Summary

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1 Notation

1. n is the number of observations.
2. p is the number of covariates.
3. $\boldsymbol{X} = \begin{pmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \\ \vdots \\ \boldsymbol{X}_n \end{pmatrix} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{np} \end{pmatrix}$
4. $A_i = a$: binary treatment where $a = 0,\ 1$
5. M_i : i^{th} continuous intermediate variable
6. Y_i : i^{th} continuous outcome
7. $\boldsymbol{\beta}_0^M = (\beta_{01}, \ldots, \beta_{0p})'$ and $\boldsymbol{\beta}_1^M = (\beta_{11}, \ldots, \beta_{1p})'$
8. $\boldsymbol{\beta}^M = (\boldsymbol{\beta}_0^M{}', \boldsymbol{\beta}_1^M{}')'$
9. $S_i = (M_i(0), M_i(1))'$
10. $\boldsymbol{\eta}_h = (\eta_{0h}, \eta_{1h})'$
11. $\boldsymbol{\beta}_{ax}^Y = (\beta_{ax1}^Y, \ldots, \beta_{axp}^Y)'$
12. $\boldsymbol{\beta}_a^Y = (\beta_{a0}^Y, \beta_{a1}^Y, \beta_{a2}^Y, \boldsymbol{\beta}_{ax}^Y{}')$
13. $Z_i = 1, \ldots, h$: cluster membership
14. $w_i,\ i = 1, \ldots, h$: mixing proportions of each cluster

2 Models

$$
\begin{aligned}
\Pr\left(S_i \mid \boldsymbol{X}_i,\ \boldsymbol{\beta}^M\right) &= \sum_{h=1}^{\infty} w_h \,\mathrm{N}\left[\begin{pmatrix}\eta_{0h} + \boldsymbol{X}_i\boldsymbol{\beta}_0^M \\ \eta_{1h} + \boldsymbol{X}_i\boldsymbol{\beta}_1^M\end{pmatrix},\ \boldsymbol{\Sigma}_h\right] \\
(\eta_{0h},\ \eta_{1h},\ \boldsymbol{\Sigma}_h) &\sim G_0 \\
G_0 &\sim \mathrm{MVN}_2\left[\begin{pmatrix}0\\0\end{pmatrix},\ 10I_2\right]\mathrm{IWS}_2\left[10,\ 20I_2\right]
\end{aligned}
$$

$$
\begin{aligned}
Y_i(0) \mid M_i(0),\ M_i(1),\ \boldsymbol{\beta}_0^Y,\ \boldsymbol{\beta}_1^Y &\sim \mathrm{N}\left[\mu_0(M_i(0), M_i(1)),\ \sigma_0^2\right] \\
Y_i(1) \mid M_i(0),\ M_i(1),\ \boldsymbol{\beta}_0^Y,\ \boldsymbol{\beta}_1^Y &\sim \mathrm{N}\left[\mu_1(M_i(0), M_i(1)),\ \sigma_1^2\right] \\
\mu_0(M_i(0), M_i(1)) &= \beta_{00}^Y + \beta_{01}^Y M_i(0) + \beta_{02}^Y M_i(1) + \boldsymbol{X}_i\boldsymbol{\beta}_{0x}^Y \\
\mu_1(M_i(0), M_i(1)) &= \beta_{10}^Y + \beta_{11}^Y M_i(0) + \beta_{12}^Y M_i(1) + \boldsymbol{X}_i\boldsymbol{\beta}_{1x}^Y
\end{aligned}
$$

