Controls Extra Project 1

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Abstract

We consider an analysis of the rotational control system required to operate a certain computer-tape-drive network. Specifically, we approach methods of stability derived from theoretical dynamic equations of motion. The system uses a DC motor to rotate a central capstan; this capstan is attached to a second capstan (load) by an elastic tap. As a result, to accurately model the rotation of the second capstan we find a relationship between the strength of the motor and the torque in the capstan network. Additionally, we consider PI controller stability methods to ensure the tape-drive operates under some bounded constraints. Therefore, our study theoretically approaches dynamic models for rotational systems, feedback control blocks to contain the velocity of the load, and an analysis of system responses due to changing controller features.

Introduction

The modeling and manipulation of the a computer-tape-drive system is considered. In particular, when building a dynamic model of the plant, three critical system elements are analyzed, the response of the motor to voltage input (e_a), the motion of the first capstan, and the motion of the load or second capstan. Leading to a state-space model, taking the position and velocity of each capstan as states and (e_a) as input. Now, because the system uses a permanent-magnetic-DC-motor we expect the torque generated by the motion to be somehow related to the current within the motor circuit. Additionally, we consider the counter-electromagnetic force (Back emf) generated by resistance from the load to feed back into the motor circuit. Therefore, we track the motion of the load or second capstan by the back emf voltage entering the circuit (e_b). And so, the motor response is expected to follow, Kirchhoff's loop rule, relating input voltage to output voltage by a sum of the voltage across each circuit component.
Additionally, we consider the capstan network; such that, the first capstan, driven by the motor, is attached to the second by an elastic tape. Because of the constraints of this type of system we consider five extraneous forces acting on each capstan, the torque from the motor, the moment of inertia for each capstan, spring force due to the elasticity in the tape, and friction. Therefore, the position of either capstan is expected to closely follow that of a coupled rotational spring-mass-damper network.

$$\begin{cases} [inertia]f_1(\ddot{\theta_1},\ddot{\theta_2}) + [friction]f_2(\dot{\theta_1},\dot{\theta_2}) + [stiffness]f_3(\theta_1,\theta_2) = f_{ext} \\ [inertia]g_1(\ddot{\theta_2},\ddot{\theta_1}) + [friction]g_2(\dot{\theta_2},\dot{\theta_1}) + [stiffness]g_3(\theta_2,\theta_1) = 0 \end{cases} \tag{1}$$

Where the position, θ of each capstan is related by some system of differential equations summing all the forces on the network.
Additionally, we consider the systems controller block and feedback loop; such that, we can alter the voltage input into the motor circuit (e_a) based on the expected system response and the desired output, ($R(s)$). This is done using a proportional-integral (PI) controller with coefficients, (k_p, k_i). Now importantly, we consider the controller in cascade with the plant; and therefore, the open-loop transfer function will multiply the controller output by our theoretical dynamic model of the capstan network.

Additionally, we tweak controller coefficients to ensure the following system response constraints,

$$\text{Ramp-Error constant: } K_v = 100, \tag{2}$$
$$\text{Rise time } < 0.02s, \tag{3}$$
$$\text{Settling time: } < 0.02s, \tag{4}$$
$$\text{Maximum overshoot: } < 1\% \text{ or at minimum.} \tag{5}$$

As a result, it is expected that the final model will both accurately model the rotation of the second capstan and be stable. We also consider a second PI controller model capable of meeting the following response characteristics.

$$\text{Ramp-Error constant: } K_v = 100, \tag{6}$$
$$\text{Phase margin: } P_m \quad \text{Is maximized.} \tag{7}$$

Procedure

We consider a computer tape-drive network driven by a permanent magnetic DC motor. We first assume the block diagram system shown in figure 1, with speed transducer feedback and cascading controller $\rightarrow$ plant open loop. Now, we assume the output of the controller, in some voltage (e_a) which feeds into the motor circuit. Additionally, we denote the error between the desired input and system response, $E(s)$, such that, $(E(s) = R(s) - K_f C(s)$, where $C(s)$ is the response of the system. Then $E(s)$ is feed into the controller block.

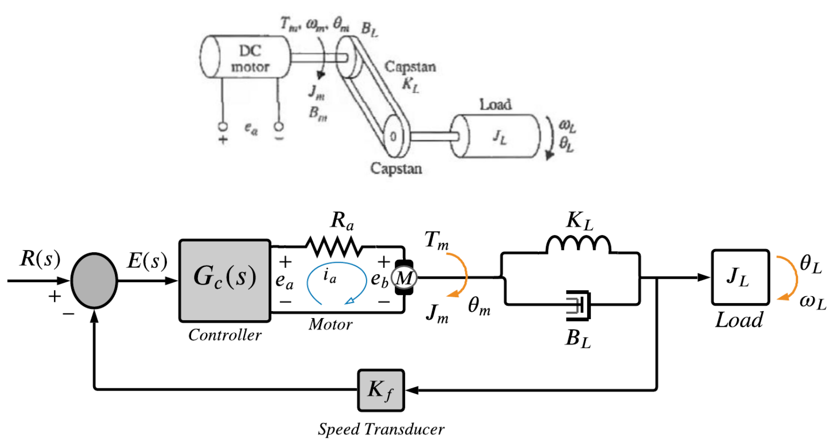

Figure 1: System Diagram: Computer-tape-drive system with block diagram

Where the following physical constraints are imposed.

