Lecture Notes

TAYLOR'S THEOREM

Taylor's theorem states that any function satisfying certain conditions may be represented by a Taylor's series.

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + ... + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \int_0^x \frac{(x-u)^{n-1}}{(n-1)} f^{(n)}(u)\,du .$$

Taylor's theorem (without the remainder term) was devised by Taylor in 1712 and published in 1715, although Gregory had actually obtained this result nearly 40 years earlier. In fact, Gregory wrote to John collris, secretary of the Royal society, on February 15-1671, to tell him of the result. The actual notes in which Gregory seems to have discovered the theorem exist on the back of a letter Gregory has received on 30 January 1671 from an Edinburgh bookseller, which is preserved in the library of the University of St. Andrew. However, it was not until almost a century after Taylor's publication that Lagrange and Cauchy derived approximations of the remainder term after a finite number of terms. These forms are now called the Lagrange remainder and Cauchy remainder. Taylor's theorem is taught in introductory level calculus courses and it is one of the central elementary tools in Mathematical analysis. Within pure mathematics it is the starting point of more advanced asymptotic analysis, and it is commonly used in more applied fields of numeric as well as in mathematical physics. Taylor's theorem also generalizes to multivariate and vector valued functions $f : R^n \to R^m$ on any dimension's n and m.

Taylor's theorem

Statement: If $y = f(x+y)$ is continuous and differentiable function of 'h' then it can be expanded in the powers of 'h' as

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + ...$$

Proof. Let $f(x+h) = A_0 + A_1 h^2 + A_3 h^3 + ... \to (1)$

Now put $h = 0$ in (1) we get

$$f(x) = A_0 .$$

Differentiate (1) with respect to 'h'

$$f'(x+h) = A_1 + 2A_2 h + 3A_3 h^2 + ...$$

Now put $h = 0$ in above we get

$$f'(x) = A_1 \Rightarrow \frac{f'(x)}{1!} = A_1 .$$

Again differentiate we get

$$f''(x+h) = 2A_2 + 6A_3 h + ...$$

Now put $h = 0$ in above we get

$$f''(x) = 2A_0 \Rightarrow A_2 = \frac{f''(x)}{2!} .$$

Again we see that

$$f'''(x+h) = 6A_3 + \cdots$$

Now put $h = 0$ in above we get

$$f'''(x) = 6A_3 \Rightarrow A_3 = \frac{f'''(x)}{3!} \ .$$

Substitute the values of $A_0, A_1, A_2, A_3, \ldots$ in (1) we get

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$$

Now put $x = a$ in above we get

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots$$

Replace h by $x - a$ in the above we get

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

Which establishes the result.

Assuming the possibility of expansion, prove the following:

Q. No. 1. $e^{x+h} = e^x \left[1 + h + \frac{h^2}{2!} + \frac{h^3}{3!} + \ldots . \right]$.

Sol. We have $f(x+h) = e^{x+h}$.

Put $h = 0$ in above we get

$$f(x) = e^x$$

$\Rightarrow \qquad f'(x) = e^x$

$\Rightarrow \qquad f''(x) = e^x$

$\Rightarrow \qquad f'''(x) = e^x$, and so on.

Now by Taylor's theorem

$$f(x+h) = f'(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \ldots .$$

$$\Rightarrow \qquad e^{x+h} = e^x + \frac{h}{1!} + \frac{h^2}{2!} + \frac{h^3}{3!} + \ldots .$$

Hence proved.

Q. No. 2. $\tan^{-1}(x+h) = \tan^{-1} + \frac{h}{1+x^2} \cdot \frac{xh^2}{(1+x^2)^2} + \ldots .$

Sol. $\qquad f(x+h) = \tan^{-1}(x+h)$

Put $h = 0$ in above we get

$$f(x) = \tan^{-1}(x)$$

$\Rightarrow \qquad f'(x) = \tan^{-1}(x)$

$\Rightarrow \qquad f''(x) = \dfrac{1}{(1+x^2)}$

$\Rightarrow \qquad f''(x) = \dfrac{1}{(1+x^2)} \times -2x$, and so on.

Now by Taylor's theorem

$$\tan^{-1}(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x)$$

$$\Rightarrow \qquad \tan^{-1}(x+h) = \tan^{-1} x + \frac{h}{(1+x^2)} + \frac{h^2}{2!(1+x^2)^2} f''(x) + ...$$

Hence proved.

