Lecture Notes

MAXIMA AND MINIMA

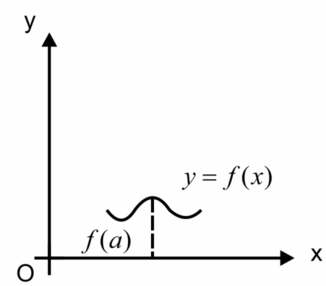
In mathematical analysis, the maxima and minima of a function known collectively as extrema (the plural of extremum) are the largest and smallest value of the function, either within a given range or on the entire domain of a function. Pierre de Fermat was one of the first mathematicians to propose a general technique, adequacy, for finding the maxima and minima of functions.

Definition: A real valued function 'f' defined on a domain X has a **maxima** point x^* if $f\left(x^*\right) \geq f\left(x\right)$ for all x in X. Similarly, the function has a **minima** point at x^* if $f\left(x^*\right) \leq f\left(x\right)$ for all x in X. The value of the function at a maxima point is called the **maximum** value of the function and the value of the function at a minima point is called the **minimum** value of the function.

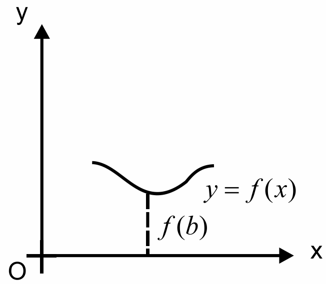
In this unit we show how differentiation can be used to find the maximum and minimum values of a function. Since the derivative provides information about the gradient or slope of the graph of a function, we can use it to locate points on a graph where the gradient is zero. We shall see that such points are often associated with the largest or smallest values of the function, at least in their immediate locality. In many applications, a scientist, engineer, or economist for example, will be interested in such points for obvious reasons such as maximizing power, or profit, or minimizing costs etc.

MAXIMA AND MINIMA

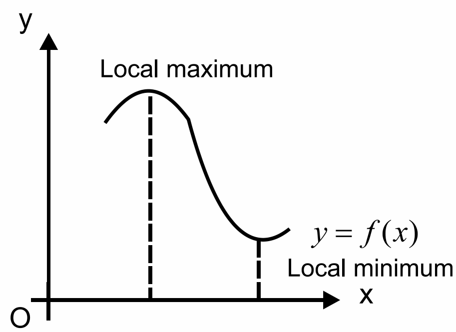
- A function $f\left(x\right)$ is said to have the **maximum value** in an interval at a point $x=a$ if $f\left(a\right) \geq f\left(x\right)\ \forall x \in 1$. $f\left(a\right)$ is known as the greatest or the maximum value of $f\left(x\right)$ in the interval concerned. The point $x=a$ is the point of maxima in that interval.



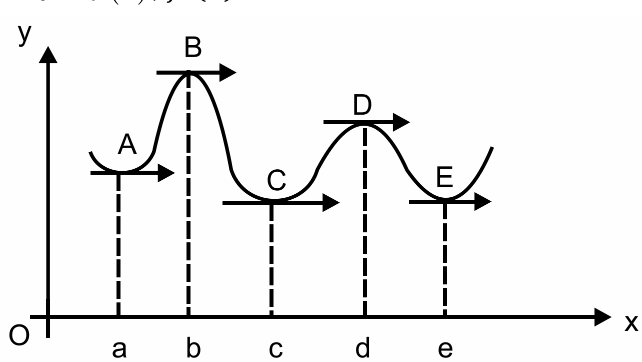
- A function $f\left(x\right)$ is said to have a **minimum value** in an interval at a point $x=b$ if $f\left(b\right) \leq f\left(x\right) \forall x \in 1.$ Here $f\left(b\right)$ is known as the smallest value or the minimum value of $f\left(x\right)$ in the said interval and the point $x=b$ is called the point of minima of $f\left(x\right)$ in that interval.



- A function $f(x)$ is said to have **local maximum** at $x = c$ if there exists $h > 0$ however small satisfying $f(x) < f(c)$ for all x for which $0 < |x - c| < h$. The value of the function at $x = c$ i.e. $f(c)$ is known as thelocal maximum value or relative maximum value of $f(x)$ and the point $x = c$ is known as local maxima.
- A function $f(x)$ is said to have a **local minimum** value at $x = d$, if there exist h>0, however small, such that $f(x) > f(d)$ $\forall x$ for which $0 < |x - d| < h$. The value of the function at $x = d$ i.e. $f(d)$ is known as the local minimum or the relative minimum value of $f(x)$ and the point $x = c$ is known as local maxima.



