Maxima & Minima of a function of one variable :-

Working Rule:- If function $f(x)$ be given, then find $f'(x)$ & equate it to zero. Solve this equation for x. Let its roots be $a_1, a_2, a_3 \ldots$.

(i) Find $f''(x)$ & hence find $f''(a_1), f''(a_2), f''(a_3) \ldots$

(ii) If $f''(a_1)$ is negative, then we have a maximum at $x = a_1$.

(iii) If $f''(a_1)$ is positive, then we have a minimum at $x = a_1$.

(iv) If $f''(a_1) = 0$, find $f'''(x)$ & then $f'''(a_1)$.

(v) If $f'''(a_1) = 0$ find $f^{iv}(x)$ & then $f^{iv}(a_1)$.

(vi) If $f^{iv}(a_1)$ is negative, then $f(x)$ is maximum at $x = a_1$ & if $f^{iv}(a_1)$ is positive then $f(x)$ is minimum at $x = a_1$.

If $f^{iv}(a_1) = 0$, then find $f^{v}(a_1)$ & so on.

Ex:- Find the maximum & minimum values of the function $f(x) = x^2 - 4x + 5$ in the interval $1 \leq x \leq 4$.

Sol^n:

Given $f(x) = x^2 - 4x + 5$

$f'(x) = 2x - 4$

For maxima & minima $f'(x) = 0$

$2x - 4 = 0 \Rightarrow x = 2$ (critical point)

Now $f''(x) = 2 \Rightarrow f''(2) = 2 > 0$.

$\therefore$ Hence function $f(x)$ has minimum values at $x = 2$.

$$\therefore [f(x)]_{x=2} = 4 - 8 + 5 = 1$$

Also, given interval $1 \leq x \leq 4$, then

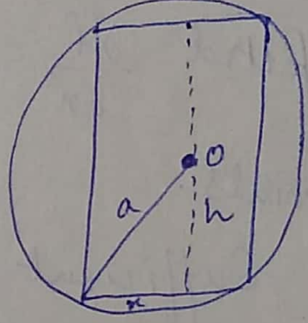
$$f(1) = 1 - 4 + 5 = 2 \quad \&$$
$$f(4) = 16 - 16 + 5 = 5$$

Thus, maximum value of $f(x)$ at $x = 4$ is 5.

Ex: Find the volume of the largest possible right circular cylinder that can be inscribed in a sphere of radius a.

Sol^n: Let radius of base of Cylinder $= x$

& height $= 2h$



$$\therefore \text{Volume } V = \pi x^2 (2h)$$
$$V = 2\pi x^2 \sqrt{a^2 - x^2}$$
$$\frac{dV}{dx} = 2\pi x^2 \, \frac{1}{2} \, \frac{(-2x)}{\sqrt{a^2 - x^2}} + 4\pi x \sqrt{a^2 - x^2}$$
$$\frac{dV}{dx} = \frac{-2\pi x^3}{\sqrt{a^2 - x^2}} + 4\pi x \sqrt{a^2 - x^2}$$

for maximum V, we put $\frac{dV}{dx} = 0$

$$\Rightarrow 0 = \frac{-2\pi x^3}{\sqrt{a^2 - x^2}} + 4\pi x \sqrt{a^2 - x^2}$$
$$\Rightarrow 2\pi x^3 = 4\pi x (a^2 - x^2)$$
$$\Rightarrow x^2 = 2(a^2 - x^2)$$
$$\Rightarrow 3x^2 = 2a^2$$
$$x = a\sqrt{\frac{2}{3}}$$

Since $h^2 = a^2 - x^2$

$$\Rightarrow h = \sqrt{a^2 - \frac{2a^2}{3}}$$
$$\Rightarrow h = \frac{a}{\sqrt{3}}$$

$$\text{Max. volume} = \pi\left(\frac{2}{3}a^2\right)\frac{2a}{\sqrt{3}}$$
$$= \frac{4\pi a^3}{3\sqrt{3}}$$

Ans

Maxima & Minima of a function of two variables :-

Working Rule :

(i) Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$

(ii) A necessary condition for maximum or minimum value of $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$.

(iii) Solve the above Simultaneous Eq^n.
Let $(a_1, b_1), (a_2, b_2)$ --- be the solutions of these equations.