3 Priors

$$
\begin{aligned}
\boldsymbol{\beta}^M &\sim \mathrm{MVN}_{2p}\left[\mathbf{0}_{2p},\ s^2 I_{2p}\right] \quad \text{where} \quad s = 20 \\
\boldsymbol{\eta}_h &\sim \mathrm{MVN}_2\left[\begin{pmatrix}0\\0\end{pmatrix},\ 10I_2\right] \\
\boldsymbol{\Sigma}_h &\sim \mathrm{IWS}_2\left[10,\ 20I_2\right]
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{\beta}_a^Y &\sim \mathrm{MVN}_{3+p}\left[\mathbf{0}_{3+p},\ s^2 I_{3+p}\right] \quad \text{where} \quad s = 20 \\
\sigma_0^2 &\sim \mathrm{IG}\left[1,\ 1\right] \\
\sigma_1^2 &\sim \mathrm{IG}\left[1,\ 1\right]
\end{aligned}
$$

4 Gibbs Sampling

1. For each i such that $A_i = 0$, draw $M_i(1)$. In this case, $M_i^{obs} = M_i(0)$ for each i.

$$
\begin{aligned}
\Pr(S_i \mid \boldsymbol{X}_i,\ \boldsymbol{\beta}^M,\ \boldsymbol{\eta}_{z_i},\ \boldsymbol{\Sigma}_{z_i}) &= \Pr((M_i(0), M_i(1))' \mid \boldsymbol{X}_i,\ \boldsymbol{\beta}^M,\ \boldsymbol{\eta}_{z_i},\ \boldsymbol{\Sigma}_{z_i}) \\
&\propto \mathrm{MVN}_2\left[\begin{pmatrix}\eta_{0z_i} + \boldsymbol{X}_i\boldsymbol{\beta}_0^M \\ \eta_{1z_i} + \boldsymbol{X}_i\boldsymbol{\beta}_1^M\end{pmatrix}, \boldsymbol{\Sigma}_{\boldsymbol{z_i}}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\Pr(M_i(1) \mid M_i(0),\ \boldsymbol{X}_i,\ \boldsymbol{\beta}^M,\ \boldsymbol{\eta}_{z_i},\ \boldsymbol{\Sigma}_{z_i}) \\
&\qquad \propto \mathrm{N}\left[\eta_{1z_i} + \boldsymbol{X}_i\boldsymbol{\beta}_1^M + \boldsymbol{\Sigma}_{z_i,21}\boldsymbol{\Sigma}_{z_i,11}^{-1}(M_i^{obs} - \eta_{0z_i} - \boldsymbol{X}_i\boldsymbol{\beta}_0^M),\ \boldsymbol{\Sigma}_{z_i,22} - \boldsymbol{\Sigma}_{z_i,21}\boldsymbol{\Sigma}_{z_i,11}^{-1}\boldsymbol{\Sigma}_{z_i,12}\right] \\
&\qquad = \mathrm{N}\left[m_1,\ v_1\right]
\end{aligned}
$$

$$
\begin{aligned}
\Pr(M_i(1) \mid -) &\propto \Pr(Y_i^{obs} \mid M_i(1),\ M_i^{obs},\ \boldsymbol{X}_i,\ \boldsymbol{\beta}_0^Y)\ \times\ \Pr(M_i(1) \mid M_i^{obs},\ \boldsymbol{X}_i,\ \boldsymbol{\beta}^M,\ \boldsymbol{\eta}_{z_i},\ \boldsymbol{\Sigma}_{z_i}) \\
&\propto \mathrm{N}\left[\beta_{00}^Y + \beta_{01}^Y M_i^{obs} + \beta_{02}^Y M_i(1) + \boldsymbol{X}_i\boldsymbol{\beta}_{0x}^Y,\ \sigma_0^2\right]\ \times\ \mathrm{N}\left[m_1,\ v_1\right] \\
&\propto \exp\left(-\frac{1}{2\sigma_0^2}\left(Y_i^{obs} - \beta_{00}^Y - \beta_{01}^Y M_i^{obs} - \beta_{02}^Y M_i(1) - \boldsymbol{X}_i\boldsymbol{\beta}_{0x}^Y\right)^2\right)\ \times\ \mathrm{N}\left[m_1,\ v_1\right] \\
&\propto \exp\left(-\frac{\beta_{02}^{Y\,2}}{2\sigma_0^2}\left(M_i(1) - \frac{Y_i^{obs} - \beta_{00}^Y - \beta_{01}^Y M_i^{obs} - \boldsymbol{X}_i\boldsymbol{\beta}_{0x}^Y}{\beta_{02}^Y}\right)^2\right)\ \times\ \mathrm{N}\left[m_1,\ v_1\right] \\
&\propto \mathrm{N}\left[\frac{Y_i^{obs} - \beta_{00}^Y - \beta_{01}^Y M_i^{obs} - \boldsymbol{X}_i\boldsymbol{\beta}_{0x}^Y}{\beta_{02}^Y},\ \frac{\sigma_0^2}{\beta_{02}^{Y\,2}}\right] \times \mathrm{N}\left[m_1,\ v_1\right] \\
&= \mathrm{N}\left[m_2,\ v_2\right] \times \mathrm{N}\left[m_1,\ v_1\right] \\
&= \mathrm{N}\left[v\left(\frac{m_1}{v_1} + \frac{m_2}{v_2}\right),\ v\right] \quad \text{where} \quad v = \frac{v_1 v_2}{v_1 + v_2}
\end{aligned}
$$