Q. No. 3. $\sin^{-1}(x+h) = \sin^{-1} + \dfrac{h}{\sqrt{(1-x^2)}} + \dfrac{x}{(1+x^2)^{3/2}} . \dfrac{h^2}{2!} + \dfrac{1+2x^2}{(1-x^2)^{3/2}} . \dfrac{h^3}{3!} + ...$

Sol. Put $h = 0$ in above we get

$$f(x) = \sin^{-1} x$$

$$\Rightarrow \qquad f'(x) = \frac{1}{\sqrt{(1-x^2)}} = (1-x^2)^{-1/2}$$

$$\Rightarrow \qquad f''(x) = \frac{-1}{2}(1-x^2)^{-3/2} \times -2x$$

$\Rightarrow \qquad f'''(x) = \dfrac{x}{(1-x^2)^{3/2}}$, and so on.

Now by Taylor's theorem

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + ...$$

$$\Rightarrow \qquad \sin^{-1}(x+h) = \sin^{-1} x + \frac{h}{1!} \frac{1}{\sqrt{1-x^2}} + \frac{h^2}{2!} \frac{x}{(1+x^2)^{3/2}} + ...$$

Hence proved.

Q. No. 4. $\log\sin(x+h) = \log \mathrm{sinx} + \mathrm{hcotx}' \dfrac{1}{2} h^2 \cos ec^2 x + \dfrac{1}{2} h^3 \cot x \cos ec^2 x + ...$

Sol. Put $h = 0$ in above we get

$$f(x) = \log\sin x$$

$$\Rightarrow \qquad f'(x) = \frac{1}{\sin x} \times \cos x = \cot x$$

$$\Rightarrow \qquad f''(x) = -\cos ec^2 x$$

$\Rightarrow \qquad f'''(x) = -\operatorname{cosec}^2 x \cot x$, and so on.

By Taylor's theorem

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + ...$$

$$\Rightarrow \qquad f(x+h) = \log \sin x + h \cot x - \frac{1}{2} h^2 \operatorname{cosec}^2 x + \frac{h^3}{3!} \cot \operatorname{cosec}^2 + ...$$

Hence proved.

Q. No. 5. $e^{a(x+h)} \sin(x+h) = e^{ax}\left[\sin mx + hr\sin(x+\phi) + \frac{1}{2}h^2 r^2 \sin(mx+2\phi) + ...\right]$.

Where $r^2 = a^2 + m^2$ and $a \tan\phi = m$. (1)

Sol. $\quad f(x+h) = e^{a(x+h)}.\sin m(x+h)$

Put $h = 0$ in above we get

$$f(x) = e^{ax}.\sin mx$$

$$\Rightarrow \qquad f'(x) = e^{ax}.a\sin mx + e^{ax} m\cos mx$$

$$= e^{ax}\left[a\sin mx + m\cos mx\right]$$

Put a = $\cos\phi$ in (1) we see that $m = \sin\phi$.

$$\Rightarrow \qquad f'(x) = e^{ax}\left[\cos\phi \sin mx + \sin\phi \cos mx\right]$$

$$\Rightarrow \qquad f''(x) = e^{ax}\sin(mx+\phi) \qquad [\because \sin a\cos b + \cos \sin b = \sin(a+b)]$$

$$\Rightarrow \qquad f''(x) = e^{ax}\sin(mx+2\phi)$$

By Taylor's theorem we have

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h}{2!} f''(x) + ...$$

$$\Rightarrow \qquad e^{a(x+h)}\sin m(x+h) = e^{ax}\sin mx + \frac{h}{1!}e^{ax}\sin\left(mx+\phi+\frac{h^2}{2}r^2\sin(mx+2\phi)\right)$$

$$\Rightarrow \qquad e^{a(x+h)}\sin m(x+h) = e^{ax}\left[\sin mx + hr\sin(mx+\phi) + \frac{1}{2}h^2r^2\sin(mx+2\phi) + ...\right]$$

Hence proved.

Q. No. 6. Expand

1) x^3 in powers of $(x-1)$.

2) $\log \sin x$ in powers of $(x-a)$.

Sol.

1) Expand x^3 in powers of $(x-1)$.

Let $f(x) = x^3$.

$\therefore f(x) = f(1+(x-1)) = f(1+h)$, where $h = (x-1)$.