- The points at which a function $f(x)$ attains either local maximum or local minimum are known as the **extreme points** of the function.
- At the points of extrema (i.e., minima or maxima), the tangents drawn to the curve are paralleled to the x-axis. Thus at each points of extrema the slope of the tangents drawn to the curve is zero, i.e. for the function $y = f(x)$, $f'(x) = 0$.



- Maximum and minimum values are also called extremum, turning or stationary values.

Rule for finding the maximum and minimum values of a function $y = f(x)$**.**

Step1. The value of $f(x)$ is found out by differentiating $y = f(x)$ with respect to x .

Step2. Put $f(x)=0$ and calculate the values of x, which are known as critical points.

Step3.

- If $f''(x) < 0$, then x is known as **maxima** and the value of function at maxima is known as **maximum**.
- If $f''(x) > 0$, then x is known as **minima** and the value of function at minima is known as **minimum**.
- If $f''(x)$=0, then x is known as point of **inflexion** that is it is neither maxima nor minima.

Exercise

Q. No. 1. $2x^3+3x^2+4$

Sol. We have $f(x)=2x^3+3x^2+4$.

$\Rightarrow \quad f'(x)=6x^2+6x$

Put $f'(x)=0$, we get

$6x^2+6x=0$

$\Rightarrow \quad 6x(x+1)=0$

$\Rightarrow \quad x=0, x=-1$.

At $x=0$, we see that

$f''(x)=12x+6$

$=12(0)+6=6>0$.

Therefore $x=0$ is the minima and $f(0) = 2(0) + 3(0) + 4 = 4$ is the minimum value of $f(x)$.

At$x = -1$, we see that

$f''(-1)=12(-1)+6$

$=-12+6 = -6 < 0$.

Therefore $x = -1$is the maxima and $f(-1) = 2(-1) + 3(0) + 4 = 4$ is the maximum value of $f(x)$.

Q. No. 2. $f(x)=2x^3-24x^2+42x+10$

Sol. We have

$f(x)=2x^3-24x^2+42x+10$

$\Rightarrow \quad f'(x)=6x^2-48x+42$

$\Rightarrow \quad f''(x)=12x-48$

Put $f'(x)=0$, we see that

$$6x^2 - 48x + 42 = 0 \Rightarrow 6\left(x^2 - 8x + 7\right) = 0$$

$\Rightarrow \quad x^2 - 8x + 7 = 0 \Rightarrow x^2 - 7x - x + 7 = 0$

$\Rightarrow \quad x(x-7) - 1(x-7) = 0$

$\Rightarrow \quad (x-1)(x-7) = 0$

$\Rightarrow \quad x = 1, x = 7$

Now at $x = 1$, we see that

$\Rightarrow \quad f''(1) = 12(1) - 48 = -36 < 0.$

Therefore $x = 1$ is the maxima and $f(1) = 2(1)^3 - 24\left(1^2\right) + 42 + 0 = 30$ is the maximum value of $f(x)$.

At $x = 7$, we see that

$$f''(7) = 12(7) - 48 = 84 - 48 = 30 > 0$$

Therefore $x = 7$ is the minima and $f(7) = 2(7)^3 - 24(7)^2 + 47(7) + 10 = -386$ is the minimum value of $f(x)$.

Q. No. 3. $f(x) = \sin x + \cos x$

Sol. $\quad f(x) = \sin x + \cos x$

$$f'(x) = \cos x - \sin x$$

$$f''(x) = -\sin x - \cos x$$

Put $f'(x) = 0$ we get

$$cosx - \sin x = 0$$

Dividing each term by $\cos x$ on both sides

$$\frac{\cos x}{\cos x} - \frac{\sin x}{\cos x} = 0$$

$\Rightarrow \quad 1 - \tan x = 0$

$\Rightarrow \quad -\tan x = -1$

$\Rightarrow \quad \tan x = 1$

$\Rightarrow \quad \tan x = \tan \frac{\pi}{4}$

$\Rightarrow \quad x = \frac{\pi}{4}$

Therefore at $x = \frac{\pi}{4}$ we

$$f''(x) = -\sin x - \cos x$$

$$f''\left(\frac{\pi}{4}\right) = -\left[\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right]$$

$$= -\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = \frac{-2}{\sqrt{2}} < 0$$

Therefore $x = \frac{\pi}{4}$ is the maxima and

$$f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} + \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \frac{\sqrt{2}.\sqrt{2}}{\sqrt{2}}$$

$$\Rightarrow \quad f\left(\frac{\pi}{4}\right) = \sqrt{2}$$

is the maximum value of $f(x)$.