$$K_L = 3000 oz - in./rad \qquad \text{Spring constant of elastic taps} \tag{8}$$
$$B_L = 10 oz - in./rad/sec \qquad \text{Viscous-friction coefficient between tape and the capstans} \tag{9}$$
$$K_i = 10 oz - in./A \qquad \text{Motor torque constant} \tag{10}$$
$$K_b = 0.0706 V/rad/sec \qquad \text{Motor back-emf constant} \tag{11}$$
$$B_m = 3 oz - in/rad/sec \qquad \text{motor friction coefficient} \tag{12}$$
$$R_a = 0.25\Omega \qquad L_a \approx 0H \tag{13}$$
$$J_L = 6 oz - in./rad/sec^2 \qquad \text{Interia of the load} \tag{14}$$
$$K_f = 1 V/rad/sec \tag{15}$$
$$J_m = 0.05 oz - in./rad/sec^2 \qquad \text{Inertia of the first capstan.} \tag{16}$$
$$\tag{17}$$

Importantly, we define our axis for either capstan as the rotational angle in radians from a theoretical origin, ($\theta = 0$). Therefore, the position with respect to time of either capstan is related to there displacement from the origin, $(\theta_m(t), \theta_L(t))$. Additionally, we assume that both capstans are originally located at their origin, $(\theta_m(0) = 0, \theta_L(0) = 0)$.
Now we consider a dynamic model of the plant network in figure 1 from two separate stages. First we approach the most general equation for voltage distribution in the motor circuit. Relating e_a to e_b. Then we consider the relationship between the position of the capstans, θ_m, θ_L and the torque generated by the motor, T_m.

Circuit Stage
As an extension of this, we examine figure 2 which displays the circuit elements within the motor system, where e_b is the back-emf voltage.

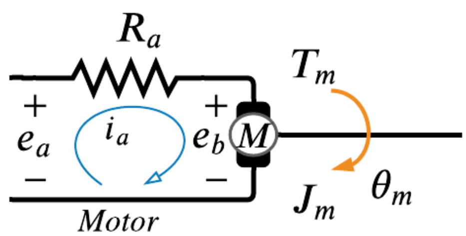

Figure 2: Motor Circuit Diagram

And so we use Kirchhoff's Voltage law and conservation of energy;

$$\textit{Kirchhoff's Voltage Law} \tag{18}$$
For any closed loop (ℓ) the with series voltage elements (v_i) is must be true that,
$$\Sigma v_i = 0$$

to find a relationship between circuit input e_a and the voltage attributed to the motor.

$$0 = e_a + R(i_a) + e_b \tag{19}$$

Now we consider the motor back-emf voltage e_b should be somehow related to the speed of rotation of the load, $\dot{\theta_m}$. And so we define,

$$e_b = K_b \dot{\theta_m}$$

Additionally, we consider the torque created by the motor circuit must also be related to the current around the loop, i_a. And therefore, we define the motor output torque, t_m,

$$T_m = K_i i_a \longrightarrow i_a = \frac{T_m}{K_i}.$$

And so rearranging our circuit loop equation,

$$0 = e_a - T_m(\frac{R_a}{K_i}) - K_b \dot{\theta_m}. \tag{20}$$

In which our only unknown value is the torque exerted by the motor. Which; intuitively, is proportional to the counter-torque from the load. And so we consider a dynamic model to approach T_m based on the tape-capstan network.

Tape-Capstan Stage
The tape-capstan portion of the system design couples two rotational capstans by an elastic tape. Importantly, because the tape is elastic, it will exert a different spring-like force based on the position of both capstans. Additionally, we consider two separate instances of viscous friction, either from the

motor or the attached final load. now, because only the first capstan is connected to the motor, we assume the second is negligibly affect by the motor friction, (B_m). And because the first motor is not directly connected to the load, we assume it is not affected by the load friction, (B_L). Finally, because of their similarities we build our dynamic model off of a coupled rotational spring system.

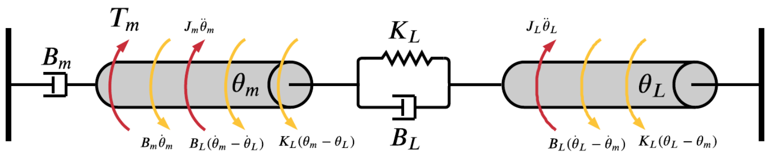

Figure 3: System Diagram: Rotational capstan Spring-Torque-Damper model.

Where, K_i is the spring coefficient for the elasticity of the tapes, and B_m, B_L are friction coefficients from either the motor or load. Importantly, we consider the rotation of torque due to the motor, T_m to be the positive direct. Additionally, we consider the damper and spring force to oppose the rotation of either capstan, and so we consider these in the negative direction. Now, because both capstans manipulate the elastic force from the taps we consider spring force from Hooke's Law, $f_{1s} = K_L(\theta_1 - \theta_2)$. This way, the total force is derived from the displacement of each object from each other, similar to a traditional coupled mass-spring-damper. Finally, we consider the friction on either capstan to directly oppose the angular velocity of rotation, and so we find the equation, $f_{friction} = B_i(\theta_i \dot{-} \theta_j)$. Where B is the friction coefficient for either the load or motor and $\theta_{i,j}$ is the pull from either capstan. Now we use Newton's second law adapted to torque (t),

Newton's Second Law:

$$\Sigma t_i = J\ddot{\theta}$$

Or the sum of torque's is exactly equal to the moment of inertia times the angular acceleration of the object. And so, analyzing the torque on both capstans reveals the equations;

$$\begin{aligned}
J_m\ddot{\theta}_m &= T_m - B_m\dot{\theta}_m - B_L(\dot{\theta}_m - \dot{\theta}_L) - K_L(\theta_m - \theta_L) \\
J_L\ddot{\theta}_L &= -B_L(\dot{\theta}_L - \dot{\theta}_m) - K_L(\theta_L - \theta_m)
\end{aligned}$$

