MEEN 432 –Automotive Engineering

Fall 2026

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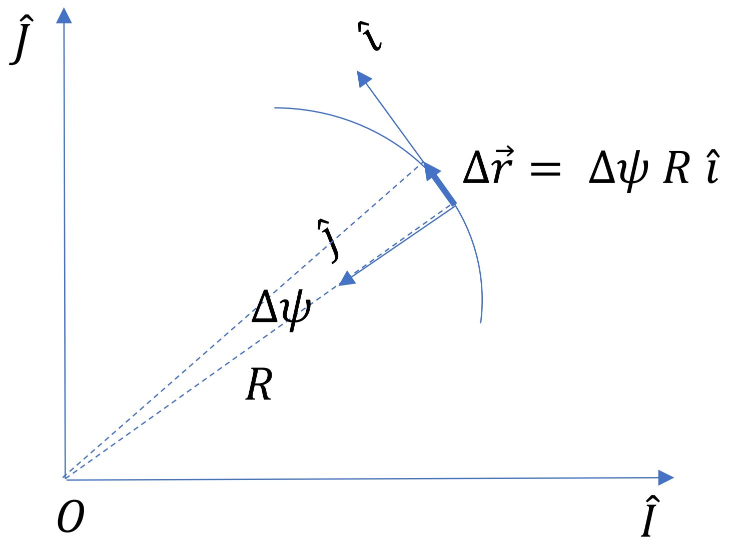
Lecture 6: Vehicle Lateral Dynamics

- Vehicle Dynamics with Simplified tire forces
- Motion in a circle, and Under-steer, over-steer and neutral-steer characterization
- Ackerman steering

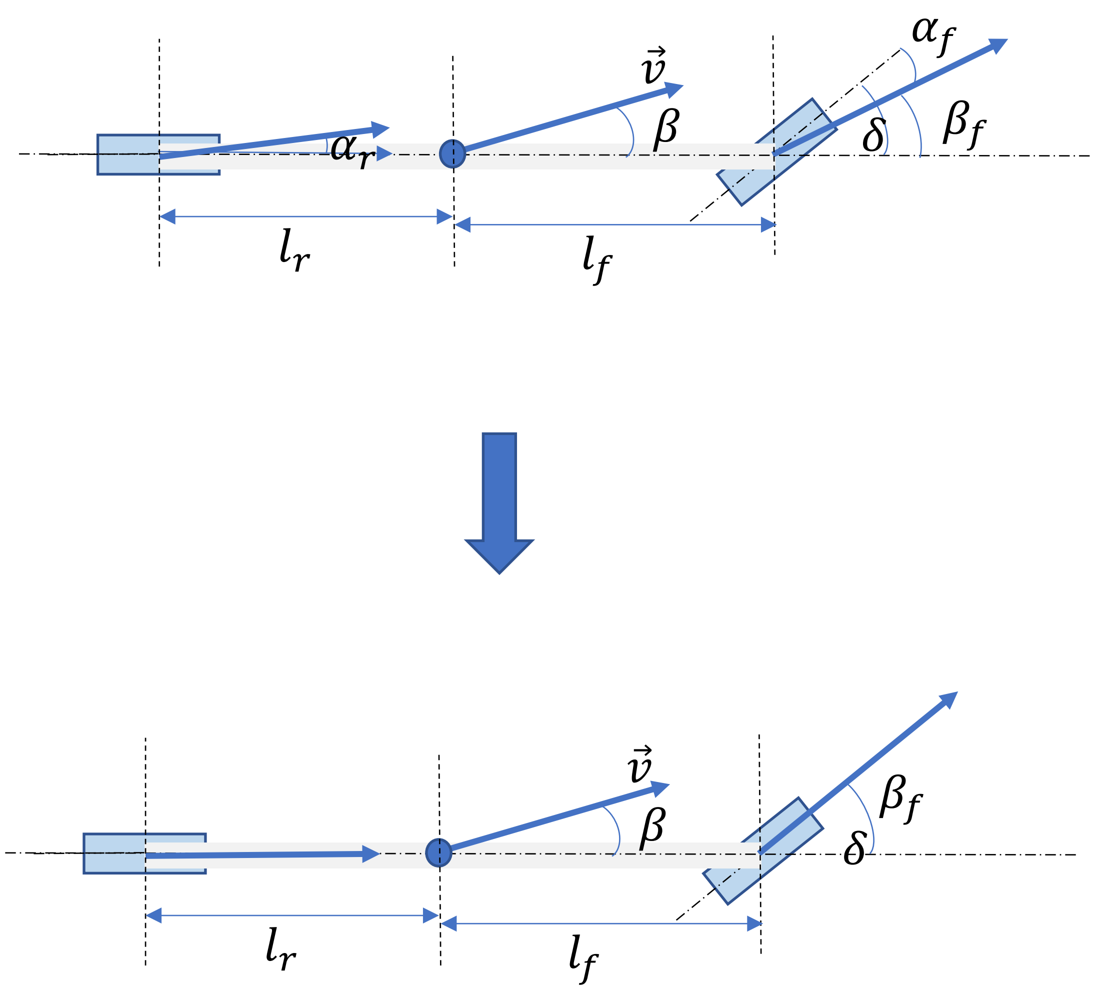
Motion in a circular arc at constant speed

