Gravity field of the Earth

- Newton's law of universal gravitation – force of gravity – proportional to the product of body masses – inversely proportional to the square of distance between them

$$F = G\frac{m_1 m_2}{r^2} \qquad \text{(Eq. 1)}$$

F = gravitational attraction between two masses

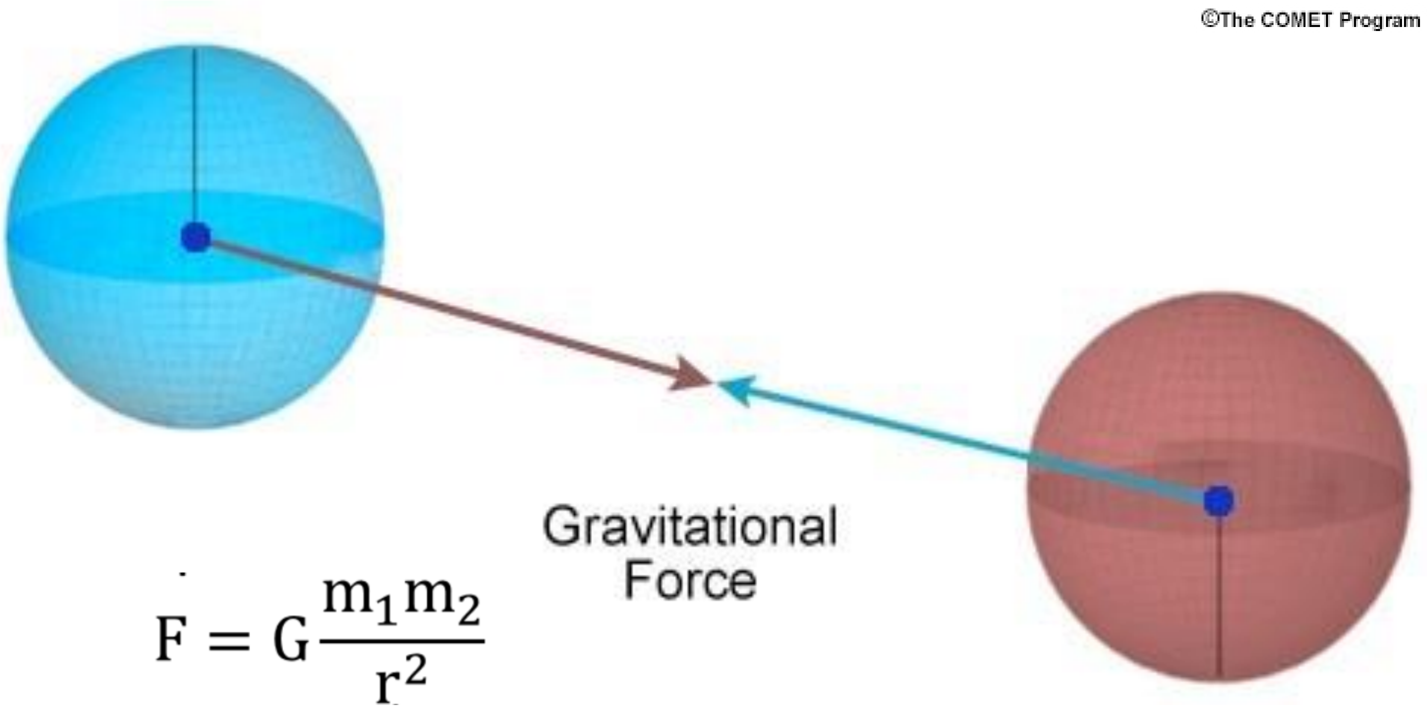
m_1, m_2 = masses of two bodies

r = distance between the centers of masses

G = universal gravitational constant

Contd....

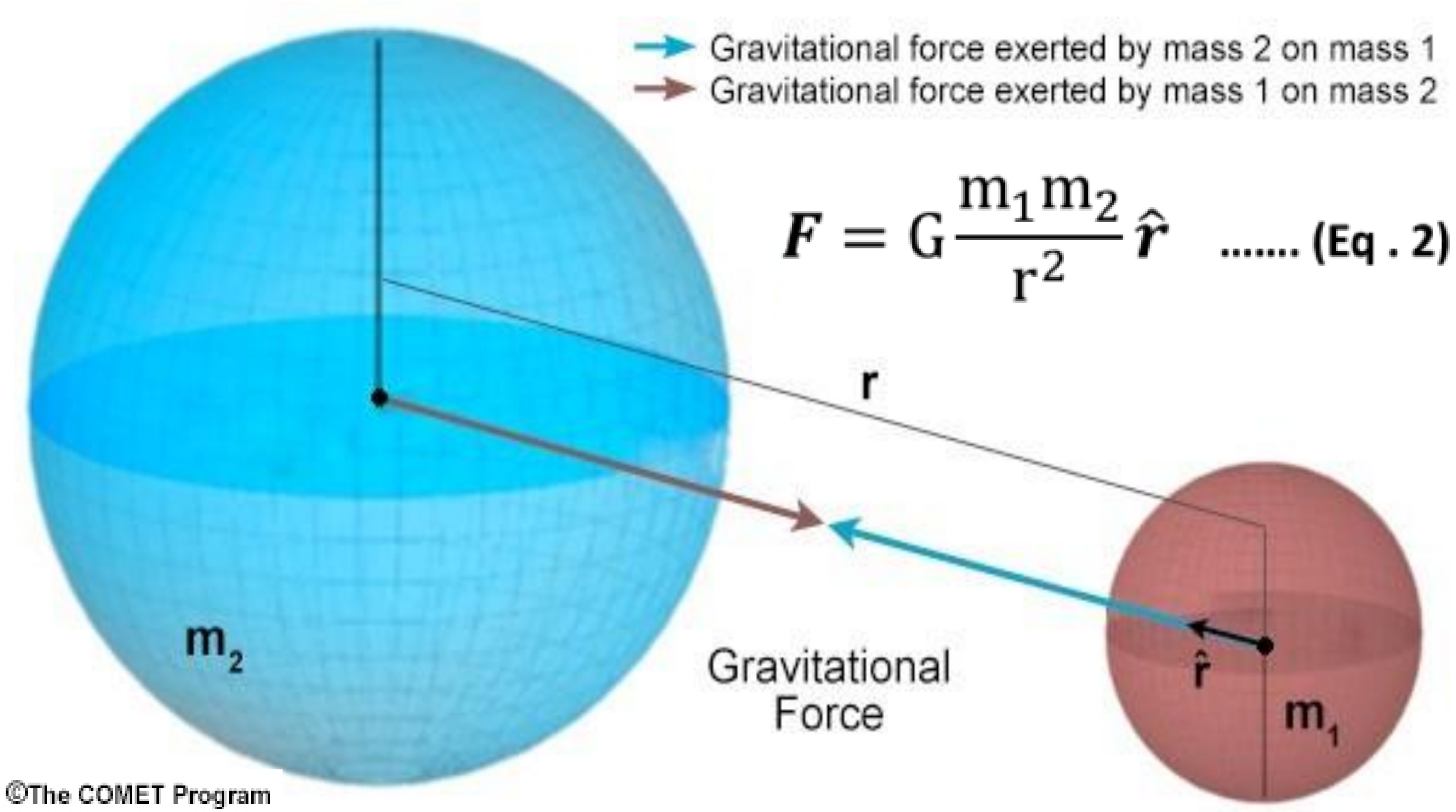
- This equation is accurate if the masses are point masses, i.e.; all the mass of a body is assigned to the point at its center.
- Not valid for real shapes – however accurate for spherical solid bodies of constant density
- For Earth – not good assumption – allow rough estimation



