Taylor's theorem:-

Let $f(x) = f(a+h)$ be a continuous & differentiable function, so that $f(x)$ can be expanded in powers of h, then

$$f(x) = f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^n}{n!} f^n(a) + \cdots$$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

Ex: Show that $\log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} + \cdots$

Sol^n: Let $f(x+h) = \log(x+h)$

By Taylor's theorem,

$$\log(x+h) = f(x+h) = f(h) + x f'(h) + \frac{x^2}{2!} f''(h) + \cdots \quad (1)$$

Here $f(h) = \log h$

$f'(h) = 1/h$, $f''(h) = -\frac{1}{h^2}$

$f'''(h) = 2/h^3$ etc,

Putting values in (1)

$$\log(x+h) = \log h + \frac{x}{h} - \frac{x^2}{2h^2} + \frac{x^3}{3h^3} - \cdots$$

Ex: Expand $\tan(x + \pi/4)$ as far as the term x^4 & evaluate $\tan 46.5^\circ$ upto four significant digits.

Sol^n: Let $f(x) = \tan x \Rightarrow f(\pi/4) = 1$

$f'(x) = \sec^2 x = 1 + \tan^2 x \Rightarrow f'(\pi/4) = 1$

$f''(x) = 2\tan x \sec^2 x = 2\tan(1+\tan^2 x)$

$f''(x) = 2\tan x + 2\tan^3 x \Rightarrow f''(\pi/4) = 4$

$f'''(x) = 2 + 8\tan^2 x + 6\tan^4 x$

$\Rightarrow f'''(\pi/4) = 16$

By Taylor's Series

$$f(x+h) = f(h) + x f'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \cdots$$

Put $h = \pi/4$, we get

$$f(\pi/4 + x) = f(\pi/4) + x f'(\pi/4) + \frac{x^2}{2!} f''(\pi/4) + \cdots$$

$$f(\pi/4 + x) = 1 + x(2) + \frac{x^2}{2!}(4) + \frac{x^3}{3!}(16) + \cdots$$

$$f(\pi/4 + x) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \cdots \quad (1)$$

Putting

$$x = 1.5^\circ = \left(\frac{3}{2}\right)^\circ = \frac{3}{2} \times \frac{\pi}{180} = 0.02618 \text{ in } (1)$$

$$f(45^\circ + 1.5^\circ) = 1 + 2(0.02618) + 2(0.02618)^2 + \frac{8}{3}(0.02618)^3 + \cdots$$

$$\Rightarrow \tan 46.5^\circ = 1.0538 \text{ (approx.)}$$ Ans

Q. Expand $\sin x$ in powers of $(x - \pi/2)$

Soln. Let $f(x) = \sin x$

$$f(x) = f\left[\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right)\right]$$

By Taylor's

$$f(x) = f\left[\pi/2 + (x - \pi/2)\right] = f(\pi/2) + (x - \pi/2) f'(\pi/2) + \frac{(x-\pi/2)^2}{2!} f''(\pi/2) + \cdots \quad (1)$$

$f(x) = \sin x$

$f'(x) = \cos x$, $f''(x) = -\sin x$, $f'''(x) = -\cos x$,

$f^{iv}(x) = \sin x$.

$\therefore f(\pi/2) = \sin \pi/2 = 1, \quad f'(\pi/2) = 0,$

$f''(\pi/2) = -1, \quad f'''(\pi/2) = 0, \quad f^{iv}(\pi/2) = 1.$

Putting in ①

$$\sin x = 1 + (x - \pi/2)\cdot 0 + \frac{(x-\pi/2)^2}{2!}(-1) + \frac{(x-\pi/2)^3}{3!}\cdot 0 + \frac{(x-\pi/2)^4}{4!}\cdot 1 + \cdots$$

$$\sin x = 1 - \frac{1}{2!}(x-\pi/2)^2 - \frac{1}{4!}(x-\pi/2)^4 + \cdots$$

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