- Consider the case when the vehicle is moving on a circular arc of radius R at some constant speed v
  - Observe that $v_x = v$
- Consider the vector $\vec{r}$ from the origin O of the inertial frame to the vehicle's center of mass
  - This vector rotates by an angle $\Delta\psi$ in a small amount of time Δt
  - The change in the vector is: $\Delta\vec{r} = \Delta\psi\, R\, \hat{\imath}$
  - The rate of change of this vector is the vehicle velocity $v_x \hat{\imath} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \dot{\psi} R\, \hat{\imath}$
  - Or, $v = v_x = \dot{\psi} R$



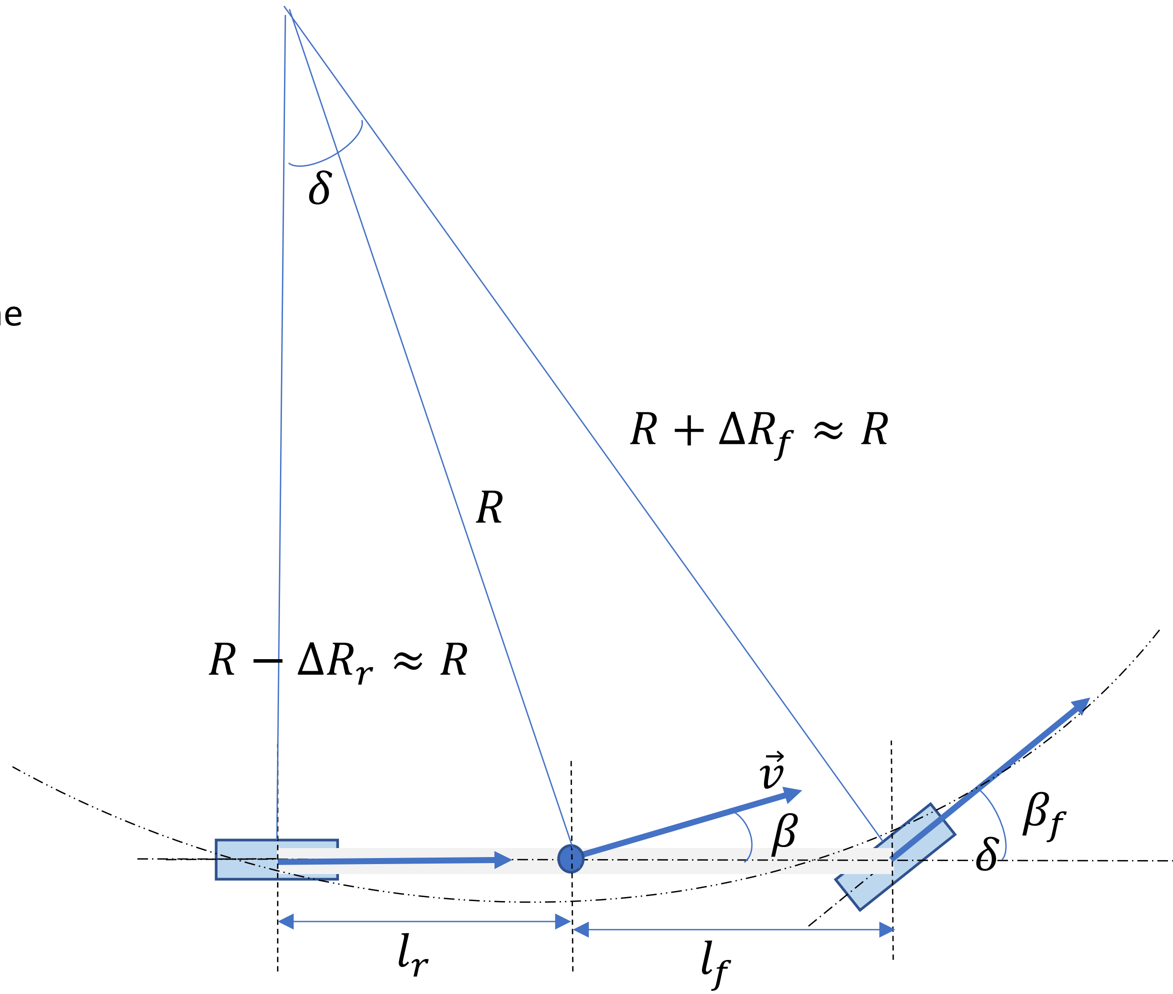
Constant Velocity Circular Motion – Kinematic Analysis

