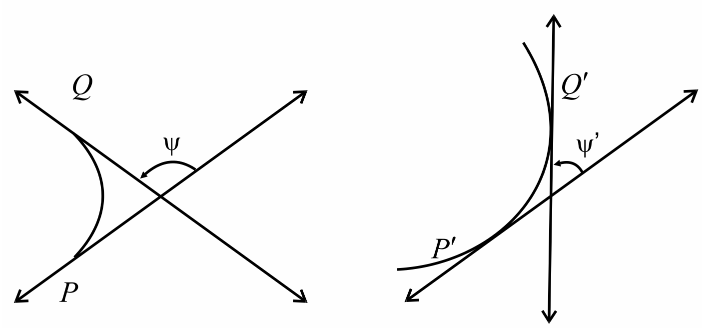
Lecture Notes

CURVATURE

The word curvature is derived from a Greek word "Kappa". Te measure of the rate of change of direction along a curve is called its curvature.

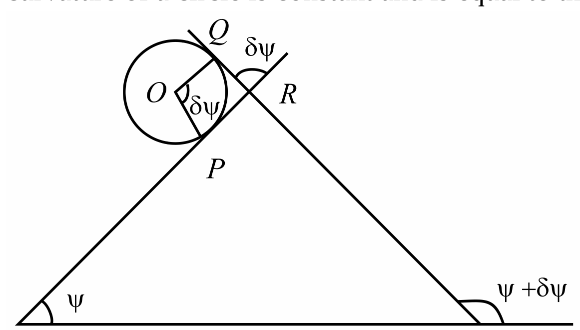


Radius of Curvature: The reciprocal of the curvature of a curve at any point, provided it is not zero is called the **radius of curvature** at the point and is denoted by P and is given by

$$\boxed{P = \frac{ds}{dp}}.$$

The radius of curvature of a circle at any point is equal to the radius of the circle

Curvature of a circle: The curvature of a circle is constant and is equal to the reciprocal of its radius.



Proof. Consider a circle with centre) and radius e let arc $PQ = \delta S$.

Let the two tangents through P and Q mark angles φ and $\delta\varphi$ with the positive direction of x -axis. Also let R the point where the two tangents intersect each other.

From trigonometry

$$\delta\varphi = \frac{\delta S}{r}$$

or $$\frac{\delta\varphi}{\delta s} = \frac{1}{r} ,$$

where $Q \to P$, we have $\frac{d\varphi}{ds} = \frac{1}{r}$. This shows that the curvature of a circle is constant and is equal to the reciprocal of its radius. It implies that curvature of a straight line is zero.

Relation between radius of curvature and curvature

We know that

$$P = \frac{1}{k}.$$

If $k = 0$, then

$$P = \frac{1}{0} = \infty.$$

Therefoe the radius of curvature is infinity.

Cartesian form

When the equation of the curve is in the form of $y = f(x)$

$$P = \frac{(1+y_1^2)^{3/2}}{y_2}.$$

Hence at the point of inflexion the curvature is zero and changes sign the radius of curvature is infinite.

Intrinsic form

When the equation of the curvature is in the form of $\delta = f(\varphi)$, therefore we have

$$\boxed{P = \frac{d\delta}{d\varphi}}.$$

Parametric form

When the equation of the curve is in the form of $x = \phi(t), y = \varphi(t)$, therefore we have

$$\boxed{P = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}}.$$

Pedal form 'or' pedal equation

When the equation of the curve is in the form of $r = f(p)$, therefore we have

$$\boxed{P = \frac{rdr}{dp}}.$$

Polar trangental form

When the equation of the curve is in the form of $P = f(\varphi)$, therefore we have

$$\boxed{P = P + \frac{d^2P}{d\psi^2}}.$$

Thus when the equation of the curve is in the form of $u = f(\theta)$, therefore we have

$$\boxed{P = \frac{(u^2 + u_1^2)^{3/2}}{u^3(u + u_2)}}.$$

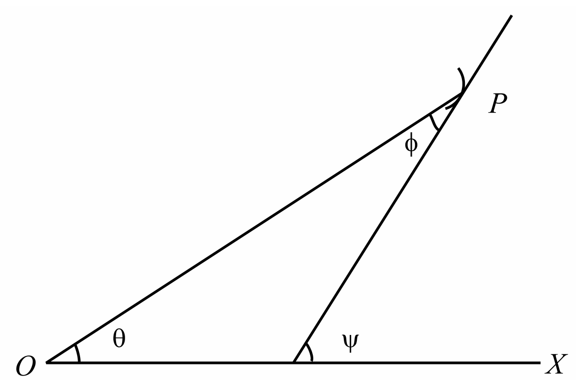
At the point of inflexion curvature vanishes and changes sign. Hence $u + u_2$ *i.e.,* $u + \frac{d^2u}{d\theta^2}$ vanishes and changes sig at a point on inflexion. Thus the curvature of the ellipse is maximum at the ends of the major axis and minimum at the ends of the minor axis.