2. Similary, for each i such that $A_i = 1$, draw $M_i(0)$. In this case, $M_i^{obs} = M_i(1)$ for each i.

$$
\Pr(M_i(0) \mid -) \ \propto\ \mathrm{N}\left[v\left(\frac{m_1}{v_1} + \frac{m_2}{v_2}\right),\ v\right]
$$

$$
\begin{aligned}
\text{where} \qquad v &= \frac{v_1 v_2}{v_1 + v_2} \\
m_1 &= \eta_{0z_i} + \boldsymbol{X}_i\boldsymbol{\beta}_0^M + \boldsymbol{\Sigma}_{z_i,12}\boldsymbol{\Sigma}_{z_i,22}^{-1}(M_i^{obs} - \eta_{1z_i} - \boldsymbol{X}_i\boldsymbol{\beta}_1^M) \\
v_1 &= \boldsymbol{\Sigma}_{z_i,11} - \boldsymbol{\Sigma}_{z_i,12}\boldsymbol{\Sigma}_{z_i,22}^{-1}\boldsymbol{\Sigma}_{z_i,21} \\
m_2 &= \frac{Y_i^{obs} - \beta_{10}^Y - \beta_{12}^Y M_i^{obs} - \boldsymbol{X}_i\boldsymbol{\beta}_{1x}^Y}{\beta_{11}^Y} \\
v_2 &= \frac{\sigma_1^2}{\beta_{11}^{Y\,2}}
\end{aligned}
$$

3. For each i, update the cluster membership Z_i as Step 2 in Appendix.

4. Update w_h's as Step 3 in section 2.3.

5. Update α as as Step 4 in Appendix.

6. Draw $\boldsymbol{\eta}_h = (\eta_{0h}, \eta_{1h})'$. Let $\boldsymbol{\mu} = (0, 0)'$. If there are no units having $Z_i = h$, draw it from the

prior.

