Computational Assignment 1 - Bungee Jumping

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EXECUTIVE SUMMARY

The objective is to analyze how the bungee cord's material properties affect the jumper's motion. The height of the jump, the weight of the person, the stiffness of the bungee cord material, the damping effect of the material, and the air resistance all affect the motion of the jumper. The equation of motion was derived by using Newton's second law and summing the forces in the vertical direction. Two main equations were formed, one considering the damping from the cord being stretched and the air drag and the other considering when the cord was not in tension, the person only experiencing damping from air resistance. Based on the general equation of motion, the natural frequency of the jumper depends on the cord stiffness (k), and the mass of the person. As expected, a higher stiffness, k, results in a higher natural frequency. A lower cord stiffness requires a higher initial jump height because it allows the cord to stretch more and the person to fall further. A higher cord stiffness would result in a smaller initial jump height required, but the person is more likely to experience injury from a higher acceleration and force acting on them.

METHOD

The first step in creating the equation of motion for the bungee jumper is to create a free-body diagram (FBD). As seen in **Figure 1**, there are three forces acting on the jumper throughout most of the fall; the jumper's weight, total damping, and the spring force in the cord. The result of Newton's Second Law equation from the FBD is **Equation 1**. Using the motion equation and the "solve_ivp" scipy function in Python, a second-order differential equation can be solved for position, velocity, and acceleration. To analyze the effect of the cord's stiffness on the motion of the jumper, 10 evenly spaced k values between 40 N/m and 240 N/m were used in the position, velocity, and acceleration equations and plotted as functions of time. Three plots were then made for the material loss factor (η), seen in **Equation 2**, with values ranging from 0.15 to 0.30. For clarity, the simulations assume an initial jump height of 100m with the cord considered taut at 82m from the ground (x=0m).

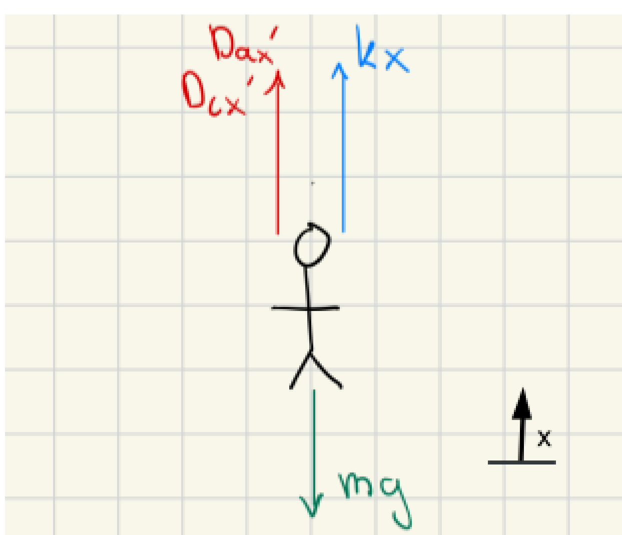

Figure 1: Free Body Diagram of Bungee Jumper

An important factor to consider with this problem is that the damping and spring force caused by the cord are inactive if the jumper is within 18 meters of the platform. 18 meters is the length of the cord so it will only be in tension after that distance. This means in **Equation 2**, D is equal to D_a within 18 meters.

Equations:
Motion Equation:

$$\Sigma F = mx'' = -mg + k(x_0 - l - x) + Dx' \qquad \text{Equation 1}$$

- x_0: Initial Position (m)
- x: Position (m)
- x': Velocity (m/s)
- x'': Acceleration (m/s^2)
- l: Length of Bungee Cord
- m: Mass (kg)
- k: Cord Stiffness (N/m)
- D: Total Damping Coefficient (Ns/m)

Total Damping Coefficient:

$$D = D_C + D_a = \frac{k\eta}{\omega_n} + D_a \qquad \text{Equation 2}$$

- D_C: Viscous Damping Coefficient (Ns/m)
- D_a: Air Resistance Equivalent Damping (8 Ns/m)

To better understand and analyze the jumper's motion, **Equation 3** showcases the natural frequency in terms of the equivalent cord stiffness and mass in the system. Meanwhile, **Equation 4** describes the cord's spring constant in terms of its material and physical properties.