Polar form

When the equation of the curvature is in the form of $r = f(\theta)$, then

$$\boxed{P = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}}.$$

Proof. Let ψ be the angle which the tangent at any point $P(r,\theta)$ makes with the x-axis. Then from figure we have



$$\psi = \theta + \phi$$

$\therefore$ $$\frac{1}{P} = \frac{d\psi}{ds}\frac{d}{ds}(\theta + \phi)\ 9$$

$$= \frac{d\theta}{ds} + \frac{d\phi}{ds}$$

$$= \frac{d\theta}{ds} + \frac{d\phi}{d\theta} \times \frac{d\phi}{ds}$$

$\Rightarrow$ $$\frac{1}{P} = \left(1 + \frac{d\phi}{d\theta}\right)\frac{d\theta}{ds}$$

$\Rightarrow$ $$\tan\phi = \frac{rd\theta}{dr}$$

$\Rightarrow$ $$\tan\phi = \frac{r/dr}{d\theta}$$

$\Rightarrow$ $$\tan\phi = \frac{r}{r_1} \qquad \left(\because r_1 = \frac{dr}{d\theta}\right)$$

$\Rightarrow$ $$r_1 = \frac{dr}{d\theta},\quad r_2 = \frac{d^2r}{d\theta^2}$$

$\Rightarrow$ $$\phi = \tan^{-1}\left(r/r_1\right)$$

$\Rightarrow$ $$\frac{d\phi}{d\theta} = \frac{1}{1 + \left(r/r_1\right)^2} \cdot \frac{r_1.r_1 - rr_2}{r_1^2}$$

$$= \frac{\cancel{r_1^2}}{r_1^2 + r^2} \cdot \frac{r_1^2 - rr_2}{\cancel{r_1^2}} = \frac{r_1^2 - rr_2}{r_1^2 + r^2}$$

Also $$\frac{ds}{d\theta} = \sqrt{(r^2 + r_1^2)}$$

$\Rightarrow$ $$\frac{d\theta}{ds} = \frac{1}{\sqrt{r^2 + r_1^2}}$$

Hence, $$\frac{1}{P} = (1 + \frac{d\phi}{d\theta})\frac{d\theta}{ds}$$

$$= \left(1 + \frac{r_1^2 - rr_2}{r_1^2 + r^2}\right)\frac{1}{\sqrt{r^2 + r_1^2}}$$

$$=\frac{r_1^2+r^2+r_1^2-rr_2}{r_1^2+r^2}\cdot\frac{1}{\sqrt{r^2+r_1^2}}$$

$$\Rightarrow \qquad \frac{1}{P}=\frac{2r_1^2+r^2-rr_2}{(r_1^2+r^2)^{3/2}}$$

$$\Rightarrow \qquad \boxed{P=\frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}}.$$

Find the radius of curvature at any point of the following

Q. No. 1. The catenary $\delta = c\tan\psi$

Sol. We have $\delta = c\tan\psi$ **.** Therefore the radius of curvature $P=\frac{d\delta}{d\psi}=c\sec^2\psi$.

Q. No. 2. The cycloid $\delta = 4a\sin\psi$

Sol. $P=\frac{d\delta}{d\psi}=4a\cos\psi$

Q. No. 3. The tractrix $\delta = c\log\sec\psi$.

Sol. $P=\frac{d\delta}{d\psi}=c\frac{1}{\sec\psi}\sec\psi\tan\psi = c\tan\psi$

Q. No. 4. The parabola $\delta = a\log(\tan\psi+\sec\psi)+a\tan\psi\sec\psi$

Sol. We have $\delta = a\log(\tan\psi+\sec\psi)+a\tan\psi\sec\psi$.

Differentiate w.r.t. ψ , we get

$$\frac{d\delta}{d\psi}=\frac{a.1}{\tan\psi+\sec\psi}(\sec^2\psi+\sec\psi\tan\psi)+a\sec^3\psi+a\tan^2\psi\sec\psi$$

$$\Rightarrow \qquad P=\frac{a\sec\psi(\sec\psi+\tan\psi)}{\tan\psi+\sec\psi}+a\sec\psi(\sec^2\psi+\tan^2\psi)$$

$$\Rightarrow \qquad P=a\sec\psi+a\sec\psi(\sec^2\psi+\sec^2\psi-1) \qquad \left(\because \tan^2\psi=\sec^2\psi-1\right)$$

$$=a\sec\psi+a\sec\psi(2\sec^2\psi-1)$$

$$=a\sec\psi(\cancel{1}+\sec^2\psi-\cancel{1})$$

$$=a\sec\psi(2\sec^2\psi)$$

$$=2a\sec^3\psi$$

Q. No. 5. $y=e\log\sec(x/c)$

Sol. We have $y=c\log\sec(x/c)$.

Differentiate w.r.t. x we get

$$\frac{dy}{dx}=c\frac{1}{\sec(x/e)}\sec(x/e)\tan(x/e).\frac{1}{e}=\tan(x/e).$$

Again Differentiate w.r.t. x , we get

$$\frac{d^2y}{dx^2} = \sec^2 x/e . \frac{1}{e} = \frac{1}{c}\sec^2 x/c$$

$$\Rightarrow \qquad P = \frac{\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left(1+\tan^2 x/e\right)^{3/2}}{\frac{1}{e}\sec^2 x/e} = \frac{e\sec^2 x/e}{\sec^2 x/e} = e\sec xe/$$

Q. No. 6. $xy = c^2$

Sol. We have the equation of the curve $y = e^2/x$

$$\therefore \qquad y_1 = \frac{-e^2}{x^2}$$

$$\Rightarrow \qquad y_2 = \frac{2e^2}{x^3}.$$

Therefore the radius of curvature at (x, y) is

$$P = \frac{\left(1+y_1^2\right)^{3/2}}{y_2}$$

$$\Rightarrow \qquad P = \frac{\left(1+e^4/x^4\right)^{3/2}}{2e^2/x^3} = \frac{\left(\frac{x^4+e^4}{x^4}\right)^{3/2}}{2e^2x^3}.$$