$$F = G\frac{m_1 m_2}{r^2}$$

Contd…

- Eq. 2 is vector form the Eq. 1, direction represented by a unit vector
- For gravitation unit vector points from smaller mass to larger



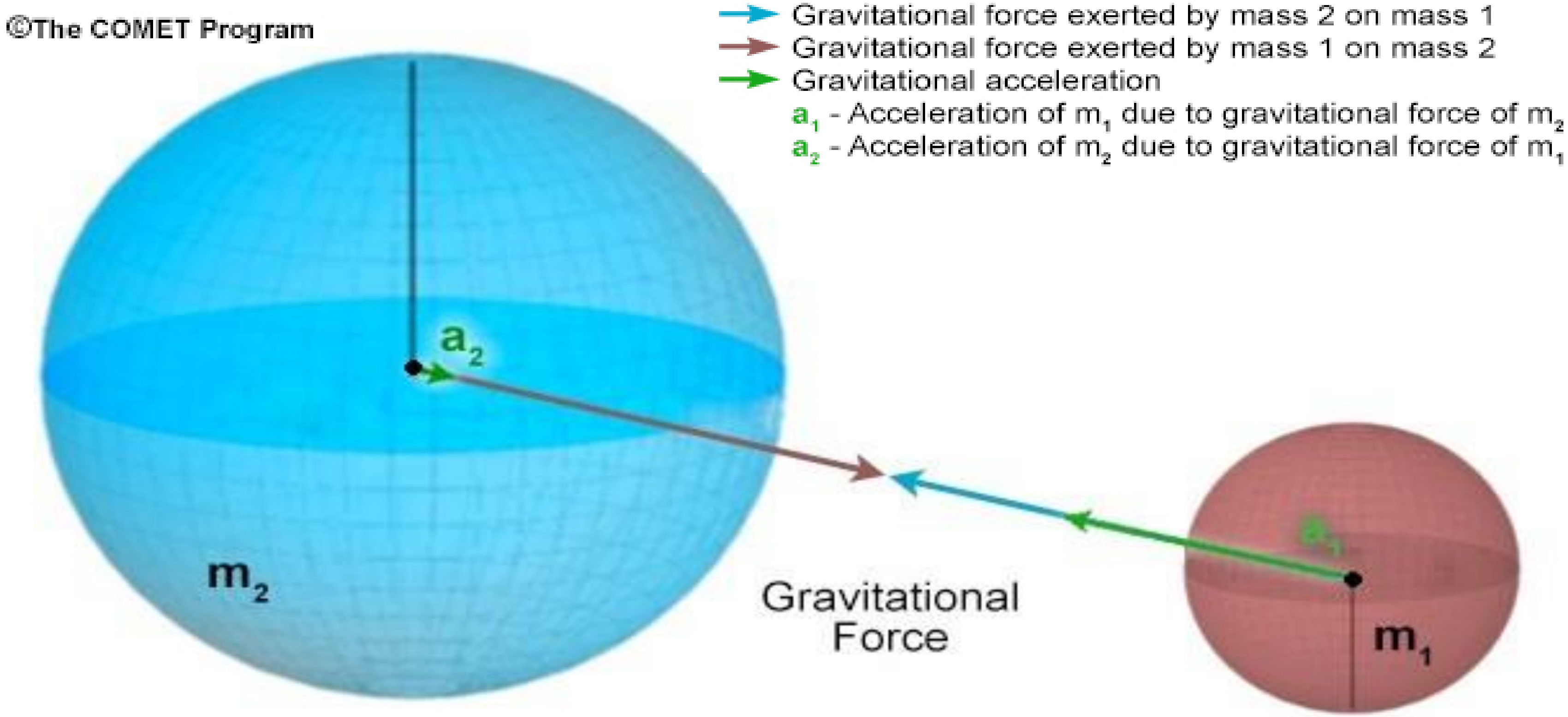
Contd…

- Newton's second law of motion: F = m x a .. (Eq.3)
- Combining equations (2) and (3) for the smaller mass

$$m_1 a_1 = G\frac{m_1 m_2}{r^2}\hat{\boldsymbol{r}} \text{ .... (Eq. 4)}$$

$$\boldsymbol{F} = G\frac{m_1 m_2}{r^2}\hat{\boldsymbol{r}} \text{ .... (Eq . 2)}$$

or, $a_1 = G\frac{m_2}{r^2}\hat{\boldsymbol{r}}$ .... (Eq. 5)

