Adv. Engineering Math Homework 6

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Problem 1:

Consider the matrix,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}$$

Find the least squares solution of $Ax = b$.

Let x be a 4×1 solution vector to, $Ax = b$.
Then, pick $\hat{x} \in \mathbb{R}^3$ to be a least squares solution to $Ax = b$, which implies,

$$||b - A\hat{x}|| \leq ||b - Ax||$$

Now, note if the column space of A is inconsistent, or that there is no x that satisfies $Ax = b$, then the closest vector b^* that is capable of satisfying the equation is the orthogonal projection of b onto the column space of A, $b^* = b_{col(A)}$.
Additionally, note that if A is consistent, then $b_{col(A)} = b$.
Now, the equation becomes,

$$A\hat{x} = b_{col(A)}$$

And so it follows,

$$A^T A\hat{x} = A^T b$$

Where,

$$A^T = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix}, \qquad A^T A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \qquad A^T b = \begin{bmatrix} 1 \\ 14 \\ 5 \end{bmatrix}$$

And because $A^T A$ is clearly invertable, a unique solution to the least squares problem becomes,

$$\hat{x} = (A^T A)^{-1} A^T b = \begin{bmatrix} \frac{1}{3} \\ \frac{14}{3} \\ \frac{5}{3} \end{bmatrix}$$

Solution

And so there is a least squares solution to $Ax = b$,

$$\hat{x} = \begin{bmatrix} \frac{1}{3} \\ \\ \frac{14}{3} \\ \\ \frac{5}{3} \end{bmatrix}$$

0.1 Find the orthogonal projection of the vector b onto the range space of A

Consider the matrix,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 6 \end{bmatrix}$$

Now let u be the range space of A, and v a matrix orthogonal to u.
Then b can be written in the form,

$$b = u + v$$

Then it follows,

$$A(A^TA)^{-1}A^Tb = A(A^TA)^{-1}A^T(u+v) = A(A^TA)^{-1}A^Tu + A(A^TA)^{-1}\underbrace{A^Tv}_{0} = A(A^TA)^{-1}A^Tu$$

Now because u is the column space of A, it follows that for some vector ω, $u = A\omega$. Or that the projection of ω onto the range space $R(A)$ is $A\omega$. Then,

$$A(A^TA)^{-1}A^Tu = A(A^TA)^{-1}(A^TA)\omega = A\omega = u$$

Therefore,

$$Proj_{R(A)}(b) = A(A^TA)^{-1}A^Tb$$

Solution

And so,

$$Proj_{R(A)}(b) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \left(\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 14 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 6 \end{bmatrix}$$

Problem 2:

A rotation of a vector in 3D can be represented with a vector $\omega = \begin{bmatrix}\omega_1 & \omega_2 & \omega_3\end{bmatrix}^T$ (referred to as the equivalent angle and axis representation). The length of ω is known as the equivalent angle, denoted by $\overline{\omega}$. and the direction of ω is known as the equivalent axis.
The vector ω can also be written as a 3×3 skew-symmetric matrix,

$$\Omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

The 'rotation matrix' R is related to Ω via $R = e^{\Omega}$ where e^{Ω} represents a 'matrix exponential', whose Taylor Series form is

$$e^{\Omega} = I + \Omega + \frac{1}{2!}\Omega^2 + ... + \frac{1}{n!}\Omega^n + ...$$

Find the expression of R in polynomial form using Cayley Hamilton.

Consider the matrix,

$$\Omega = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$$

Then by Taylor series expansion,

$$e^{\Omega} = \sum_{n=0}^{\infty} \frac{\Omega^n}{n!}$$

Then, by Cayley Hamilton theorem,

$$\Omega^k = c_0(k) I_{3\times 3} + c_1(k)\Omega + c_2(k)\Omega^2$$

And so the exponential can be written in the form,

$$e^{\Omega} = \sum_{n=0}^{\infty} \frac{\Omega^n}{n!} = I_{3\times 3}\sum_{n=0}^{\infty}\frac{1}{n!}c_0(k) + \Omega\sum_{n=0}^{\infty}\frac{1}{n!}c_1(k) + \Omega^2\sum_{n=0}^{\infty}\frac{1}{n!}c_2(k)$$