Q. No. 7. $x = a\cos t, \quad y = b\sin t$

Sol. $x = a\cos t, \qquad y = b\sin t$

$x' = -a\sin t \qquad y' = b\cos t$

$x'' = -a\cos t \qquad y'' = -b\sin t$

Therefore $P = \dfrac{\left(x'^2 + y'^2\right)^{3/2}}{x'y'' - x''y'}$

$$\Rightarrow \qquad P = \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab\sin^2 t + ab\cos^2 t}$$

$$= \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab(\sin^2 t + \cos^2 t)}$$

$$= \frac{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}}{ab}. \qquad [\because \sin^2 x + \cos^2 x]$$

Q. No. 8. The catenary $x = e \log\{\delta + \sqrt{(e^2 + \delta^2)}\}$ and $y = \sqrt{(e^2 + \delta^2)}$.

Sol. We have $\frac{dx}{ds} = e.\frac{1}{\delta + \sqrt{(e^2+\delta^2)}}\left(1 + \cancel{1/2}\frac{\cancel{2}\delta}{\sqrt{(e^2+\delta^2)}}\right)$

$$= \frac{e}{\cancel{\delta + \sqrt{e^2+\delta^2}}}\frac{\left(\cancel{\sqrt{e^2+\delta^2}+\delta}\right)}{\sqrt{e^2+\delta^2}} = \frac{e}{\sqrt{e^2+\delta^2}}$$

and $\frac{dy}{ds} = \frac{1}{2}\frac{2\delta}{\sqrt{e^2+\delta^2}} = \frac{\delta}{\sqrt{e^2+\delta^2}}$

$\Rightarrow$ $\frac{dy}{ds} = \frac{dy}{d\delta} \times \frac{d\delta}{dx} = \frac{\delta}{\sqrt{e^2+\delta^2}} \times \frac{\sqrt{e^2+\delta^2}}{e}\frac{\delta}{e}$

or $\tan\psi = \delta/e$ t, $\qquad [\because \frac{dy}{dx} = \tan\psi]$

that is $\delta = e\tan\psi$

$\Rightarrow$ $P = \frac{d\delta}{d\psi} = e\sec^2\psi$

$$= e(1 + \tan^2\psi) = e(1 + \delta^2/e^2) = e\left(\frac{e^2+\delta^2}{e^2}\right)$$

$\Rightarrow$ $P = \frac{e^2+\delta^2}{e}$.

Find the radius of curvature of the following curves

Q. No. 9. $x^3 + y^3 = 2$ at $(1,1)$.

Sol. Differentiate w.r.t. $'x'$, we get

$3x^2 + 3y^2y^1 = 0$

$\Rightarrow$ $3\{x^2 + y^2y_1\} = 0$

$\Rightarrow$ $x^2 + y^2y_1 = 0$

$\Rightarrow$ $y_1 = -x^2/y^2$.

Now at (1,1)

$y_1 = -1$

$\Rightarrow$ $y'' = \frac{(-2x)(y^2) + x^2 2yy_1}{y^4}$

$\Rightarrow$ $y'' = \frac{-2\times1\times1 + 1^2\times2\times1\times-1}{1}$

$\Rightarrow \qquad y'' = -2 - 2 = -4$

$$\Rightarrow \qquad P = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+1)^{3/2}}{-4} = -\frac{2^{3/2}}{4}$$

$$= -\frac{2^{3/2}}{2^2} = -2^{-1/2} = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \qquad P = \frac{1}{\sqrt{2}}.$$

Q. No. 10. $(x^2+y^2)^2 = a^2(y^2-x^2)$ at (0,9).

Sol. Please try yourself.

Q. No. 11. Find the curvature at any point of the catenary $y = e\cos h(x/e)$ ans show that it varies inversely as the square of the ordinate.

Sol. We have $y = e\cos h(x/e)$.

$$\Rightarrow \qquad y_1 = e\sin h(x/e).\frac{1}{e} = \sin h\, x/e$$

$$\Rightarrow \qquad y_2 = \frac{1}{e}\cos h\, x/e$$

$$\Rightarrow \qquad P = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+\sin^2 h\, x/e)^{3/2}}{1/e \cos h\, x/e}$$

$$= \frac{e(\cos h^2\, x/e)^{3/2}}{\cos h\, x/e} = c\cos h^2\, x/e \qquad [\because \cos^2 h\, x/e - \sin^2 h^2\, x/e = 1]$$

$$\Rightarrow \qquad K = \frac{1}{P}$$

$$\Rightarrow \qquad K = \frac{1}{e^2 \cos h^2\, x/e}$$

$$\Rightarrow \qquad K = \frac{e}{e^2 \cos h^2\, x/e}$$

$$\Rightarrow \qquad K = \frac{e}{y^2}$$

$$\Rightarrow \qquad K \propto \frac{1}{y^2}.$$

Hence K is inversely proportional as the square of the ordinate.

Q. No. 12. In the cycloid $x = a(\theta + \sin\theta), y = a(1-\cos\theta)$. Prove that the radius of curvature is $ya\cos \frac{1}{2}\theta$.