- Consider a near perfect tire that will resist any lateral movement i.e.:
  - Cornering Stiffness → infinity)
  - Or $\alpha_f = \alpha_r = 0$
  - And $v_y = 0$ or $v = v_x$
  - Then the tires can only have a velocity along the longitudinal axis
- Then, motion on a circular path requires the tires to be tangential to the



Constant velocity circular motion – Kinematic Analysis

- Let the circular path be of radius R
  - Subtended angle at center of rotation is δ
- Assume radius is much larger than the wheel base of the car:
  - $R \gg l_f + l_r$
  - Then we can assume that the distance from the center of rotation to the wheels are the same and equal to R
- From kinematics,
  - $v = R\,\dot{\psi}$
- From Geometry
  - $l_f + l_r = \delta$ R Or $\delta = \frac{(l_f + l_r)}{R}$ - called the Ackermann angle
- Thus:
  - $\delta = \frac{(l_f + l_r)\dot{\psi}}{v}$
    - Given a linear velocity, steering angle is proportional to angular velocity
    - Alternatively, for a desired angular velocity, we can directly determine how much steering is required



Dynamics in motion in a circular arc at constant speed

- $$\begin{bmatrix} \dot{v}_x \\ \dot{v}_y \\ \dot{\omega} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} v_y\omega + \Sigma F_x/m \\ -v_x\omega + \Sigma F_y/m \\ \frac{\Sigma M}{I} \\ \omega \end{bmatrix}$$ (recall)
- For the special case of motion in a circular arc at constant speed:
  - $\dot{v}_x = 0; \dot{v}_y = v_y = 0; \dot{\omega} = \frac{\dot{v}_x}{R} = 0$
- Thus we have:
  - $\Sigma F_x = 0$ i.e. net force along longitudinal axis should be 0
    - Does it mean no power required from the engine?
  - $\Sigma M = 0$ i.e. net moment about the vehicle should be 0
  - $\Sigma F_y = m\, v_x\omega = m\, v\, \omega = m\frac{v^2}{R}$ centripetal force!
    - Where is this force coming from?