(iv) find $\frac{\partial^2 f}{\partial x^2} = r$, $\frac{\partial^2 f}{\partial x \partial y} = s$, $\frac{\partial^2 f}{\partial y^2} = t$ at these points.

(v) A Sufficient Condition for maximum or minimum value is $rt - s^2 > 0$.

(vi) ⓐ If $r > 0$ at one or more points then function f is minimum.
ⓑ If $r < 0$ at one or more points then function f is maximum.

(vii) If $rt - s^2 = 0$, Such Case is undecided & further investigation is needed.

(viii) If $r = 0$, nothing Can be said about the maxima or minima. It requires further investigation.

Remarks:— * If $rt - s^2 < 0$ then $f(x,y)$ has neither maxima nor minima at (a, b) and point (a,b) is called Saddle point.

① A maximum or minimum value of a given function is called extreme value.

② $f(a,b)$ is to be a stationary value of $f(x,y)$, if $f_x(a,b) = 0$ & $f_y(a,b) = 0$.
Thus extreme value is stationary value but the Converse may not be true.

Ex: Find the maximum or minimum values of the function $x^3y^2(1-x-y)$.

Sol^n: Let $u = x^3y^2(1-x-y)$

for maxima or minima,

$$\frac{\partial u}{\partial x} = 3x^2y^2(1-x-y) - x^3y^2 = 0 \quad \text{——— (1)}$$

$$\frac{\partial u}{\partial y} = 2x^3y(1-x-y) - x^3y^2 = 0 \quad \text{——— (2)}$$

Subtracting (2) from (1)

$$x^2y(1-x-y)(3y-2x) = 0$$

$$\Rightarrow y = \frac{2}{3}x \quad \text{——— (3)}$$

Put in (1)

$$3x^2 \cdot \frac{4}{9}x^2\left(1-x-\frac{2}{3}x\right) - x^3\frac{4}{9}x^2 = 0$$

$$\Rightarrow \frac{4}{9}x^4[3-5x-x] = 0$$

$$\Rightarrow \frac{4}{9}x^4[3-6x] = 0$$

$$\Rightarrow x = 1/2$$

Put in (3)

$$\therefore y = \frac{1}{3}$$

Now $r = \frac{\partial^2 u}{\partial x^2} = 6xy^2 - 12x^2y^2 - 6xy^3$

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value of r at the point $\left(\frac{1}{2}, \frac{1}{3}\right) = -\frac{1}{9}$

$$t = \frac{\partial^2 u}{\partial y^2} = 2x^3 - 2x^4 - 6x^2 y$$

The value of t at the point $\left(\frac{1}{2}, \frac{1}{3}\right) = -\frac{1}{8}$

$$s = \frac{\partial^2 u}{\partial x \partial y} = -6x^2 y - 8x^3 y - 9x^2 y^2.$$

The value of s at the point $\left(\frac{1}{2}, \frac{1}{3}\right) = -\frac{1}{12}$

$$\therefore rt - s^2 = \left(-\frac{1}{9}\right)\left(-\frac{1}{8}\right) - \left(\frac{-1}{12}\right)^2 = +\text{ive} \ \& \ r < 0$$

Hence the function u has a maximum value ~~of~~ at $x = \frac{1}{2}$, $y = \frac{1}{3}$.

$$\therefore \text{Max. value of } u = \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{2} - \frac{1}{3}\right)$$

$$= \frac{1}{432} \quad \text{Ans}$$

Ex: Discuss the maxima & minima of $u = \sin x + \sin y + \sin(x+y)$.

Sol^n: For maxima or minima of u, we have

$$\frac{\partial u}{\partial x} = \cos x + \cos(x+y) = 0 \quad \text{— (1)}$$

$$\frac{\partial u}{\partial y} = \cos y + \cos(x+y) = 0 \quad \text{— (2)}$$

Solving (1) & (2) we get critical points $\left(\frac{\pi}{3}, \pi/3\right)$ & (π, π).