Sol. $\qquad x = a(\theta+\sin\theta) \qquad y = a(1+\cos\theta)$

$$x' = a(1+\cos\theta) \qquad y' = a\sin\theta$$

$$x'' = -a\sin\theta \qquad y''a\cos\theta$$

Therefore $P = \dfrac{(x'^2 + y'^2)^{3/2}}{x'y'' - x''y'}$

$$= \frac{(a^2(1+\cos\theta)^2 + a^2\sin^2\theta)^{3/2}}{(a+a\cos\theta)a\cos\theta + a^2\sin^2\theta}$$

$$= \frac{(a^2(1+\cos\theta)^2 + a^2\sin^2\theta)^{3/2}}{(a+a\cos\theta)a\cos\theta + a^2\sin^2\theta)}$$

$$= \frac{a^3(1+1+2\cos\theta)^{3/2}}{a^2(\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \frac{a(2+2\cos\theta)^{3/2}}{(1+\cos\theta)} = \frac{2^{3/2}a(1+\cos\theta)^{3/2}}{(1+\cos\theta)}$$

$$= 2^{3/2}a(1+\cos\theta)^{1/2}$$

$$= 2^{3/2}a\left(2\cos^2\frac{\theta}{2}\right)^{1/2} \qquad [\because 1+\cos\theta = 2\cos^2\theta/2]$$

$$= 2^{3/2}.2^{1/2}a\cos\frac{\theta}{2} = 2^{4/2}a\cos\theta/2$$

$$= 4a\cos\frac{1}{2}\theta.$$

Q. No. 13. Prove that for the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $P = \dfrac{a^2b^2}{P^3}, P$ being the perpendicular from the centre upon tangent at (x, y).

Sol. Any point (x, y) on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $(a\cos\theta, b\sin\theta)$ or simply $'\theta'$. Now parametric equations of the ellipse are

$$x = a\cos\theta \qquad y = b\sin\theta$$

$$x' - a\sin\theta \qquad y' = b\cos\theta$$

$$x'' = -a\cos\theta \qquad y'' = -b\sin\theta.$$

Therefore P at $'\theta' = \dfrac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$

$$= \frac{(a^2\sin\theta + b^2\cos^2\theta)^{3/2}}{ab\sin^2\theta + ab\cos^2\theta}$$

$$= \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab}.$$

$\Rightarrow \qquad P_{ab} = (a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}$.

Taking cube root on bothsides we get

$$(P_{ab})^{1/3} = (a^2\sin^2\theta + b^2\cos^2\theta)^{1/2}. \qquad (1)$$

Now equation of tangent is given by

$$\frac{xx_1}{a^2} + \frac{yy_1}{b_2} = 1$$

$$\Rightarrow \qquad \frac{xa\cos\theta}{a^2} + \frac{yb\sin\theta}{b^2} = 1$$

$$\Rightarrow \qquad \frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1.$$

Now perpendicular distance from centre (0,0) on tangent is

$$P = \frac{1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}} = \frac{ab}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}}$$

$$\Rightarrow \qquad (a^2\sin^2\theta + b^2\cos^2\theta)^{1/2} = \frac{ab}{P}. \qquad (2)$$

From(1) and (2), we get

$$(P_{ab})^{1/3} = \frac{ab}{P}.$$

Cubing on bothsides we get

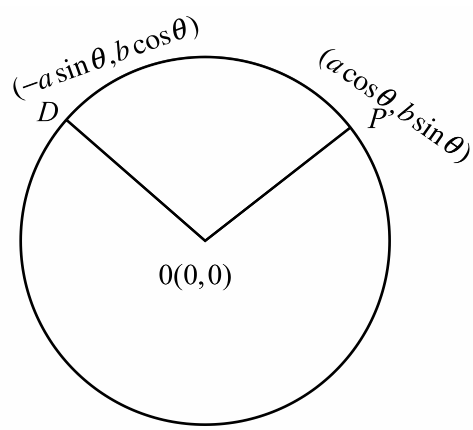
$$P_{ab} = \frac{a^3b^3}{P^3}$$

$$\Rightarrow \qquad P = \frac{a^2b^2}{P^3}.$$

Q. No. 14.

(i) If P_1, P_2 be the radii of curvature at the extremities of two conjugate diameters of an ellipse, prove that

$$\left(P_1^{2/3} + P_2^{2/3}\right)(ab)^{2/3} = a^2 + b^2$$



Sol. Theequation of ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Put $\qquad x = a\cos\theta \qquad$ and $\qquad y = b\sin\theta$

$\Rightarrow \qquad x' = -a\sin\theta \qquad\qquad y' = b\cos\theta$

$\qquad\qquad x'' = -a\cos\theta \qquad\qquad y'' = -b\sin\theta$

Let P_1 be radii of curvature on semi-diameter CP, then

$$P_1 = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$$

$$\Rightarrow \quad P_1 = \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab} \qquad (1)$$

Let P_2 be radii of curvature on semi-diameter CD, then

$$P_2 = \frac{(a^2\cos^2\theta + b^2\sin^2\theta)^{3/2}}{ab} \qquad (2)$$

$$\Rightarrow \quad P_1 ab = (a^2\sin^2\theta + b^2\cos^2\theta)^{3/2} \text{ a}$$

$$\Rightarrow \quad P_1 ab = (a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}$$

$$\Rightarrow \quad P_1 ab = (a^2\sin^2\theta + b^2\cos^2\theta)^3$$

$$\Rightarrow \quad (P_1 ab)^{2/3} = a^2\sin^2\theta + b^2\cos^2\theta \qquad (3)$$

Similarly we have

$$(P_2 ab)^{2/3} = a^2\cos^2\theta + b^2\sin^2\theta\,. \qquad (4)$$

Adding(3)and (4), we get

$$(P_1 ab)^{2/3} + (P_2 ab)^{2/3} = a^2 + b^2$$

$$\Rightarrow \quad \left(P_1^{2/3} + P_2^{2/3}\right)(ab)^{2/3} = a^2 + b^2\,.$$

(ii) If CP, CD be a pair of conjugate semi-diameters of an ellipse with semi-axes of lengths a and b, prove that the radius of curvature at $P = CD^3/ab$.