Vehicle Dynamics (Recall)

- $$\begin{bmatrix}\dot{v}_x\\ \dot{v}_y\\ \dot{\omega}\\ \dot{\psi}\end{bmatrix} =$$

$$\begin{bmatrix} v_y\omega + (F_{x_{Wf}}(\lambda_f)\cos\delta - F_{y_{Wf}}(\alpha_f)\sin\delta + \ F_{x_{Wr}}(\lambda_r) - F_{drag} - F_{gravity}\ )/m \\ -v_x\omega + (F_{x_{Wf}}(\lambda)\sin\delta\ + F_{y_{Wf}}(\alpha)\cos\delta + F_{y_{Wr}}(\alpha)\ )/m \\ ((F_{x_{Wf}}(\lambda)\sin\delta\ + F_{y_{Wf}}(\alpha)\cos\delta)\, l_f \qquad - \ F_{y_{Wr}}(\alpha)\cos\delta\ \ l_r)/I \\ \omega \end{bmatrix}$$

- $\alpha_f = \delta - \beta_f = \delta - \tan^{-1}\frac{v_y + l_f\dot{\psi}}{v_x}$
- $\alpha_r = -0 - \ \beta_r = -\tan^{-1}\frac{v_y - l_r\dot{\psi}}{v_x}$

Small Steering and slip Angle approximation

- Assume small $\delta, \alpha_f, \alpha_r$ → small β_f, β_r → $\beta_f \approx \tan^{-1} \beta_f, \beta_r \approx \tan^{-1} \beta_r$
- Then:
  - $\alpha_f = \delta - \beta_f = \delta - \tan^{-1} \frac{v_y + l_f \dot{\psi}}{v_x} = \delta - \frac{v_y + l_f \dot{\psi}}{v_x}$
  - $\alpha_r = -0 - \ \beta_f = -\tan^{-1} \frac{v_y - l_r \dot{\psi}}{v_x} = -\frac{v_y - l_r \dot{\psi}}{v_x}$
- Then
  - $\alpha_f - \alpha_r = \delta - \frac{v_y + l_f \dot{\psi} - v_y + l_r \dot{\psi}}{v_x} = \delta - \frac{(l_f + l_r)\,\dot{\psi}}{v_x} \coloneqq \delta - \frac{L\dot{\psi}}{v_x}$ where L is the wheel base

Constant Circular Motion

- Assume constant longitudinal velocity $v_x > 0$ and constant angular velocity $\dot{\psi}$
  - Recall: This implies a motion in a circle of radius $R = \frac{v_x}{\dot{\psi}}$
- Then
  - $\alpha_f - \alpha_r = \delta - \frac{L\dot{\psi}}{v_x} = \delta - \frac{L}{R}$

This is called the Ackerman Angle

- Or,
  - $\delta = \frac{L}{R} + (\alpha_f - \alpha_r)$
- For a given vehicle design (i.e. the vehicle base) ,and a desired radius of curvature of a turn, the steering is determined by the Ackerman Angle and the difference in the sideslip between the front and rear wheels

Constant Circular Motion (Cornering)

