Controls Inverted Pendulum Stability Analysis

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Abstract

Chaotic and unpredictable behavior can be a fundamental element of real-world events and thereby a unique challenge is posed to mathematical modeling and controls specialists. Specifically, the response of these systems is largely suspect to their initial conditions and small mechanical variations during operation. Which can drastically impact the expected response. Therefore, building models to describe the behavior of more complicated networks can reveal methods to control the response of chaotic systems. As a result we consider a particular problem which places an inverted vertical pendulum on a moving cart. Such that, an input force is applied to the cart which causes both the cart to move and the pendulum to oscillate. Specifically, we approach a comprehensive dynamic model of both the motion of the cart and the position of the pendulum center of mass. Additionally, we allow real world constraints to our model and attempt to stabilize the theoretical adaption of the physical system under some characteristic parameters. Ultimately, leading to a method of designing a controller capable of stabilizing both the carts position and the pendulums center of mass.

Introduction: The Inverted Pendulum Problem

Throughout our experiment we consider the effect of chaotic oscillatory motion on control system analysis. Specifically, we consider a particular inverted pendulum problem which places a rigid-body pendulum on top of a moving cart (figure 1). Such that, for any force input f_{ext} applied to the cart, via a motor placed on a horizontal plane, we will be able to model both the position of the cart (x) and center of mass of the pendulum system (P). Therefore, we desire a functional response $F : f_{ext} \to (x, P)$. Which also implies our system will be single input multi-output (SIMO) and so when designing a control system $(G_c(s))$ we will need to account its impact on both transfer functions. More generally to the problem specifications, we consider the pendulum bob to be attached to the cart by a rigid bar such that the displacement from the cart to the center of mass of the pendulum is ℓ. And so we see system diagram 1:

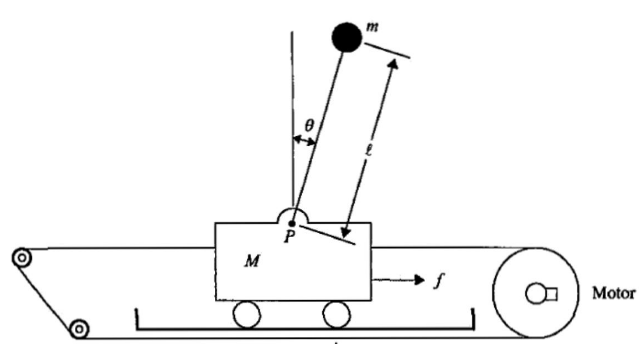

Figure 1: System Diagram of the Inverted Pendulum Problem.

Now because the input force will affect both the position of the cart and pendulum we intend to design a controller capable of stabilizing both positions under the following characteristics.

$$\max\theta < 0.05 radians \quad \text{Maximum displacement of the pendulum from equilibrium} \tag{1}$$
$$t_s < 5 seconds \quad \text{Settling time for both mass and pendulum} \tag{2}$$
$$t_r < 0.5 seconds \quad \text{Rise time for the cart} \tag{3}$$
$$M_p < 20 degrees \quad \text{Overshoot of the pendulum} \tag{4}$$

Returning to our dynamic model, we assume the rigid bar connecting the pendulum bob to the cart has 'simple' rotation, such that we neglect friction force from the cart to the bob. Additionally, we assume the mass of the pendulum system and distance from the cart to center of mass is constant throughout the motion and as a result we can define both the mass of the pendulum and length to center of mass as rigid bodies. We also note this type of system is highly subject to small mechanical variations and as a result we consider building our dynamic equations of motion by the Euler-Lagrangian Relation. However, we also note that the same equations could be build by taking sum of forces for each system element using Newton's Second Law.

Procedure

First allow the position of the cart on some horizontal plane to be called x, such that rightward motion is in the positive direction and leftward motion is negative. Now consider a physical pendulum with mass (m) and center of mass from the cart (ℓ). Additionally, call the angle made by the wire and vertical equilibrium point θ. (see figure 1). Next allow for a theoretical equilibrium point $\theta = 0$ to be directly above the carts current location; such that, if we take x to be the carts location, $\theta_0 = x$ or for any x the equilibrium position θ_0 is equal to x. Now consider the motion of the pendulum is in an arc centered along a vertical line across the systems equilibrium point, then we can define a reference axis by the arc created my the motion, such that, the position of the pendulum is equal to the surface area of the arc from the equilibrium point. Additionally, we consider motion to the right of the equilibrium to be position while motion to the left to be negative surface area. Therefore, we can define the position of the mass relative to time by,

$$S = \ell\theta(t).$$

Therefore, if we allow P to be the exact position of the pendulum then, we must also account for the position of the cart:

$$P = S + x \longrightarrow P = \ell\theta(t) + x \tag{5}$$

And so we arrive at the physical system model:

