hence, the approximate root to three places of decimal is 2.094

(d) Let $f(x) = 2x^3 - 5x + 2$

Here, a = 1, b = 2

$f(1) = 2 \times 1^3 - 5 \times 1 + 2 = -1$

$f(2) = 2 \times 2^3 - 5 \times 2 + 2 = 8$

Since $f(1) \times f(2) = -1 \times 8 = -8 < 0$, a root lies between 1 and 2.

a	b	$m = \frac{a+b}{2}$	f(a)	f(b)	f(m)
1	2	1.5	-1	8	1.25
1	1.5	1.25	-1	1.25	-0.34375
1.25	1.5	1.375	-0.34375	1.25	0.32421
1.25	1.375	1.3125	-0.34375	0.3242	-0.04052
1.3125	1.375	1.34375	-0.04052	0.3242	0.13397
1.3125	1.34375	1.32813	-0.04052	0.13397	0.04478
1.3125	1.32813	1.32032	-0.04052	0.04478	0.00165

Here $|f(m)| = |0.00165| = 0.00165 < 10^{-2}$

So, the required root is 1.32032

(e) Given $\cos x = 3x - 1$

or: $\cos x - 3x + 1 = 0$

Let $f(x) = \cos x - 3x + 1$

$f(0) = \cos 0 - 3 \times 0 + 1 = 2$

and $f(1) = \cos 1 - 3 \times 1 + 1 = -1.459698$

Since f(0) and f(1) have opposite sign.

so a root lies between 0 and 1.

a	b	$m = \frac{a+b}{2}$	f(a)	f(b)	f(m)
0	1	0.5	2	−1.459698	0.377582
0.5	1	0.75	0.377582	−1.459698	−0.518311
0.5	0.75	0.625	0.377582	−0.518311	−0.064037
0.5	0.625	0.5625	0.377582	−0.064037	0.158424
0.5625	0.625	0.59375	0.158424	−0.064037	1.235098
0.59375	0.625	0.609375	1.235098	−0.064037	−0.008119
0.59375	0.609375	0.601563	1.235098	−0.008119	0.019765
0.601563	0.609375	0.605469	0.019765	−0.008119	0.005828
0.605469	0.609375	0.607422	0.005828	−0.008119	−0.001144
0605469	0.607422	0.606446	0.005828	−0.001144	0.002343
0.606446	0.607422	0.606934	0.002343	−0.001144	0.000599
0.606934	0.607422	0.607178	0.000599	−0.001144	−0.000273
0.606934	0.607178	0.607056	0.000599	−0.000273	0.000163
0.607056	0.607178	0.607117	0.000163	−0.000273	−0.000055
0.607056	0.607117	0.6070865	0.000163	−0.000055	0.000054
0.6070865	0.607117	0.60710175	0.000054	−0.000055	−0.0000036
0.6070865	0.60710175	0.607094125	0.000054	−0.0000036	0.00002686
0.607094125	0.60710175	0.6070979375	0.00002686	−0.0000036	0.000013248
0.6070979375	0.60710175	0.6070998438	0.000013248	−0.0000036	0.00000644
0.6070998438	0.60710175	0.6071007969	0.00000644	−0.0000036	0.000003039
0.6071007969	0.60710175		0.000003039	−0.0000036	

Since a and b have same value upto 4 places of decimal, so the required root is 0.6071

(f) Given $e^{-x} = 10x$

or, $e^{-x} - 10x = 0$

Let $f(x) = e^{-x} - 10x$

Now, $f(0) = e^{-0} - 10 \times 0 = 1$

$f(1) = e^{-1} - 10 \times 1 = -9.6321$

Since f(0) and f(1) have opposite sign, so a root lies between 0 and 1.

Now,

a	b	$m = \frac{a+b}{2}$	f(a)	f(b)	f(m)
0	1	0.5	1	−9.6321	−3.351279
0	0.5	0.25	1	−3.351279	−1.215975
0	0.25	0.125	1	−1.215975	−0.116852
0	0.125	0.0625	1	−0.116852	0.439494
0.0625	0.125	0.09375	0.439494	−0.116852	0.160785
0.09375	0.125	0.109375	0.160785	−0.116852	0.0218306
0.109375	0.125	0.1171875	0.0218306	−0.116852	−0.0475447
0.109375	0.1171875	0.11328125	0.0218306	−0.0475447	−0.0128656
0.109375	0.11328125	0.111328125	0.0218306	−0.0128656	0.00448036
0.111328195	0.11328125	0.112304687	0.00448036	−0.0128656	−0.004193165
0.111328195	0.112304687	0.111816406	0.00448036	−0.004193165	0.000143467
0.111816406	0.112304687	0.1120605465	0.000143467	−0.004193165	−0.00202488
0.111816406	0.1120605465	0.1119384763	0.000143467	−0.00202488	−0.00094071
0.111816406	0.1119384763	0.1118774412	0.000143467	−0.00094071	−0.000398626
0.111816406	0.1118774412		0.000143467	−0.000398626	