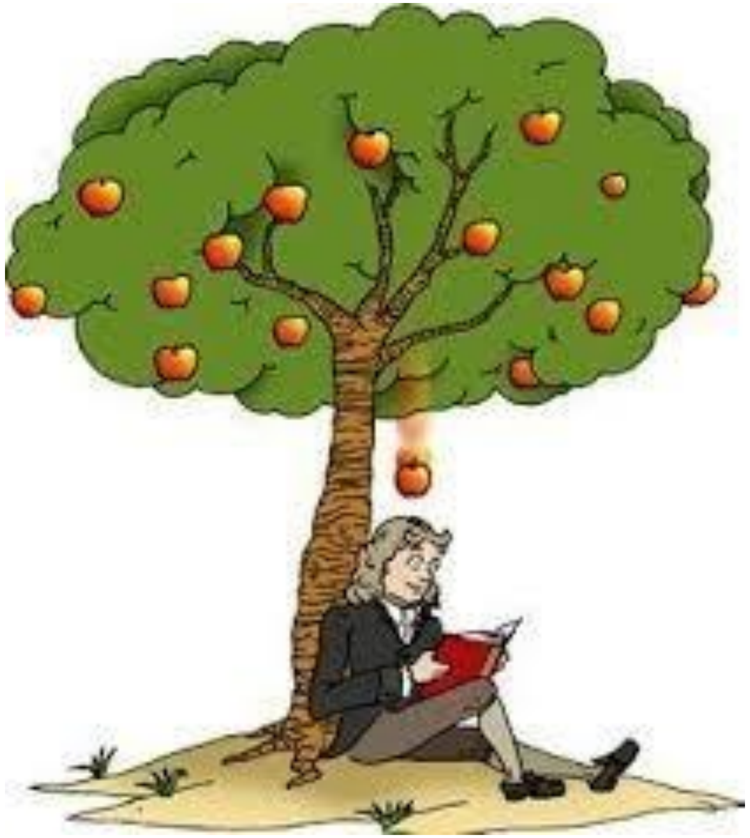


Contd…

- Similarly, $a_2 = G\frac{m_1}{r^2}\hat{\boldsymbol{r}}$ ………. (Eq. 6)

- Refer to equations (5) and (6) implies if m_1 is much less than m_2, so

  a_2 is much less than a_1 and vice versa



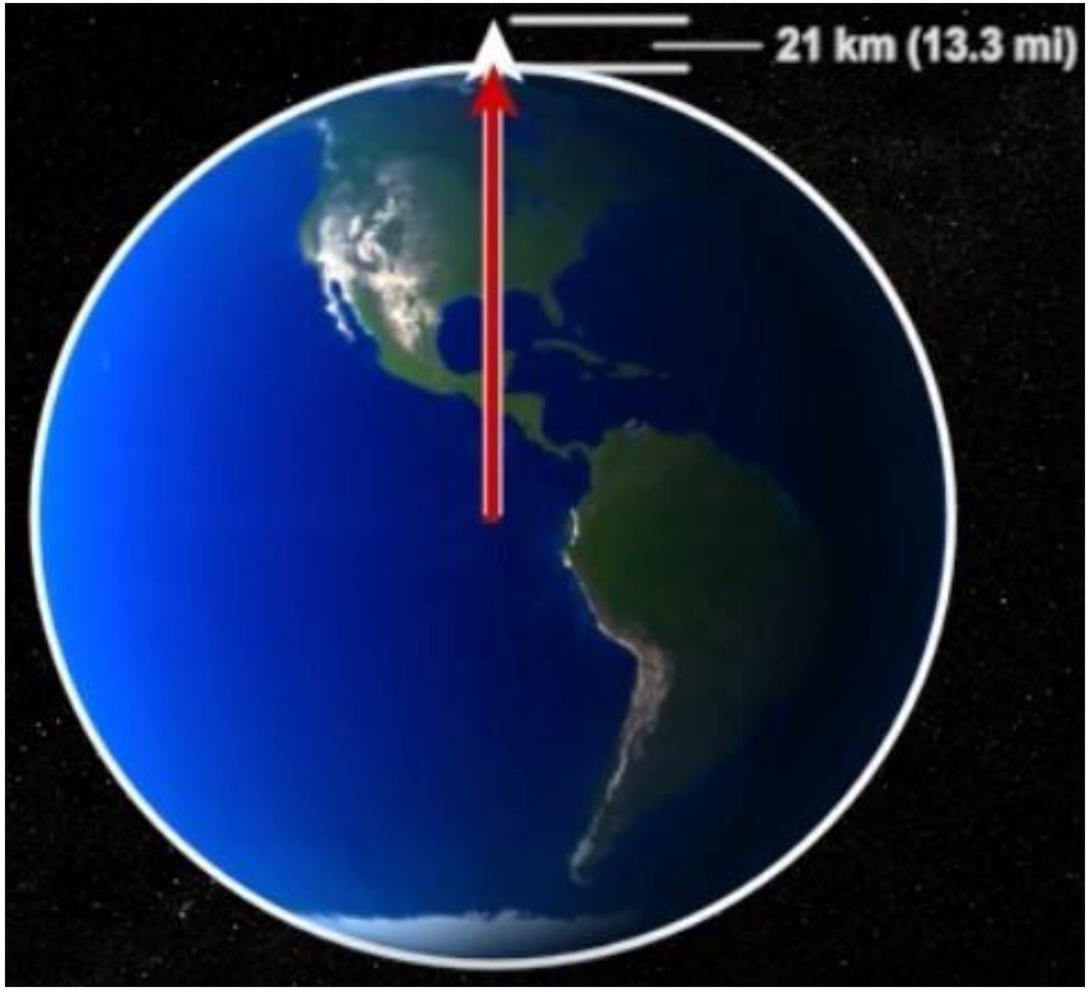
Contd…

- For spherical earth
  - $G = 6.673 \times 10^{-11}\ m^3/s^2kg$
  - $m = 5.972 \times 10^{24}$ kg