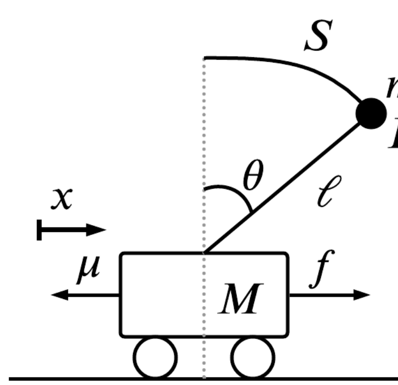

Figure 2: System Model: Inverted Pendulum Problem with updated axis.

Additionally, we consider the following constraints:

$$M = 0.5kg \qquad \text{Mass of the cart,} \tag{6}$$
$$m = 0.2kg \qquad \text{Mass of the pendulum,} \tag{7}$$
$$\mu = 0.1N/m/sec \qquad \text{Friction of the cart,} \tag{8}$$
$$\ell = 0.3m \qquad \text{Length of the pendulum center of mass,} \tag{9}$$
$$I = 0.006kgm^2 \qquad \text{Inertia of the pendulum.} \tag{10}$$

Now we approach a dynamic model for the motion of both of the cart and pendulum system. Importantly, we note that it is assumed the pendulum is friction-less and therefore is a closed energy system. Therefore, we can use the Euler-Lagrange Theorem to estimate the position of the bob in relation to some force input on the cart.

Euler-Lagrange Theorem for 2-dimensions

Consider the behavior of a closed physical system with arbitrarily small fluctuations in sub-optimal paths between beginning and final states must obey a differential equation arriving from the total derivative of the Lagrangian, a function which can describe the difference between kinetic and potential energy,

$$\begin{aligned}
&\exists L(x,y,z), \quad L = T - U, \\
&\qquad\qquad \textit{Where } T \textit{ is kinetic energy and } U \textit{ is potential energy} \\
&\textit{Under some constraint, } \lambda : (y,z) \longrightarrow y, \\
&\textit{Then the total derivative of } L \textit{ defines,} \\
&\qquad\qquad \frac{d}{dx}\frac{\partial L}{\partial y'} - \frac{\partial L}{\partial y} = 0
\end{aligned}$$

Additionally, we note that the cart is subject to some friction force pulling energy out of the system and so it does not meet the 'closed' requirements for the Lagrangian. However, we can consider the Rayleigh Dissipation function, such that under some net energy loss within a system Q_i, we can define the change to the system by some axis q, st. $Q_i = -\frac{\partial F}{\partial \dot{q}_i}$. Which allows us to compute our Lagrangian,

Rayleigh Dissipation function

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \frac{\partial F}{\partial \dot{q}_i} = 0 \tag{11}$$

And so we can consider the change on the system energy by the difference between any two states of motion, $E_0 - E_t$, $E = energy$. Such that for an initial force input f, assuming the cart is initially at rest, we see that, $E_0 = f$ then any extraneous force, caused by friction must influence the total energy. So if we consider friction to be an opposing force with respect to the velocity of the cart, $f_{friction} = \mu(\dot{x})$ then we arrive at an expression for the energy leaving the system:

$$Q_i = -\frac{\partial F}{\partial \dot{q}_i} = f - \mu\dot{x} \tag{12}$$

As a result of the above, we can redefine our system without friction or initial input, and therefore, as closed. This allows us to compute the total derivative of the Lagrangian by finding equations for kinetic and potential energy. And so using standard energy equations we see that:

Kinetic Energy

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\dot{P}^2 + \frac{1}{2}I\dot{\theta}^2 \tag{13}$$

$$st. \quad P = x + \ell\theta \longrightarrow \dot{P} = \dot{x} + \ell\dot{\theta}$$

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x} + \ell\dot{\theta})^2 + \frac{1}{2}I\dot{\theta}^2$$

$$T = \frac{1}{2}(M+m)\dot{x}^2 - \ell m\dot{x}\dot{\theta} + \frac{1}{2}m\ell^2\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2. \tag{14}$$

Now to find potential energy we make the assumption that the cart is initially at rest, and that there is sufficient normal force to gravity on the platform to neglect vertical force on the cart. Additionally, we note by our choice of P we can define the vertical component of the pendulum by $P_y = \ell cos(\theta)$.