Now simplifying and solving for standard form;

$$\ddot{\theta}_m = \frac{1}{J_m}(T_m) - \frac{B_m}{J_m}\dot{\theta}_m - \frac{B_L}{J_m}\dot{\theta}_m - \frac{K_L}{J_m}\theta_m + \frac{B_L}{J_m}\dot{\theta}_L + \frac{K_L}{J_m}\theta_L \tag{21}$$

$$\ddot{\theta}_L = -\frac{B_L}{J_L}\dot{\theta}_L - \frac{K_L}{J_L}\theta_L + \frac{B_L}{J_L}\dot{\theta}_m + \frac{K_L}{J_L}\theta_m \tag{22}$$

Next it becomes possible to rearrange our circuit loop equation for T_m.

$$\begin{aligned}
0 &= e_a - T_m(\frac{R_a}{K_i}) - K_b\dot{\theta}_m \\
\longrightarrow T_m &= \frac{K_i}{R_a}e_a - \frac{K_iK_b}{R_a}\dot{\theta}_m.
\end{aligned}$$

And substituting into our equation for the torque on the first capstan,

$$\ddot{\theta}_m = \frac{1}{J_m}\underbrace{(\frac{K_i}{R_a}e_a - \frac{K_iK_b}{R_a}\dot{\theta}_m)}_{T_m} - \frac{B_m + B_L}{J_m}\dot{\theta}_m - \frac{K_L}{J_m}\theta_m + \frac{B_L}{J_m}\dot{\theta}_L + \frac{K_L}{J_m}\theta_L$$

$$\rightarrow \quad \ddot{\theta}_m = -\frac{K_iK_b + B_mR_a + B_LR_a}{J_mR_a}\dot{\theta}_m - \frac{K_L}{J_m}\theta_m + \frac{B_L}{J_m}\dot{\theta}_L + \frac{K_L}{J_m}\theta_L + \frac{K_i}{J_mR_a}e_a \tag{23}$$

Finally, we consider an expression for the angular velocity of either θ_m or θ_L, such that we define, ω_i,

$$\omega_m = \dot{\theta}_m \tag{24}$$

$$\omega_L = \dot{\theta}_L \tag{25}$$

And now we have a system of four equations relating the position and velocity of both capstans to the voltage input, e_a.

$$\begin{cases} \omega_m = \dot{\theta}_m \\ \ddot{\theta}_m = -\frac{K_i K_b + B_m R_a + B_L R_a}{J_m R_a}\dot{\theta}_m - \frac{K_L}{J_m}\theta_m + \frac{B_L}{J_m}\dot{\theta}_L + \frac{K_L}{J_m}\theta_L + \frac{K_i}{J_m R_a}e_a \\ \omega_L = \dot{\theta}_L \\ \ddot{\theta}_L = -\frac{B_L}{J_L}\dot{\theta}_L - \frac{K_L}{J_L}\theta_L + \frac{B_L}{J_L}\dot{\theta}_m + \frac{K_L}{J_L}\theta_m \end{cases}$$

And substituting values for $\dot{\omega}$,

$$\begin{cases} \omega_m = \dot{\theta}_m \\ \omega_L = \dot{\theta}_L \\ \dot{\omega}_m = -\frac{K_i K_b + B_m R_a + B_L R_a}{J_m R_a}\omega_m - \frac{K_L}{J_m}\theta_m + \frac{B_L}{J_m}\omega_L + \frac{K_L}{J_m}\theta_L + \frac{K_i}{J_m R_a}e_a \\ \dot{\omega}_L = -\frac{B_L}{J_L}\omega_L - \frac{K_L}{J_L}\theta_L + \frac{B_L}{J_L}\omega_m + \frac{K_L}{J_L}\theta_m \end{cases} \tag{26}$$

Now, motivating our state-space model, we pick $\theta_m, \theta_L, \omega_m, \omega_L$ as states,

State-Space Model:

$$\dot{x} = Ax + B\mu$$

Where A, B are transition matrices and μ is an input signal, $\mu = e_a$. Such that we can define,

$$x = \begin{pmatrix} \theta_m \\ \theta_L \\ \omega_m \\ \omega_L \end{pmatrix} \qquad \text{Which implies,} \qquad \dot{x} = \begin{pmatrix} \omega_m \\ \omega_L \\ \dot{\omega}_m \\ \dot{\omega}_L \end{pmatrix} \tag{27}$$

Now we can define our transition matrices by the equations from system (25),

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{K_L}{J_m} & \frac{K_L}{J_m} & -\frac{K_i K_b + B_m R_a + B_L R_a}{J_m R_a} & \frac{B_L}{J_m} \\ \frac{K_L}{J_L} & -\frac{K_L}{J_L} & \frac{B_L}{J_L} & -\frac{B_L}{J_L} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{K_i}{J_m R_a} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Resulting in a final state-space model,

State-Space Model: (28)

$$\begin{pmatrix} \omega_m \\ \omega_L \\ \dot{\omega}_m \\ \dot{\omega}_L \end{pmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{K_L}{J_m} & \frac{K_L}{J_m} & -\frac{K_i K_b + B_m R_a + B_L R_a}{J_m R_a} & \frac{B_L}{J_m} \\ \frac{K_L}{J_L} & -\frac{K_L}{J_L} & \frac{B_L}{J_L} & -\frac{B_L}{J_L} \end{bmatrix} \begin{pmatrix} \theta_m \\ \theta_L \\ \omega_m \\ \omega_L \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{K_i}{J_m R_a} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ e_a \end{pmatrix}$$