    $r = 6,371,000.79$ m
  - a = gravitational acceleration of any object near to the Earth is $9.82\ m/s^2$
- Usually rounded to $10\ m/s^2$ or $9.8\ m/s^2$ directed the Earth's center

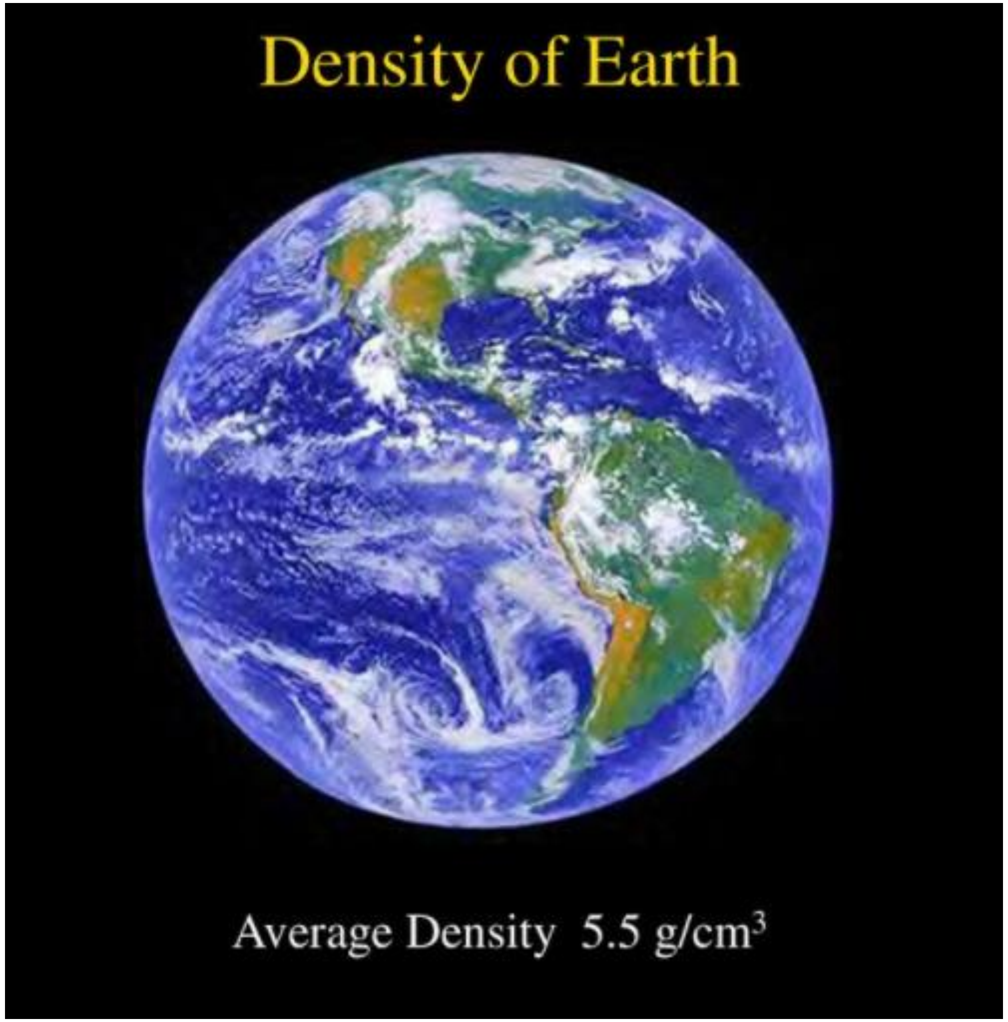
Contd…

- For rough estimate – spherical earth is fine
- Better estimate – with ellipsoid
- Equatorial radius > polar radius
- Difference = 21 km (13.3 miles)
- Two radii – key parameters to define modern ellipsoidal datum
- The difference implies that gravitation acceleration vary with location



Newtonian Gravitational Attraction

- putting $M = \rho * v$
  - ρ: mean density of the Earth
  - v: volume of the Earth
- $g = G \rho v/r^2$
- Taking
  - the mean density of the earth as 5.517 +/- 0.004,
  - g is only dependent on the value of r, the radius of the earth which is larger at equator than at poles



Contd….

- The gravitational attraction has units m/s^2.
- In geodesy, normally the unit Gal, named after Galileo

$$1\,\mathrm{Gal} = 10^{-2}\,\mathrm{m/s^2} = 1\,\mathrm{cm/s^2}$$

$$1\,\mathrm{mGal} = 10^{-5}\,\mathrm{m/s^2}$$

$$1\,\mu\mathrm{Gal} = 10^{-8}\,\mathrm{m/s^2}\,.$$

Contd…

- So the value of g at equator (~978cm/sec^2) is less than that at poles (~983cm/sec^2) at sea levels.
- In the open air above sea level the value of g decreases with height by about 1 gal per 3250 meters.
- The density distribution of the earth is not even due to the topographic as well as the interior structure of the earth so the value of g varies with the variation in density.
- When the observations are made on land the space between the sea level and the ground level is not filled with air but with rock which increases the value of g

So what is Gravity???