Next, note the eigen values of Ω,

$$|\Omega - \lambda I_{3\times 3}| = 0$$

$$0 = |\Omega - \lambda I_{3\times 3}| = \begin{vmatrix} -\lambda & -\omega_3 & \omega_2 \\ \omega_3 & -\lambda & -\omega_1 \\ -\omega_2 & \omega_1 & -\lambda \end{vmatrix} = -\lambda^3 - \lambda\omega_1^2 - \lambda\omega_3^2 - \lambda\omega_2^2 = -\lambda(\lambda^2 + \omega_1^2 + \omega_3^2 + \omega_2^2)$$

Implies,

$$\lambda = 0, \quad \pm\sqrt{\omega_1^2 + \omega_3^2 + \omega_2^2}i = \pm\overline{w}i$$

Lastly, because λ must also satisfy its exponential sum, it follows,

$$e^{\lambda_i} = b_0 + b_1\lambda_i + b_2\lambda^2$$

Implies,

i) $\lambda = 0$

$$e^0 = 1 = b_0 + b_1(0) + b_2(0)^2 \longrightarrow b_0 = 1$$

ii) $\lambda = \overline{w}i$

$$e^{\overline{w}i} = 1 + b_1(\overline{w}i) - b_2(\overline{w}^2)$$

$$b_1 = -\frac{e^{\overline{w}i} - 1}{\overline{w}}i - b_2\overline{w}i$$

iii) $\lambda = -\overline{w}i$

$$e^{-\overline{w}i} = 1 - b_1(\overline{w}i) - b_2(\overline{w}^2)$$

$$e^{-\overline{w}i} = 1 - (-\frac{e^{\overline{w}i} - 1}{\overline{w}}i - b_2\overline{w}i)(\overline{w}i) - b_2(\overline{w}^2)$$

$$b_2 = -\frac{e^{-i\overline{w}} + e^{i\overline{w}} - 2}{2\overline{w}^2}$$

Then lastly, substituting in b_1 equation,

$$b_1 = -\frac{e^{\overline{w}i} - 1}{\overline{w}}i - b_2\overline{w}i$$

$$b_1 = -\frac{e^{\overline{w}i} - 1}{\overline{w}}i - (-\frac{e^{-i\overline{w}} + e^{i\overline{w}} - 2}{2\overline{w}^2})\overline{w}i$$

$$b_1 = \frac{-ie^{i\overline{w}} + ie^{-i\overline{w}}}{2\overline{w}}$$

Therefore, it follows from Cayley hamilton,

$$e^{\Omega} = I_{3\times 3} + \frac{-ie^{i\overline{w}} + ie^{-i\overline{w}}}{2\overline{w}}\Omega - \frac{e^{-i\overline{w}} + e^{i\overline{w}} - 2}{2\overline{w}^2}\Omega^2$$

Solution

And so for $\omega = \begin{bmatrix}\omega_1 & \omega_2 & \omega_3\end{bmatrix}^T$ the matrix $R = e^{\Omega}$ can be written in polynomial form by,

$$e^{\Omega} = I_{3\times 3} + \frac{-ie^{i\overline{w}} + ie^{-i\overline{w}}}{2\overline{w}}\Omega - \frac{e^{-i\overline{w}} + e^{i\overline{w}} - 2}{2\overline{w}^2}\Omega^2$$

Problem 3:

Measurement failure detection: A group of n parameters (data points), $x_1, ..., x_n$ is measured by a group of m sensors giving m measurements $y_1, ..., y_n$ The sensing model is given as $y = Ax$ where $A \in R^{m \times n}$ where A_{ij} contains the sensitivity of the i^{th} sensor to the j^{th} unknown data point. The sensitivity matrix A is known, $x \in R^n$ is to be found from the measurements. However, some sensors may be faulty, giving a vector $z \in R^m$ that does not match the data and the sensing model. Assume that at most one sensor may have failed (we don't know which one has failed, or whether any have failed). Thus, the vector we have obtained from the sensors, $z \in R^m$ may differ from the true measurement $y \in R^m$ in one entry at most. If all sensors are operating correctly, we have $z = y$. If the k^{th} sensor fails (where $k \in \{1, 2, ..., m\}$ we have $z_i = y_i$ for all $i \neq k$ except for $i = k$. The numerical values of the matrix A and vector Z are given in the file $sensor_data$ posted on Canvas. Determine which sensor has failed, or if no sensors have failed. Explain your reasoning and submit the MATLAB code that identifies the faulty sensor.