Sol. By distance formula we have

$$CD = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(-a\sin\theta - 0)^2 + (b\cos\theta - 0)^2}$$

$$= \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$\Rightarrow \quad CD = (a^2\sin^2\theta + b^2\cos^2\theta)^{1/2}\,.$$

Cubing on bothsides we get

$$CD^3 = (a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}\,.$$

From (3), we have

$$(\mathrm{a}^2\sin^2\theta + b^2\cos^2\theta)^{3/2} = Pab$$

$$\Rightarrow \quad CD^3 = Pab$$

$$\Rightarrow \quad P = CD^3/ab\,.$$

Q. No. 15. Prove that at a point of inflexion, the circle of curvature degenerates into a straight line.

Sol. Since at the point of inflexion $y_2 = 0$.

But $\quad P = \dfrac{(1+y_1^2)^{3/2}}{y_2}$

$\Rightarrow \quad K = \dfrac{1}{P} = \dfrac{y_2}{(1+y_1^2)^{3/2}}$

$\Rightarrow \quad K = 0$. $\qquad [\because y_2 = 0]$

$K = 0$ means that it is a straight line.

Q. No. 16. Show that the radius of curvature at the point $(a\cos^3\theta, a\sin^3\theta)$ on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is $3a\sin\theta\cos\theta$?

Sol. Here $x = a\cos^3\theta \qquad y = a\sin^3\theta$

$\Rightarrow \quad \dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \qquad \dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

$\Rightarrow \quad \dfrac{dy}{dx} = \dfrac{dy}{d\theta}\times\dfrac{d\theta}{dx} = -\dfrac{3a\sin^2\theta\cos\theta}{3a\cos^2\theta\sin\theta} = -\tan\theta$

$\Rightarrow \quad \dfrac{d^2y}{dx^2} = -\sec^2\theta\dfrac{d\theta}{dx} = -\sec^2\theta\times -\dfrac{1}{3a\cos^2\theta\sin\theta}$

$= \dfrac{1}{3a\cos^4\theta\sin\theta}$.

Radius of curvature

$$P = \frac{(1+\tan^2\theta)^{3/2}}{3a\cos^4\theta\sin\theta}$$

$= (\sec^2\theta)^{3/2}.3a\cos^4\theta\sin\theta \qquad [\because 1+\tan^2\theta = \sec^2\theta]$

$= 3a\sec^3\theta\cos^4\theta\sin\theta$

$= 3a\dfrac{1}{\cos^3\theta}\cos^4\theta\sin\theta$

$= 3a\cos\theta\sin\theta$.

Find the radius of curvature at any point of the following curves.

Q. No. 1. $r = a(1-\cos\theta)$.

Sol. We have $r = a(1-\cos\theta)$

$\Rightarrow \quad r' - a\sin\theta,\ r'' = a\cos\theta$

Therefore $P = \dfrac{(a^2(1-\cos\theta)^2 + a^2\sin^2\theta)^{3/2}}{a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a(\cos\theta - \cos^2\theta)}$

$= \dfrac{a^3(1+\cos^2\theta - 2\cos\theta + \sin^2\theta)^{3/2}}{a^2(1+\cos^2\theta - 2\cos\theta + 2\sin^2\theta - \cos\theta + \cos^2\theta}$

$= \dfrac{a(1+1-2\cos\theta)^{3/2}}{(1+2\cos^2\theta + 2\sin^2\theta - 3\cos\theta)}$

$$=\frac{a(2-2\cos\theta)^{3/2}}{1+2-3\cos\theta}=\frac{2^{3/2}a(1-\cos\theta)^{3/2}}{3-3\cos\theta}$$

$$=\frac{2^{3/2}a(1-\cos\theta)^{3/2}}{3(1-\cos\theta)}=\frac{2^{3/2}a(1-\cos\theta)^{3/2}}{3-3\cos\theta}$$

$$=\frac{2^{3/2}a(1-\cos\theta)^{3/2}}{3(1-\cos\theta)}=\frac{2^{3/2}a)1-\cos\theta)^{1/2}}{3}$$

$$=\frac{2^{3/2}a(2\sin^2\theta/2)^{1/2}}{3}=\frac{2^{3/2}.2^{1/2}a\sin\theta/2}{3}$$

$$=\frac{4a}{3}\sin\theta/2\,.$$

Q. No. 2. $r=a\cos m\theta$.

Sol. Here $r=a\cos m\theta$.

$\Rightarrow$ $r'=-ma\sin m\theta,\ r'=-m^2a\cos m\theta$

$$\Rightarrow \quad P=\frac{(a^2\cos^2 m\theta+m^2a^2\sin^2 m\theta)^{3/2}}{a^2\cos^2 m\theta+2a^2m^2\sin^2 m\theta+a^2m^2\cos^2 m\theta}$$

$$=\frac{(a^2\cos^2 m\theta+a^2m^2(1-\cos^2 m\theta))^{3/2}}{a^2\cos^2 m\theta+2a^2m^2(1-\cos^2 m\theta)+a^2m^2\cos^2 m\theta}$$

$$=\frac{(a^2\cos^2 m\theta+a^2m^2-a^2m^2\cos^2 m\theta)^{3/2}}{a^2\cos^2 m\theta+2a^2m^2-2a^2m^2\cos^2 m\theta+a^2m^2\cos^2 m\theta}$$

$$=\frac{(a^2\cos^2 m\theta+a^2m^2-a^2m^2\cos^2 m\theta)^{3/2}}{a^2\cos^2 m\theta-a^2m^2\cos^2 m\theta+2a^2m^2}$$

$$=\frac{(r^2+a^2m^2-r^2m^2)^{3/2}}{r^2-r^2m^2+2a^2m^2}\,.$$

Q. No. 3.

(i) The cardioids $2ap^2=r^3$.