$$
\begin{aligned}
\Pr\left(\boldsymbol{\eta}_h \mid -\right) &\propto \Pr\left(\boldsymbol{\eta}_h\right) \times \prod_{i:z_i=h} \Pr\left(S_i \mid \boldsymbol{X}_i,\ Z_i,\ \boldsymbol{\beta}^M,\ \boldsymbol{\eta}_h,\ \boldsymbol{\Sigma}_h\right) \\
&\propto \mathrm{MVN}_2\left[\begin{pmatrix}0\\0\end{pmatrix},\ 10I_2\right] \ \times \prod_{i:z_i=h} \mathrm{MVN}_2\left[\begin{pmatrix}\eta_{0z_i} + \boldsymbol{X}_i\boldsymbol{\beta}_0^M\\ \eta_{1z_i} + \boldsymbol{X}_i\boldsymbol{\beta}_1^M\end{pmatrix}, \boldsymbol{\Sigma}_{\boldsymbol{z_i}}\right] \\
&\propto \exp\left[-\frac{1}{2}\left(\boldsymbol{\eta}_h - \boldsymbol{\mu}\right)'\left(\boldsymbol{\eta}_h - \boldsymbol{\mu}\right)/10\right] \\
&\quad \times \exp\left[-\frac{1}{2}\sum_{i:z_i=h}\left(S_i - \boldsymbol{\eta}_h - \begin{pmatrix}\boldsymbol{X}_i\boldsymbol{\beta}_0^M\\ \boldsymbol{X}_i\boldsymbol{\beta}_1^M\end{pmatrix}\right)'\boldsymbol{\Sigma}_{z_i}^{-1}\left(S_i - \boldsymbol{\eta}_h - \begin{pmatrix}\boldsymbol{X}_i\boldsymbol{\beta}_0^M\\ \boldsymbol{X}_i\boldsymbol{\beta}_1^M\end{pmatrix}\right)\right] \\
&\propto \exp\Bigg[\boldsymbol{\eta}_h'\Big(I_2/10 + \sum_{i:z_i=h}\boldsymbol{\Sigma}_{z_i}^{-1}\Big)\boldsymbol{\eta}_h - \boldsymbol{\eta}_h'\Big(\boldsymbol{\mu}/10 + \sum_{i:z_i=h}\boldsymbol{\Sigma}_{z_i}^{-1}\Big(S_i - \begin{pmatrix}\boldsymbol{X}_i\boldsymbol{\beta}_0^M\\ \boldsymbol{X}_i\boldsymbol{\beta}_1^M\end{pmatrix}\Big)\Big) \\
&\qquad\qquad - \Big(\boldsymbol{\mu}'/10 + \sum_{i:z_i=h}\Big(S_i - \begin{pmatrix}\boldsymbol{X}_i\boldsymbol{\beta}_0^M\\ \boldsymbol{X}_i\boldsymbol{\beta}_1^M\end{pmatrix}\Big)'\boldsymbol{\Sigma}_{z_i}^{-1}\Big)\boldsymbol{\eta}_h\Bigg] \\
&\sim \mathrm{MVN}_2\left[V\left(\frac{\boldsymbol{\mu}}{10} + \sum_{i:z_i=h}\boldsymbol{\Sigma}_{z_i}^{-1}\left(S_i - \begin{pmatrix}\boldsymbol{X}_i\boldsymbol{\beta}_0^M\\ \boldsymbol{X}_i\boldsymbol{\beta}_1^M\end{pmatrix}\right)\right),\ V\right] \text{ where } V = \left(\frac{I_2}{10} + \sum_{i:z_i=h}\boldsymbol{\Sigma}_{z_i}^{-1}\right)^{-1}
\end{aligned}
$$