Note, if z is in the range space of A then $z = y$. Also, if it is not, then there must be a fault sensor.
And so, the fault sensor can be found by removing output rows of z and the corresponding sensor sensitivity data in A one by one and checking if the modified vector z^* is in the range space of modified A^*.
Iterating through each row, because there can only be exactly one fault filter, the first row index removed that is incident with z^* being in the range space of A^* is the fault sensor.

```
%% Problem 3

A = [ ...
    1.164953510500657 -0.360029625711573 0.375041023696104 -0.557093642241282 -0.507700386669636 0.924159404893175 0.039884852845797 -0.620214209475792 0.438705097860831 0.670291996969230;
    0.626839082632431 -0.135576294466487 1.125161817875028 -0.336705699002853 0.885299448191509 -1.814114702851241 -2.482842514256541 0.237148765008739 -1.247344316401997 0.420146041651794;
    0.075080154677683 -1.349338480385175 0.728641591773905 0.415227462723156 -0.248093553237236 0.034973320285167 1.158654705247901 -1.586846990031033 0.324666916936102 -2.872751269668520;
    0.351606902768522 -1.270449896283403 -2.377454293765433 1.557813537123208 -0.726248999742084 -1.807862060321251 -1.026279466693260 -0.401484809800359 0.390070410090458 1.685874080406989;
    -0.696512535163682 0.984570272925253 -0.273782415743900 -2.444298897865560 -0.445040300996161 1.028192546045777 1.153486988237923 -0.770692268923938 -0.405138316773605 0.027924553523994;
    1.696142480747077 -0.044880613828856 -0.322939921204497 -1.098195387799324 -0.612911120338436 0.394600308811932 -0.786456613020222 -0.262680506066512 0.292314877283450 -0.902030581228208;
    0.059059777981351 -0.798944516671106 0.317987915650739 1.122647857944875 -0.209144084593638 0.639405642088516 0.634808587961936 0.976489543659970 2.565910242123806 -2.053257491526201;
    1.797071783694818 -0.765172428787515 -0.511172207780700 0.581667258045274 0.562147834450359 0.874212894863609 0.820409761532064 0.977815041129280 -0.457815643580367 0.089086297675464;
    0.264068528817227 0.861734897324192 -0.002041345349433 -0.271354295524753 -1.063922887881042 1.752401730329559 -0.176026510455600 1.170021110265055 -1.610827014289158 2.087099131649750;
    0.871673288690637 -0.056225124358975 1.606510961119237 0.414191307229504 0.351588948379816 -0.320050826432138 0.562473874646301 0.159310862415417 -2.669523782410902 0.365118460310679;
    -1.446171539339335 0.513478173674302 0.847648634500925 -0.977814227461400 1.132999926008681 -0.137413808144866 -0.127442875395491 0.499520851464531 -0.759696648513815 0.846105526166482;
    -0.701165345682908 0.396680865935824 0.268100811901575 -1.021466173866152
```

```
        0.149994248007729    0.615769628086716    0.554171560978313   -1.055375070659330
        -0.674720856431937   -0.184537657075523;
    1.245982120437819    0.756218970285488   -0.923489085784077    0.317687979852042
        0.703144053247466    0.977894069845197   -1.097344319221644   -0.450743202815186
        -1.171687194533551    1.030714423869546;
   -0.638976995013557    0.400486023191097   -0.070499387778694    1.516107798150034
        -0.052411584998689   -1.115347712205141   -0.731301400074801    1.270378242169987
        2.032930016155204   -1.527622652429381;
    0.577350218771609   -1.341380722378574    0.147891351014747    0.749432452588256
        2.018496124007770   -0.550021448804486    1.404731919616814    0.898693600923036
        0.968481047964462    0.964938959209115];

ytilde = [ ...
    0.293700010391366;
   -0.548030198505630;
    0.003532110461013;
    1.375859546156174;
   -6.752682998496523;
    1.190484875889765;
    8.782196150345506;
    1.911972855063559;
   -1.462868211077097;
   -4.433624854460799;
   -1.723404706120404;
   -4.547493026790328;
   -0.109245813786955;
    8.033526684801210;
    1.782619515709060];


if rank([A ytilde]) ~= rank(A)
    fault_index = 0;
    for i =  1:length( A )-2
        y = [ytilde(1:i) ; ytilde(i+2:length(ytilde))];
        B=[A(1:i, :) ; A(i+2:length(A), :)];
        if rank([B y]) == rank(B)
            fault_index = i+1;
            break;
        end
    end
end

disp(fault_index);
```