Potential Energy

$$U = mgP_y = mg\ell cos(\theta) \tag{15}$$

Now we can define our closed system Lagrangian:

$$L = T - U$$

$$L = \frac{1}{2}(M+m)\dot{x}^2 - \ell m\dot{x}\dot{\theta} + \frac{1}{2}m\ell^2\dot{\theta}^2 + \frac{1}{2}I\dot{\theta}^2 - mg\ell cos(\theta) \tag{16}$$

Now approaching a relationship for the pendulum system, we consider the total derivative of the Lagrangian with respect to θ.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0 \tag{17}$$

$$-m\ell\ddot{x} + m\ell^2\ddot{\theta} + I\ddot{\theta} - mg\ell sin(\theta) = 0$$

$$\underline{(I + m\ell^2)\ddot{\theta} - mg\ell sin(\theta) = m\ell\ddot{x}} \tag{18}$$

Importantly, because we only want linear time-invariant models for our system and because we only allow for small displacement of the pendulum from equilibrium as part of our initial constraints, we linearize the above expression using a small angle approximation.

Small Angle Approximation (19)

For $\theta \approx 0 : \qquad sin(\theta) \approx \theta$

So we arrive at the following approximation:

$$\underline{(I + m\ell^2)\ddot{\theta} - mg\ell\theta = m\ell\ddot{x}} \tag{20}$$

Now we must also consider the total derivative of the Lagrangian with respect to x; however, because this system was not originally closed we include the Rayleigh Dissipation function adaption.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} + \frac{\partial F}{\partial \dot{x}} = 0 \tag{21}$$

$$(M+m)\ddot{x} - m\ell\ddot{\theta} + \underbrace{\mu x - f}_{-\frac{\partial F}{\partial \dot{x}}} = 0$$

$$\underline{(M+m)\ddot{x} + \mu\dot{x} - m\ell\ddot{\theta} = f} \tag{22}$$

And so we arrive at the following dynamic equations of motion:

$$\begin{cases} (I + m\ell^2)\ddot{\theta} - mg\ell\theta = m\ell\ddot{x} \\ (M+m)\ddot{x} + \mu\dot{x} - m\ell\ddot{\theta} = f \end{cases} \tag{23}$$

Critically, we want to find some controller that will be able to manipulate the input signal enough to deliver some desired output, for both the cart and pendulum. It is more convenient to design these controllers in the frequency domain, so approach transfer functions, $\frac{X(s)}{F(s)}$ and $\frac{\Theta(s)}{F(s)}$.

Taking the Laplace transfer of the above equations:

$$\begin{cases}(I+m\ell^2)s^2\Theta - mg\ell\Theta = m\ell s^2 X\\(M+m)s^2X + \mu sX - m\ell s^2\Theta = F\end{cases} \tag{24}$$

Now simplifying the first equation we see that,

$$X(s) = \Big[\frac{(I+m\ell^2)}{m\ell} - \frac{g}{s^2}\Big]\Theta(s)$$

Substituting into the second equation:

$$(M+m)s^2\Big[\frac{(I+m\ell^2)}{m\ell} - \frac{g}{s^2}\Big]\Theta(s) + \mu s\Big[\frac{(I+m\ell^2)}{m\ell} - \frac{g}{s^2}\Big]\Theta(s) - m\ell s^2\Theta(s) = F(s) \tag{25}$$

We arrive at the transfer function,

$$\frac{\Theta(s)}{F(s)} = \frac{m\ell s}{[I(M+m)+Mm\ell^2]s^3 + \mu(I+m\ell^2)s^2 - gm\ell(M+m)s - \mu gm\ell} \tag{26}$$

And again we note that,

$$X(s) = [\frac{(I+m\ell^2)s^2 - gm\ell}{m\ell s^2}]\Theta(s) \longrightarrow \Theta(s) = \frac{m\ell s^2}{(I+m\ell^2)s^2 - gm\ell}X(s)$$

Therefore, we get a transfer function for the cart:

$$\frac{X(s)}{F(s)} = \frac{(I+m\ell^2)s^2 - gm\ell}{s([I(M+m)+Mm\ell^2]s^3 + \mu(I+m\ell^2)s^2 - gm\ell(M+m)s - \mu gm\ell)} \tag{27}$$

Now, we have two transfer functions relating an input force $F(s)$ to the position of the cart and pendulum.