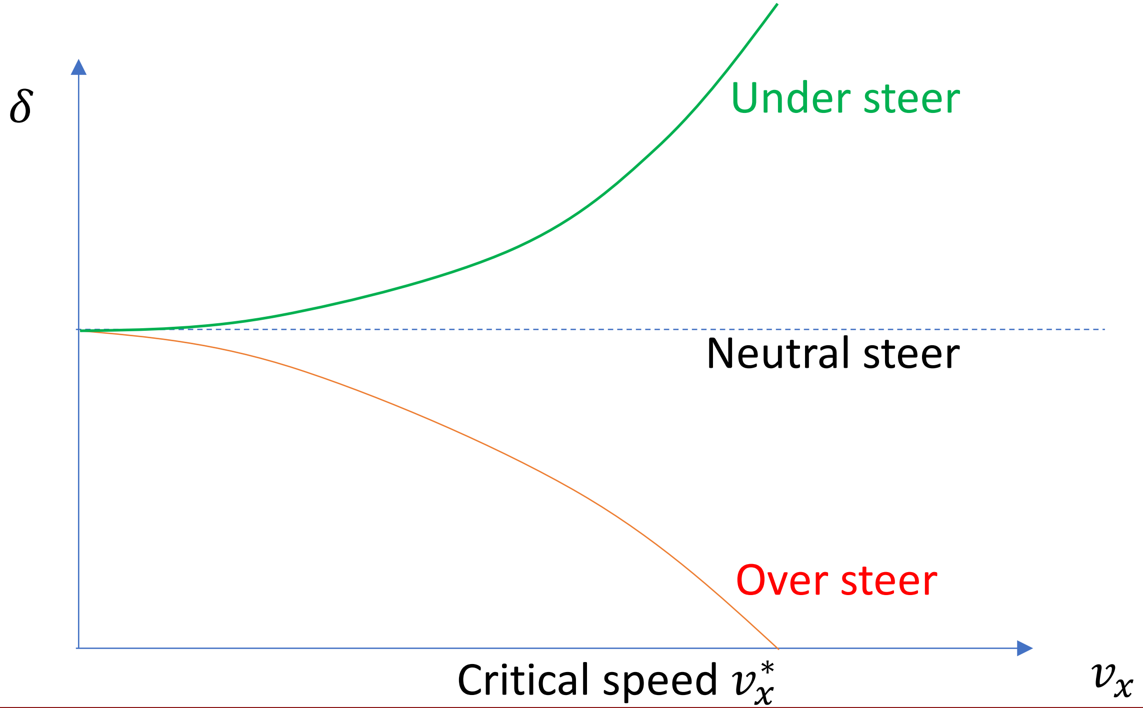
- The small side slip angle and constant velocity assumption also yields the following approximations:
  - linearized representation of tire forces: $F_{yW} = C_\alpha \alpha$
  - Constant velocity implies small F_{x_W} and so we can say $F_{x_W} \sin\delta + F_{y_W} \cos\delta \approx F_{y_W} = C_\alpha \alpha$
- Thus, the dynamics can be written as:
- $$\begin{bmatrix} \dot{v}_y \\ \dot{\omega} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} -v_x\omega + \frac{F_{x_{Wf}}(\lambda)\sin\delta + F_{y_{Wf}}(\alpha)\cos\delta + F_{y_{Wr}}(\alpha)}{m} \\ \frac{(F_{x_{Wf}}(\lambda)\sin\delta + F_{y_{Wf}}(\alpha)\cos\delta)l_f - F_{y_{Wr}}(\alpha)\cos\delta\, l_r}{I} \\ \omega \end{bmatrix} = \begin{bmatrix} -v_x\omega + \frac{C_{\alpha_f}\alpha_f + C_{\alpha_r}\alpha_r}{m} \\ \frac{C_{\alpha_f}\alpha_f l_f - C_{\alpha_r}\alpha_r l_r}{I} \\ \omega \end{bmatrix}$$
- Circular motion assumption: $v_y = 0; \dot{v}_y = 0; \dot{\omega} = 0$
- Then:
  - $C_{\alpha_f}\alpha_f + C_{\alpha_r}\alpha_r = m v_x \omega$ and $C_{\alpha_f}\alpha_f l_f = C_{\alpha_r}\alpha_r l_r$

Constant Circular Motion

- $C_{\alpha_f}\alpha_f + C_{\alpha_r}\alpha_r = mv_x\omega$ and $C_{\alpha_f}\alpha_f l_f = C_{\alpha_r}\alpha_r l_r$
  - $\rightarrow C_{\alpha_f}\alpha_f + C_{\alpha_f}\alpha_f \frac{l_f}{l_r} = mv_x\omega$ Or, $C_{\alpha_f}\alpha_f\left(1+\frac{l_f}{l_r}\right) = \frac{C_{\alpha_f}\alpha_f L}{l_r} = mv_x\omega$
  - Or, $\alpha_f = \frac{m}{L}\ \frac{l_r}{C_{\alpha_f}}\ \omega v_x$
  - Similarly: $\alpha_r = \frac{m}{L}\ \frac{l_f}{C_{\alpha_r}}\ \omega v_x$
- Then: $\delta = \frac{L\omega}{v_x} + \alpha_f - \alpha_r = \frac{L\omega}{v_x} + \frac{m}{L}\left(\frac{l_r}{C_{\alpha_f}} - \frac{l_f}{C_{\alpha_r}}\right)\omega v_x$
- Observing that $v_x = R\,\omega$ we can write:
  - $\delta = \frac{L}{R} + \frac{m}{LR}\left(\frac{l_r}{C_{\alpha_f}} - \frac{l_f}{C_{\alpha_r}}\right) v_x^2$

