Partial Derivatives :-

Let u denote the function of independent variables of x & y i.e $u = u(x,y)$. At point (x,y), the partial derivative of u with respect to x & y are defined by

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(x+h,y) - u(x,y)}{h} \quad , \quad \frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{u(x,y+k) - u(x,y)}{k}$$

provided the limit exist.

we may use the following Notation:

$$\frac{\partial u}{\partial x} = u_x \, , \quad \frac{\partial u}{\partial y} = u_y \, , \quad \frac{\partial^2 u}{\partial x^2} = u_{xx} \, , \quad \frac{\partial^2 u}{\partial y^2} = u_{yy} \, , \quad \frac{\partial^2 u}{\partial x \partial y} = u_{xy}$$

$$p = \frac{\partial u}{\partial x} \, , \quad q = \frac{\partial u}{\partial y} \, , \quad r = \frac{\partial^2 u}{\partial x^2} \, , \quad s = \frac{\partial^2 u}{\partial x \partial y} \, , \quad t = \frac{\partial^2 u}{\partial y^2} .$$

Q. If $u = e^{xyz}$, then show that

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)\, e^{xyz}$$

Sol^n. Given $u = e^{xyz}$

Diff^n. w.r to z,

$$\frac{\partial u}{\partial z} = e^{xyz} \cdot xy$$

Now, differentiating w.r to y

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial y}\left[e^{xyz} \cdot xy\right]$$

$$= x \frac{\partial}{\partial y}\left(y\, e^{xyz}\right)$$

$$= x\left[e^{xyz} + y\, e^{xyz}\, xz\right]$$

$$= x(1 + xyz)\, e^{xyz}$$

$$= (x + x^2yz)\, e^{xyz}$$

Again diff^n. w.r to x

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial y \partial z}\right) = \frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 2xyz)\, e^{xyz} + e^{xyz}\, yz(x + x^2yz)$$

$$= e^{xyz}\left[1 + 2xyz + xyz + x^2y^2z^2\right]$$

$$= e^{xyz}\,(1 + 3xyz + x^2y^2z^2).$$

Ex: If $u = \log(x^3+y^3+z^3-3xyz)$ then prove that

(i) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$

(ii) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$

Solⁿ: Given $u = \log(x^3+y^3+z^3-3xyz)$

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz} \quad \text{——— (1)}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz} \quad \text{——— (2)}$$

& $$\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz} \quad \text{——— (3)}$$

Adding (1), (2) & (3)

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2+y^2+z^2-yz-xz-xy)}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3\cancel{(x^2+y^2+z^2-yz-xz-xy)}}{(x+y+z)\cancel{(x^2+y^2+z^2-yz-xz-xy)}}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$$

(ii) $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$$

Q. If $x^x y^y z^z = c$ then show that $\frac{\partial^2 z}{\partial x \partial y} = -(x\log_e x)^{-1}$, here z is the function of x & y.

Sol:

Given $x^x y^y z^z = c$

Taking log of both sides

$$x\log x + y\log y + z\log z = \log c$$

Diff. partially w.r to x by noting that z is a function of x & y, we get

$$\left[x\cdot\frac{1}{x} + \log x\cdot 1\right] + \left[z\cdot\frac{1}{z} + \log z\cdot 1\right]\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log z)}$$

Similarly, $\frac{\partial z}{\partial y} = -\frac{(1+\log y)}{1+\log z}$

Now $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[-\frac{(1+\log y)}{(1+\log z)}\right]$

$$= -(1+\log y)\frac{\partial}{\partial x}\left[(1+\log z)^{-1}\right]$$

$$= -(1+\log y)\left[-(1+\log z)^{-2}\frac{1}{z}\cdot\frac{\partial z}{\partial x}\right]$$

$$= +\frac{(1+\log y)}{z(1+\log z)^2}\left\{-\frac{(1+\log x)}{(1+\log z)}\right\}$$