$$\begin{cases}Plant_{Pendulum}: & \frac{\Theta(s)}{F(s)} = \frac{m\ell s}{[I(M+m)+Mm\ell^2]s^3+\mu(I+m\ell^2)s^2-gm\ell(M+m)s-\mu gm\ell}\\ \\ Plant_{Cart}: & \frac{X(s)}{F(s)} = \frac{(I+m\ell^2)s^2-gm\ell}{s([I(M+m)+Mm\ell^2]s^3+\mu(I+m\ell^2)s^2-gm\ell(M+m)s-\mu gm\ell)}\end{cases} \tag{28}$$

And substituting initial constraints,

$$\begin{cases}Plant_{Pendulum}: & \frac{\Theta(s)}{F(s)} = \frac{0.06s}{0.0132s^3+0.0024s^2-0.42s-0.06}\\ \\ Plant_{Cart}: & \frac{X(s)}{F(s)} = \frac{0.024s^2-0.6}{s(0.0132s^3+0.0024s^2-0.42s-0.06)}\end{cases} \tag{29}$$

Now it becomes possible to design a controller for the pendulum system such that for some input impulse force, the pendulums displacement from equilibrium θ does not exceed $0.05rad \longrightarrow \forall f, \theta \leq 0.05rads$. Additionally, to prevent oscillatory behavior and ensure the pendulum returns to vertical position we impose a restraint on the settling time: $t_s < 5seconds$.
Importantly, we first consider the stability of the pendulum transfer function without the affect of a controller. By finding the location of the poles we see that;

$$\begin{aligned}&Plant_{Pendulum}: \quad \frac{\Theta(s)}{F(s)} = \frac{0.06s}{0.0132s^3+0.0024s^2-0.42s-0.06}\\&\text{Poles} \longrightarrow p_i = -5.66, -0.143, 5.62\end{aligned} \tag{30}$$

Noting the pole position at s = 5.62, we see that the system is originally unstable, and so to find a stable controller we must be able to cancel the effect or move this pole to the left half s-plane. This can be accomplished by making small updates to the force on the cart f for any current pendulum position. And so we place a PID controller in the feedback loop of the closed system $G_c(s)$ with equation,

$$\text{PID} \qquad G_c(s) = k_p + \frac{k_i}{s} + k_d s \tag{31}$$

For unknown coefficients k_p, k_i, k_d. With the following implementation; (figure 3).

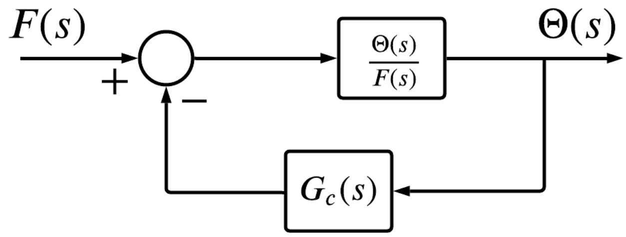

Figure 3: Pendulum Response with controller unit and feedback loop

Which results in a closed loop transfer function,

$$T(s) = \frac{\frac{\Theta(s)}{F(s)}}{1 + G_c(s)\frac{\Theta(s)}{F(s)}}$$

Now we can tweak the values k_p, k_i, k_d to find a stable model that meets the system restraints to an impulse input. Because of the complexity of the calculations this is best done using some modeling software, we consider the use of MATLAB's PID Tuning toolbox, which gives the following coefficients.

$$k_p = 80 \qquad k_i = 1 \qquad k_d = 20 \tag{32}$$

Which results in the following impulse response;