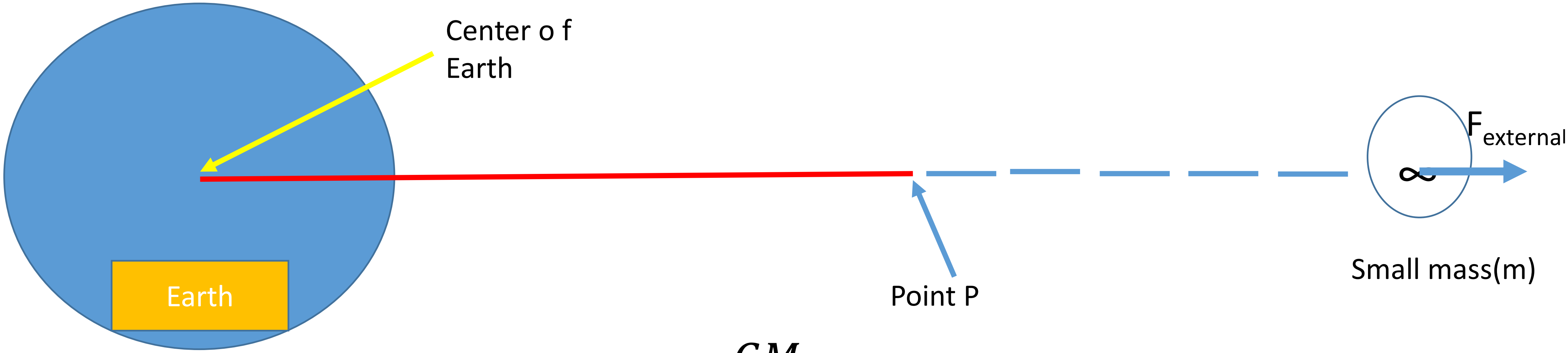
- The Earth's gravity is the sum of gravitational attraction and centrifugal force that the Earth exerts on an object on or near its surface.
  - i.e. *Gravity = gravitational attraction + centrifugal attraction*

- Its strength is quoted as an object acceleration being approximately $9.8 m/s^2$.
- The precise strength and direction of the Earth's gravity vary from point to point

Gravitational Potential

- Gravitational Potential *V* is equal to the work done in taking a unit mass from infinity to a point P without any changes in kinetic energy (i.e speed remains constant).

Why ΔK.E = 0



$$Vp = -\frac{GM}{r}$$

Why -ve

Note:

- Gravitational potential is a scalar quantity

Gravitational acceleration and Gravitational potential

- And to get the gravitational acceleration (b) from the potential we have to take the gradient of the potential (V)

i.e b $= \nabla V \;\; or\ (grad\ V)$

so , $\nabla V(x, y, z) = V\frac{\partial}{\partial x} + V\frac{\partial}{\partial y} + V\frac{\partial}{\partial z}$

Centrifugal acceleration and Centrifugal potential

- The centrifugal force arises as a result of the rotation of the earth about its axis
- Angular velocity ω about the rotational axis of earth
- So, the centrifugal acceleration (z)

$$z = \omega^2 p$$

where, ω = 7.29215 * 10 ^-5 rad/s

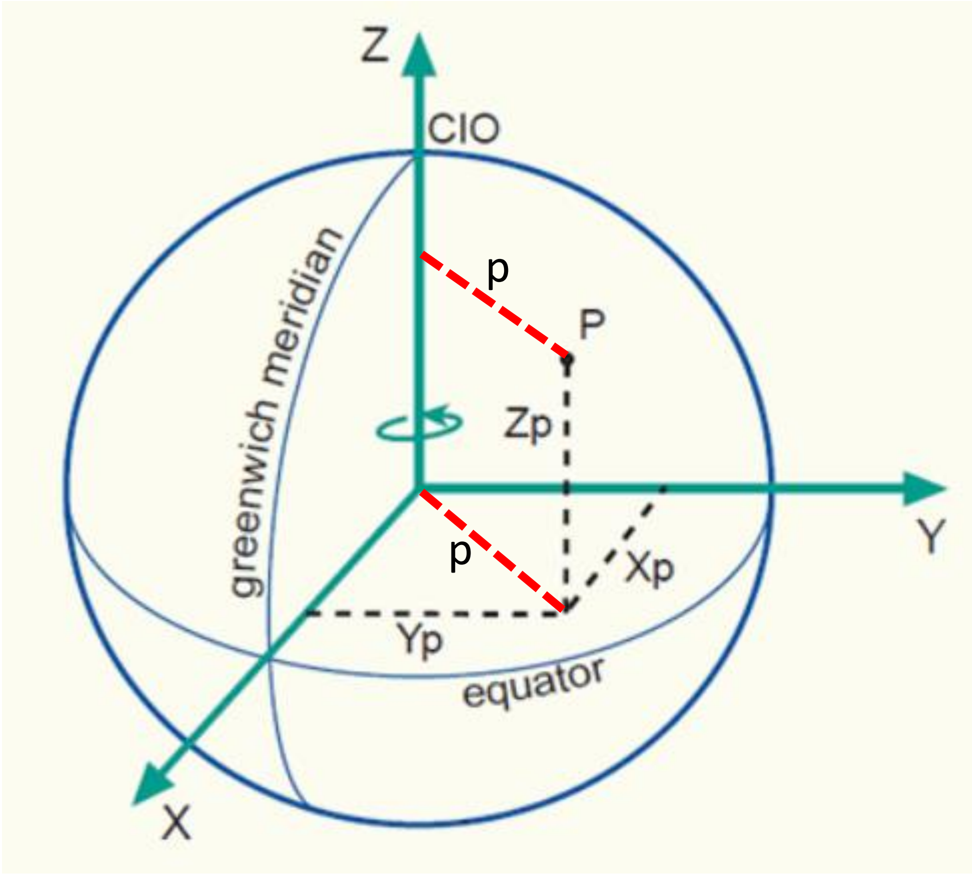
- From fig,

p = $\sqrt{x^2 + y^2}$

- Again, centrifugal acceleration can be obtained by applying gradient over centrifugal potential (Φ)

where, $\Phi = \frac{1}{2}\omega^2 p^2$

Therefore,
z = grad Φ



Fig

Gravity acceleration and Gravity potential

- Gravity acceleration or simply gravity (g) is the resultant of gravitation (b) and centrifugal acceleration (z)