7. Draw $\boldsymbol{\Sigma}_h$. If there are no units having $Z_i = h$, draw it from the prior.

$$
\begin{aligned}
\Pr(\boldsymbol{\Sigma}_h \mid -) &\propto \Pr(\boldsymbol{\Sigma}_h) \ \times \prod_{i:z_i=h} \Pr(S_i \mid \boldsymbol{X}_i,\ Z_i,\ \boldsymbol{\beta}^M,\ \boldsymbol{\eta}_h,\ \boldsymbol{\Sigma}_h) \\
&\propto \mathrm{IWS}_2\left[10,\ 20I_2\right] \times \prod_{i:z_i=h} \mathrm{MVN}_2\left[\begin{pmatrix}\eta_{0z_i} + \boldsymbol{X}_i\boldsymbol{\beta}_0^M\\ \eta_{1z_i} + \boldsymbol{X}_i\boldsymbol{\beta}_1^M\end{pmatrix},\ \boldsymbol{\Sigma}_{\boldsymbol{z_i}}\right] \\
&\propto |\boldsymbol{\Sigma}_h|^{-(10+2+1)/2}\exp\left[-\frac{1}{2}\,\mathrm{tr}\left(20\boldsymbol{\Sigma}_h^{-1}\right)\right] \\
&\quad \times \prod_{i:z_i=h}|\boldsymbol{\Sigma}_{z_i}|^{-\frac{1}{2}}\exp\left[-\frac{1}{2}\sum_{i:z_i=h}\mathrm{tr}\left(\left(S_i - \boldsymbol{\eta}_h - \begin{pmatrix}\boldsymbol{X}_i\boldsymbol{\beta}_0^M\\ \boldsymbol{X}_i\boldsymbol{\beta}_1^M\end{pmatrix}\right)\left(S_i - \boldsymbol{\eta}_h - \begin{pmatrix}\boldsymbol{X}_i\boldsymbol{\beta}_0^M\\ \boldsymbol{X}_i\boldsymbol{\beta}_1^M\end{pmatrix}\right)'\boldsymbol{\Sigma}_{z_i}^{-1}\right)\right] \\
&\propto \mathrm{IWS}_2\left[10 + \sum_{i:z_i=h} 1,\ V\right] \\
&\quad \text{where } V = 20I_2 + \sum_{i:z_i=h}\left(S_i - \boldsymbol{\eta}_h - \begin{pmatrix}\boldsymbol{X}_i\boldsymbol{\beta}_0^M\\ \boldsymbol{X}_i\boldsymbol{\beta}_1^M\end{pmatrix}\right)\left(S_i - \boldsymbol{\eta}_h - \begin{pmatrix}\boldsymbol{X}_i\boldsymbol{\beta}_0^M\\ \boldsymbol{X}_i\boldsymbol{\beta}_1^M\end{pmatrix}\right)'
\end{aligned}
$$

8. Update $\boldsymbol{\beta}^M$. Define

$$\tilde{\boldsymbol{X}}_i \ = \ \begin{pmatrix}\boldsymbol{X}_i & \boldsymbol{0}_p'\\ \boldsymbol{0}_p' & \boldsymbol{X}_i\end{pmatrix}$$

Then,

$$
\begin{aligned}
\Pr\left(\boldsymbol{\beta}^M \mid -\right) &\propto \Pr\left(\boldsymbol{\beta}^M\right) \;\times\; \prod_{i=1}^{n} \Pr\left(S_i \mid \boldsymbol{X}_i,\ Z_i,\ \boldsymbol{\beta}^M,\ \boldsymbol{\eta}_{\boldsymbol{z_i}},\ \boldsymbol{\Sigma}_{\boldsymbol{z_i}}\right) \\
&\propto\ \mathrm{MVN}_{2p}\left[\mathbf{0}_{2p},\ s^2 I_{2p}\right] \;\times\; \prod_{i=1}^{n} \mathrm{MVN}_2\left[\begin{pmatrix}\eta_{0z_i} + \boldsymbol{X}_i\boldsymbol{\beta}_0^M \\ \eta_{1z_i} + \boldsymbol{X}_i\boldsymbol{\beta}_1^M\end{pmatrix},\ \boldsymbol{\Sigma}_{z_i}\right] \\
&\propto\ \mathrm{MVN}_{2p}\left[\mathbf{0}_{2p},\ s^2 I_{2p}\right] \;\times\; \prod_{i=1}^{n} \exp\left[-\frac{1}{2}\left(S_i - \boldsymbol{\eta}_{z_i} - \tilde{\boldsymbol{X}}_i\boldsymbol{\beta}^M\right)' \boldsymbol{\Sigma}_{z_i}^{-1}\left(S_i - \boldsymbol{\eta}_{z_i} - \tilde{\boldsymbol{X}}_i\boldsymbol{\beta}^M\right)\right] \\
&\propto\ \mathrm{MVN}_{2p}\left[V\sum_{i=1}^{n} \tilde{\boldsymbol{X}}_i'\boldsymbol{\Sigma}_{z_i}^{-1}\left(S_i - \boldsymbol{\eta}_{z_i}\right),\ V\right] \quad \text{where} \quad V = \left(I_{2p}/s^2 + \sum_{i=1}^{n} \tilde{\boldsymbol{X}}_i'\boldsymbol{\Sigma}_{z_i}^{-1}\tilde{\boldsymbol{X}}_i\right)^{-1}
\end{aligned}
$$