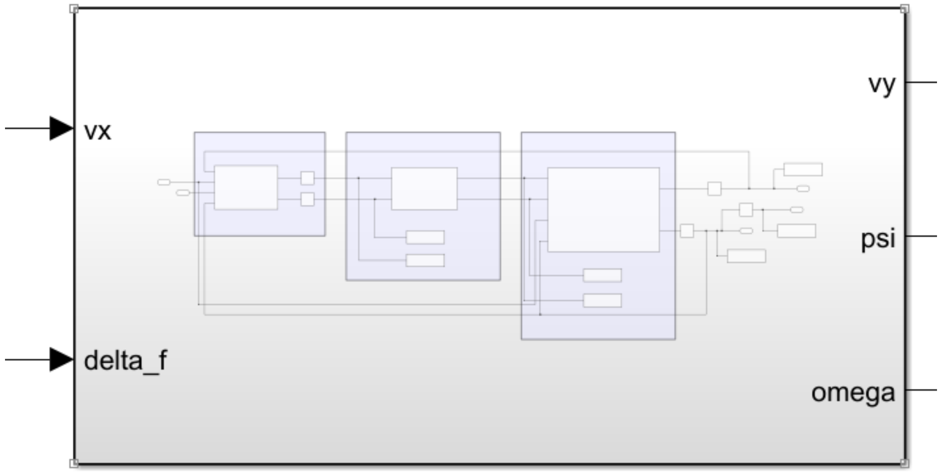
Under, Over and Neutral Steer

- $\delta = \frac{L}{R} + \frac{m}{LR}\left(\frac{l_r}{C_{\alpha_f}} - \frac{l_f}{C_{\alpha_r}}\right)v_x^2$
  - This is the steering required to achieve constant speed circular motion
- Relative cornering stiffness between front and rear, and weight distribution impact steer
- Critical Speed:
  - $0 = \frac{L}{R} + \frac{m}{LR}\left(\frac{l_r}{C_{\alpha_f}} - \frac{l_f}{C_{\alpha_r}}\right){v_x^*}^2$

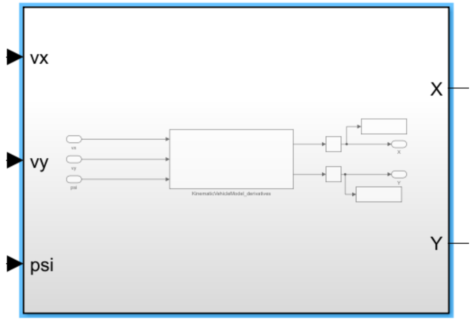


Lateral dynamics simulation

- $\dot{v}_x = 0$ and we will set $v_x = v_{xd}$
- $\begin{bmatrix} \dot{v}_y \\ \dot{\omega} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} -v_x\omega + \dfrac{C_{\alpha_f}\alpha_f(\delta) + C_{\alpha_r}\alpha_r}{m} \\ \dfrac{C_{\alpha_f}\alpha_f(\delta) l_f \ - C_{\alpha_r}\alpha_r l_r}{I} \\ \omega \end{bmatrix}$
    - assumes small angle approximation
- $\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = R(-\psi) \begin{bmatrix} v_x \\ v_y \end{bmatrix}$
- With the above you can simulate the (lateral) vehicle dynamics as long as you have the steering angle δ
    - This is your choice! Your controller!



