In view of COVID-19
This PPT is prepared using online data available on different website and is prepared for the purpose to give an overview of Graph theory.
Hope you all are safe and fine.
Beware; Stay Safe;

Graph Theory

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Overview

Introduction

Graph Theory began with Leonhard Euler in his study of the Bridges of Konigsburg problem(Can you take a walk through the town, visiting each part of the town and crossing each bridge only once? figure 1). Since Euler solved this very first problem in Graph Theory, the field has exploded, becoming one of the most important areas of applied mathematics we currently study. Generally speaking, Graph Theory is a branch of Combinatorics but it is closely connected to Applied Mathematics, Optimization Theory and Computer Science.

Introduction

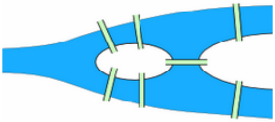


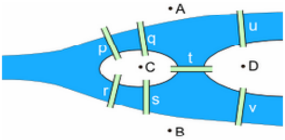
Figure 1

There are four areas of the town - on the mainland north of the river, on the mainland south of the river, on the island and on the peninsula (the piece of land on the right)

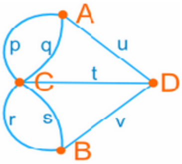
Let us label them A, B, C and D:

To "visit each part of the town" you should visit the points **A, B, C and D**.

And you should cross each bridge **p, q, r, s, t, u and v** just once.

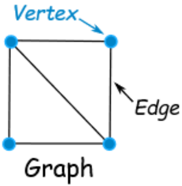


And we can further simplify it to this:



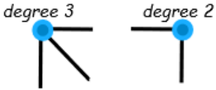
Words used in graph theory

- A point is called a **vertex** (plural vertices)
- A line is called an **edge**.
- The whole diagram is called a **graph**.



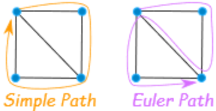
Yes, it is called a "Graph"... but it is **NOT this kind of graph**:

They are both called "graphs".
But they are different things. Just how it is.



- The number of edges that lead to a vertex is called the **degree**.

- A route around a graph that visits **every vertex** once is called a **simple path**.
- A route around a graph that visits **every edge** once is called an **Euler path**.



Basic Definitions used in Graph Theory

Graph Theory

Graph Theory is a sub-field of mathematics which deals with graphs: diagrams that involve points and lines and which often pictorially represent mathematical truths. Graph theory is the study of the relationship between edges and vertices. Formally, a graph is a pair (V, E), where V is a finite set of vertices and E a finite set of edges.

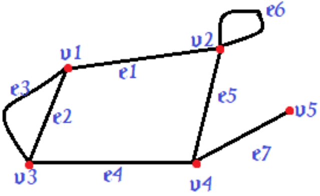
Self loop

In graph theory a loop is an edge of a graph which starts and ends at the same vertex.

Parallel edges

In graph theory, multiple edges (also called parallel edges or a multi-edge), are two or more edges that are incident to the same two vertices. A simple graph has no multiple edges.

Types of Graph



Graph with five vertices and seven edges

- **Simple Graph:** A graph with no loops and no parallel edges is called a simple graph.
  Note 1: The maximum number of edges possible in a single graph with 'n' vertices is nC2 where nC2 $= n(n-1)/2$.
  Note 2: The number of simple graphs possible with 'n' vertices = 2nc2 = 2n(n-1)/2.
  Example: In the following graph, there are 3 vertices with 3 edges which is maximum excluding the parallel edges and loops.

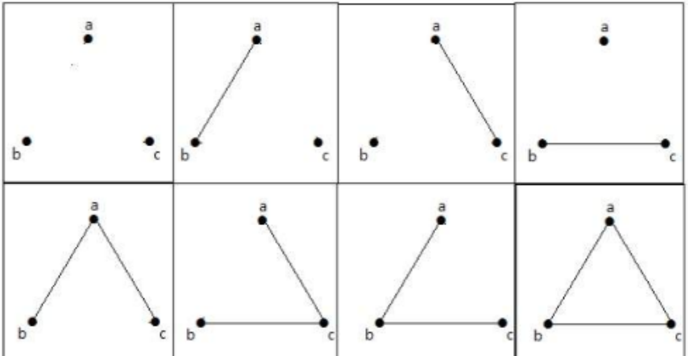
The maximum number of edges with n=3 vertices −

```
ⁿC₂ = n(n−1)/2
    = 3(3−1)/2
    = 6/2
    = 3 edges
```

The maximum number of simple graphs with n=3 vertices −

$$
\begin{aligned}
2^{^nC_2} &= 2^{n(n-1)/2} \\
&= 2^{3(3-1)/2} \\
&= 2^3 \\
&= 8
\end{aligned}
$$

These 8 graphs are as shown below −



Multiple Graph: In some directed as well as undirected graphs,we may have pair of nodes joined by more than one edges, such edges are called multiple or parallel edges.