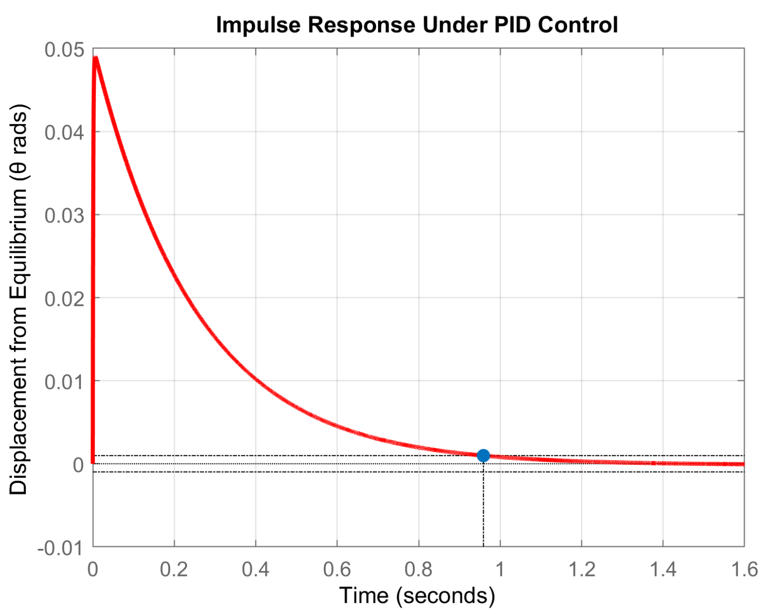

Figure 4: Impulse Response Plot: Pendulum response due to feedback controller

Importantly, this design only accounts for controlling the pendulum by applying some force to the cart; and therefore, if implemented will not allow the cart to move freely. I.e. if we desire for the cart to maneuver to some x coordinate then stop all while ensuring the motion of the pendulum meets the system constraints we will need to design a different controller. One capable of controlling both the cart and pendulum simultaneously. Which results in a single input multiple output (SIMO) design. Importantly, this will be difficult with anything similar to our current model (figure 3) because the feedback loop will need information about both the pendulum and cart. As a result we consider a full state controller that feeds the displacement of the pendulum, position of the cart, and their respective velocities back as states. This will allow us to design a proportional gain controller K capable of directly manipulating the poles of the system, thereby affecting stability and response characteristics. Additionally, we note that we implement this controller in the time domain using a state-space model for the original dynamic equations of motion. And so we consider the state-space model:

$$\dot{\boldsymbol{w}} = \boldsymbol{A}\boldsymbol{w} + \boldsymbol{B}\boldsymbol{f} \tag{33}$$
$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{w} + \boldsymbol{D}\boldsymbol{f} \tag{34}$$

for A, B as transition matrix's, f as an input variable, or the force on the cart, and $\dot{w}, w$ as states.(Note: we take state variable dummies to be w in replace of the standard x because of our choice of cart axis label.)
Therefore, we can define;

$$w = \begin{pmatrix} \theta \\ x \\ \dot{\theta} \\ \dot{x} \end{pmatrix} \longrightarrow \dot{w} = \begin{pmatrix} \dot{\theta} \\ \dot{x} \\ \ddot{\theta} \\ \ddot{x} \end{pmatrix} \tag{35}$$

Now we desire to find some matrix A, B that satisfy the state-space equation given states, $w, \dot{w}$. And so we take our dynamic equations of motion and find an expression for $\ddot{\theta}, \ddot{x}$;

$$(I + m\ell^2)\ddot{\theta} - mg\ell\theta = m\ell\ddot{x}$$
$$(M + m)\ddot{x} + \mu\dot{x} - m\ell\ddot{\theta} = f$$
$$\downarrow$$

$$\ddot{\theta} = \frac{m\ell}{I(M+m) + Mm\ell^2} f + \frac{mg\ell(M+m)}{I(M+m) + Mm\ell^2}\theta - \frac{\mu m\ell}{I(M+m) + Mm\ell^2}\dot{x}$$
$$\ddot{x} = \frac{g(m\ell)^2}{I(M+m) + Mm\ell^2}\theta - \frac{\mu(I + m\ell^2)}{I(M+m) + Mm\ell^2}\dot{x} + \frac{I + m\ell^2}{I(M+m) + Mm\ell^2} f$$

Which by inspection leads to transition matrix, A, B:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{mg\ell(M+m)}{I(M+m)+Mm\ell^2} & 0 & 0 & -\frac{\mu m\ell}{I(M+m)+Mm\ell^2} \\ \frac{g(m\ell)^2}{I(M+m)+Mm\ell^2} & 0 & 0 & -\frac{\mu(I+m\ell^2)}{I(M+m)+Mm\ell^2} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \frac{m\ell}{I(M+m)+Mm\ell^2} \\ \frac{I+m\ell^2}{I(M+m)+Mm\ell^2} \end{bmatrix} \tag{36}$$

Which leads to state-space equation:

$$\begin{pmatrix} \dot{\theta} \\ \dot{x} \\ \ddot{\theta} \\ \ddot{x} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{mg\ell(M+m)}{I(M+m)+Mm\ell^2} & 0 & 0 & -\frac{\mu m\ell}{I(M+m)+Mm\ell^2} \\ \frac{g(m\ell)^2}{I(M+m)+Mm\ell^2} & 0 & 0 & -\frac{\mu(I+m\ell^2)}{I(M+m)+Mm\ell^2} \end{bmatrix} \begin{pmatrix} \theta \\ x \\ \dot{\theta} \\ \dot{x} \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{m\ell}{I(M+m)+Mm\ell^2} \\ \frac{I+m\ell^2}{I(M+m)+Mm\ell^2} \end{bmatrix} f \tag{37}$$