Sol. We have $2ap^2=r^3$.

$$\Rightarrow \quad P=r\frac{dr}{dp}$$

$$\Rightarrow \quad 4ap=3r^2\frac{dr}{dp}$$

$$\text{or} \quad 3r^2\frac{dr}{dp}=4ap$$

$$\Rightarrow \quad r\frac{dr}{dp}=\frac{4ap}{3r}\,.$$

(ii) The lemniscates $Pa^2 = r^3$.

Sol. Here $Pa^2 = r^3$.

Differentiate w. r. t. P we have

$$a^2 = 3r^2 \frac{dr}{dp}$$

$$\Rightarrow \qquad 3r^2 \frac{dr}{dp} = a^2$$

$$\Rightarrow \qquad r\frac{dr}{dp} = \frac{a^2}{3r}.$$

Q. No. 4.

(i) The ellipse $r^2 = a^2 + b^2 - \left(\frac{a^2b^2}{P^2}\right)$.

Sol. Here $r^2 = a^2 + b^2 - \left(\frac{a^2b^2}{P^2}\right)$.

Differentiate w.r.t. P we have

$$2r\frac{dr}{dp} = 0 + 2\frac{(a^2b^2)}{P^3}$$

$$\Rightarrow \qquad r\frac{dr}{dp} = \cancel{2}\frac{(a^2b^2)}{\cancel{2}P^3} = \frac{(a^2b^2)}{P^3}.$$

(ii) The hyperbola $r^2 = a^2 - b^2 + \left(\frac{a^2b^2}{P^2}\right)$

Sol. Here $r^2 = a^2 - b^2 + \left(\frac{a^2b^2}{P^2}\right)$.

Differentiate w.r.t, P we have

$$2r\frac{dr}{dp} = 0 - 2\frac{(a^2b^2)}{p^3}$$

$$\Rightarrow \qquad r\frac{dr}{dp} = -\cancel{2}\frac{(a^2b^2)}{\cancel{2}P^3} = -\left(\frac{a^2b^2}{P^3}\right).$$

Q. No. 5.

(i) The parabola $P\sin\psi = a$.

Sol. Here $P\sin\psi = a$.

$$\Rightarrow \qquad P = {}^{a}\!\!/_{\sin\psi} = a\cos\mu\psi$$

$$\Rightarrow \qquad \frac{dp}{d\psi}\sin\psi + P\cos\psi = 0$$

$$\Rightarrow \qquad \frac{dp}{d\psi}\sin\psi = -P\cos\psi$$

$\Rightarrow \quad \frac{dp}{d\psi} = -P\cot\psi$

$\Rightarrow \quad \frac{d^2P}{d\psi^2} = P\operatorname{cosec}^2\psi - \frac{dp}{d\psi}\cot\psi$

$\Rightarrow \quad \frac{d^2P}{d\psi^2} = a\operatorname{cosec}\psi\operatorname{cosec}^2\psi + P\cot\psi.\cot\psi$

$= a\operatorname{cosec}^3\psi + P\cot^2\psi$

$= a\operatorname{cosec}^3\psi + a\operatorname{cosec}\psi(\operatorname{cosec}^2\psi - 1)$

$= a\operatorname{cosec}^3\psi + a\operatorname{cosec}^3\psi - a\operatorname{cosec}\psi$

$= 2a\operatorname{cosec}^3\psi - a\operatorname{cosec}\psi$

$\Rightarrow \quad P = P + \frac{d^2p}{d\psi^2}$

$= a\cancel{\operatorname{cosec}\psi} + 2a\operatorname{cosec}^3\psi - a\cancel{\operatorname{cosec}\psi}$

$= 2a\operatorname{cosec}^3\psi$.

(ii) The circle $P = a(1+\sin\psi)$.

Sol.Here $P = a(1+\sin\psi)$.

Differentiate w.r.t, P we have

$\frac{dp}{d\psi} = a\cos\psi$

$\Rightarrow \quad \frac{d^2P}{d\psi^2} = -a\sin\psi$

$\Rightarrow \quad P = P + \frac{d^2P}{d\psi^2} = a(1+\sin\psi) - a\sin\psi$

$= a + a\sin\psi - a\sin\psi = a$.

(iii) The hyperbola $P^2 = a^2\cos^2\psi - b^2\sin^2\psi$

Sol. Try yourself.

Q. No. 6. Show that for the cardioids $r = a(1+\cos\theta), P = \frac{4}{3}a\cos\frac{1}{2}\theta$.

Sol. We have

$r = a(1+\cos\theta)$

$\Rightarrow \quad r' = -a\sin\theta, \quad r'' = -a\cos\theta$

$\Rightarrow \quad P = \frac{(a^2(1+\cos\theta)^2 + a^2\sin^2\theta)^{3/2}}{a^2(1+\cos^2\theta) + 2a^2\sin^2\theta + a^2(\cos\theta + \cos^2\theta)}$

$= \frac{a^3(1^2 + \cos^2\theta + 2\cos\theta + \sin^2\theta)^{3/2}}{a^2(1+\cos^2\theta + 2\cos\theta) + 2\sin^2\theta + \cos\theta + \cos^2\theta}$

$$= \frac{a(1+1+2\cos\theta)^{3/2}}{(1+2\cos^2\theta+2\sin^2\theta+3\cos\theta)}$$

$$= \frac{a(2+2\cos\theta)^{3/2}}{(1+2+3\cos\theta)}$$

$$= \frac{2^{3/2}a(1+\cos\theta)^{3/2}}{3+3\cos\theta} = \frac{2^{3/2}a(1+\cos\theta)^{3/2}}{3(1+\cos\theta)}$$

$$= \frac{2^{3/2}a(1+\cos\theta)^{1/2}}{3}$$

$$= \frac{2^{3/2}a(2\cos^2\theta/2)^{1/2}}{3} \qquad [\because 1+\cos\theta = 2\cos^2\theta/2]$$

$$= \frac{2^{3/2}a2^{1/2}\cos\theta/2}{3} = \frac{2^2 0\cos\theta/2}{3}$$

$$= \frac{4}{3}a\cos 1/2\,\theta.$$

Q. No. 7. Show that in the rectangular hyperbola $r^2\cos 2\theta = a^2, P = r^3/a^2$.

Sol. We have $r^2\cos 2\theta = a^2$.