Now $f(x)=x^3 \quad \therefore f(1)=1$

$f'(x)=3x^2 \quad \therefore f(1)=3$

$f''(x)=6x \quad \therefore f(1)=6$

$f'''(x)=6 \quad \therefore f(1)=6$.

Therefore by Taylor's theorem we have

$$\therefore x^3 = f(1)+hf'(3)+\frac{h^2}{2!}f''(6)+\frac{h^3}{3!}f'''(6)+\cdots$$

$$\Rightarrow x^3 = 1+(x-1)3+\frac{(x-1)^2}{2!}.6+\frac{(x-1).6^3}{3!}+\cdots$$

$$\Rightarrow x^3 = 1+3(x-1)+(x-1)^2.3+(x-1)^2.2!+\cdots$$

2) $\log\sin x$ in powers $(x-a)$.

Sol. $f(x)=\log\sin x$

$\Rightarrow f(x-a+a)=\log\sin(x-a+a)$

$=\log\sin(a+(x-a))$

$=\log\sin(a+h)$,

where $h=x-a$.

Now $f(x)=\log\sin x \qquad \therefore f(a)=\log\sin a$

$\Rightarrow \quad f'(x)=\cot x \qquad \therefore f'(a)=\cot a$

$\Rightarrow \quad f''(x)=-\cos ec^2 x \qquad \therefore f''(a)=-\cos ec^2 a$

$\Rightarrow \quad f'''(x)=2\cos ec^2 x\cot x \qquad \therefore f'''(a)=2\cos ec^2 a\cot a$

Now by Taylor's theorem

$$f(x+h)=f(x)+\frac{h}{1!}f'(x)+\frac{h^2}{2!}f''(x)+....$$

$$\Rightarrow \quad \log\sin(a+(x-a))=\log\sin x+(x-a)\cot a-\tfrac{(x-a)^2}{2}\cos ec^2 a+\cdots$$

<u>McLaurin's Theorem</u>

Statement: $y=f(x)$ is continuous and differentiable then

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f''(0)+\cdots$$

Proof. Let $f(x)=A_0+A_1x+A_2x^2+A_3x^3+...$ (1)

$\Rightarrow \quad f'(x)=A_1+2A_2x+3A_3x^2+...$

$\Rightarrow \quad f''(x)=2A_2+6A_3x+...$

$\Rightarrow \quad f'''(x) = 6A_3 + ...$

Put $x = 0$ in above equations we get

$$f(0) = A_0,\ f'(0) = A_1,\ f''(0) = 2A_2,\ f'''(0) = 6A_3, \text{ and so on}$$

$$\Rightarrow \quad A_0 = f(0), A_1 = f'(0), A_2 = f''\frac{(0)}{2!}, A_3 = f'''\frac{(0)}{3!}, \text{ and so on.}$$

Using in (1) we get

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots.$$

Which establishes the result.

Assuming the possibility of expansion, obtain the following series.

Q. No. 1. 1) $e^{mx} = 1 + mx + \frac{m^2x^2}{2!} + \frac{m^2x^3}{3!} + ...$

Sol. Let $f(x) = e^{mx} \qquad \therefore f(0) = 1$

$$f'(x) = me^{mx} \qquad \therefore f(0) = m$$

$$f''(x) = m^2e^{mx} \qquad \therefore f(0) = m^2$$

$$f'''(x) = m^3e^{mx} \qquad \therefore f(0) = m^3.$$

Now by McLaurin's theorem we have

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ...$$

$$\Rightarrow \quad e^{mx} = 1 + mx + \frac{x^2}{2!}m^2 + \frac{x^3}{3!}m^3 + ...$$

Hence proved.

ii) $a^{mx} = 1 + (m\log a)x + \frac{(m\log a)^2 x^2}{2!} + \frac{(m\log a)^3 x^3}{3!} + ...$

Sol. $f(x) = a^{mx} \qquad \therefore f(0) = 1$

$$\Rightarrow \quad f'(x) = m\log a a^{mx} \qquad \therefore f'(0) = m\log a$$

$$\Rightarrow \quad f''(x) = a^{mx}m^2(\log a)^2 \qquad \therefore f''(0) = (m\log a)^2$$

$$\Rightarrow \quad f'''(x) = a^{mx}m^3(\log a)^3 \qquad \therefore f'''(0) = (m\log a)^3.$$

And so on.