And substituting initial conditions

$$\begin{pmatrix} \dot{\theta} \\ \dot{x} \\ \ddot{\theta} \\ \ddot{x} \end{pmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 31.18 & 0 & 0 & -0.455 \\ 2.672 & 0 & 0 & -0.1818 \end{bmatrix} \begin{pmatrix} \theta \\ x \\ \dot{\theta} \\ \dot{x} \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ 4.54 \\ 1.82 \end{bmatrix} f \tag{38}$$

Now to approach the stability of the system, we consider the eigen-values of the transition matrix A which correspond to the location of the poles for the system transfer function.

$$\text{Eigenvalues:} \qquad \lambda = 0, -0.14, -5.6, 5.57 \tag{39}$$

Now we note that because we have at least one eigen value with a positive real component the system must be unstable, by Routh-Hurwitz Polynomial Criterion. Therefore, we will need to design state feedback controller capable of moving the poles of the system into the negative real half plane. However, first we must ensure that a particular controllability matrix is capable of spanning the state space, or reaching each state variable, $\theta, \dot{\theta}, x, \dot{x}$. Therefore, for controllability matrix,

$$M_c = [B, AB, A^2B, ..., A^{n-1}B]$$

if the system is controllable it must be true that M_c has a rank of 4, or has full rank. Now using MATLAB to compute the controllability matrix, we see that,

$$M_c = \begin{bmatrix} 0 & 1.8 & -0.3 & 12.2 \\ 1.8 & -0.3 & 12.2 & -4.4 \\ 0 & 4.5 & -0.83 & 141.88 \\ 4.5 & -0.83 & 141.88 & -31.32 \end{bmatrix} \qquad \longrightarrow Rank(M_c) = 4 \qquad \textit{Full rank.} \tag{40}$$

Therefore, there exists a proportional controller K capable of spanning the states and manipulate the position of the system poles.
And so we consider redefining our input vector f around some proportion of the states.

$$f = -Kw \tag{41}$$

Or in general we arrive at a state space model,

$$\dot{\boldsymbol{x}} = (\boldsymbol{A} - \boldsymbol{BK})\boldsymbol{x} \tag{42}$$

Implies that the eigenvalues and thereby system poles are entirely dependent on matrix K.
Additionally, we note with this kind of feedback system, we will need to fix some desired final reference output. Thereby, we take reference states as,

$$w_{ref} = \begin{bmatrix} \theta_f \\ x_f \\ \dot{\theta}_f \\ \dot{x}_f \end{bmatrix} = \begin{bmatrix} 0 \\ 0.4 \\ 0 \\ 0 \end{bmatrix} \tag{43}$$

Which implies at steady-state we want the position of the cart to be 0.4m to the right from the starting position. Additionally, the displacement $\theta_{ref} = 0$ and the velocity of both the cart and pendulum will be 0.
Next we again consider our system constraints, such that the settling time for both x and θ is less than 5 seconds, the rise time for x is less than 0.5 seconds, and the overshoot for θ is less than 20%.
Critically, now we are able to place the poles of the system anywhere in the complex plane and change the response characteristics. As an example, using the place function in MATLAB we consider pole locations,

$$\lambda = -1, -2, -3, -4 \tag{44}$$

Which results in a response curve, (figure 5).