Pseudograph: A graph in which loops and multiple edges are allowed is called psuedograph.

Null Graph: A graph G=(V,E) where E=0 is said to be Null or Empty graph.

Trivial Graph: A graph with One vertex and no edge is called as a trivial graph.

Regular Graph: A graph G is said to be regular, if all its vertices have the same degree. In a graph, if the degree of each vertex is 'k', then the graph is called a 'k-regular graph'.

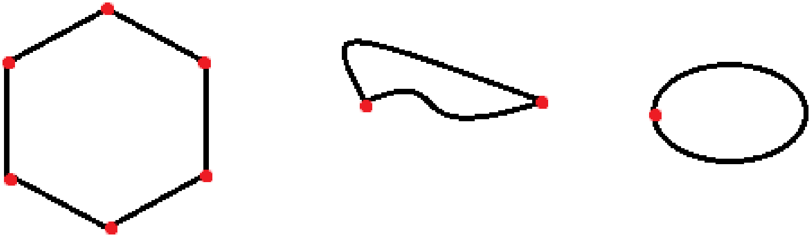
Complete Graph: A simple graph with 'n' mutual vertices is called a complete graph and it is denoted by 'Kn'. In the graph, a vertex should have edges with all other vertices, then it called a complete graph.
In other words, if a vertex is connected to all other vertices in a graph, then it is called a complete graph.

Walk/ Path/ Open Walk/ Closed Walk

- Walk: In graph theory, A walk is defined as a finite length alternating sequence of vertices and edges. The total number of edges covered in a walk is called as Length of the Walk.
- Path: A path is defined as an open walk in which neither vertices (except possibly the starting and ending vertices) are allowed to repeat nor edges are allowed to repeat.
- Open walk: A path is defined as an open walk in which neither vertices (except possibly the starting and ending vertices) are allowed to repeat nor edges are allowed to repeat.
- Closed walk: A walk is called as a Closed walk if length of the walk is greater than zero and the vertices at which the walk starts and ends are same.

Circuit

A closed walk in which no vertex (except the initial and final vertex) appears more than once is called a circuit. That is circuit is a closed non intersecting walk.

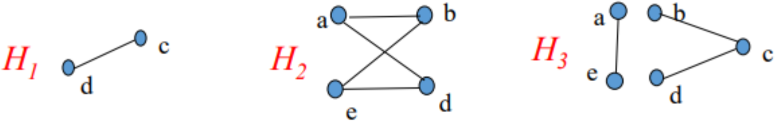


Connected and Disconnected Graph

Connected and Disconnected Graph: In connected graph there exists at least one path between two vertices and disconnected otherwise.

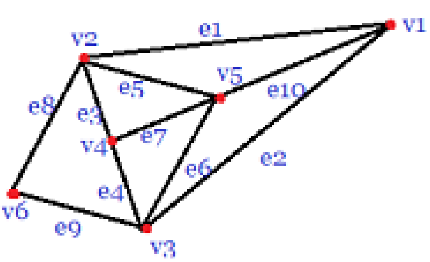
Example:

- H1 and H2 are connected
- H3 is disconnected



Subgraphs

A Graph is said to be a subgraph of a graph G if all the vertices and all the edges of 'g' are in 'G', and each edge of 'G' has the same end vertices in 'g'as in 'G'.
'or'
we can say g is subset of set G.



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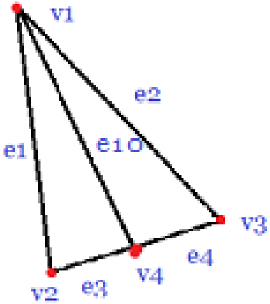

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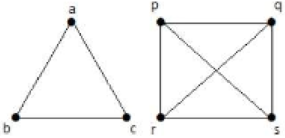
Definition

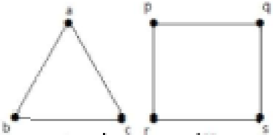
Complete Graph: A simple graph with 'n' mutual vertices is called a complete graph and it is denoted by 'Kn'. In the graph, a vertex should have edges with all other vertices, then it called a complete graph. In other words, if a vertex is connected to all other vertices in a graph, then it is called a complete graph. **Example:** In the following graphs, each vertex in the graph is connected with all the remaining vertices in the graph except by itself.



Complete Graph

Regular Graph: A graph G is said to be regular, if all its vertices have the same degree. In a graph, if the degree of each vertex is 'k', then the graph is called a 'k-regular graph'.

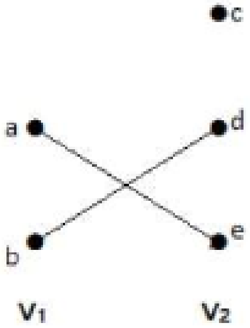
Example: In the following graphs, all the vertices have the same degree. So these graphs are called regular graphs.