Since a and b are same upto 4 places of decimal, so the required root is 0.1118

3. **Show that the equation $f(x) = x^3 - 10 = 0$ has no negative root and one positive root. Find the positive root using bisection method with $a = 2$, $b = 3$ and $\varepsilon = 0.01$.**

Solution

Let $f(x) = x^3 - 10$

Here, a = 2, b = 3, $\varepsilon = 0.01$

$f(3) = 2^3 - 10 = -2$

$f(4) = 3^3 - 10 = 17$

Since $f(3) \times f(4) < 0$, a real root lies between 2 and 3.

Since f(x) has one change in sign, so it has a positive root

Also, $f(-x) = -x^3 - 10$ has no change in sign, so it has no negative roots.

a	b	$m = \frac{a+b}{2}$	f(a)	f(b)	f(m)
2	3	2.5	-2	17	5.625
2	2.5	2.25	-2	5.626	1.390625
2	2.25	2.125	-2	1.390625	-0.404297
2.125	2.25	2.1875	-0.404297	1.390625	0.467529
2.125	2.1875	2.15625	-0.404297	0.467529	0.0252991
2.125	2.15625	2.140625	-0.404297	0.0252991	-0.191067
2.140625	2.15625	2.148438	-0.191067	0.0252991	-0.083277
2.148438	2.15625	2.152344	-0.083277	0.0252991	-0.029084
2.152344	2.15625	2.154297	-0.029084	0.0252991	-0.0019172

Here, $|f(m)| = |-0.0019172| = 0.0019172 < 0.01$

The required root is 2.154297.

4. **How many iterations are required to get the root with the tolerance 10^{-4} in the interval [1, 2]?**

Solution

Here, a = 1, b = 2

Error tolerance $(\varepsilon i) = 10^{-4}$

We have, $\frac{|b-a|}{2^i} < \varepsilon i$

or, $\frac{2-1}{2^i} < 10^{-4}$

or, $2^{-i} < 10^{-4}$

or, $-i \log_{10} 2 < -4 \log_{10} 10$

or, $i > \frac{4}{\log_{10} 2} = 13.2877$

Hence the required no. of iterations = 14

Objective Questions

1. **Which of the following is an approximate number?**

(a) 2 (b) π

(c) $\sqrt{4}$ (d) 5

Ans: b

2. **The percent error is given by**

(a) Relative error × 100 (b) $\frac{\text{Relative error}}{100}$

(c) $\frac{\text{Relative error}}{\text{Exact value}} \times 100$ (d) Exact value × 100

Ans: a

3. **Which of the following is transcendental equation?**

(a) $3x + 4y = 5$ (b) $x^2 + 5x + 6 = 0$

(c) $2x^2 + 7xy + 5y^2 = 0$ (d) $\cos x = x$

Ans: d

4. **The bisection method of finding roots of non-linear equation is**

(a) open method (b) random method

(c) graphical method (d) bracketing method

Ans: d

5. **If f is continuous on [a, b] with $f(a) \cdot f(b) < 0$ then $f(x)$ has ... in [a, b]**

(a) no roots (b) one root

(c) at least one root (d) an undeterminable number of roots

Ans: c

6. Why we can't use bisection method to the function $f(x) = x^2$ though it has a root $x = 0$?
 (a) Because it is a polynomial function
 (b) Because it has a slope equal to zero at $x = 0$
 (c) Because it has repeated root at $x = 0$
 (d) Because it is always non-negative.