Natural Frequency:

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \text{Equation 3}$$

Cord Stiffness:

$$k = \frac{EA}{l} \qquad \text{Equation 4}$$

- E: Elastic Modulus (MPa)
- A: Cross-sectional area of Bungee Cord (m^2)
- l: Length of Bungee Cord (m)

PROCEDURE

A model was constructed to simulate the motion of the bungee jumper using the initial value problem (IVP) solver built into the Python Scipy library. First, a set of constants and initial values were set such that the simulation could use some values as constants and vary others over a provided range and step. Arrays were constructed to contain the values for each combination of the spring constant and loss factor. Each simulation was run and the arrays were propagated using the solution found with the IVP solver. A free fall curve was plotted to show motion without elastic force or damping resistance. A horizontal line representing the point of zero elastic deflection was plotted. The stored values were then printed to the console and stored as a CSV for processing.

The equilibrium points approach the point of zero elastic deflection (the rope length from the jump height) as the k value of the spring increases. This phenomenon follows the understanding that stiffer springs will deflect less under equal loading conditions.

In the reported times of maximum acceleration, the third reported value aligning with k=40 and η=0.30 shows the maximum acceleration at t=0. This discrepancy can be attributed to the increasing η value which led to the first oscillation having a lower maximum magnitude of acceleration than the free fall region. The reported time value represents the initial acceleration due to gravity, the point of minimum velocity, and before the bungee cord is engaged. This is to be expected because, during the first oscillation, the peak remains below the equilibrium point of the spring. Therefore, the body remains in damped harmonic motion after the initial free fall. As seen in the free fall region of the acceleration chart of **Figure A**, the acceleration begins only due to gravity; however, as the body falls and accelerates, the air resistance increases proportionally to the velocity, leading to an exponential decrease in the magnitude of the acceleration as the downward velocity of the jumper leads to a positive force on the body.

This implies several things: that the acceleration is at a maximum in the free fall region, that once the body enters the damped harmonic motion region it can never re-enter the free fall region, and that within the initial free fall region, the acceleration is at its maximum at the initial state. Therefore, the reported time value for the acceleration must be correct.

For reported values, both parameters were varied simultaneously, but to aid in visualizing trends, each was plotted while the other was fixed, as depicted in **Figures 2 and 3**. The simulations were run for 20 seconds as these would allow for quicker computing times, and the jumper would be considered to be moving slow enough to be recovered safely.

RESULTS and DISCUSSION

With the equations of motion, velocity, and acceleration solved using Python's matplotlib library, **Figures 2 and 3** show the position of the jumper as a function of time with varying K and η values. As per the assumptions, these figures include a "free fall" for the first 18m of descent from an initial position of 100m from the ground (x=0m), as well as a sample "free fall" with no bungee cord for reference. The 18m when the cord begins to act on the jumper are also marked by a dotted line in both graphs.