Now by McLaurin's theorem we have

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ...$$

$$\Rightarrow \qquad a^{mx} = 1 + x(m\log a) + \frac{(m\log a)^2 x^2}{2!} + \frac{(m\log a)^3 x^3}{3!} + ...$$

Hence proved.

iii) $\log(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 + ...$

Sol. $f(x) = \log(1-x) \qquad \therefore\ f(0) = 0$

$\Rightarrow f'(x) = \frac{1}{(1-x)} \times -1 \qquad \therefore\ f'(0) = -1$

$\Rightarrow f''(x) = \frac{-1}{(1-x)^2} \qquad \therefore\ f''(0) = -1$

$\Rightarrow f'''(x) = \frac{-1}{(1-x)^3} \qquad \therefore\ f'''(0) = -1.$

And so on.
Now by McLaurin's theorem we have

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ...$$

$$\Rightarrow \qquad \log(1-x) = -x - \frac{x^2}{2!} - \frac{1}{3}x^3 + \cdots$$

Hence proved.

iv) $\text{Sin}\, ax = ax - \frac{a^3x^3}{3!} + \frac{a^5x^5}{5!} + \cdots$

Sol. We have to show that $\sin ax = ax - \frac{a^3x^3}{3!} + \frac{a^5x^5}{5!} + \cdots$

Here $\quad f(x) = \sin ax \qquad \therefore f(0) = 0$

$\Rightarrow f'(x) = a\cos ax \qquad \therefore f(0) = a$

$\Rightarrow f''(x) = -\sin ax.a^2 \qquad \therefore f''(0) = 0$

$\Rightarrow f'''(x) = -\cos ax.a^3 \qquad \therefore f'''(0) = -a^3$

$\Rightarrow f^{iv}(x) = \sin axa^4 \qquad \therefore f^{iv}(0) = 0$

$\Rightarrow f^{v}(x) = \cos axa^5 \qquad \therefore f^{v}(0) = a^5$

and so on.
Now by McLaurin's theorem we have

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ...$$

$$\Rightarrow \sin ax = ax - \frac{a^3x^3}{3!} + \frac{a^5x^5}{5!} + ...$$

Hence proved.

Q. No. 2. Prove that $e^{ax}.\cos bx = 1 + ax + \frac{a^2-b^2}{2!}x^2 + \frac{a\left(a^2-36^2\right)}{3!}x^2 + ...$

Sol. Let $f(x) = e^{ax}\cos bx$, then $f(0) = 1$. Therefore we know that $f^n(x) = r^n e^{ax}\cos(bx + n\phi)$, where r = $\sqrt{a^2+b^2}, \phi = \tan^{-1}\frac{b}{a}$. Therefore $f^n(0) = r^n \cos n\phi$.

Now $\quad f'(0) = r\cos\phi = \sqrt{a^2+b^2}.\frac{a}{\sqrt{a^2+b^2}} = a$

$$\Rightarrow \quad f''(0) = r^2\cos 2\phi = \left(a^2+b^2\right)\left(\frac{1-\tan^2\phi}{1+\tan^2\phi}\right) = a^2-b^2$$

$$\Rightarrow \quad f'''(0) = r^3\cos\phi = \left(a^2+b^2\right)^{3/2}.\left(4\cos^3 - 3\cos\phi\right)$$

$$= \left(a^2+b^2\right)^{3/2}\cos\phi\left(4\cos^2\phi - 3\right)$$

$$= \left(a^2+b^2\right)^{3/2}.\frac{a}{\sqrt{a^2+b^2}}.\left(\frac{4a^2}{a^2+b^2} - 3\right)$$

$$= a\left(a^2+b^2\right)\left(\frac{a^2-3b^2}{a^2+b^2}\right)$$

$$= a\left(a^2-3b^2\right).$$

and so on.
Now by McLaurin's theorem we have

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ...$$

$$\therefore e^{ax}\cos bx = 1 + ax + \frac{a^2-b^2}{2!}x^2 + \frac{a\left(a^2-3b^3\right)}{3!}x^3 + ...$$

Hence proved.