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Putting $x = y = z$, we get

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{(1+\log x)^2}{x(1+\log x)^3} = \frac{-1}{x(1+\log x)}$$

(Proved)

Q: If $u = f(r)$, where $r^2 = x^2 + y^2$, then prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{f'(r)}{r}.$$

Homogeneous Functions :-

A function in which every term is of the same degree is known as a homogeneous function of that degree. Thus, every homogeneous function of x & y of degree n can be expressed in the form

$$f(x,y) = a_0 x^n + a_1 x^{n-1} y + a_2 x^{n-2} y^2 + \cdots + a_n y^n$$
$$= x^n F\left(\frac{y}{x}\right)$$

Euler's Theorem on homogeneous Functions :

Th. 1 If u is a homogeneous function of x & y of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

Pf:- Since u is a homogeneous function of x & y of degree n.

$$\therefore u = x^n F(y/x) \quad \text{—— (1)}$$

Diff. w. r to x

$$\frac{\partial u}{\partial x} = n x^{n-1} F(y/x) + x^n F'(y/x)(-y/x^2)$$

$$\Rightarrow x\frac{\partial u}{\partial x} = n x^{n-\cancel{1}+\cancel{1}} F(y/x) - x^{n-1} y F'(y/x)$$

$$\Rightarrow x\frac{\partial u}{\partial x} = nu - x^{n-1} y F'(y/x) \quad \text{—— (2)}$$

Again differentiating (1) $\frac{\partial u}{\partial y} = x^n F'(y/x)\left(\frac{1}{x}\right)$

$$\Rightarrow y\frac{\partial u}{\partial y} = x^{n-1} y F'(y/x) \quad \text{—— (3)}$$

Adding (2) & (3)

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad \text{(Proved)}$$

Relation betn. second order Derivative of homogeneous functions :-

Th② If u be a homogeneous function of degree n, then

(i) $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = (n-1)\frac{\partial u}{\partial x}$

(ii) $x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}$

(iii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$

Th③ Deductions of Euler's Theorem :- If u is not a homogeneous function of x & y, but $f(u)$ is homogeneous function of degree 'n' then

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$ [By Euler's th. $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$, $z = f(u)$, $\frac{\partial z}{\partial x} = f'(u)\frac{\partial u}{\partial x}$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u) - 1]$

where $g(u) = \frac{nf(u)}{f'(u)}$.

Ex: If $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$, then, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \tan u$$

Soln: Given $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$

Clearly, u is not homogeneous function.

$$\sin u = \frac{x^2+y^2}{x+y} = f(u)$$

i.e $f(u) = \frac{x^2[1+(y/x)^2]}{x[1+(y/x)]} = x^1 F(y/x)$

$\Rightarrow$ $f(u) = \sin u$ is a homogeneous function of x & y of degree $n=1$. Hence by using deduction of Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)} = 1\cdot\frac{\sin u}{\cos u} = \tan u.$$

Ex: If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ then show that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\cos 3u \cdot \sin u$.

Soln: (i) Given $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

Clearly, u is not homogeneous.

$\therefore \tan u = \frac{x^3+y^3}{x-y} = f(u)$

$\therefore f(u) = \frac{x^3[1+(y/x)^3]}{x[1-(y/x)]} = x^2 F(y/x)$

$\therefore$ degree $= 2$

Hence by using deduction of Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{n f(u)}{f'(u)}$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2 \cdot \frac{\tan u}{\sec^2 u} = 2\sin u \cos u = \sin 2u.$$

(ii) We know that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)[g'(u)-1] \quad \text{--- (1)}$$

Here $g(u) = \frac{n f(u)}{f'(u)} = \sin 2u$

$\Rightarrow g'(u) = 2\cos 2u$.

Hence (1) becomes

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 2u[2\cos 2u - 1]$$

$$= 2\sin 2u \cos 2u - \sin 2u$$

$$= \sin 4u - \sin 2u$$

$$= 2\cos 3u \cdot \sin u. \quad \text{(Proved)}$$