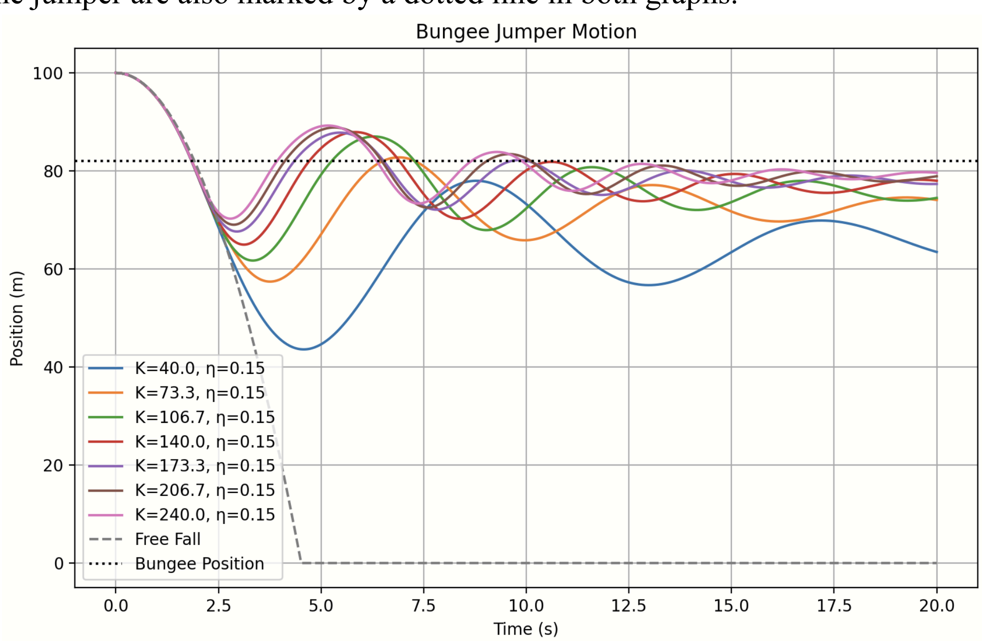

Figure 2: Bungee Jumper position as a function of time with varying K values

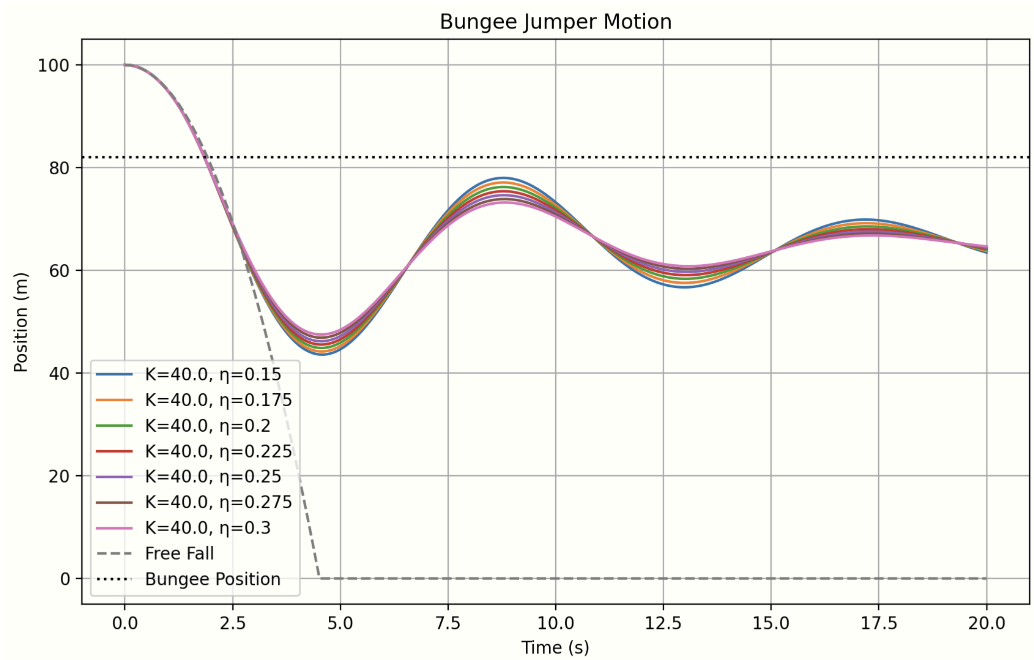

Figure 3: Bungee Jumper position as a function of time with varying η values

From **Figure 2**, it can be seen that there is a lot of variation in both the amplitude and period of the oscillatory motion of the jumper. As these values of k are bounded between 40N/m and 240N/m and η between 0.15 and 0.30, a minimum fall position, maximum velocity, and maximum acceleration can all be identified using **Table A** and interpreting **Figures 2, 3, and A**. At k=40N/m and η = 0.15, the absolute lowest point the jumper reaches is 45.8m from the ground after 4.8s, his maximum acceleration of 11.4m/s^2 is obtained after 4.4s and his maximum absolute velocity is 18.7m/s at 2.5s. In terms of his position, this would mean that maximum acceleration is when he is being pulled up before the first bungee recoil and maximum speed is at the end of the freefall. Furthermore, all of these maxima emerge from the first ½ period of motion, further insinuating the importance of the cord's stretching abilities to dictate the subsequent oscillations. Additionally, from varying k and η coefficients, k was noticed to have a larger impact on the behavior of the jump following a trend of lower maximum acceleration and bigger fall height as k decreased. From **Table A**, this is seen by the k=240N/m value resulting in a maximum acceleration of 27.1m/s^2 and a minimum tower height of 28.2m, which are significantly more dangerous than the previously seen k=40N/m values. Following this line of reasoning, it would be reasonable to assume that is the reason why jumpers can fall without sustaining serious injury from a high tower. The force experienced at any moment in time by the jumper is slowly being "lost" in the cord the more it stretches and results in a less violent jolt when it comes time to reverse the direction of their trajectory. Meanwhile, maximum velocity remained more or less constant throughout all trials (around -18m/s) at the tail end of the freefall.

As such, the simulation showed that an increasing K coefficient would lead to a higher maximum acceleration, making k=40N/m the "safest" option while still allowing for the jumper to experience the longest fall. This is consistent, as k depends on the elastic modulus, cross-sectional area, and the original length of the cable. The original cable length is given as 18m. Natural rubber or other elastomers are typically used for the core of the cable. The cable is typically covered by a woven synthetic material such as nylon to protect the rubber from heat/sunlight. Based on these two considerations, the material selection for the core is narrowed down to natural rubber, which has an elastic modulus in the range of 3.45Mpa to 24.1Mpa [1].