  i.e $g = b+z$

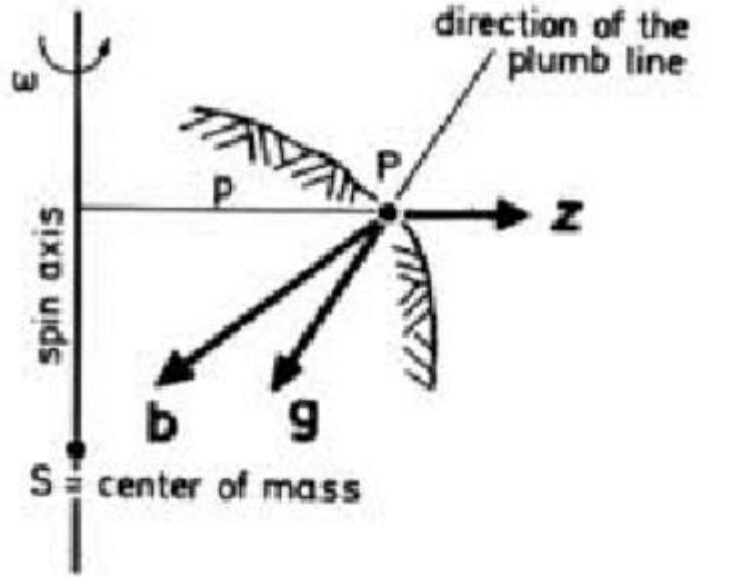


Fig : Gravitation (b), centrifugal acceleration (z), gravity acceleration (g)

- And the gravity potential of the earth is given as

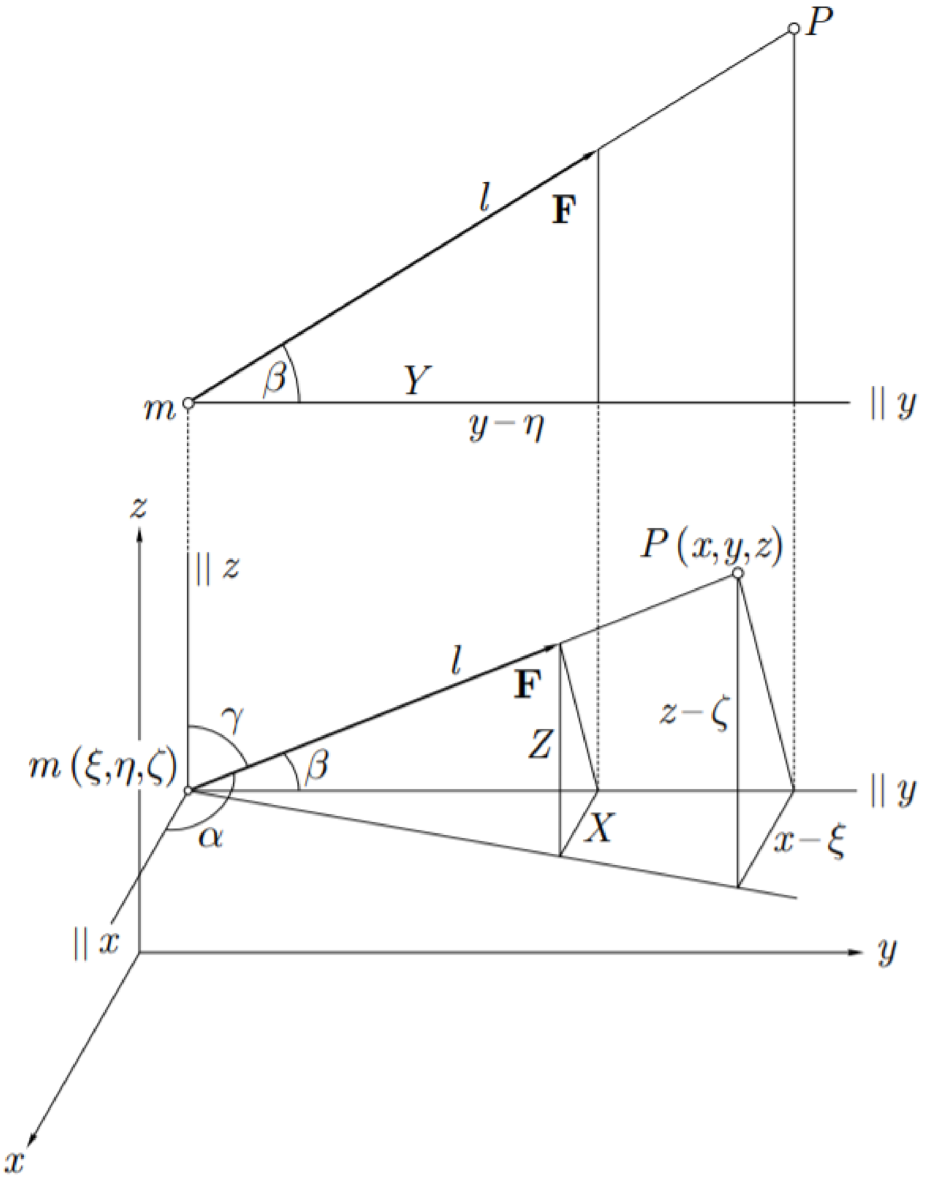
  $W = V + \Phi$

- So applying the gradient over the gravity potential we derive the gravity acceleration

  i.e $g = \text{grad } W$

Gravitational Potential

- We have $F = G\dfrac{m}{l^2}$ Which express the force exerted by a mass m on a body P at a distance of l



- Let the rectangular coordinate system xyz and denote the coordinate of attracting mass m by ξ, η, ζ and the coordinate of attracted point P by x, y, z.
- The component of force F can be represented as

$$X = -F\cos\alpha = -\frac{G\,m}{l^2}\,\frac{x-\xi}{l} = -G\,m\,\frac{x-\xi}{l^3}\,;$$

$$Y = -F\cos\beta = -\frac{G\,m}{l^2}\,\frac{y-\eta}{l} = -G\,m\,\frac{y-\eta}{l^3}\,;$$

$$Z = -F\cos\gamma = -\frac{G\,m}{l^2}\,\frac{z-\zeta}{l} = -G\,m\,\frac{z-\zeta}{l^3}\,,$$

where,

$$l = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$$

Contd....