Regular Graph

Bipartite Graph

Bipartite Graph: A simple graph G = (V, E) with vertex partition V = V1, V2 is called a bipartite graph if every edge of E joins a vertex in V1 to a vertex in V2.
In general, a Bipartite graph has two sets of vertices, let us say, V1 and V2, and if an edge is drawn, it should connect any vertex in set V1 to any vertex in set V2. **Example:**



Useful Theorem

Prove that in any graph, the number of vertices of odd degree is always even.

refer book

refer book

Theorem: The maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

refer book

A Tree with n vertices has exactly (n-1) edges.

refer book

Prove that a simple graph with 'n' vertices and 'k' components can have at most $\frac{(n-k)(n-k+1)}{2}$ edges

refer book

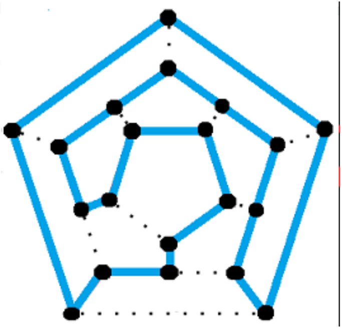
Hamiltonian Graph

Hamiltonian cycle in graph G is a cycle that passes through each vertex exactly ones.

Hamiltonian walk in graph G is a walk that passes through each vertex exactly ones.

If a graph has a Hamiltonian cycle, it is called a Hamiltonian graph.

If a graph has a Hamiltonian walk, it is called a semi Hamiltonian graph.



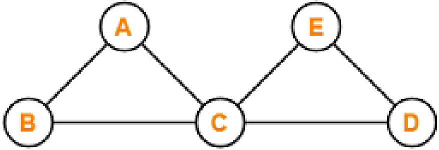
Euler Graph

Any connected graph is called as an Euler Graph if and only if all its vertices are of even degree. 'or'

If some closed walk in a graph contains all the edges of the graph, then the walk is called Euler Line and the graph an Euler Graph

If there exists a walk in the connected graph that starts and ends at the same vertex and visits every edge of the graph exactly once with or without repeating the vertices, then such a walk is called as an Euler circuit.

Here,



Euler Graph

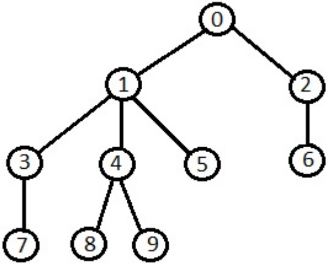
- This graph is a connected graph and all its vertices are of even degree.
- Therefore, it is an Euler graph.

Alternatively, the above graph contains an Euler circuit BACEDCB, so it is an Euler graph.

Trees and its Properties

Trees

A tree is a connected graph without any circuits. example: The geneology of a family is often represented by means of a tree.
Note: A Tree has to be a simple graph, that is having neither a self loop nor a parallel edges.



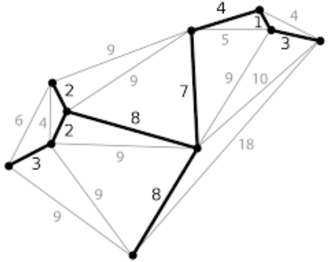
Some Properties of Trees

Properties

- There is only one path between every pair of vertices in a Tree T.
- In a Graph G there is only one path between every pair of vertices in a Tree T.
- A Tree with n vertices has n-1 edges.
- Any connected graph with n vertices and n-1 edges is a tree.
- A Graph is a Tree if and only if it is minimally connected.
- A Graph with n vertices, n-1 edges, and no circuit is connected.

Spanning Tree

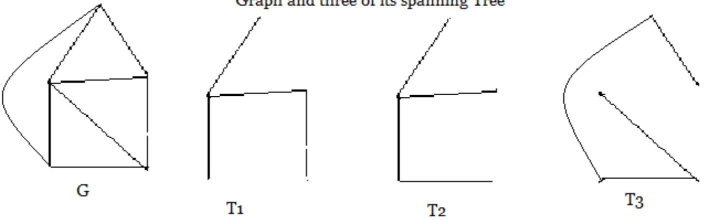
A tree T is said to be a spanning tree of a connected graph G if T is a subgraph of G and T contains all vertices of G. For instance, the subgraph with dark lines is a spanning tree of the graph shown.



- Every connected graph has at least one spanning tree.
- With respect to any of its spanning tree, a connected graph of 'n' vertices and 'e' edges has n-1 tree branches and e-n+1 chords.

Example:

Graph and three of its spanning Tree



Referred Book by Narsingh Deo: Graph Theory with application to Engineering and Computer Sciences.