Taking log on both sides we get

$$\log r^2\cos 2\theta = \log a^2$$

$$\Rightarrow \quad \log r^2 + \log\cos\theta = \log a^2$$

$$\Rightarrow \quad 2\log r + \log\cos 2\theta = 2\log a$$

Differentiate w.r.t. θ, we get

$$\frac{2}{r}\frac{dr}{d\theta} + \frac{1}{\cos 2\theta} - 2\sin 2\theta = 0$$

$$\Rightarrow \quad \frac{2}{r}\frac{dr}{d\theta} - 2\tan 2\theta = 0$$

$$\Rightarrow \quad \frac{\cancel{2}}{r}\frac{dr}{d\theta} = \cancel{2}\tan 2\theta$$

$$\Rightarrow \quad \frac{1}{r}\frac{dr}{d\theta} = \tan 2\theta$$

$$\Rightarrow \quad r\frac{dr}{d\theta} = \frac{1}{\tan 2\theta}$$

$$\Rightarrow \quad \tan\phi = \cot 2\theta$$

$$\Rightarrow \quad \cancel{\tan}\,\phi = \cancel{\tan}\left(\pi/2 - \theta\right)$$

$$\Rightarrow \quad \phi = \pi/2 - 2\theta$$

$\Rightarrow \qquad P = r\sin\phi$

$\qquad = r\sin(\pi/2 - 2\theta) \ = r\cos 2\theta$

$\Rightarrow \qquad P = r\dfrac{a^2}{r^2} = \dfrac{a^2}{r}.$

Differentiate w.r.t. P, we get

$$1 = \frac{-a^2}{r^2}\frac{dr}{dp} \qquad \Rightarrow \frac{dr}{dp} = \frac{-r^2}{a^2}.$$

We know that $P = r\dfrac{dr}{dp}$

$\Rightarrow \qquad P = r.\left(-r^2/a^2\right) = -r^3/a^2$

$\Rightarrow \qquad P = \dfrac{r^3}{a^2}$ in magnitude.

Q. No. 8. Show that for the curve $r^n = a^n\cos n\theta$ the radius of curvature is $\dfrac{a^n r^{1-n}}{(n+1)}$.

Sol. We have $r^n = a^n\cos n\theta$.

$\Rightarrow \qquad n\log r = \log a^n + \log\cos n\theta$

$\Rightarrow \qquad \dfrac{n}{r}\dfrac{dr}{d\theta} = 0 + \dfrac{1}{\cos n\theta}\times -n\sin n\theta$

$\Rightarrow \qquad \dfrac{n}{r}\dfrac{dr}{d\theta} = -n\tan n\theta$

$\Rightarrow \qquad \dfrac{1}{r}\dfrac{dr}{d\theta} = -\tan n\theta \ \Rightarrow \dfrac{dr}{d\theta} = -r\tan n\theta$

or $\qquad r' = -r\tan n\theta$

$\Rightarrow \qquad r'' = -(r\tan n\theta + e\sec^2 n\theta.n)$

or $\qquad r'' = r\tan^2 n\theta - r\sec^2 n\theta.n$

$\qquad = r(\tan^2 n\theta - n\sec^2 n\theta)$

$\Rightarrow \qquad P = \dfrac{(r^2 + r^2\tan^2 n\theta)^{3/2}}{r^2 + 2r^2\tan^2 n\theta - r^2\tan^2 n\theta + nr^2\sec^2 n\theta}$

$\qquad = \dfrac{\cancel{r}^3(1+\tan^2 n\theta)^{3/2}}{\cancel{r}^2(1+2\tan^2 n\theta - \tan^2 n\theta + n\sec^2 n\theta)}$

$\qquad = \dfrac{r\sec^3 n\theta}{1+\tan^2 n\theta + n\sec^2\theta} = \dfrac{r\sec^3 n\theta}{\sec^2 n\theta + n\sec^2 n\theta}$

$\Rightarrow \qquad P = \dfrac{r\sec^3 n\theta}{\sec^2 b\theta(1+n)} = \dfrac{r\sec n\theta}{1+n}$

$\Rightarrow \qquad P = \dfrac{r}{(1+n)\cos n\theta} = \dfrac{ra^n r^{-n}}{(1+n)} \qquad \left(\because \cos n\theta = \dfrac{r^n}{a^n}\right)$

$\Rightarrow \qquad P = \dfrac{a^n r^{1-n}}{(1+n)}$.

Q. No. 9. Show that for the curve $a^n P = r^{n+1}, P$ varies inversely as the $(n-1)$ the power of the radius vector.

Sol. We have $a^n P = r^{n+1}$.

