LOGICAL OPERATORS & BOOLEAN ALGEBRA

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Introduction

- Developed by English Mathematician George Boole in between 1815-1864.
- It is described as an algebra of logic or an algebra of two values i.e. True or False.
- The term logic means a statement having binary decisions i.e True/Yes or False/ No.

Application of Boolean Algebra:

- IT IS USED TO PERFORM THE LOGICAL OPERATIONS IN DIGITAL COMPUTER.
- IN DIGITAL COMPUTER TRUE REPRESENT BY '1' (HIGH VOLT) AND FALSE REPRESENT BY '0' (LOW VOLT)
- LOGICAL OPERATIONS ARE PERFORMED BY LOGICAL OPERATORS. THE FUNDAMENTAL LOGICAL OPERATORS ARE:

I. NOT (NEGATION/COMPLEMENT) : **(-/~)**

II. AND (CONJUNCTION) : **(∧)**

III. OR (DISJUNCTION)) : **(∨)**

IV. CONDITIONAL (IF….. THEN) : $\rightarrow$ / $\longrightarrow$

V. BI-CONDITIONAL (IF AND ONLY IF) : $\leftrightarrow$/ $\Longleftrightarrow$

Truth Table: A *truth table* shows how the truth or falsity of a compound statement depends on the truth or falsity of the simple statements from which it's constructed. So we'll start by looking at truth tables for the five logical connectives (Negation, Conjunction, Disjunction, Conditional, Bi-conditional).

Table for Negation or NOT (~):

This table is easy to understand.
If 'P' is true, its negation '~P' is false. If 'P' is false, then '~P' is true.

P	~P
T	F
F	T

REMARK: ~~P=P

AND (conjunction) : (^)

$P \wedge Q$ should be *true* when both P and Q are *true*, and *false* otherwise:

P	Q	P ^ Q
T	T	T
T	F	F
F	T	F
F	F	F

OR (DiSjunction) : (V)

$P \vee Q$ is *true* if either P is *true* or Q is *true* (or both --- remember that we're using "or" in the inclusive sense). It's only *false* if both P and Q are *false*.

P	Q	P V Q
T	T	T
T	F	T
F	T	T
F	F	F

Conditional (if $- -$*then*) $P \rightarrow Q$:

To understand the table, consider the following example of Father and his son:

If son passes 2^{nd} semester with 80+ marks in all subject, then father buy Bike for his son."
The statement will be *true* if father keeps his promise and *false* if he don't.

Suppose it's *true* that you got 80+ marks in all subject and it's *true* that father give him bike. Since both kept their promises, the implication is *true*. This corresponds to the first line in the table.

Suppose it's *true* that you got 80+marks but it's *false* that Father give you a bike. Since father *didn't* kept his promise, the implication is *false*. This corresponds to the second line in the table.

What if it's false that the son got 80+marks in all subjects? Whether or not Father gave him bike, I haven't broken my promise. Thus, the implication can't be false, so (since this is a two-valued logic) it must be true. This explains the last two lines of the table.

P	Q	P →Q
T	T	T
T	F	F
F	T	T
F	F	T

Bi-Conditional (if and only if)

$P \leftrightarrow Q$ means that P and Q are *equivalent*. So the double implication is *true* if P and Q are both *true* or if P and Q are both *false*; otherwise, the double implication is false.

P	Q	P↔Q
T	T	T
T	F	F
F	T	F
F	F	T

You'll use these tables to construct tables for more complicated sentences. It's easier to demonstrate what to do when to describe it in words, so you'll see the procedure worked out in the examples.

#Construct a truth table for the formula $\sim P \wedge (P \rightarrow Q)$

Solution:

- First, list all the alternatives for P and Q.
- Next, in the third column, list the values of $\sim P$ based on the values of P. Use the truth table for negation: When P is true $\sim P$ is false, and when P is false, $\sim P$ is true.
- In the fourth column, list the values for $P \rightarrow Q$. Check for yourself that it is only false ('F') if P is true ('T') and Q is false ('F').
- The fifth column gives the values for compound expression $\sim P \wedge (P \rightarrow Q)$. It is an "and" of $\sim P$ and $P \rightarrow Q$. An 'and' is true only if both parts of the 'and' are true, otherwise, it is false. So, look at the third and fourth columns; if both are true, put T in the fifth column, otherwise put F.

P	Q	~P	P→Q	~P^(P→Q)
T	T	F	T	F
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Tautology: A *tautology* is a formula which is "always true" that is, it is true for every assignment of truth values to its simple components.

Contradiction: The opposite of a tautology is a *contradiction*, a formula which is "always false". In other words, a contradiction is false for every assignment of truth values to its simple components.