Ans: d

EXERCISE 6 (G)

1. **If $f(x) = x^2 - 2$ and $x_0 = 1$, find x_1, x_2 and x_3 using Newton–Raphson method.**

Solution

Given, $f(x) = x^2 - 2$

$f'(x) = 2x$

$x_0 = 1.$

We have, $x_{n+1} = x_n - \dfrac{(x_n)}{f'(x_n)}$

$$\therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1^2 - 2}{2 \times 1} = 1 + \frac{1}{2} = 1.5$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.5 - \frac{(1.5)^2 - 2}{2 \times 1.5} = 1.4166$$

$$x_3 = x_2 - \frac{(x_2)}{f(x_2)} = 1.4166 - \frac{(1.4166)^2 - 2}{2 \times 1.4166} = 1.4142$$

2. **Find (a) square root of 51 (b) cube roots of 123 correct to 3 places of decimals using Newton–Raphson method.**

Solution

Let x be the square root of 51.

Then, $x^2 = 51$

or, $x^2 - 51 = 0$

Let $f(x) = x^2 - 51$

Here, $f(7) = 7^2 - 51 = -2 < 0$

$f(8) = 8^2 - 51 = 13 > 0$

So, a root lines between 7 and 8.

Take initial guess $(x_0) = 7$.

Here, a = 51

We have,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$

$$\therefore \quad x_1 = \frac{1}{2}\left(x_0 + \frac{a}{x_0}\right) = \frac{1}{2}\left(7 + \frac{51}{7}\right) = 7.142857$$

$$x_2 = \frac{1}{2}\left(x_1 + \frac{a}{x_1}\right) = 7.141428$$

$$x_3 = \frac{1}{2}\left(x_2 + \frac{a}{x_2}\right) = 7.141428429$$

Since x_2 and x_3 are same upto 3 places of decimal, so required root = 7.141

(b) Let x be the cube root of 123.

Then, $x^3 = 123$

or, $x^3 - 123 = 0$

Let $f(x) = x^3 - 123$

$f'(x) = 3x^2$

Take, initial guess $(x_0) = 10$

We have,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 10 - \frac{10^3 - 123}{3 \times 10^2} = 5.615$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 5.043755415 \qquad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 4.97417249$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 4.973190027 \qquad x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 4.973189833$$

Since x_4 and x_5 are same upto 3 plaeces of decimals, so the required root is 4.973

3. **Using Newton–Raphson method, find a root of**

(a) **$x^3 - x - 4 = 0$ between 1 and 2 to three places of decimal.**

(b) **$x^3 - 2x - 5 = 0$ lying between 2 and 3 correct to 3 places of decimals.**

(c) **$2x^2 - 3x - 1 = 0$ taking $x_0 = 1$ with error less than 10^{-4}.**

(d) **$x - \cos x = 0$ to 3 places of decimals.**

(e) **$x + \log x = 20$ to 3 places of decimals.**

Solution

(a) Given equation is $x^3 - x - 4 = 0$

Let $f(x) = x^3 - x - 4$

$f(1) = 1^3 - 1 - 4 = -4$

$f(2) = 2^3 - 2 - 4 = 2$

Since $f(1)\,f(2) = (-4) \times 2 = -8 < 0$, a real root lies between 1 and 2.

$f'(x) = 3x^2 - 1$

Let the initial guess $x_0 = 1$.

By Newton Raphson's method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{(-4)}{2} = 3$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3 - \frac{20}{26} = 2.2307$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.2307 - \frac{4.8693}{13.928} = 1.8811$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.8811 - \frac{0.7752}{9.6156} = 1.8004$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 1.8004 - \frac{0.0354}{8.7243} = 1.7963$$

$$x_6 = x_5 - \frac{f(x_5)}{f'(x_5)} = 1.7963 - \frac{0.00019}{8.68} = 1.7962$$

Comparing the values of x_5 and x_6, we find that the digits in the first three places of decimal are same. Hence, the required root is 1.796.

(b) Let $f(x) = x^3 - 2x - 5$

Here, a = 2, b = 3

$f(2) = 2^3 - 2 \times 2 - 5 = -1$

$f(3) = 3^3 - 2 \times 3 - 5 = 16$

Since f(2) and f(3) have opposite sign, so one root lies between 2 and 3

a	b	$m = \frac{a+b}{2}$	f(a)	f(b)	f(m)
2	3	2.5	–1	16	5.625
2	2.5	2.25	–1	5.625	1.8906
2	2.25	2.125	–1	1.8906	0.3457

2	2.125	2.0625	−1	0.3457	−0.3513
2.0625	2.125	2.0938	−0.3513	0.3457	−0.0084
2.0938	2.125	2.1094	−0.0084	0.3457	0.1671
2.0938	2.1094	2.1016	−0.0084	0.1671	0.0790
2.0938	2.1016	2.0977	−0.0084	0.0790	0.0352
2.0938	2.0977	2.0958	−0.0084	0.0352	0.0139
2.0938	2.0958	2.0948	−0.0084	0.0139	0.0028
2.0938	2.0948	2.0943	−0.0084	0.0028	−0.0028
2.0943	2.0948		−0.0028	0.0028	

The values of a and b are same to three places of decimal in last row. Hence, the approximate root to three places of decimal is 2.094.