Finally, substituting our initial constraints,

State-Space Model: (29)

$$\begin{pmatrix} \omega_m \\ \omega_L \\ \dot{\omega}_m \\ \dot{\omega}_L \end{pmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -60,000 & 60,000 & -316.48 & 200 \\ 500 & -500 & 1.66 & -1.66 \end{bmatrix} \begin{pmatrix} \theta_m \\ \theta_L \\ \omega_m \\ \omega_L \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 800 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ e_a \end{pmatrix}$$

Now we have established the controller input related dynamics of the plant with respect to time. However, when designing a controller that ensures this response is stable it is convenient to also analyze the system in the frequency domain. Therefore, we approach transfer functions, $\frac{\Omega_m(s)}{E_a(s)}$, and $\frac{\Omega_L(s)}{E_a(s)}$.

Revisiting our equations of motion for either capstan and the motor circuit,

Motor Circuit:

$$0 = e_a - T_m(\frac{R_a}{K_i}) - K_b\dot{\theta_m}.$$

Capstan 1:

$$J_m\ddot{\theta}_m = T_m - B_m\dot{\theta}_m - B_L\dot{\theta}_m + B_L\dot{\theta}_L - K_L\theta_m + K_L\theta_L$$

Capstan 2:

$$J_L\ddot{\theta}_L = -B_L\dot{\theta}_L + B_L\dot{\theta}_m - K_L\theta_L + K_L\theta_m$$

we can start our analysis by rearranging our loop equation for the motor circuit:

Motor Circuit

$$0 = e_a - T_m(\frac{R_a}{K_i}) - K_b\dot{\theta_m}.$$

$$0 = \mathcal{L}\{e_a - T_m(\frac{R_a}{K_i}) - K_b\dot{\theta_m}\}$$

$$\underline{0 = E_a - T_m(\frac{R_a}{K_i}) - K_b s\Theta_m(s)}$$

From here the only unknown again is T_m. So analyzing our first capstan we see that:

Capstan 1:

$$T_m = J_m\ddot{\theta}_m + B_m\dot{\theta}_m + B_L\dot{\theta}_m + K_L\theta_m - B_L\dot{\theta}_L - K_L\theta_L$$

$$\mathcal{L}\{T_m\} = \mathcal{L}\{J_m\ddot{\theta}_m + B_m\dot{\theta}_m + B_L\dot{\theta}_m + K_L\theta_m - B_L\dot{\theta}_L - K_L\theta_L\}$$

$$\longrightarrow T_m = J_m s^2\Theta_m + (B_m + B_L)s\Theta_m + K_L\Theta_m - B_L s\Theta_L - K_L\Theta_L$$

$$\underline{\text{And then, } T_m = \Theta_m(J_m s^2 + (B_m + B_L)s + K_L) - \Theta_L(B_L s + K_L).}$$

Unfortunately; now we have introduced another unknown, Θ_L, and so we also need to analyze the second capstan.

Capstan 2:

$$J_L\ddot{\theta}_L = -B_L\dot{\theta}_L + B_L\dot{\theta}_m - K_L\theta_L + K_L\theta_m$$

$$0 = J_L\ddot{\theta}_L + B_L\dot{\theta}_L + K_L\theta_L - B_L\dot{\theta}_m - K_L\theta_m$$

$$0 = \mathcal{L}\{J_L\ddot{\theta}_L + B_L\dot{\theta}_L + K_L\theta_L - B_L\dot{\theta}_m - K_L\theta_m\}$$

$$0 = J_L s^2\Theta_L + B_L s\Theta_L + K_L\Theta_L - B_L s\Theta_m - K_L\Theta_m$$

$$\underline{\text{And so, } \Theta_L = \frac{B_L s + K_L}{J_L s^2 + B_L s + K_L}\Theta_m.}$$

And plugging this into our equation for motor torque, T_m.

$$T_m = \Theta_m(J_m s^2 + (B_m + B_L)s + K_L) - (B_L s + K_L)\frac{B_L s + K_L}{J_L s^2 + B_L s + K_L}\Theta_m.$$

$$\underline{T_m = \Theta_m(J_m s^2 + (B_m + B_L)s + K_L - \frac{(B_L s + K_L)^2}{J_L s^2 + B_L s + K_L})}$$

Therefore, our final equation connecting, $E_a \to \Theta_m$:

$$0 = E_a - T_m(\frac{R_a}{K_i}) - K_b\Theta_m(s)$$

$$0 = E_a - (\frac{R_a}{K_i})\underbrace{(J_m s^2 + (B_m + B_L)s + K_L - \frac{(B_L s + K_L)^2}{J_L s^2 + B_L s + K_L})}_{T_m}\Theta_m(s) - K_b\Theta_m(s)$$

$$0 = E_a - \Theta_m(s)\left(\frac{R_a}{K_i}(J_m s^2 + (B_m + B_L)s + K_L - \frac{(B_L s + K_L)^2}{J_L s^2 + B_L s + K_L}) - K_b s\right)$$

$$\frac{\Theta_m}{E_a} = \frac{K_i}{R_a J_m s^2 + (R_a B_m + R_a B_L)s + R_a K_L - \frac{R_a(B_L s + K_L)^2}{J_L s^2 + B_L s + K_L} - K_b K_i s}$$