Solution

Iterating through the entire vector, sensor 11 was found to be faulty.

Problem 4:

Consider the matrix,

$$S = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 4 & 3 \\ 0 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix}$$

0.2 Find the rank of this matrix

Consider the maximum rank of S is 3, also note that $rank(S) = rank(S^T)$. now proceeding by row reduction on S^T,

$$\begin{bmatrix} 1 & 3 & 5 & 0 & 1 \\ 2 & 4 & 4 & 2 & 3 \\ 3 & 5 & 3 & 4 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 0 & 1 \\ 0 & 2 & 6 & -2 & -1 \\ 0 & 4 & 12 & -4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 0 & 1 \\ 0 & 2 & 6 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

And so the matrix S has two linearly independent columns, and thus has rank 2.

Solution

$$Rank(S) = 2$$

0.3 Compute the matrices S^TS and SS^T then find the eigenvalues and eigenvectors (Note: use MATLAB to perform the required operations; see command eig in Matlab).

Consider the matrices S^TS and SS^T by matrix multiplication,

$$SS^T = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 4 & 3 \\ 0 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 & 0 & 1 \\ 2 & 4 & 4 & 2 & 3 \\ 3 & 5 & 3 & 4 & 5 \end{bmatrix} = \begin{pmatrix} 14 & 26 & 22 & 16 & 22 \\ 26 & 50 & 46 & 28 & 40 \\ 22 & 46 & 50 & 20 & 32 \\ 16 & 28 & 20 & 20 & 26 \\ 22 & 40 & 32 & 26 & 35 \end{pmatrix}$$

And

$$S^TS = \begin{bmatrix} 1 & 3 & 5 & 0 & 1 \\ 2 & 4 & 4 & 2 & 3 \\ 3 & 5 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 4 & 3 \\ 0 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix} = \begin{pmatrix} 36 & 37 & 38 \\ 37 & 49 & 61 \\ 38 & 61 & 84 \end{pmatrix}$$

Now using matlab to compute eigen vectors and eigen values,

```
S = [1  2  3
     3  4  5
     5  4  3
     0  2  4
     1  3  5];


[V1, D1] = eig(S'*S);

[V2, D2] = eig(S*S');
```

Results in the following eigen values and eigen vectors.

1. S^TS :

$$\lambda_1 = 0, \qquad \lambda_2 = 15.433 \qquad \lambda_3 = 153.567$$

With corresponding eigen vector matrix,

$$V = \begin{bmatrix} -0.4082 & -0.8160 & 0.4093 \\ 0.8165 & -0.1259 & 0.5635 \\ -0.4082 & 0.5642 & 0.7176 \end{bmatrix}$$

2. SS^T :

$$\lambda_1 = 0, \qquad \lambda_2 = 0 \qquad \lambda_3 = 0 \qquad \lambda_4 = 15.433 \qquad \lambda_5 = 153.567$$

With corresponding eigen vector matrix,

$$V = \begin{bmatrix} 0.8191 & -0.4133 & -0.2107 & 0.1591 & 0.2977 \\ -0.5133 & -0.6391 & 0.0388 & -0.0332 & 0.5705 \\ 0.1229 & 0.3971 & -0.1208 & -0.7359 & 0.5207 \\ -0.1979 & 0.3795 & -0.6725 & 0.5104 & 0.3226 \\ 0.1064 & 0.3451 & 0.6980 & 0.4143 & 0.4590 \end{bmatrix}$$

0.4 Using your answer in part b), find the SVD for the matrix S (Don't use the MATLAB command SVD unless to compare with your answer).