- Now we have $V = \frac{G\,m}{l}$ which is called Gravitational potential.
- the component of X, Y, Z of the gravitational force F are then given by

$$X = \frac{\partial V}{\partial x}, \quad Y = \frac{\partial V}{\partial y}, \quad Z = \frac{\partial V}{\partial z}$$

- In vector notation it can be represented as,

$$\mathbf{F} = [X,\ Y,\ Z] = \operatorname{grad} V$$

- Thus if we have a system of several point masses m1, m2,……….mn, the potential of the system is the sum of the individual contribution

$$V = \frac{G\,m_1}{l_1} + \frac{G\,m_2}{l_2} + \cdots + \frac{G\,m_n}{l_n} = G\sum_{i=1}^{n}\frac{m_i}{l_i}$$

Contd….

- If we assume that point masses are distributed continuously over a volume v with density $\varrho = \frac{dm}{dv}$

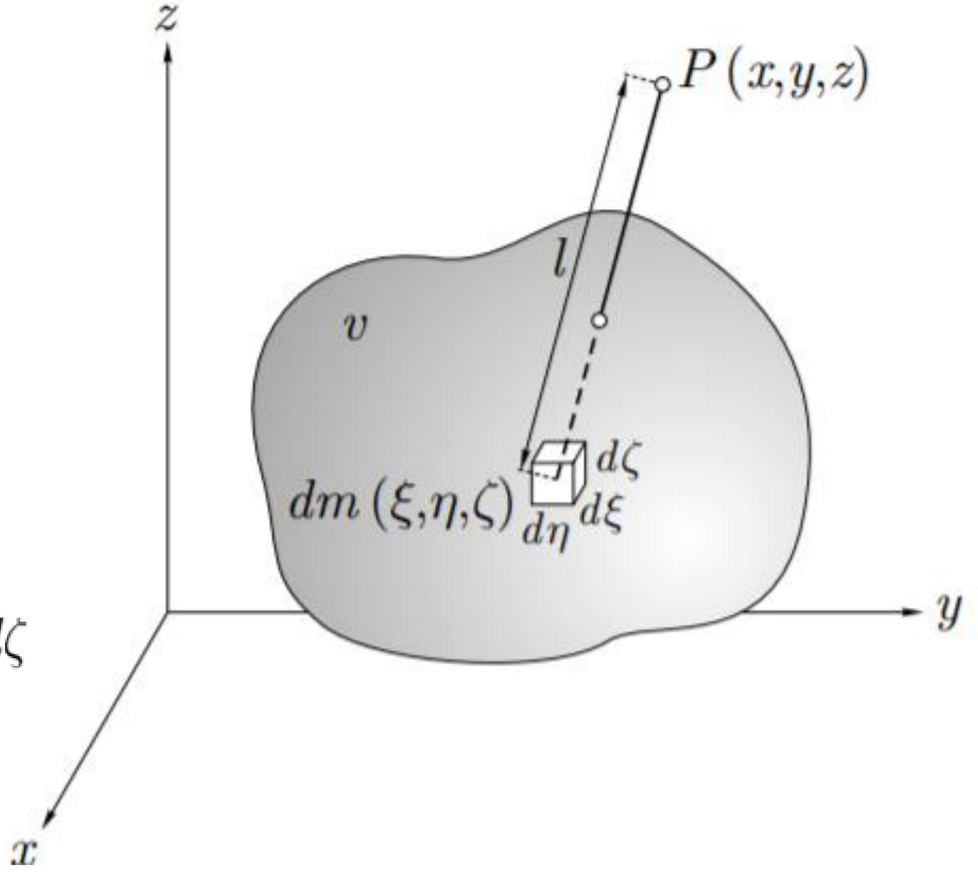
Where dv is an element of volume and dm is an element of mass

- Then total contribution become ,

$$V = G \iiint_v \frac{dm}{l} = G \iiint_v \frac{\varrho}{l}\, dv$$

- It can be written as

$$V(x,y,z) = G \iiint_v \frac{\varrho(\xi,\eta,\zeta)}{\sqrt{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2}}\, d\xi\, d\eta\, d\zeta$$



Contd…

- The component of force can be represented as,

$$X = \frac{\partial V}{\partial x} = G\,\frac{\partial}{\partial x}\iiint\limits_{v} \frac{\varrho(\xi,\eta,\zeta)}{l}\,d\xi\,d\eta\,d\zeta$$

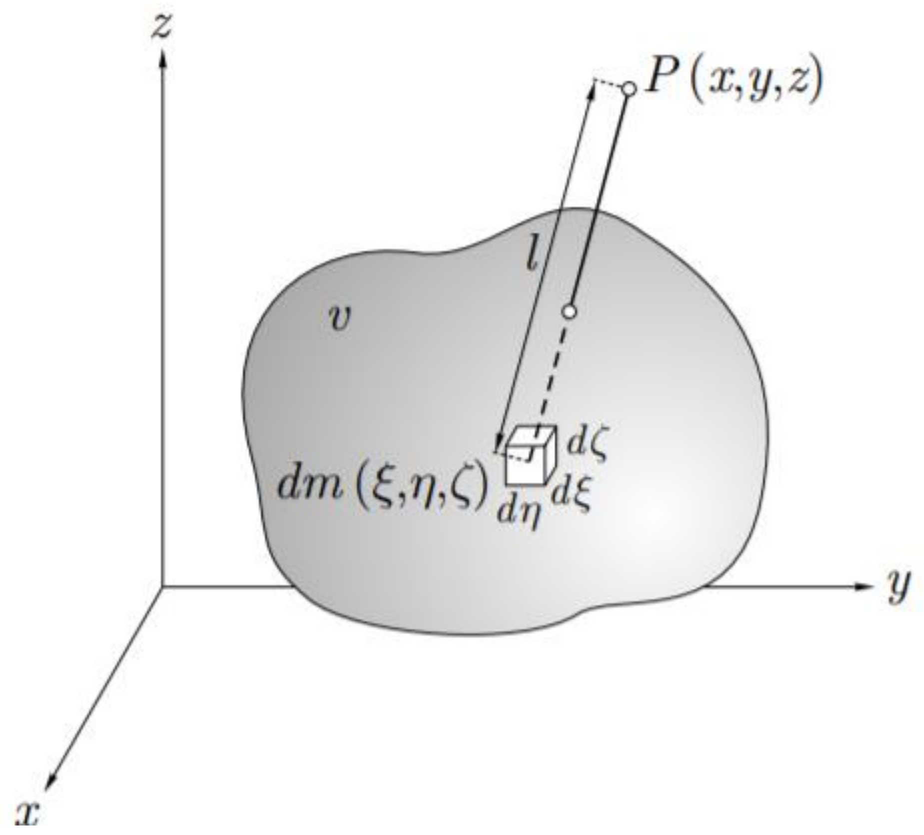
$$= G\iiint\limits_{v} \varrho(\xi,\eta,\zeta)\,\frac{\partial}{\partial x}\left(\frac{1}{l}\right)d\xi\,d\eta\,d\zeta\,.$$