Q. No. 3. $e^{ax}\sin bx = bx + abx^2 + \frac{3a^2b-b^3}{3!}x^3$

Sol. Let $y = e^{ax}\sin bx$, then $y(0) = 0$. Now we know that $y_n = r^n e^{ax}\sin\left(bx+n\phi\right)$ t, where r = $\sqrt{a^2+b^2}, \phi = \tan^{-1}\frac{b}{a}$. Therefore $y_n(0) = r^n \sin n\phi$. Thus we have

$$\therefore y_1(0) = r\sin\phi = \sqrt{a^2+b^2}\frac{b}{\sqrt{a^2+b^2}} = b$$

$$\Rightarrow y_2(0) = r^2\left(\sin 2\phi\right) = \left(a^2+b^2\right)2\sin\phi\cos\phi$$

$$= 2\left(a^2+b^2\right)\left(\frac{b}{\sqrt{a^2+b^2}}\right)\left(\frac{a}{\sqrt{a^2+b^2}}\right)$$

$$=2ab$$

$$\Rightarrow y_3(1)=r^3\sin 3\phi=\left(a^2+b^2\right)^{3/2}\left(3\sin\phi-4\sin^3\phi\right)$$

$$=(a^2+b^2)^{3/2}\frac{b}{\sqrt{a^2+b^2}}\left(\frac{3-4b^2}{a^2+b^2}\right)$$

$$=b\left(a^2+b^2\right)\left(\frac{3a^2-b^2}{a^2+b^2}\right)$$

$$=b\left(3a^2-b^2\right)$$

and so on.

Now by McLaurin's theorem we have

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+....$$

Therefore $y=e^{ax}\sin bx=y_0+xy_1+\frac{x^2}{2!}y_2(0)+\frac{x^3}{3!}y_3(0)+....$

$$\Rightarrow e^{ax}\sin bx=bx+abx^2+\frac{3a^2b-b^3}{3!}x^3+....$$

Hence proved.

Q. No. 4. $\sin\left(m\sin^{-1}x\right)=mx+\frac{m\left(1^2.m^2\right)}{3!}x^2+\frac{m\left(1^2-m^2\right)\left(3^2-m^2\right)x^5}{5!}+\cdots$

Sol. $f(x)=\sin\left(m\sin^{-1}x\right)$ $\qquad\qquad \therefore\ f(0)=0$

$$\Rightarrow f'(x)=\cos\left(m\sin^{-1}x\right)\frac{m}{\sqrt{1-x^2}}$$

Or $\quad y_1\sqrt{1-x^2}=m\cos\left(m\sin^{-1}x\right).$ $\qquad\qquad \therefore y_1(0)=m$

Squaring both sides we get

$$\Rightarrow\left(1-x^2\right)y_1^{\,2}=m^2\cos^2\left(m\sin^{-1}x\right)$$

$$\Rightarrow\left(1-x^2\right)y_1^{\,2}=m^2\left\{1-\sin^2\left(m\sin^{-1}x\right)\right\}$$

$$\Rightarrow\left(1-x^2\right)y_1^{\,2}=m^2\left\{1-y^2\right\}.$$

Now differentiating again we get

$$\Rightarrow f''(0)=2y_1y_2\left(1-x^2\right)+y_1^{\,2}\left(-2x\right)=-2m^2yy_1$$

$$\Rightarrow\left(1-x^2\right)y_2-xy_1+m^2y=0 \qquad \therefore y_2(0)=0$$

Differentiating n times, we have

$$\left(1-x^2\right)y_{n+2}+{}^{n}y_{n+1}\left(-2x\right)+\frac{n(n-1)}{2!}y_n\left(-2\right)-xy_{n+1}-{}^{n}y_nm^2y_n=0$$

Or $\left(1-x^2\right)y_{n+2}-\left(2n+1\right)xy_{n+1}+\left(m^2-n^2\right)y_n=0$

Now put $x=0$ in above we get

$\Rightarrow y_{n+2}\left(0\right)=\left(n^2-m^2\right)y_n\left(0\right).$ (1)

Also

$y\left(0\right)=0, y_1(0)=m, y_1(0)=0,\ldots$

$\therefore$ Putting $n=1,2,3,\ldots$ in (1), we get

$\Rightarrow y_3\left(0\right)=\left(1^2-m^2\right)y_1\left(0\right)=m\left(1^2-m^2\right)$

$\Rightarrow y_4\left(0\right)=\left(2^2-m^2\right)y_2\left(0\right)=0$

$\Rightarrow y_5\left(0\right)=\left(3^2-m^2\right)y_3\left(0\right)=m\left(1^2-m^2\right)\left(3^2-m^2\right)$

and so on.