Simplifying;

$$\underline{\frac{\Theta_m}{E_a} = \frac{K_i(J_L s^2 + B_L s + K_L)}{\big(J_L s^2 + B_L s + K_L)(R_a J_m s^2 + (R_a B_m + R_a B_L - K_b K_i)s + R_a K_L\big) - R_a(B_L s + K_L)^2}}$$

And simplifying we arrive at the equation;

$$\left\{ \frac{\Theta_m}{E_a} = \frac{K_i(J_L s^2 + B_L s + K_L)}{s(R_a J_m J_L s^3 + (R_a J_m B_L + R_a J_L B_L + R_a B_m J_L - K_b K_i J_L)s^2 + (R_a K_L J_L + R_a J_m K_L + R_a B_m B_L - K_b K_i B_L)s + (R_a K_L J_m - K_b K_i J_m))} \right.$$

Now we note our equation for $\Theta_L \to \Theta_m$:

$$\Theta_L = \frac{B_L s + K_L}{J_L s^2 + B_L s + K_L}\Theta_m$$

And so, $\Theta_m = \dfrac{J_L s^2 + B_L s + K_L}{B_L s + K_L}\Theta_L$ Then, $\dfrac{\Theta_m}{E_a} = \dfrac{J_L s^2 + B_L s + K_L}{B_L s + K_L}\left(\dfrac{\Theta_L}{E_a}\right)$

And so our final transfer function:

$$\underline{\frac{\Theta_L}{E_a} = \frac{(B_L s + K_L)K_i}{\big(J_L s^2 + B_L s + K_L)(R_a J_m s^2 + (R_a B_m + R_a B_L - K_b K_i)s + R_a K_L\big) - R_a(B_L s + K_L)^2}}$$

Again simplifying,

$$\left\{ \frac{\Theta_L}{E_a} = \frac{(B_L s + K_L)K_i}{s(R_a J_m J_L s^3 + (R_a J_m B_L + R_a J_L B_L + R_a B_m J_L - K_b K_i J_L)s^2 + (R_a K_L J_L + R_a J_m K_L + R_a B_m B_L - K_b K_i B_L)s + (R_a K_L J_m - K_b K_i J_m))} \right.$$

As a result, we have motivated a dynamic model for the plant the interprets the angular position of each capstan by some voltage input E_a. However, in reference to the intended operations of the system we consider it more convenient to describe the motion of each capstan as a velocity. Which allows us to predict the speed at which each capstan in the tape driver is rotating. And so we approach a model of angular velocity, Ω. Such that,

$$\Omega_i = \mathscr{L}\{\omega_i\} = \mathscr{L}\{\dot{\theta}_i\} = s\Theta_i$$
$$\longrightarrow \Theta_i = \frac{\Omega_i}{s} \tag{30}$$

Which can be used to motivate transfer functions, $\frac{\Omega_m}{E_a}$ and $\frac{\Omega_L}{E_a}$. By implying,

$$\frac{\Theta_m}{E_a} = \frac{\Omega_m}{sE_a}$$
$$\frac{\Theta_L}{E_a} = \frac{\Omega_L}{sE_a}$$

Therefore, we find our final transfer function for the plant with respect to velocity to be,

$$\boxed{\begin{cases} \frac{\Omega_m}{E_a} = \frac{K_i(J_L s^2 + B_L s + K_L)}{R_a J_m J_L s^3 + (R_a J_m B_L + R_a J_L B_L + R_a B_m J_L - K_b K_i J_L)s^2 + (R_a K_L J_L + R_a J_m K_L + R_a B_m B_L - K_b K_i B_L)s + (R_a K_L J_m - K_b K_i J_m)} \\ \\ \frac{\Omega_L}{E_a} = \frac{K_i(B_L s + K_L)}{R_a J_m J_L s^3 + (R_a J_m B_L + R_a J_L B_L + R_a B_m J_L - K_b K_i J_L)s^2 + (R_a K_L J_L + R_a J_m K_L + R_a B_m B_L - K_b K_i B_L)s + (R_a K_L J_m - K_b K_i J_m)} \end{cases}}$$

And substituting our system constraints,

$$\begin{cases} \dfrac{\Omega_m}{E_a} = \dfrac{800s^2+1333s+400{,}000}{s^3+318.1s^2+60{,}700s+58{,}200} \\ \\ \dfrac{\Omega_L}{E_a} = \dfrac{1333s+400{,}000}{s^3+318.1s^2+60{,}700s+58{,}200} \end{cases} \tag{31}$$

Now, that we have a simplified model the plant, we can reduce the block diagram in figure 1 and begin designed a stability controller.

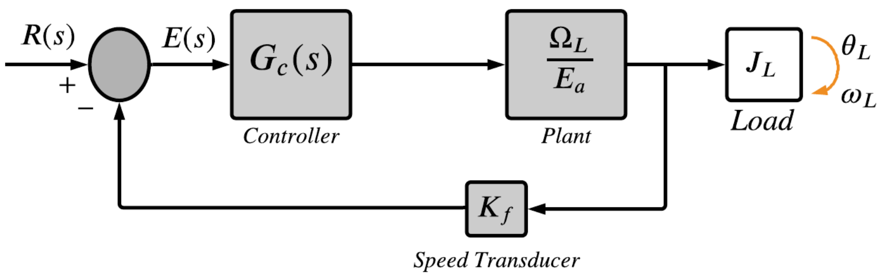

Figure 4: Reduced System Block Diagram

Designing a controller

We have motivated an equation for the speed of each second capstan based on a particular voltage input. Now we note the only capstan connected to the load has transfer function, $\frac{\Omega_L}{E_a}$ and because this transfer function is already in terms of input E_a, the other capstan, $\frac{\Omega_m}{E_a}$ no longer has an effect on the output of the system. Therefore, it is neglected. Additionally, we consider affect of the transducer in the feedback loop, $K_f = 1V/rad/sec$, such that for feedback signal, $C(s)$ the output of the transducer should be related to the velocity of the load $C(s)$. Additionally, we note that we have directed the output of our plant to be in terms of speed, and therefore, we can consider K_f to be unity feedback.
Finally, analyzing our systems response we note we will need to design a controller in cascade with the plant as a means of manipulating the final output to meet system constraints. And so we consider a PI (Proportional Integral) controller with unknown coefficients, k_p, k_i, such that,