$\cos x + \cos(x+y) = 0$
$\cos y + \cos(x+y) = 0$
$\cos y = -\cos(x+y)$
$\cos x - \cos y = 0$
$\cos x = \cos y$
$x = y$
$\cos x + \cos 2x = 0$
$\cos x + 2\cos^2 x - 1 = 0$
$2\cos^2 x + 2\cos x - \cos x - 1 = 0$
$2\cos x(\cos x + 1) - (\cos x + 1) = 0$

$(2\cos x - 1)(\cos x + 1) = 0$
$\cos x = 1/2, \quad \cos x = -1$
$x = \pi/3 \qquad x = \pi$

Now $r = \frac{\partial^2 u}{\partial x^2} = -\sin x - \sin(x+y)$

$$s = \frac{\partial^2 u}{\partial x \partial y} = -\sin(x+y)$$

$$\& \; t = \frac{\partial^2 u}{\partial y^2} = -\sin y - \sin(x+y)$$

At point $x = y = \pi/3$

$$r = -\sin(\pi/3) - \sin(2\pi/3) = -\sqrt{3}$$

$$s = -\sin(\pi/3 + \pi/3) = -\frac{\sqrt{3}}{2}$$

$$t = -\sin(\pi/3) - \sin(2\pi/3) = -\sqrt{3}$$

$$\& \; rt - s^2 = (-\sqrt{3})(-\sqrt{3}) - \left(-\frac{\sqrt{3}}{2}\right)^2 = \frac{9}{4} = +\text{ive}$$

Since $rt - s^2$ is +ive & r is -ive.

$\Rightarrow$ u has a maxima at $x = y = \pi/3$.

At point $x = y = \pi$

$$r = 0, \; s = 0, \; t = 0$$

$$\therefore rt - s^2 = 0$$

This case is doubtful and hence we shall leave it.

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Q. Find the maxima & minima of the function $u = \sin x \cdot \sin y \cdot \sin(x+y)$.

Solⁿ: For maxima or minima of u

$$\frac{\partial u}{\partial x} = \cos x \sin y \sin(x+y) + \sin x \sin y \cos(x+y)$$

$$\frac{\partial u}{\partial x} = \sin y \sin(2x+y) = 0 \quad \text{--- (1)}$$

& $$\frac{\partial u}{\partial y} = \sin x \cos y \sin(x+y) + \sin x \sin y \cos(x+y)$$

$$\frac{\partial u}{\partial y} = \sin x \sin(2y+x) = 0 \quad \text{--- (2)}$$

from (1) $\sin y = 0$ or $\sin(2x+y) = 0$

$\Rightarrow y = 0$ or $2x+y = \pi$ or 2π --- (3)

& from (2) $x = 0$ or $2y+x = \pi$ or 2π --- (4)

Solving (3) & (4) we get critical point $(\pi/3, \pi/3)$ & $(2\pi/3, 2\pi/3)$.

Now $r = \frac{\partial^2 u}{\partial x^2} = 2\sin y \cos(2x+y)$

$s = \frac{\partial^2 u}{\partial x \partial y} = \sin(2x+2y)$

$t = \frac{\partial^2 u}{\partial y^2} = 2\sin x \cos(x+2y)$

At point $(\pi/3, \pi/3)$

$r = 2(\sqrt{3}/2)(-1) = -\sqrt{3}$

$s = \sin(4\pi/\sqrt{3}) = (-\sqrt{3}/2)$

$t = 2\left(\frac{\sqrt{3}}{2}\right)(-1) = -\sqrt{3}$

$\therefore rt - s^2 = 3 - 3/4 = 9/4 > 0$

$\& \; r = -\sqrt{3} < 0.$

Hence u is maximum at $x = \pi/3, \; y = \pi/3.$

$$u_{max} = \sin(\pi/3)\sin(\pi/3)\sin(2\pi/3) = \frac{3\sqrt{3}}{8}$$

At point $(2\pi/3, 2\pi/3)$

$r = 2(\sqrt{3}/2)(1) = \sqrt{3}$

$t = \sqrt{3}, \quad s = \sqrt{3}/2$

$\therefore rt - s^2 = 3 - (3/4) = 9/4 > 0 \; \& \; r = \sqrt{3} > 0$

$\Rightarrow u$ is minimum at $x = y = 2\pi/3.$

$$u_{min} = \sin(2\pi/3)\sin(2\pi/3)\sin(4\pi/3) = -\frac{3\sqrt{3}}{8}.$$