9. Update $\boldsymbol{\beta}_0^Y$. Let $\mathbf{X}_0$ be the matrix of covariates of control group. Let $\mathbf{M}_0(a)$ be the vector of potential outcomes $M_i(a)$ for units assigned to control group, and define $\mathbf{Y}_0(a)$ analogously. Let k be the number of units in the control group.

$$
\Pr\left(\mathbf{Y}_0(0) \mid \mathbf{X}_0,\ \mathbf{M}_0(0),\ \mathbf{M}_0(1),\ \boldsymbol{\beta}_0^Y,\ \sigma_0^2\right) \sim \mathrm{MVN}_k\left[\begin{pmatrix}1 & \mathbf{M}_0(0) & \mathbf{M}_0(1) & \mathbf{X}_0\end{pmatrix}\boldsymbol{\beta}_0^Y,\ \sigma_0^2 I_k\right]
$$

$$
\begin{aligned}
\Pr\left(\boldsymbol{\beta}_0^Y \mid -\right) &\propto\ \Pr\left(\boldsymbol{\beta}_0^Y\right) \;\times \prod_{i:A_i^{obs}=0} \Pr\left(Y_i^{obs} \mid S_i,\ \boldsymbol{X}_i,\ \boldsymbol{\beta}_0^Y\right) \\
&=\ \Pr\left(\boldsymbol{\beta}_0^Y\right) \;\times\; \Pr\left(\mathbf{Y}_0(0) \mid \mathbf{X}_0,\ \mathbf{M}_0(0),\ \mathbf{M}_0(1),\ \boldsymbol{\beta}_0^Y,\ \sigma_0^2\right) \\
&\propto\ \mathrm{MVN}_{3+p}\left[\mathbf{0}_{3+p},\ s^2 I_{3+p}\right] \;\times\; \mathrm{MVN}_k\left[\begin{pmatrix}1 & \mathbf{M}_0(0) & \mathbf{M}_0(1) & \mathbf{X}_0\end{pmatrix}\boldsymbol{\beta}_0^Y,\ \sigma_0^2 I_k\right] \\
&\propto\ \exp\left[-\frac{1}{2}\boldsymbol{\beta}_0^{Y\prime}\boldsymbol{\beta}_0^Y/s^2\right] \;\times\; \exp\Bigg[-\frac{1}{2}\left(\mathbf{Y}_0(0) - \begin{pmatrix}1 & \mathbf{M}_0(0) & \mathbf{M}_0(1) & \mathbf{X}_0\end{pmatrix}\boldsymbol{\beta}_0^Y\right)' \\
&\qquad\qquad \left(\mathbf{Y}_0(0) - \begin{pmatrix}1 & \mathbf{M}_0(0) & \mathbf{M}_0(1) & \mathbf{X}_0\end{pmatrix}\boldsymbol{\beta}_0^Y\right)/\sigma_0^2\Bigg] \\
&\propto\ \mathrm{MVN}_{3+p}\left[V\left(\frac{\begin{pmatrix}1 & \mathbf{M}_0(0) & \mathbf{M}_0(1) & \mathbf{X}_0\end{pmatrix}'\mathbf{Y}_0(0)}{\sigma_0^2}\right),\ V\right] \\
&\qquad \text{where} \quad V = \left(\frac{\begin{pmatrix}1 & \mathbf{M}_0(0) & \mathbf{M}_0(1) & \mathbf{X}_0\end{pmatrix}'\begin{pmatrix}1 & \mathbf{M}_0(0) & \mathbf{M}_0(1) & \mathbf{X}_0\end{pmatrix}}{\sigma_0^2} + \frac{I_{3+p}}{s^2}\right)^{-1}
\end{aligned}
$$