Q. No. 4. $3x^5 - 25x^3 + 60x$

Sol. $f(x) = 3x^5 - 25x^3 + 60x$

$$f'(x) = 15x^4 - 75x^2 + 60$$

$$f''(x) = 60x^3 - 150x$$

Put $f'(x) = 0$ we see that

$$\Rightarrow \quad 15x^4 - 75x^2 + 60 = 0$$

$$\Rightarrow \quad 5\left(3x^4 - 15x^2 + 12\right) = 0$$

$$\Rightarrow \quad 3x^4 - 15x^2 + 12 = 0$$

$$\Rightarrow \quad 3\left(x^4 - 5x^2 + 4\right) = 0$$

$$\Rightarrow \quad x^4 - 5x^2 + 4 = 0$$

$$\Rightarrow \quad x^4 - x^2 - 4x^2 + 4 = 0$$

$$\Rightarrow \quad x^2\left(x^2 - 1\right) - 4\left(x^2 - 1\right) = 0$$

$$\Rightarrow \quad \left(x^2 - 4\right)\left(x^2 - 1\right) = 0$$

$$\Rightarrow \quad x = \pm 2, x = \pm 1$$

Now

$$f''(2) = 480 - 300 = 80 > 0$$

Thus $x = 2$ is minima.

$$f''(-2) = -480 + 300 = -80 < 0$$

Thus $x = -2$ is maxima.

$$f''(1) = 60 - 150 = -90 < 0$$

Thus $x = 1$ is maxima.

$$f''(-1) = -60 + 150 = 90 > 0$$

Thus $x = -1$ is maxima.

Note: Calculate the corresponding maximum and minimum values of the function.

Q. No. 5. $2x^3 - 21x^2 + 36x - 20$

Sol. $f(x) = 2x^3 - 21x^2 + 36x - 20$

$\Rightarrow \quad f'(x) = 6x^2 - 42x + 36$

$\Rightarrow \quad f''(x) = 12x - 42$

Put $f'(x) = 0$, we get

$6x^2 - 42x + 36 = 0$

$\Rightarrow \quad 6(x^2 - 7x + 6) = 0$

$\Rightarrow \quad x^2 - 7x + 6 = 0$

$\Rightarrow \quad x^2 - 6x - x + 6 = 0$

$\Rightarrow \quad x(x-6) - 1(x-6) = 0$

$\Rightarrow \quad (x-1)(x-6) = 0$

$\Rightarrow \quad x = 1, x = 6$

At $x = 1$

$f''(1) = 12(1) - 42 = -30 < 0$

Thus $x = 1$ is maxima.

At $x = 6$

$f''(6) = 12(6) - 42 = 72 - 42 = 30 > 0$

Thus $x = 6$ is maxima.

Note: Calculate the corresponding maximum and minimum values of the function.

Q. No. 6. $(x-3)^5 (x+1)^4$

Sol. $f(x) = (x-3)^5 (x+1)^4$

$\Rightarrow \quad f'(x) = 5(x-3)^4 (x+1)^4 + (x-3)^5 4(x+1)^3$

$\Rightarrow \quad f'(x) = 5(x-3)^4 (x+1)^4 + 4(x-3)^5 (x+1)^3$

$\Rightarrow \quad f''(x) = 20(x-3)^3 (x+1)^4 + 20(x-3)^4 (x+1)^3 + 12(x-3)^5 (x+1)^2 + 20(x+1)^3 (x-3)^4$

Put $f'(x) = 0$ we see that

$5(x-3)^4 (x+1)^4 + 4(x-3)^5 (x+1)^3 = 0$

$\Rightarrow \quad (x-3)^4 (x+1)^3 \left[5(x+1) + 4(x-3)\right] = 0$

$\Rightarrow \quad (x-3)^4 (x+1)^3 (5x + 5 + 4x - 12) = 0$

$\Rightarrow \quad (x-3)^4 (x+1)^3 (9x - 7) = 0$

Either $(x-3)^4 = 0 \qquad (x+1)^3 = 0 \qquad 9x - 7 = 0$

$$x-3=0 \quad x+1=0 \quad 9x=7$$

$$x=3 \quad x=-1 \quad x=\frac{7}{9}$$

At $x=3$,we have

$$f''(3)=0.$$

Thus $x=3$ is point if inflexion that is it is neither maxima nor minima.

At $x=-1$, we have

$$f''(-1)=0.$$

Thus $x=-1$is point if inflexion that is it is neither maxima nor minima.

At$x=\frac{7}{9}$, we have

$$f''\left(\frac{7}{9}\right)<0.$$

Thus $x=\frac{7}{9}$ is maxima.

Note: Calculate the corresponding maximum and minimum values of the function.