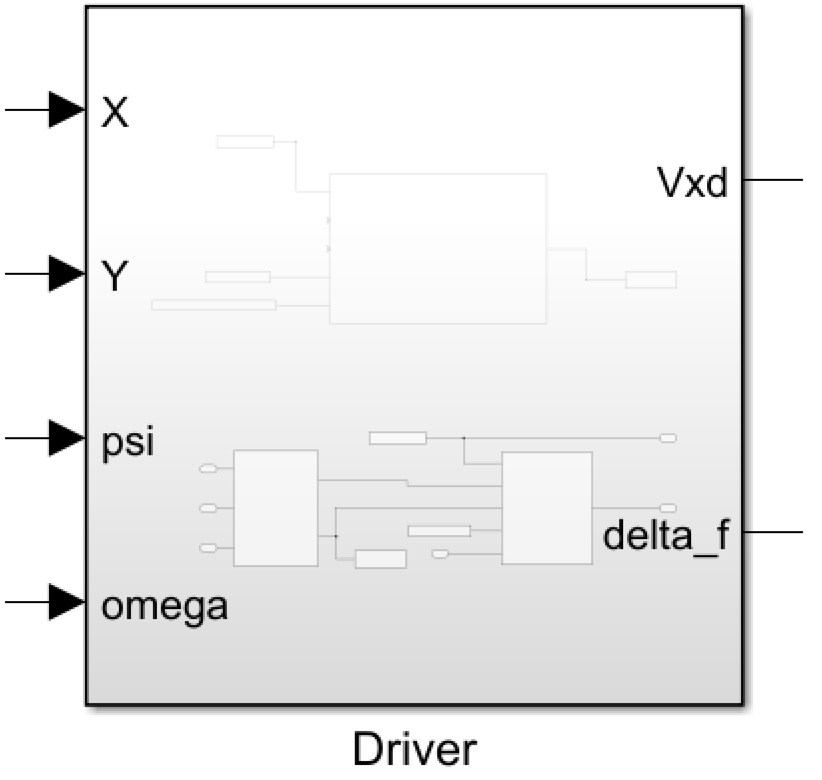
CarLateralDynamicsBodyFrame



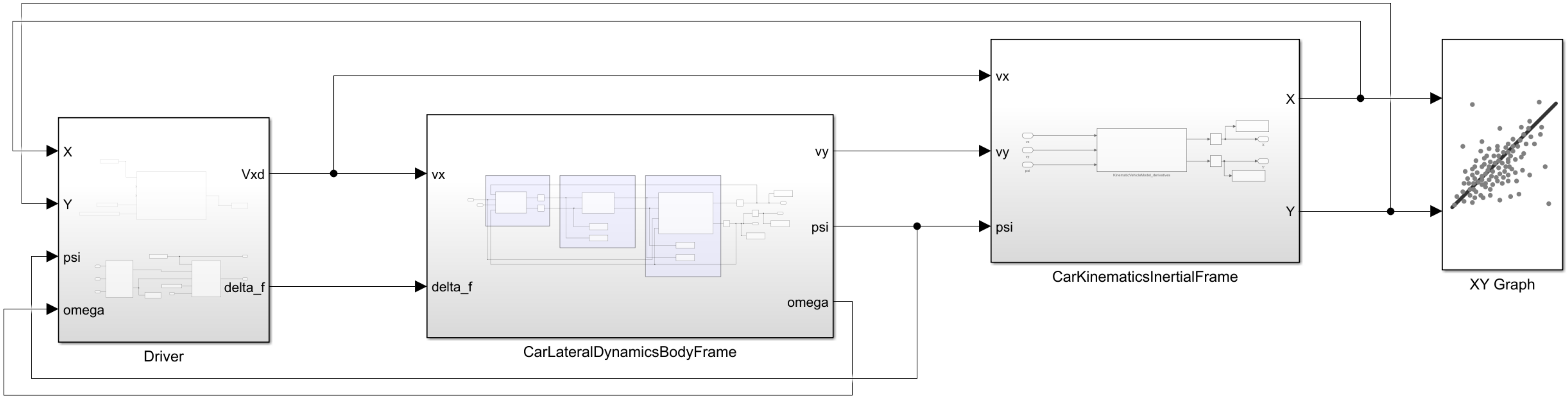
CarKinematicsInertialFrame

Lateral Dynamics Simulation

- The Driver model represents you!
  - You consider all the information available to you (from your model) to decide what the inputs to your car (model) will be
- Objective:
  - To travel along the track at some desired speed v_d
- Inputs:
  - Global Position X, Y
  - Heading and Yaw Rate $\psi, \dot{\psi}$ or ω
- Outputs:
  - Commanded velocity v_{xd}
  - Commanded (front wheel) steering δ_f



Full System Model



- Simplest Driver Model
  - Set commanded velocity to desired velocity
  - If on straight-away set steering to 0
  - If on curve, set steering to some $\delta_d > 0$