10. Simalary, update $\boldsymbol{\beta}_1^Y$.

$$
\begin{aligned}
\Pr\left(\boldsymbol{\beta}_1^Y \mid -\right) &\propto\ \mathrm{MVN}_{3+p}\left[V\left(\frac{\begin{pmatrix}1 & \mathbf{M}_1(0) & \mathbf{M}_1(1) & \mathbf{X}_1\end{pmatrix}'\mathbf{Y}_1(1)}{\sigma_1^2}\right),\ V\right] \\
&\qquad \text{where} \quad V = \left(\frac{\begin{pmatrix}1 & \mathbf{M}_1(0) & \mathbf{M}_1(1) & \mathbf{X}_1\end{pmatrix}'\begin{pmatrix}1 & \mathbf{M}_1(0) & \mathbf{M}_1(1) & \mathbf{X}_1\end{pmatrix}}{\sigma_1^2} + \frac{I_{3+p}}{s^2}\right)^{-1}
\end{aligned}
$$

11. Update σ_0^2.

$$
\begin{aligned}
\Pr\left(\sigma_0^2 \mid -\right) &\propto \Pr\left(\sigma_0^2\right) \times \prod_{i:A_i^{obs}=0} \Pr\left(Y_i^{obs} \mid S_i, \boldsymbol{X}_i, \boldsymbol{\beta}_0^Y, \sigma_0^2\right) \\
&\propto \left(\sigma_0^2\right)^{-a-1} \times \exp\left[-\frac{b}{\sigma_0^2}\right] \\
&\qquad\qquad \times \prod_{i:A_i^{obs}=0} \left(\sigma_0^2\right)^{-\frac{1}{2}} \exp\left[-\frac{1}{2\sigma_0^2}\left(Y_i^{obs} - \beta_{00}^Y - \beta_{01}^Y M_i(0) - \beta_{02}^Y M_i(1) - \boldsymbol{X}_i \boldsymbol{\beta}_{0x}^Y\right)^2\right] \\
&\propto \left(\sigma_0^2\right)^{-(a+n_0/2)-1} \exp\left[-\frac{1}{\sigma_0^2}\left(\frac{\sum_{i:A_i^{obs}=0}\left(Y_i^{obs} - \beta_{00}^Y - \beta_{01}^Y M_i(0) - \beta_{02}^Y M_i(1) - \boldsymbol{X}_i \boldsymbol{\beta}_{0x}^Y\right)^2}{2} + b\right)\right] \\
&\sim \mathrm{IG}\left[a + \frac{n_0}{2},\ \frac{\sum_{i:A_i^{obs}=0}\left(Y_i^{obs} - \beta_{00}^Y - \beta_{01}^Y M_i(0) - \beta_{02}^Y M_i(1) - \boldsymbol{X}_i \boldsymbol{\beta}_{0x}^Y\right)^2}{2} + b\right]
\end{aligned}
$$

12. Similarly, update σ_1^2.

$$
\Pr\left(\sigma_1^2 \mid -\right) \sim \mathrm{IG}\left[a + \frac{n_1}{2},\ \frac{\sum_{i:A_i^{obs}=1}\left(Y_i^{obs} - \beta_{10}^Y - \beta_{11}^Y M_i(0) - \beta_{12}^Y M_i(1) - \boldsymbol{X}_i \boldsymbol{\beta}_{1x}^Y\right)^2}{2} + b\right]
$$