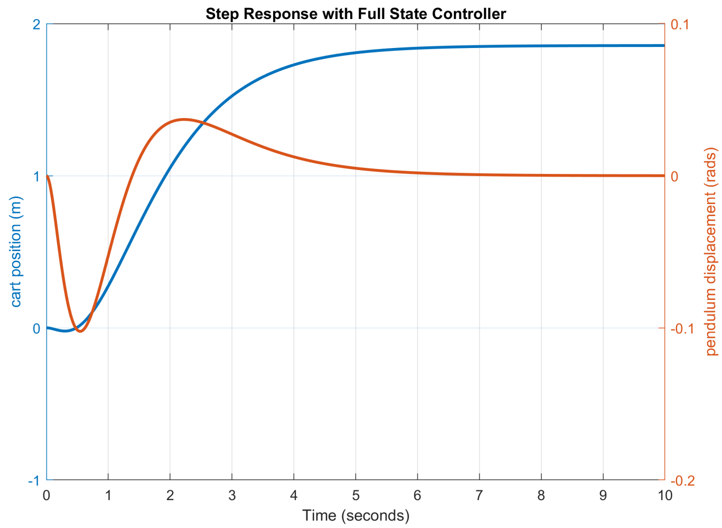

Figure 5: Cart and Pendulum Response to step input under full state controller

Now examining figure 5 we notice the movement of the cart kind of attempts to counter-act the position of the pendulum while gradually moving towards the reference state at $x = 0.4m$. However, It is also clear that the system parameters are not met. Therefore, we must reconsider the pole positions.
Importantly, in our design we must also consider the effort of the controller, or how much energy is required to create a particular response. If we move the poles further into the negative real half complex plane our system response will increase. However, we will need to expend a lot more energy to facilitate the design, which our physical system (motor) might not be able to maintain. As a result we consider a linear-quadratic regulator which works as a middle ground for the optimal response to controller energy. Specifically, we redefine the proportional gain component;

Linear-Quadratic Regulator

For quadratic cost function for the controller for any controllable state-space model the proportional gain can be define

$$K = R^{-1}(B^T P(t) + N^T)$$

For, R, P, N being boundary equations for the controller cost or behavioral weights.

And now we can use the 'lqr' command in MATLAB to automatically generate a linear-quadratic regulator for transition matrix A, B. Which results in the response figure 6.

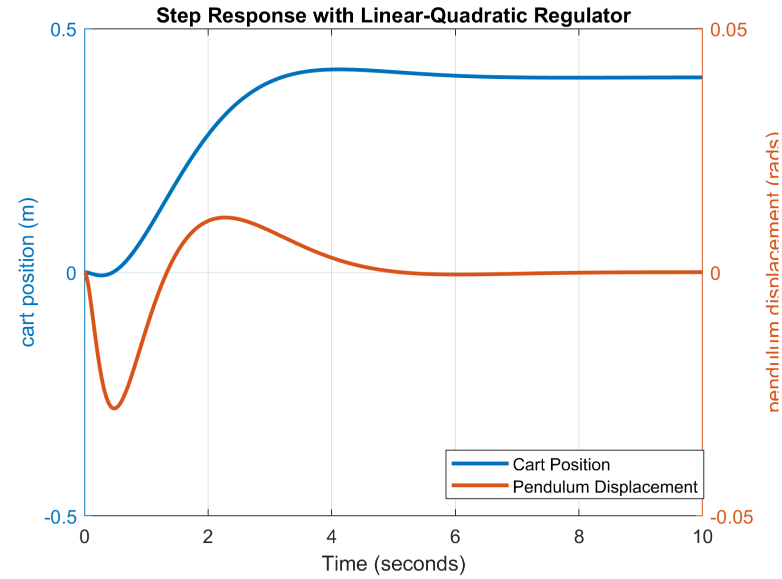

Figure 6: Step Response with Linear Quadratic Regulator

This response is significantly closer to the desired behavior, with both the cart and pendulum converging to the correct steady-state values. Additionally, we can see the cart attempting to counter-act the movement of the pendulum before settling. However, the rise time is still not within system requirements. Now reiterating, moving the poles further into the left half complex plane will increase the speed of the systems response; however, this will also make the action more expensive in terms of energy. And so instead we consider redefining the weights placed on the behavior of the cart. Specifically, we increase the carts initial desire to the stabilize the pendulum, which ideally, allows the cart to begin moving towards the desired location $x = 0.4m$ faster.

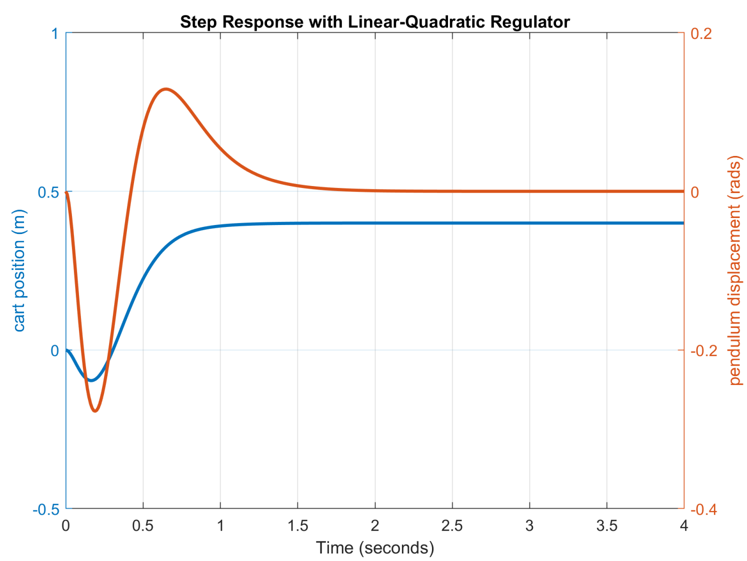

Figure 7: Modified System response

Results & Conclusions

We considered a complex oscillatory system that easily becomes unstable. Such that a pendulum is balance in vertical position on-top a moving cart. Specifically, we desired to be able to control the motion of the cart to ensure the pendulum stays at vertical equilibrium and the cart be able to reach some final destination. Importantly, our study included designing dynamic equations of both the carts positions and the center of mass of the pendulum, creating a SIMO model. This was done using the Lagrangian techniques and therefore, was expected to be consistent with normal forced oscillatory motion. Which was demonstrated by equations (28)(29).