Now by McLaurin's theorem we see by assuming $f(x)=y$ and $f^n(x)=y_n$.

$$\Rightarrow \sin\left(m\sin^{-1}x\right)=mx+m\left(1^2-m^2\right)\frac{x^3}{3!}+\frac{m\left(1^2-2^2\right)\left(3^2-m^2\right)x^5}{5!}+\cdots$$

Hence proved.

Q. No. 5. $\tan^{-1}x=x-\frac{1}{3}x^3+\frac{1}{5}x^5\ldots.$

Sol. Let $y=\tan^{-1}$ $\therefore y(0)=0$

Now $y_1=\frac{1}{1+x^2}$ $\therefore y_1\left(0\right)=1$

$\Rightarrow y_1=\left(1+x^2\right)=1-x^2+x^4-x^6+\ldots.$

$\Rightarrow$ $y_2=-2x+4x^3-6x^5+\ldots.$ $\therefore y_2\left(0\right)=0$

$\Rightarrow$ $y_3=-2+12x^2-30x^4$ $\therefore y_3\left(0\right)=-2$

$\Rightarrow$ $y_4=24x-120x^3$ $\therefore y_4\left(0\right)=0$

$\Rightarrow$ $y_5=24-360x^2$ $\therefore y_5\left(0\right)=24$

and so on.

Now by McLaurin's theorem we have

$$f\left(x\right)=f\left(0\right)+\frac{x}{1!}f'\left(0\right)+\frac{x^2}{2!}f''\left(0\right)+\frac{x^3}{3!}f'''\left(0\right)+\ldots.$$

$$\Rightarrow \quad \tan^{-1}x=x-\frac{2x^3}{3!}+\frac{24x^5}{5!}+\cdots$$

Hence proved.

Q. NO. 5. $\frac{e^x}{\cos x}=1+x+\frac{2x^2}{2!}+\frac{4x^3}{3!}+\ldots.$

Sol. Let $f\left(x\right)=\dfrac{e^{x}}{\cos x}\Rightarrow f\left(0\right)=1$.

Or $f(x)=e^{x}\sec x$

$\Rightarrow f'\left(x\right)=e^{x}\sec x+e^{x}\sec x\tan x$

$\Rightarrow f''\left(x\right)=f'\left(x\right)+f'\left(x\right)\tan x+f\left(x\right)\sec^{2}x$ $\qquad\therefore f''\left(0\right)=1+1\times 0+1=1+1=2$

$\Rightarrow f'''\left(x\right)=f''\left(x\right)+f''\left(x\right)\tan x+f'\left(x\right)\sec^{2}+f\left(x\right)2\sec^{2}x\tan x+f'\left(x\right)\sec^{2}x$

$\qquad=f''\left(x\right)+f''\left(x\right)\tan x+2f'\left(x\right)\sec^{2}x+2f\left(x\right)\sec^{2}x\tan x$

$\qquad\therefore f'''\left(0\right)=2+2\times 0+2\times 1\times 1+2\times 1=2+4=4$

and so on.

Now by McLaurin's theorem we have

$$f\left(x\right)=f\left(0\right)+\frac{x}{1!}f'\left(0\right)+\frac{x^{2}}{2!}f''\left(0\right)+\frac{x^{3}}{3!}f'''\left(0\right)+....$$

$$\Rightarrow\qquad \frac{e^{x}}{\cos x}=1+x+\frac{x^{2}}{2!}\left(2\right)+\frac{x^{3}}{3!}\left(4\right)+....$$

Hence proved.

Q. NO. 7. $\left(1+x\right)^{1+x}=1+x+x^{2}+\dfrac{1}{2}x^{3}+....$

Sol. Let $y=\left(1+x\right)^{1+x}$.

Taking log on both sides we get

$\Rightarrow\log y=\left(1+x\right)\log\left(1+x\right)$

$$=\left(1+x\right)\left(x-\frac{x^{2}}{2}+\frac{x^{3}}{3}-\frac{x^{4}}{4}+....\right)\qquad\left[\because\log\left(1+x\right)=x-\frac{x^{2}}{2}+\frac{x^{3}}{3}-\frac{x^{4}}{4}+....\right]$$

$$=x-\frac{x^{2}}{2}+\frac{x^{3}}{3}-\frac{x^{4}}{4}+x^{2}-\frac{x^{3}}{2}+\frac{x^{4}}{3}-\frac{x^{5}}{4}+....$$