$$G_c(s) = k_p + \frac{k_i}{s} \tag{32}$$

And therefore, we arrive at the modified block diagram,

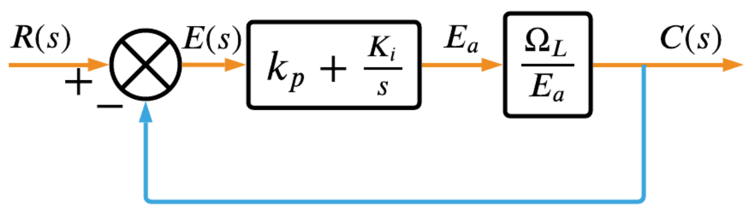

Figure 5: Block Diagram with updated controller and transducer values

Note: because our system response constraints are subject to the initial physical limitations of each component when designing the controller we consider our dynamic equation for the plant after substituting initial values. (equation 30). Now, to reiterate we must find coefficients, k_p, k_i such that our system meet;

$$\begin{aligned}&\text{Ramp-Error constant: } K_v = 100,\\&\text{Rise time } < 0.02s,\\&\text{Settling time: } < 0.02s,\\&\text{Maximum overshoot: } < 1\% \text{ or at minimum.}\end{aligned}$$

And so because our controller is in cascade with our plant we find an overall open loop transfer function,

$$T(s) = (k_p + \frac{k_i}{s})\frac{\Omega_L}{E_a} = \frac{(k_p s + k_i)\Omega_L}{sE_a}$$

Which results in;

$$T(s) = \frac{(k_p s + k_i)(1333s + 400,000)}{s(s^3 + 318.1s^2 + 60,700s + 58,200)}$$

Now approaching the ramp-error coefficient, K_v for the modified system we consider the steady-state error function in the frequency domain, $E(s) = R(s) - k_f C(s) = R(s) - C(s)$, where $R(s) = \frac{1}{s^2}$ unit ramp function.
Additionally, we approach the system 'type' from the open loop transfer function, from $0 \to 2$, such that for a system with transfer function;

$$T = \frac{K(s - z_1)(s - z_2)...}{s^n(s - p_1)(s - p_2)...}$$

we define the type (n) by the power of the zero pole, s^n.
And because the denominator is completely factored by inspection we consider the $n = 1$, and the system to be type 1.
As a result we can find the static ramp-error constant, k_v using the steady-state error for ramp input, $r(t) = t\mu(t) \to R(s) = \frac{1}{s^2}$.

$$K_v = \lim_{s\to 0} sT(s)$$

$$\begin{aligned}&= \lim_{s\to 0} s\Big(\frac{(k_p s + k_i)(1333s + 400,000)}{s(s^3 + 318.1s^2 + 60,700s + 58,200)}\Big)\\&= \lim_{s\to 0} \frac{(k_p s + k_i)(1333s + 400,000)}{s(s^3 + 318.1s^2 + 60,700s + 58,200)}\\&K_v = \frac{400,000k_i}{58,200}\end{aligned}$$

Now by our initial constraint, $K_v = 100$,

$$100 = \frac{400,000k_i}{58,200}$$
$$\longrightarrow k_i = 14.55.$$

And so we can write our transfer function,

$$\frac{(k_p s + 14.55)(1333s + 400,000)}{s(s^3 + 318.1s^2 + 60,700s + 58,200)}$$

Next we can approach elements of the time response of the system by considering the closed loop transfer function generated by,

$$\frac{C(s)}{R(s)} = \frac{T(s)}{1 + T(s)}$$

And because $T(s)$ is in the form, $T(s) = \frac{N(s)}{D(S)}$ where $N(s), D(s)$ are numerator and denominator polynomials we arrive at the following,

$$\frac{C(s)}{R(s)} = \frac{N(s)}{D(s)+N(s)}$$

$$\frac{C(s)}{R(s)} = \frac{(k_p s + 14.55)(1333s + 400,000)}{s(s^3 + 318.1s^2 + 60,700s + 58,200) + (k_p s + 14.55)(1333s + 400,000)} \tag{33}$$

Importantly, we notice by root locus methods that the region in which the system meets the time requirements, does not cover the dominant poles and zeros of the closed loop transfer function, specifically, the zero at, $1333s + 400,000 = 0$. Where the shaded region identifies a space the poles and zeros must

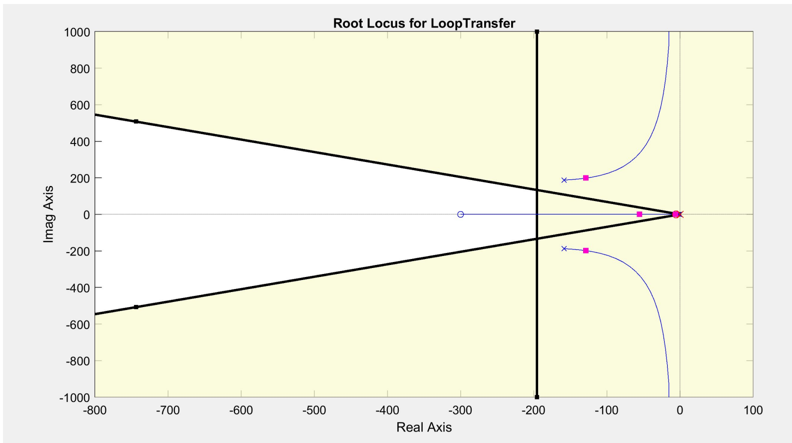

Figure 6: Root Locus of closed loop transfer function with PI controller, and time domain requirements.