Another important consideration is the possibility of stress relaxation and hysteresis of the bungee cable, which would result in permanent deformation and weakening of the cable and safety concerns. "Hysteretic" would refer to the uneven forces being exerted on the person between falling downward and upward. It also is dependent on the uneven compression and tension rope friction properties (i.e. the rope can compress easier than stretch or vice-versa). Since cord manufacturing affects η and K, the mechanical properties of the rope affect the studied displacement, velocity, and acceleration. Therefore, the manufacturing process of the rubber should involve vulcanization of the rubber to reduce the possibility of stress relaxation. Vulcanization introduces cross-links between the polymer chains, strengthening the material. Research proves that more crosslinks produced during the vulcanization process result in the material being capable of withstanding more stress without permanent deformation[2]. Vulcanization also leads to higher elasticity (E=1.5Mpa [3]), and consequently, a lower stiffness(k). Furthermore, vulcanization improves the cable's ability to withstand hysteresis, because the cable will also be stronger under compression after vulcanization[2]. In particular, carbon-containing cross-links are proven to increase the lifespan of vulcanizates[2] compared to other materials typically used to vulcanize natural rubber. Assuming the diameter of the cord is around 2.5cm, the resulting stiffness value for a fixed cord length of 18m, using **Equation 4**, is 41 N/m. The k value corresponds to the lowest k value from previous evaluations, indicating that it will meet the vertical distance constraints too. Based on the figures, the k value of 40 N/m results in enjoyable conditions for the jumper, and the jumper will experience minimal amounts of upward acceleration. The maximum acceleration the jumper will reach is just under $15 m/s^2$, or less than 1.5g, which is less than a person will experience on a typical roller coaster[4], but still more than they experience on a daily basis. They would also experience only around 3 ups/downs within 20 seconds, the assumed time it takes for their velocity to be low enough to be safely recoverable. Thus, the jumper will have fun without putting their safety at risk.

Finally, ride expectations also affect the model in unaccounted-for ways. They have to do with the behavior of the person dropping. As this is a person, they can flail their arms in/out to decrease/expand their surface area in contact with the air. This could make the assumed constant damping coefficient of the air (Da=8) change at a moment's notice - impacting the simulation by slowing or quickening descent.

CONCLUSIONS

In conclusion, the jumper can complete the jump without sustaining injury by observing the parameters that influence maximum acceleration and velocity. Such parameters include the cord stiffness (k), the cord length, the jumper's vertical starting position, the cord's loss factor, and the manufacturing and mechanical properties of the cord. Ultimately, a lower K value and higher η value would mean that the jumper experiences less maximum acceleration while also coming to the equilibrium position as quickly as possible. This would ensure the jumper has a smoother experience that also comes to an end quicker as opposed to violently being pulled back and suspended in oscillation for longer. Overall, taking into account the safety and enjoyment of the person jumping, it is most ideal to choose a vulcanized natural rubber cord with a stiffness of around 40N/m, and for the jumper to start at an absolute minimum height of 55 meters. Then, the jumper will not be injured due to acceleration or due to hitting the ground.

