LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

❖ **Def ^{n} :**

A diff. equation $\dfrac{d^n y}{dx^n} + a_1 \dfrac{d^{n-1} y}{dx^{n-1}} + a_2 \dfrac{d^{n-2} y}{dx^{n-2}} + ... + a_n y = X$

Where $a_1, a_2, ..., a_n$ are constants

While X is a function of x.

is called Linear differential equation of nth order first degree.

Above equation can be written by

$D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + ... + a_n y = X$

Where $D = \dfrac{d}{dx}$ and $a_1, a_2, ..., a_n$ are constants

While X is a function of x.

❖ **General Solution of Differential Equation:**

General solution = Complementary function + Particular integral

G.S. = **C.F.** + **P.I.**

❖ **Complementary function (C.F.)of Differential Equation**

The solution which contains a number of arbitrary constants equal to the order of the differential equation is called the **complementary function (C.F.)** of a Differential equation.

❖ **Auxiliary Equation:**

An equation

$D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + ... + a_n y = 0$

i.e. $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n) y = 0$

is called Auxiliary Equation of differential equation

$D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + ... + a_{n1} y = X$

❖ **Method to find C.F. :**

Solve equation

$D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n = 0$ for values of D.

Say factorization is $(D - m_1)(D - m_2)(D - m_3)............(D - m_n) = 0$.

Then we can classify roots and we can find C.F. of L.D.E. by following way

Sr. No.	Classification of roots	C.F.
1	If roots D = $m_1, m_2, m_3, \ldots\ldots\ldots, m_n$ of Auxiliary equation are real and distinct	$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$
2	Two roots are equal (real roots) i.e. If roots are $m_1 = m_2, m_3 \ldots\ldots, m_n$	$y = (c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$
3	If three roots are equal. (real roots) i.e. If roots are $m_1 = m_2 = m_{3,} m_4, \ldots\ldots, m_n$	$y = (c_1 + c_2 x + c_3 x^2)e^{m_1 x} + \ldots\ldots + c_n e^{m_n x}$
4	If two roots are complex numbers (i.e. complex conjugate) i.e. If roots are $a \pm ib, m_{3,} m_4, \ldots\ldots, m_n$	$y = e^{a_1 x}(A \cos bx + B \sin bx) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$
5	If complex conjugate are equals i.e. If roots are $a \pm ib, a \pm ib, m_{3,} m_4, \ldots\ldots, m_n$	$y = e^{a_1 x}\{(Ax + B)\cos bx + (Cx + D)\sin bx\}$ $+ c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$

❖ **Method to find P.I. :**

Assuming
$$f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_n$$
in $D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_n y = X$
Differential equation will be
$$\textbf{f (D) y} = \textbf{X}$$
$$\text{i.e. } \textbf{y} = \frac{1}{f(D)} X$$

Above value of **y** will be **P.I.** of given **L.D.E.**

❖ **To Evaluate** $\dfrac{1}{f(D)} X$ **Following Results Can be Used :**

➤ $\dfrac{1}{D} X = \int X \, dx$

➤ If $f(D) = (D - m_1)(D - m_2)(D - m_3)\ldots\ldots\ldots(D - m_n)$

  Then $\dfrac{1}{f(D)} X = \dfrac{1}{(D - m_1)(D - m_2)(D - m_3)\ldots\ldots\ldots(D - m_n)} X$

➤ $\dfrac{1}{D - m} X = e^{mx} \int X e^{-mx} \, dx$ ………………………….………(1)

❖ **General method to find P.I.:**

If $f(D) = (D - m_1)(D - m_2)(D - m_3)\ldots\ldots\ldots(D - m_n)$

$$\text{Then } \frac{1}{f(D)}X = \frac{1}{(D-m_1)(D-m_2)(D-m_3)............(D-m_n)}X$$

$$= \left\{ \frac{1}{(D-m_1)(D-m_2)(D-m_3)............} \right\} \frac{1}{D-m_1}X$$

$$= \left\{ \frac{1}{(D-m_1)(D-m_2)(D-m_3)............} \right\} e^{m_1 x} \int X\, e^{-m_1 x}\, dx$$

Continue above process for all factors of $f(D)$. And get P.I. for y.

❖ **Partial fraction method:**

We can use partial fraction method for factors of $f(D)$.

$$\frac{1}{(D-m_1)(D-m_2)(D-m_3)............(D-m_n)}X$$

$$= \left\{ \frac{A_1}{(D-m_1)} + \frac{A_2}{(D-m_2)} + ... + \frac{A_n}{(D-m_n)} \right\}X$$

Use $\dfrac{1}{D-m}X = e^{mx}\displaystyle\int X\, e^{-mx}\, dx$ to get P.I. for $y = \dfrac{1}{f(D)}X$.

❖ **Short cut methods to find P.I.:**

Let given **L.D.E.** is $f(D)y = x$.
We can use following short cut methods to find P.I.

(1) $\dfrac{1}{f(D)}e^{ax} = \dfrac{1}{f(a)}e^{ax} \qquad if\ f(a) \neq 0$.