Example: Show that $(P \rightarrow Q) \lor (Q \rightarrow P)$ is a tautology.

Solution:

Given $(P \rightarrow Q) \lor (Q \rightarrow P)$ and we have to show that the formula is always true.

P	Q	$(P \rightarrow Q)$	$(Q \rightarrow P)$	$(P \rightarrow Q) \lor (Q \rightarrow P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

The last column contains only T’s. Therefore, the formula is a Tautology.

Prove that (P V~(Q^R)) is equivalent to ((P V~Q)V ~R)

Solution:

P I	Q II	R III	~Q IV using II	~R V using III	(Q^R) VI using II-III	~(Q^R) VII using VI	P V~(Q^R) VIII using I-VII	(P V ~Q) IX using I-IV	(P V ~Q)V~R X using IX-V
T	T	T	F	F	T	F	T	T	T
T	T	F	F	T	F	T	T	T	T
T	F	T	T	F	F	T	T	T	T
T	F	F	T	T	F	T	T	T	T
F	T	T	F	F	T	F	F	F	F
F	T	F	F	T	F	T	T	T	T
F	F	T	T	F	F	T	T	F	T
F	F	F	T	T	F	T	T	T	T

Since, column VIII and column X are equal.

We can say P V~(Q^R) is equivalent to ((P V~Q)V ~R).

Construct a truth table for $(P \rightarrow Q) \wedge (Q \rightarrow R)$

Solution: We have to find $(P \rightarrow Q) \wedge (Q \rightarrow R)$

P I	Q II	R III	$P \rightarrow Q$ IV using I-II	$Q \rightarrow R$ V using II-III	$(P \rightarrow Q) \wedge (Q \rightarrow R)$ VI using IV-V
T	T	T	T	T	T
T	T	F	T	F	F
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

Law of Algebra of Propositions:

1) Commutative Laws

I. $p \vee q \Leftrightarrow q \vee p$

II. $p \wedge q \Leftrightarrow q \wedge p$

2) Associative Las:

I. $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$

II. $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

3) Distributive Laws:

I. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (q \vee r)$

II. $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

4)* Demorgan'S Laws:

I. $\sim(p \vee q) \Leftrightarrow (\sim p) \wedge (\sim q)$

II. $\sim(p \wedge q) \Leftrightarrow (\sim p) \vee (\sim q)$

5) Identity Laws:

I. $p \wedge t \Leftrightarrow p$ or $t \wedge p \Leftrightarrow p$

II. $p \vee f \Leftrightarrow p$ or $f \vee p \Leftrightarrow p$

6) Complement Laws:

I. $p \vee (\sim p) \Leftrightarrow t$

II. $p \wedge (\sim p) \Leftrightarrow f$

Demorgan’S Laws:

I. ~(p v q) ⇔ (~p)^ (~q)

p	q	~p	~q	(pvq)	~(pvq)	(~p)^ (~q)	~(pvq) ⟷ (~p)^ (~q)
T	T	F	F	T	F	F	T
T	F	F	T	T	F	F	T
F	T	T	F	T	F	F	T
F	F	T	T	F	T	T	T

II. ~(p ^ q) ⇔ (~p) v (~q)

p	q	~p	~q	(p^q)	~(pvq)	(~p) v (~q)	~(p^q) ⟷ (~p)^ (~q)
T	T	F	F	T	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	F	T	T	F	T	T	T

Boolean Algebra: Function of two values {1,0} or {true, false}.

Definition: Let B={a,b,c,…..} be a non-empty set with two binary operations '+' and '.' and a unary operation " ' " then the algebraic structure (B,+, ., ') is called a Boolean algebra, if it satisfies the following laws:

Closure Laws: For any two elements a and b of B, a+b and a, b are unique elements of B, i.e.,

i) a+b ϵ B, for all a, b ϵ B.

ii) a.b ϵ B, for all a, b ϵ B.

Commutative Laws:

i) a+b =b+a, for all a, b ϵ B.

ii) a.b =b.a, for all a, b ϵ B.

Distributive laws:

i) a+b =b+a, for all a, b ϵ B.

ii) a.b =b.a, for all a, b ϵ B.

Refer book for proofs

Identity Laws:

There exist two elements 0 and 1 in B such that for every a ϵ B , we have

i) a+0 =0+a=a

ii) a.1 =1.a=a

0 is the identity element of B with respect to ‘ +’ and 1 identity element of B with respect to ‘ . ’

Refer book for proofs

Complementary Laws or laws of inverse:

For each a ϵ B, there exist a’ ϵ B, such that

I. a+a’ = a’+a = 1

II. a.a’ = a’.a = 0

Refer book for proofs

STAY 

Thank you!!