APPENDIX

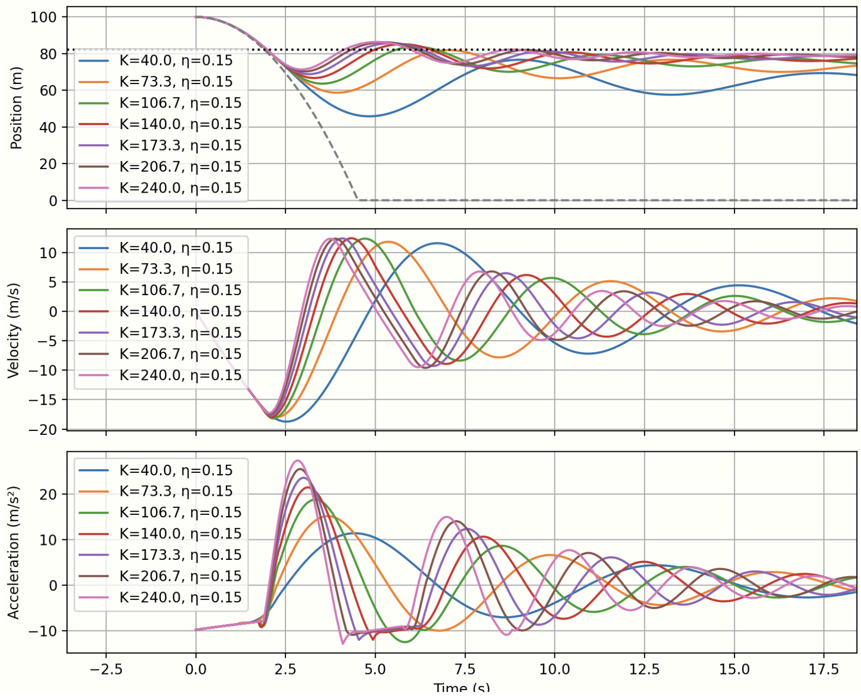

Figure A: Modeled Equations of Motion, Velocity, and Acceleration using varied K and η values.

Table A: Numerical values of select cases in the simulation with accompanying maximum acceleration, velocity, and deflection

Eta	*K (N/m)*	*Max_acc (m/s^2)*	*Max_acc _time (s)*	*Max_vel (m/s)*	*Max_vel_ time (s)*	*Min_pos (m)*	*Min_pos_ time (s)*	Minimum Tower Height req. (m)	Cord Stretched Length (m)
0.150	*40.0*	*11.4*	*4.4*	*18.7*	*2.5*	*45.8*	*4.8*	54.2	36.2
0.225	*40.0*	*10.5*	*4.3*	*18.1*	*2.5*	*47.8*	*4.8*	52.2	34.2
0.300	*40.0*	*9.8*	*0.0*	*17.6*	*2.4*	*49.5*	*4.8*	50.5	32.5
0.150	*68.6*	*14.7*	*3.7*	*18.0*	*2.3*	*57.5*	*4.0*	42.5	24.5
0.300	*68.6*	*12.5*	*3.6*	*16.8*	*2.2*	*60.3*	*4.0*	39.7	21.7
0.150	*97.1*	*18.0*	*3.4*	*18.3*	*2.2*	*62.4*	*3.6*	37.6	19.6
0.300	*97.1*	*15.9*	*3.2*	*17.8*	*2.1*	*64.3*	*3.6*	35.7	17.7
0.150	*240.0*	*27.4*	*2.8*	*17.3*	*2.0*	*71.3*	*2.9*	28.7	10.7
0.300	*240.0*	*27.1*	*2.7*	*18.0*	*2.1*	*71.8*	*2.9*	28.2	10.2

REFERENCES

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