Additionally, we considered the affect on the pendulum of an impulse voltage input to the driving motor (pushes the cart). Specifically, we designed a PID controller capable of stabilizing the pendulum to vertical equilibrium after the impulse. PID coefficients were found, $k_p = 80, k_i = 1, k_d = 20$ which resulted in a maximum overshoot of $M_p < 0.05$ or within system parameter requirements. Now to better facilitate the SIMO design, the controller responsible for both moving the cart and stabilizing the pendulum was built from state space models. Importantly, this involved ensuring the system was controllable by some transition matrix. We verified this by showing the controllability matrix M_c spans the state space or has full rank. equation (40). Therefore, we found matrix A, B, C, D related to the state-space form that is capable of positioning the system poles anywhere in the s-plane. This is demonstrated in figure 5.

Therefore, is must be possible to theoretically satisfy any response parameters involving any of the states. However, with more demanding poles positions the work done by the controller unit (or the driving motor in this case) grows potentially to large to achievable by the physical system. Therefore, a LQR controller is used to balance the work done by the controller and the desired response of the system. figure 6,7. Then applying a step input to the cart at $0.4\mu(t)$ we desired the cart to move $0.4m$ to the right of its current position, while stabilizing the pendulum. The closest result to the system requirements is shown in figure 7, such that in the transient portion of the carts motion it directly attempts to counter-act the motion of the pendulum, then gradually moves to $0.4m$. Which is consistent with the expected behavior of the system.

Appendix

Listing 1: MATLAB CODE

```
M = 0.5;
m = 0.2;
b = 0.1;
l = 0.3;
I = 0.006;
g = 9.8;

p = I*(M+m)+M*m*l^2;

A = [0      1                0              0;
     0  -(I+m*l^2)*b/p   (m^2*g*l^2)/p     0;
     0      0                0              1;
     0  -(m*l*b)/p        m*g*l*(M+m)/p     0];

B = [     0;
     (I+m*l^2)/p;
          0;
        m*l/p];

C = [1  0  0  0;
     0  0  1  0];

D = [0;
     0];



states = {'x' 'x_dot' 'theta' 'theta_dot'};
inputs = {'u'};
outputs = {'x'; 'phi'};
%sys = ss(A,B,C,D);

  sys_ss = ss(A,B,C,D,'statename',states,'inputname',inputs,'outputname',outputs);

poles = eig(A);

co = ctrb(sys_ss);

controllability = rank(co);

%eigs = [   -5.5978 + 0.4070i,  -5.5978 - 0.4070i,  -0.8494 + 0.8323i,  -0.8494 -
    0.8323i];

Q = C'*C;
Q(1,1) = 1000;
Q(3,3) = 20;
R = 1;

%K = place(A,B,eigs);

K = lqr(A,B,Q,R);
```

```
%Cn = [1  0  0  0];
%sys_ss = ss(A,B,Cn,0);
%N = rscale(sys_ss,K);

Ac = [(A-B*K)];
Bc = [B];
Cc = [C];
Dc = [D];

states = {'x' 'x_dot' 'phi' 'phi_dot'};
inputs = {'r'};
outputs = {'x'; 'phi'};


%sys_cl = ss(Ac,Bc*N,Cc,Dc,'statename',states,'inputname',inputs,'outputname',
    outputs);
sys_cl = ss(Ac,Bc,Cc,Dc,'statename',states,'inputname',inputs,'outputname',outputs)
    ;


t = 0:0.01:3;
r = 0.4*ones(size(t));
[y,t,x]=lsim(sys_cl,r,t);
[AX,H1,H2] = plotyy(t,y(:,1),t,y(:,2),'plot');
set(get(AX(1),'Ylabel'),'String','cart position (m)')
set(get(AX(2),'Ylabel'),'String','pendulum displacement (rads)')
xlabel('Time (seconds)');
title('Step Response with Linear-Quadratic Regulator');
```