be within to meet system requirements. Notice, at least one zero is located outside the shaded region; and therefore, the time specifications cannot be met.
However, using the PID design tool in MATLAB we select,

$$K_p = 14,$$

to arrive at a reasonable approximation of the time requirements. Such that,

Rise time: $t_r \approx 0.014$
Settling time: $t_s \approx 0.04$ Does not meet system requirements.
Maximum Overshoot: $M_p \approx 0.2\% < 1\%$.

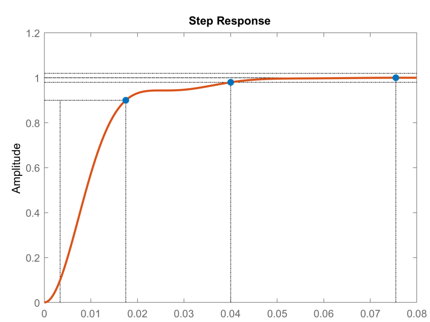

Figure 7: Step response in time domain of closed loop transfer function with updated PI controller.

Importantly Now we approach methods to nullify the zero outside of the required space. Which will allow us to achieve the required time characteristics. In particular we consider replacing the PI controller with a PID controller.

$$\text{PID: } G_c(s) = \frac{k_D s^2 + k_p s + k_i}{s} \tag{34}$$

And so we arrive at the following open loop transfer function;

$$T(s) = \frac{(k_D s^2 + k_p s + k_i)(1333s + 400,000)}{s(s^3 + 318.1s^2 + 60,700s + 58,200)}$$

Now because our system is still type 1, we again take $k_i = 14.55$ to ensure our ramp error constant, $k_p = 100$. Now tweaking our values for k_D, k_p we find,

$$k_D = 10$$
$$k_p = 10$$

Which results in the following time domain characteristics from step response.

Rise time: $t_r \approx 0.00016$
Settling time: $t_s \approx 0.00030$
Maximum Overshoot: $M_p \approx 0.02\% < 1\%$.

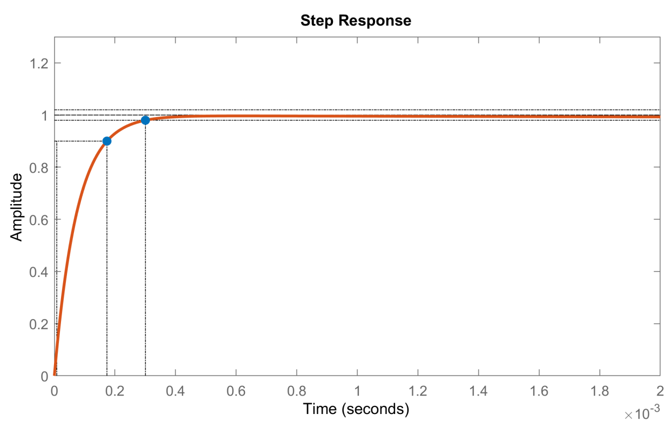

Figure 8: Step response for closed loop transfer function with PID controller

Now returning to our PI controller model, we can approximate different phase margins, (PM) gain margins (GM) and bandwidth characteristics for different values of k_p. Such that,

$$k_i = 14.55 \quad \text{Is fixed.}$$

Now if we allow k_p to vary, the gain/phase margins will change. And specifically, we are interested in a k_p that will maximize the phase margin. And so plotting the Bode plots for different k_p, we approach a range in acceptable values.

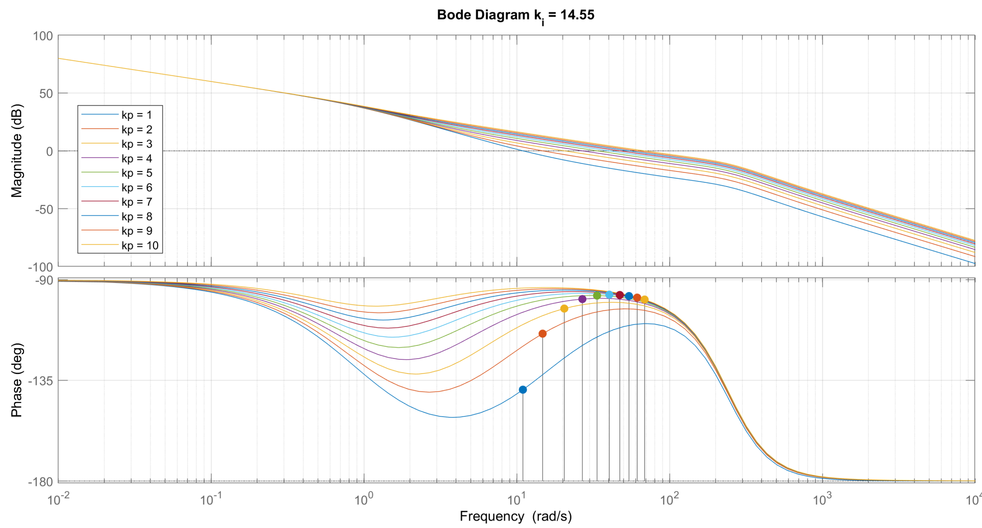

Figure 9: Bode diagram with varying k_p PI controller

Now it is clear that the curve corresponding to $k_p = 6$ (teal) is at the maximum phase margin of the selected inputs. A more refined search reveals an optimal value,

$$k_p = 6.13 \tag{35}$$

And so we get PI controller;

$$k_p = 6.13, \qquad k_i = 14.55,$$

Such that the static ramp error coefficient is $k_v = 1$ and the phase margin is maximized.