$$=x+\frac{x^{2}}{2}-\frac{x^{3}}{6}+\frac{x^{4}}{12}+....$$

$$\therefore\log y=z\quad\text{where }z=\left(x+\frac{x^{2}}{2}-\frac{x^{3}}{6}+....\right)$$

$$\therefore y=e^{z}$$

$$=1+z+\frac{z^{2}}{2!}+\frac{z^{3}}{3!}+....$$

$$=1+\left(x+\frac{x^{2}}{2}-\frac{x^{3}}{6}+....\right)+\left(x+\frac{x^{2}}{2}-\frac{x^{3}}{6}+....\right)+\left(x+\frac{x^{2}}{2}-\frac{x^{3}}{6}+....\right)\frac{1}{6}+....$$

$$=1+x+x^2+\frac{x^2}{2}+....$$

$$\Rightarrow \quad (1+x)^{1+x}=1+x+x^2+\frac{x^3}{2}+....$$

Hence proved.

Q. No. 8. $\log\sec x=\frac{1}{2}x^2+\frac{1}{12}x^4+\frac{1}{45}x^6+....$

Sol. Let $f(x)=\log\sec x\Rightarrow f(0)=\log(1)=0$.

$\Rightarrow f'(x)=\frac{1}{\sec x}.\tan x\sec x=\tan x \qquad \therefore f'(0)=0$

$\Rightarrow f''(x)=\sec^2 x \qquad \therefore f''(0)=1$

$\Rightarrow f'''(x)=2\sec^2 x\tan x$

$=2f''(x)f'(x) \qquad \therefore f'''(0)=0$

$\Rightarrow f^{iv}(x)=2\left[f'''(x)f'(x)+f''(x)f''(x)\right]$

$=2\left[f'''(x)f'(x)+\left\{f''(x)\right\}^2\right] \qquad \Rightarrow f^{iv}(0)=2[0+1]=2$

$\Rightarrow f^{v}(x)=2\left[f^{iv}(x)f'(x)+f''(x)f'''(x)+2f''(x)f'''(x)\right]$

$\Rightarrow f^{v}(x)=2[0\times 0+1\times 0+2\times 1\times 0]=0$

and so on.

Now by McLaurin's theorem we have

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+....$$

$$\therefore \log\sec x=\frac{x^2}{2!}+\frac{2x^4}{4!}+\cdots$$

<u>Series expansion of common functions</u>

$\frac{1}{1-x}=1+x+x^2+x^3+x^4+x^5+\ldots$ for $-1<x<1$ **(1)**

$\cos x=1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6+\ldots$ for $-\infty<x<\infty$ **(2)**

$\cos^{-1}x=\frac{1}{2}\pi-x-\frac{1}{6}x^3-\frac{3}{40}x^5-\frac{5}{112}x^7-\ldots$ **(3)**

$\cosh x=1+\frac{1}{2}x^2+\frac{1}{24}x^4+\frac{1}{720}x^6+\frac{1}{40320}x^8+\ldots$ **(4)**

$\cot^{-1}x=\frac{1}{2}\pi-x+\frac{1}{3}x^3-\frac{1}{5}x^5+\frac{1}{7}x^7-\frac{1}{9}x^9+\ldots$ **(5)**

$e^x=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\ldots$ for $-\infty<x<\infty$ **(6)**

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots \quad \text{for } -1 < x \le 1 \tag{7}$$

$$\ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \frac{2}{7}x^7 + \dots \quad \text{for } -1 < x < 1 \tag{8}$$

$$\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \dots \tag{9}$$

$$\operatorname{sech} x = 1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 - \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \dots \tag{10}$$

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \frac{1}{5040}x^7 + \dots \quad \text{for } -\infty < x < \infty \tag{11}$$

$$\sin^{-1} x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \frac{35}{1152}x^9 + \dots \tag{12}$$

$$\sinh x = x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \frac{1}{362880}x^9 + \dots \tag{12}$$

$$\sinh^{-1} x = x - \frac{1}{6}x^3 + \frac{3}{40}x^5 - \frac{5}{112}x^7 + \frac{35}{1152}x^9 - \dots \tag{13}$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \dots \tag{14}$$

$$\tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots \quad \text{for } -1 < x < 1 \tag{15}$$

$$\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \dots \tag{16}$$

$$\tanh^{-1} x = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \frac{1}{9}x^9 + \dots. \tag{17}$$