(c) Here, $f(x) = 2x^2 - 3x - 1$

$f(1) = 2 \times 1^2 - 3 \times 1 - 1 = -2$

$f(2) = 2 \times 2^2 - 3 \times 2 - 1 = 1$

Since f(1) and f(2) have opposite signs, so there is a real root between 1 and 2.

$f'(x) = 4x - 3$

Let us take initial guess $x_0 = 1$

By Newton Raphson's method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{(-2)}{1} = 3$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 3 - \frac{8}{9} = 2.1111$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.1111 - \frac{1.5802}{5.4444} = 1.8209$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.8209 - \frac{0.1684}{4.2836} = 1.7816$$

$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 1.7816 - \frac{0.0033}{4.1264} = 1.7808$$

$$x_6 = x_5 - \frac{f(x_5)}{f'(x_5)} = 1.7808 - \frac{0.0001}{4.1232} = 1.7808$$

Here, $|f(1.7808)| = |2(1.7808)^2 - 3(1.7808) - 1| = 0.00009728 < 0.0001$

Hence, the required root is 1.7808.

(d) Given $x - \cos x = 0$

Let $f(x) = x - \cos x$

$f'(x) = 1 + \sin x$

Take, initial guess $(x_0) = 0$

We have,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{f(0)}{f'(0)} = 0 - \frac{0 - \cos 0}{1 + \sin 0} = 1$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1 - \frac{f(1)}{f'(1)} = 0.7503638$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.7391128909$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 0.7390851334$$

Since x_3 and x_4 are same upto 3 places of decimals, so required root = 0.739

(e) $x + \log_{10} x = 2$

or, $x + \log_{10} x - 2 = 0$

Let $f(x) = x + \log_{10} x - 2$

$$f'(x) = 1 + \frac{1}{x}$$

Initial guess $(x_0) = 1$

We have,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 1 - \frac{f(1)}{f'(1)} = 1.5$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 1.694345245 \qquad x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.742548565$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.752883931 \qquad x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 1.755025245$$

$$x_6 = x_5 - \frac{f(x_5)}{f'(x_5)} = 1.755465677$$

Since x_5 and x_6 have same value upto 3 places of decimals, so required root = 1.755

Objective Questions

1. The general formula for Newton–Raphson method is

(a) $x_{n+1} = x_n - \frac{f'(x_n)}{f(x_n)}$ (b) $x_n = x_{n+1} - \frac{f'(x_n)}{f(x_n)}$

(c) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ (d) $x_{n+1} = x_n + \frac{f(x_n)}{f'(x_n)}$

Ans: c

2. If a be the square root of x then Newton–Raphson's formula for square root is

(a) $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ (b) $x_{n+1} = \frac{1}{2}\left(x_n - \frac{a}{x_n}\right)$

(c) $x_{n+1} = \frac{1}{2}\left(x_n + \frac{x_n}{a}\right)$ (d) $x_{n+1} = \frac{1}{2}\left(x_n - \frac{x_n}{a}\right)$

Ans: a

3. The Newton–Raphson method of finding roots of non-linear equation is

(a) open method (b) random method

(c) graphical method (d) bracketing method

Ans: a

4. The next iterative value of the root of $x^2 - 4 = 0$ by Newton–Raphson method if the initial guess is 1 is

(a) 1.5 (b) 2.5

(c) 3.5 (d) 4.5

Ans: b

$x_0 = 1$

$f(1) = 1^2 - 4 = -3$

$f'(x) = 2x$

$f'(1) = 2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{(-3)}{2} = \frac{2+3}{2} = 2.5$$

5. Which of the following is not the condition for measuring error?

(a) $|x_n - x_{n-1}| < \varepsilon$ (b) $\frac{|x_n - x_{n-1}|}{|x_n|} < \varepsilon$

(c) $|f(x_n)| < \varepsilon$ (d) $|f(x_n)| > \varepsilon$

Ans: d

□