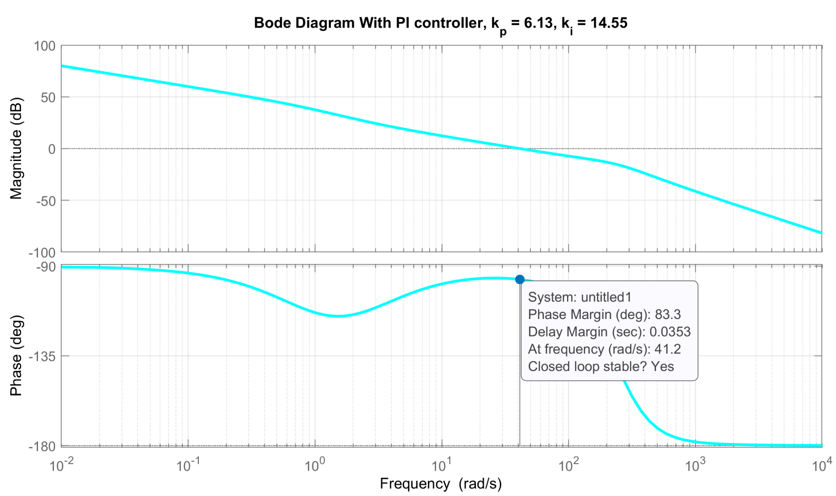

Figure 10: Bode plot of system under PI controller, $k_p = 6.13$, $k_i = 14.55$.

Conclusions

We considered a particular physical rotation control system to operate a computer-tape-drive network. Our study included finding dynamic equations of motion to predict the speed of either capstan for some voltage input on the driving motor. Along with the design of a voltage controller unit capable of both stabilizing the system and meeting pre-specified parameters. In particular, we found third order transfer functions that are consistent with our expected model,

$$\begin{cases} [inertia]f_1(\ddot{\theta}_1, \ddot{\theta}_2) + [friction]f_2(\dot{\theta}_1, \dot{\theta}_2) + [stiffness]f_3(\theta_1, \theta_2) = f_{ext} \\ [inertia]g_1(\ddot{\theta}_2, \ddot{\theta}_1) + [friction]g_2(\dot{\theta}_2, \dot{\theta}_1) + [stiffness]g_3(\theta_2, \theta_1) = 0 \end{cases} \tag{36}$$

Such that for the closed loop system we found, equation (29). Additionally, in our design of a controller, we approached two sets of systems constraints, optimizing time domain characteristics, (rise time, settling time, maximum overshoot), or frequency components, (phase margin) all under a static error coefficient, $k_v = 100$. Unfortunately, we found that we could not meet the time domain specific parameters with a PI controller because we have at least one pole outside the applicable region on the root locus, and the restrictions of PI, don't allow us to move the pole into a region that would meet the requirements. As a resolution, we either relax the parameters to a level fillable by the PI controller,

Rise time: $t_r \approx 0.014$
Settling time: $t_s \approx 0.04$ Does not meet system requirements.
Maximum Overshoot: $M_p \approx 0.2\% < 1\%$.

or design a full theoretical PID which has enough control over the poles to grantee the desired parameters.

Rise time: $t_r \approx 0.00016$
Settling time: $t_s \approx 0.00030$
Maximum Overshoot: $M_p \approx 0.02\% < 1\%$.

Finally, we consider a PI controller to maximize phase margins at static error coefficients $k_v = 100$. And so we maintained integral gain value $k_i = 14.55$, and found a proportional gain value which maximized the desired parameters. It was found that,

$$k_p = 6.13 \longrightarrow P_m = 83.3°.$$

maximized the phase margin for the local inputs $k_p \in (1, 10)$, that were tested.

Appendix

Listing 1: MATLAB Code

```
t = 0:1e-4:40*pi;


num = [0,0,1333,400000];
den = [1,318.1,60700,58200];

sys = tf(num,den);

c = pid(6.13,14.55);

bode(c*sys);

kp = 1;
hold on
%for i = 1:10
   %c1 = pid(kp,14.55);
  % kp =kp+1;
 %  bode(c1*sys);
   %margin(c1*sys);
%end
%legend('kp = 1','kp = 2','kp = 3','kp = 4','kp = 5','kp = 6','kp = 7','kp = 8','kp
    = 9','kp = 10')
[Gm,Pm,Wcg,Wcp] = margin(c*sys);

%step(feedback(c*sys,1));


%impulse(feedback(sys,c));

%sys_tf = [sys2 ; sys];

hold on
%impulse(feedback(sys2,c)*sys);

%impulse(feedback(sys,c));

%step(feedback(feedback(sys,c)*sys2,1));

%rlocus(sys*c);

%nyquist(sys);

%bode(sys);




%rlocus(c*sys);


%step(feedback(sys,1));

%c = pid(1,25.44);
```

```
%sys_cl = feedback(c*sys,1);

%rlocus(sys);
%step(sys_cl);

%pzmap(sys);
grid off
%rlocus(sys);
hold on
```