Consider the singular value decomposition of S takes the form,

$$S = U\Sigma V^T$$

Where U is the left singular vectors of S, Σ is the singular values of S, and V is the right singular values of S. And so,

$$S = \begin{bmatrix} \sigma_1 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \sigma_n \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \sqrt{\lambda_n} \end{bmatrix}$$

Where $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n$.
And note, from part b, the non zero eigen values are,

$$\lambda_1 = 15.4333, \qquad \lambda_2 = 153.567$$

And so,

$$\sigma_2 = \sqrt{15.4333} = 3.935, \qquad \sigma_1 = \sqrt{153.567} = 12.392$$

Then,

$$\Sigma = \begin{bmatrix} 12.392 & 0 & 0 \\ 0 & 3.935 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Next, V is the corresponding eigen space of S^TS, and so again by part B,

$$V = \begin{bmatrix} 0.4093 & -0.8160 & -0.4082 \\ 0.5635 & -0.1259 & 0.8165 \\ 0.7176 & 0.5642 & -0.4082 \end{bmatrix}$$

Lastly, U is the eigen values of SS^T,

$$U = \begin{bmatrix} 0.2977 & 0.1591 & 0.8191 & -0.4133 & -0.2107 \\ 0.5705 & -0.0332 & -0.5133 & -0.6391 & 0.0388 \\ 0.5207 & -0.7359 & 0.1229 & 0.3971 & -0.1208 \\ 0.3226 & 0.5104 & -0.1979 & 0.3795 & -0.6725 \\ 0.4590 & 0.4143 & 0.1064 & 0.3451 & 0.6980 \end{bmatrix}$$

And so it follows that,

$$U\Sigma V^T = \begin{bmatrix} 0.2977 & 0.1591 & 0.8191 & -0.4133 & -0.2107 \\ 0.5705 & -0.0332 & -0.5133 & -0.6391 & 0.0388 \\ 0.5207 & -0.7359 & 0.1229 & 0.3971 & -0.1208 \\ 0.3226 & 0.5104 & -0.1979 & 0.3795 & -0.6725 \\ 0.4590 & 0.4143 & 0.1064 & 0.3451 & 0.6980 \end{bmatrix} \begin{bmatrix} 12.392 & 0 & 0 \\ 0 & 3.935 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0.4093 & -0.8160 & -0.4082 \\ 0.5635 & -0.1259 & 0.8165 \\ 0.7176 & 0.5642 & -0.4082 \end{bmatrix}^T$$
$$= \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 5 & 4 & 3 \\ 0 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix}$$

0.5 Set the smaller singular value to 0 and compute the 1-dimensional approximation to the matrix S (this is sometimes done for dimensionality reduction of large matrices. Setting the l smallest singular values to 0 then we can also eliminate the corresponding l columns of U and V.

Consider if the smaller singular value is set to 0 then the Σ matrix becomes,

$$\begin{bmatrix} 12.392 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

And so the partial decomposition because,

$$S \approx \sigma_1 U_1 V_1^T = 12.392 \begin{bmatrix} 0.2977 \\ 0.5705 \\ 0.5207 \\ 0.3226 \\ 0.45 \end{bmatrix} \begin{bmatrix} 0.4093 & 0.5635 & 0.7176 \end{bmatrix} = \begin{pmatrix} 1.50994 & 2.07880 & 2.64729 \\ 2.89360 & 3.983739886 & 5.07317 \\ 2.64101 & 3.63599 & 4.63032 \\ 1.63624 & 2.25268 & 2.86872 \\ 2.28242052 & 3.1423014 & 4.00162464 \end{pmatrix}$$

0.6 How much energy of the original singular values is retained by the 1-dimensional approximation? (Energy can be represented as the sum of the squares of the singular values).

Consider first the total energy of the system,

$$\sum_{\sigma \in \Sigma} \sigma_i^2 = 12.392^2 + 3.935^2 = 169$$

Additionally,

$$12.392^2 = 153.561664$$

And so the retained energy of the 1 dimensional approximation is,

$$\frac{153.561664}{169} \times 100 = 90.86\%$$

And so about 90.86% of the original energy is retained by the reduced approximation.