- The portion $\frac{\partial}{\partial x}\left(\frac{1}{l}\right)$ on differentiation yields

$$\frac{\partial}{\partial x}\left(\frac{1}{l}\right) = -\frac{1}{l^2}\,\frac{\partial l}{\partial x} = -\frac{1}{l^2}\,\frac{x-\xi}{l} = -\frac{x-\xi}{l^3}$$

- So,

$$X = -G\iiint\limits_{v} \frac{x-\xi}{l^3}\,\varrho\,dv$$



Contd....

- The potential V is continuous throughout the whole space and vanishes at infinity i.e $\frac{1}{l}$ for l tends to infinity.

- The first derivative of V, that is the force components are also continuous throughout the space, but on calculating the second derivative it is not so because density changes discontinuously.
- Thus in such a case potential V is represented by Poisson's equation

$$\Delta V = -4\pi\, G\, \varrho\,,$$

Where Δ is a called a laplacian operator and it has a form of

$$\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Contd…

- So, the piosson's equation

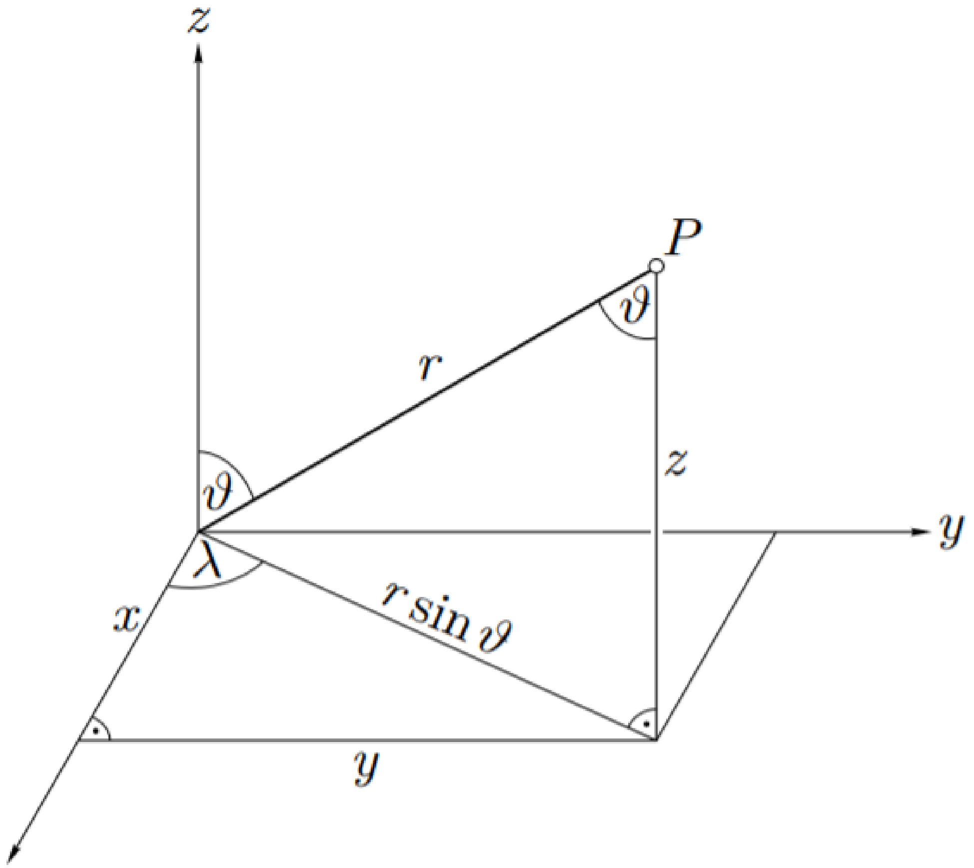
$$\Delta V = -4\pi\, G\, \varrho\,,$$

$$\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

- It is satisfied when second derivative of potential V is discontinuous with density ϱ
- But outside the attracting body in empty space the density is zero thus we get, $\Delta V = 0$
- This is called Laplace's equation. Its solution is called harmonic function
- Thus the potential of gravitation is a harmonic function outside the attracting masses but not inside the masses, there it satisfies Poisson's equation.

Contd…

- The spherical harmonics considering the spherical coordinate



- Let the spherical coordinate r, v, **λ**
- From figure,

$$x = r \sin \vartheta \cos \lambda$$

$$y = r \sin \vartheta \sin \lambda \,;$$

$$z = r \cos \vartheta \,;$$

- We have laplace equation $\Delta V = 0$

$$\Delta V \equiv \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \vartheta^2} + \frac{\cot \vartheta}{r^2}\frac{\partial V}{\partial \vartheta} + \frac{1}{r^2 \sin^2 \vartheta}\frac{\partial^2 V}{\partial \lambda^2} = 0$$