Phys 512 Assignment 8

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1. We begin with the leapfrog scheme

$$\frac{f(t+dt,x)-f(t-dt,x)}{2dt} = -v\frac{f(t,x+dx)-f(t,x-dx)}{2dx}, \tag{1}$$

which may be written as

$$f(t+dt,x) = f(t-dt,x) - v\frac{dt}{dx}\left(f(t,x+dx)-f(t,x-dx)\right), \tag{2}$$

where we may define $\alpha \equiv v\frac{dt}{dx}$ as the CFL condition. Substituting $f(t,x)=\xi^t e^{ikx}$, we find

$$\xi^{t+dt}e^{ikx} = \xi^{t-dt}e^{ikx} - \alpha(\xi^t e^{ik(x+dx)} - \xi^t e^{ik(x-dx)}), \tag{3}$$

and dividing by $\xi^t e^{ikx}$,

$$\xi^{dt} = \xi^{-dt} - \alpha(e^{ikdx} - e^{-ikdx}), \tag{4}$$

which further reduces to

$$\xi^{dt} = \xi^{-dt} - 2i\alpha\sin(kdx). \tag{5}$$

Multiplying by ξ^{dt}, we find

$$\xi^{2dt} + \xi^{dt}2i\alpha\sin(kdx) - 1 = 0, \tag{6}$$

which is a quadratic with solutions

$$\xi^{dt} = \frac{-2i\alpha\sin(kdx) \pm \sqrt{-4\alpha^2\sin^2(kdx)+4}}{2}. \tag{7}$$

This reduces to

$$\xi^{dt} = -i\alpha\sin(kdx) \pm \sqrt{1-\alpha^2\sin^2(kdx)}. \tag{8}$$

Where could this go crazy? $\sin^2(kdx)$ has a maximum value of 1. If we have $\alpha > 1$, there will be some instances (some values of k) in which $\alpha^2\sin^2(kdx) > 1$. In this case, ξ^{dt} is entirely imaginary:

$$\xi^{dt} = i\left(-\alpha\sin(kdx) \pm \sqrt{\alpha^2\sin^2(kdx)-1}\right), \tag{9}$$

and its modulus is

$$|\xi^{dt}| = -\alpha \sin(kdx) \pm \sqrt{\alpha^2 \sin^2(kdx) - 1}. \tag{10}$$

Since $\alpha > 1$, $\sqrt{\alpha^2 \sin^2(kdx) - 1} > 1$ for some values of k, suggesting that $|\xi^{dt}| > 1$. This leads to a solution that continues to grow beyond our control over time! Therefore, the leapfrog scheme only conserves energy if the CFL condition is satisfied, where $\alpha \leq 1$.

2. Unforunately, I had many conflicts since this problem set was assigned that prevented me from giving any time to this question, namely a project worth %40 of my grade in another class and preparation for a research trip. I understand and accept that this will be marked as a 0, although I expect this will become my dropped assignment. Thanks for understanding!