If $f(a) = 0$ then (D- a) is one factor of $f(D)$

Say $f(D) = (D\text{-}a)^{\,r}\,\phi(D)$

Then $\dfrac{1}{f(D)}e^{ax} = \dfrac{x^r e^{ax}}{r!\,\phi(a)} \qquad$ Where **r** is power of (D-a)

(2) $\dfrac{1}{f(D^2)}\sin ax = \dfrac{1}{f(-a^2)}\sin ax \qquad if\ f(-a^2) \neq 0$

and

$$\frac{1}{f(D^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax \quad if \ f(-a^2) \neq 0$$

If $f(-a^2) = 0$ then $(D^2 + a^2)$ is one factor of $f(D^2)$

Say $f(D^2) = (D^2 + a^2)^r \ \phi(D^2)$

Then

$$\frac{1}{f(D^2)}\sin ax = \frac{(-1)^r \ x^r}{(2a)^r \ r! \ \phi(-a^2)} \ \sin\left(ax + r\frac{\pi}{2}\right)$$

Where **r** is power of $(D^2 + a^2)^r$

And

$$\frac{1}{f(D^2)}\cos ax = \frac{(-1)^r \ x^r}{(2a)^r \ r! \ \phi(-a^2)} \ \cos\left(ax + r\frac{\pi}{2}\right)$$

Where **r** is power of $(D^2 + a^2)^r$

(3) $\dfrac{1}{f(D)}x^m = \dfrac{1}{1 + \phi(D)}x^m$

Use $\dfrac{1}{1+t} = 1 - t + t^2 - t^3 + ...$ Or $\dfrac{1}{1-t} = 1 + t + t^2 + t^3 + ...$

On $\dfrac{1}{1 + \phi(D)}$ and use $\dfrac{1}{D}X = \int X \, dx$ to get P.I.

(4) $\dfrac{1}{f(D)}e^{ax}v = e^{ax}\dfrac{1}{f(D+a)} \ v$ Then use method for $\dfrac{1}{f(D+a)} \ v$

(5) $\dfrac{1}{f(D)}x^m \sin ax$ = imaginary Part of $\dfrac{1}{f(D)}x^m \ e^{iax}$

$\dfrac{1}{f(D)}x^m \cos ax$ = Real Part of $\dfrac{1}{f(D)}x^m \ e^{iax}$

❖ **Method of Variation of Parameters to find P.I.:**

This method is useful to find P.I. for second order linear differential equations.

Consider a L.D.E. of second order $\quad a_0 \dfrac{d^2 y}{dx^2} + a_1 \dfrac{dy}{dx} + a_n y = X$

let $y = c_1 y_1 + c_2 y_2$ be C.F. of above equation then we can find P.I using following method

P.I. $= -y_1 \displaystyle\int \dfrac{y_2 X}{W} dx + -y_2 \displaystyle\int \dfrac{y_1 X}{W} dx$ Where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ called Wronskian

❖ **Method of undetermined multipliers to find P.I.:**

This method is useful to find P.I. for second order linear differential equations.

Consider a L.D.E. of second order $\quad a_0 \dfrac{d^2 y}{dx^2} + a_1 \dfrac{dy}{dx} + a_n y = X$

here to find P.I we will use following table of derivatives family in several cases of function *X*.

Sr. No	X	Family of derivatives of X
1	$X = K$	$\{1\}$
2	$X = e^{ax}$	$\{e^{ax}\}$
3	$X = sinax$	$\{\cos ax, \sin ax\}$
4	$X = x^n$	$\{x^n, x^{n-1}, ...x, 1\}$

In case, if any of two functions above given are multiplied we will use family of new function as multiplied family (each member of first family with every member of second family) of above multiplied functions.

We will use P.I.$= y =$ linear combination of functions which are in family of X
We will put above P.I. in given equation and we will find out constants which are used in linear combination to write P.I.
And finally we will write solution of given equation.

❖ **Linear dependent and Linear independent functions:**

Let $y_1, y_2, ...y_n$ be given functions of *x*.

we can say they are, Linearly independet if wronskian, $W = \begin{vmatrix} y_1 & y_2 .... y_n \\ y_1' & y_2' .... y_n' . \\ . & . \\ . & . \\ y_1^{(n)} & y_2^{(n)} .... y_n^{(n)} \end{vmatrix} \neq 0$

we can say they are, Linearly dependet if wronskian , $W = \begin{vmatrix} y_1 & y_2 \text{..........} y_n \\ y_1' & y_2' \text{.........} y_n' & . \\ . & . \\ . & . \\ y_1^{(n)} & y_2^{(n)} \text{.........} y_n^{(n)} \end{vmatrix} = 0.$

❖ **Method to find second solu and general solution of second order Homogeneous Linear differential equation when one solution is given:**

Let y_1 be one given solution of second order Homogeneous Linear differential equation $y'' + Py' + Qy = 0.$.

To find other solution y_2 which is independet with y_1 we can use following

formula $y_2 = u_1 y_1$ where $u_1 = \int \dfrac{1}{y_1^2} e^{\int -Pdx} dx$.

And general solution can be obatin as $y = c_1 y_1 + c_2 y_2$.