$\Rightarrow \qquad a^n = n+1 r^n \dfrac{dr}{dP}$

$\Rightarrow \qquad \dfrac{a^n}{(n+1)} = r^{n-1} r \dfrac{dr}{dP}$

$\Rightarrow \qquad \dfrac{a^n}{(n+1)r^{n-1}} = r\dfrac{dr}{dp}$

$\Rightarrow \qquad P = \dfrac{a^n}{(n+1)r^{n-1}}$

$\Rightarrow \qquad P = \dfrac{K}{r^{n-1}} \qquad \left(\because K = \dfrac{a^n}{(n+1)}\right)$

$\Rightarrow \qquad P \propto \dfrac{1}{r^{n-1}}$.

Q. No. 10. In the conic $\dfrac{l}{r} = 1 + e\cos\theta,$ show that $P = \dfrac{l(1+2e\cos\theta+e^2)^{3/2}}{(1+e\cos\theta)^3}$.

Sol. We have $\dfrac{l}{r} = 1+e\cos\theta$.

Put $\dfrac{1}{r} = u$ in above we get

$\Rightarrow \qquad ul = 1+e\cos\theta$

$\Rightarrow \qquad u = \dfrac{1+e\cos\theta}{l}$

$\Rightarrow \qquad Pu_1 = -e\sin\theta$

$\Rightarrow \qquad u_1 = -\dfrac{e\sin\theta}{P}$

$\Rightarrow \qquad Pu_2 = -e\cos\theta$

$\Rightarrow \qquad u_2 = -\dfrac{e\cos\theta}{P}$

$\Rightarrow \qquad P = \dfrac{(u^2+u_1^2)^{3/2}}{u^3(u+u_2)}$

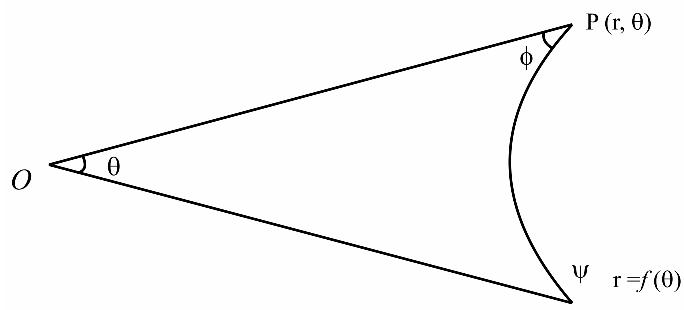
$$= \frac{\left(\left(\frac{a+e\cos\theta}{l}\right)^2 + \left(\frac{-e\sin\theta}{l}\right)^2\right)^{3/2}}{\left(\frac{1+e\cos\theta}{l}\right)^3\left(\frac{1+e\cos\theta}{l} - \frac{e\cos\theta}{P}\right)}$$

$$= \frac{\dfrac{(1^2 + e^2\cos^2\theta + 2e\cos\theta + e^2\sin^2\theta)^{3/2}}{l^3}}{\dfrac{(1+e\cos\theta)^3}{P^3}\,\dfrac{(1+e\cancel{\cos\theta} - e\cancel{\cos\theta})}{P}}$$

$$= \frac{(1+e^2+2e\cos\theta)^{3/2}}{P^3} \times \frac{l^4}{(1+e\cos\theta)^3}$$

$$= \frac{l(1+e^2+2e\cos\theta)^{3/2}}{(1+e\cos\theta)^3}\,.$$

Q. No. 11. If ϕ is the angle which the radians vector of the curve $r = f(0)$ makes with the tangent prove that $P = \dfrac{r\,\text{cosec}\,\phi}{1+\left(\dfrac{d\phi}{d\theta}\right)}$ which P is the radians of curvature.



Sol. We have $r = f(0)$. We know that the sum of two interior angles is equal to the exterior angle.*i.e.*, $\psi = \theta + \phi$.

Differentiate it w.r.t. S, we get

$$\frac{d\psi}{ds} = \frac{d(\theta+\psi)}{d\theta}\frac{d\theta}{ds}$$

$$\Rightarrow \quad \frac{d\psi}{ds} = \left(\frac{d\phi}{d\theta}+1\right)\frac{\sin\theta}{r} \qquad \left(\text{where}\,\frac{d\theta}{ds} = \frac{\sin\theta}{r}\right)$$

$$\Rightarrow \quad \frac{ds}{d\psi} = \frac{r\,\text{cosec}\,\theta}{\left(1+\dfrac{d\phi}{d\theta}\right)}$$

$$\Rightarrow \quad \boxed{P = \frac{r\,\text{cosec}\,\theta}{1+\dfrac{d\phi}{d\theta}}}\,.$$

Q. No. 12. Show that at any point of the equiangular spiral $r = ae\cot x, P = r\,\text{cosec}\,x$.

Sol. We have $r = a_e \cot x$.

Differentiate it w.r.t. θ, we get

$$\frac{dr}{d\theta} ae\theta \cot x.\cot x = r\cot x$$

$\Rightarrow \quad \tan\phi = r\dfrac{d\theta}{dr}$

$= \cancel{r}.\dfrac{1}{\cancel{r}\cot x}$

$\Rightarrow \quad \cancel{\tan}\phi = \cancel{\tan} x$

$\Rightarrow \quad \boxed{\phi = x}$.

$\Rightarrow \quad P = e\sin\phi = r\sin x$

Differentiate w.r.t. P, we get

$1 = \sin x\dfrac{dr}{dp}$

$\Rightarrow \quad \dfrac{dr}{dp} = \dfrac{1}{\sin x}$

$\Rightarrow \quad \dfrac{dr}{dp} = \operatorname{cosec} x$

$\Rightarrow \quad r\dfrac{dr}{dp} r\operatorname{cosec} x$

$\Rightarrow \quad \boxed{P = r\operatorname{cosec} x}$.