AI6102: Machine Learning Methodologies & Applications

L7: Bayesian Classifiers

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Supervised Learning: Recall

In mathematics

- Given: a set of $\{\boldsymbol{x}_i, y_i\}$ for $i = 1, \ldots, N$, where $\boldsymbol{x}_i = [x_{i1}, x_{i2}, \ldots, x_{im}]$ is m-dimensional vector of numerical values, and y_i is a scalar
- Aim to learn a mapping $f: \boldsymbol{x} \rightarrow y$ by requiring $f(\boldsymbol{x}_i) = y_i$
- The learned mapping f is expected to make precise predictions on any unseen $\boldsymbol{x}^*$ as $f(\boldsymbol{x}^*)$

In Probability Point of View

- The mapping $f: \boldsymbol{x} \rightarrow y$ can be considered as a conditional probability $P(y|\boldsymbol{x})$
- Given a test data instance $\boldsymbol{x}^*$

$$y^* = c^* \text{ if } c^* = \arg\max_c P(y = c|\boldsymbol{x}^*), c \in \{0, \ldots, C-1\}$$

- In logistic regression, the conditional probabilities of different classes are assumed to be expressed as specific forms in terms of parameters $\boldsymbol{w}$, e.g., for binary classification (0 or 1)

$$P(y = 1|\boldsymbol{x}) = \frac{1}{1 + \exp(-\boldsymbol{w}^T\boldsymbol{x})} \qquad P(y = 0|\boldsymbol{x}) = \frac{\exp(-\boldsymbol{w}^T\boldsymbol{x})}{1 + \exp(-\boldsymbol{w}^T\boldsymbol{x})}$$

- Today, we introduce another way to estimate $P(y|\boldsymbol{x})$

Probability Review

- Let A be a random variable (a feature / a label in machine learning)
- Marginal probability

$$0 \leq P(A = a) \leq 1$$

$$P(A = a)$$

refers to the probability that variable $A = a$

$$\sum_{a_i} P(A = a_i) = 1$$

Probability Review (cont.)

- Let A and B be a pair of random variables (features/labels in machine learning).
- Their joint probability

$$P(A = a, B = b)$$

  refers to the probability that variable $A = a$, and at the same time variable $B = b$

Probability Review (cont.)

- Conditional probability

$$P(B = b|A = a)$$

refers to the probability that variable B will take on the value b, given that the variable A is observed to have the value a

$$\sum_{b_i} P(B = b_i|A = a) = 1$$

Sum Rule

- The connection between joint probability of A and B and marginal probability of A:

$$P(A=a)=\sum_{b_i} P(A=a, B=b_i) \quad \textbf{OR} \quad P(A)=\sum_{B} P(A,B)$$

$$P(A=a)=\sum_{c_j}\sum_{b_i} P(A=a, B=b_i, C=c_j)$$

OR

$$P(A)=\sum_{C}\sum_{B} P(A,B,C)$$

Product Rule

- The connections between marginal, joint and conditional probabilities of A and B:

$$P(A = a, B = b) = P(B = b|A = a) \times P(A = a)$$

$$= P(A = a|B = b) \times P(B = b)$$

OR

$$P(A, B) = P(B|A) \times P(A)$$

$$= P(A|B) \times P(B)$$

Bayes Rule / Bayes Theorem

$$P(A|B) = \frac{P(A,B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$$

- The 1st and the 2nd equations are both based on product rule

$$P(A,B) = P(A|B) \times P(B) = P(B|A) \times P(A)$$

- Can be generalized to the case when **A** and **B** are a set of variables

$$P\left(A_1 \ldots A_k \middle| B_1 \ldots B_p\right) = \frac{P\left(B_1 \ldots B_p \middle| A_1 \ldots A_k\right) P(A_1 \ldots A_k)}{P(B_1 \ldots B_p)}$$

Bayesian Classifiers

- To estimate $P(y|\boldsymbol{x})$ from the training set $\{\boldsymbol{x}_i, y_i\}$ $i = 1, \dots, N$, we can use the Bayes Rule

$$P(y|\boldsymbol{x}) = \frac{P(y, \boldsymbol{x})}{P(\boldsymbol{x})} = \frac{P(\boldsymbol{x}|y)P(y)}{P(\boldsymbol{x})}$$

- Recall that to make a prediction on $\boldsymbol{x}^*$

$$y^* = c^* \text{ if } c^* = \arg\max_c P(y = c|\boldsymbol{x}^*), c \in \{0, \dots, C-1\}$$

$$= \arg\max_c \frac{P(\boldsymbol{x}^*|y = c)P(y = c)}{P(\boldsymbol{x}^*)}$$

$P(\boldsymbol{x}^*)$ is a constant w.r.t. different c

$$= \arg\max_c P(\boldsymbol{x}^*|y = c)P(y = c)$$

An Example

- Suppose we aim to predict the class label (Repay = Yes or No) of the following data instance

Fixed Assets	Occupation	Income	Repay
Yes	Manager	125K	?

- That is we need to compute

$$P(\text{Rapy=Yes} \mid \text{Assets=Yes,Occ.=Manager,Income=125K})$$

$$= \frac{P(\text{Assets=Yes,Occ.=Manager,Income=125K} \mid \text{Rapy=Yes})P(\text{Rapy=Yes})}{\cancel{P(\text{Assets=Yes,Occ.=Manager,Income=125K})}}$$

v.s.

$$P(\text{Rapy=No} \mid \text{Assets=Yes,Occ.=Manager,Income=125K})$$

$$= \frac{P(\text{Assets=Yes,Occ.=Manager,Income=125K} \mid \text{Rapy=No})P(\text{Rapy=No})}{\cancel{P(\text{Assets=Yes,Occ.=Manager,Income=125K})}}$$

Bayesian Classifiers (cont.)

$$P(y|\boldsymbol{x}) = \frac{P(y, \boldsymbol{x})}{P(\boldsymbol{x})} = \frac{P(\boldsymbol{x}|y)P(y)}{P(\boldsymbol{x})}$$

$$y^* = c^* \text{ if } c^* = \arg\max_c P(y = c|\boldsymbol{x}^*), c \in \{0, \dots, C-1\}$$

$$= \arg\max_c \frac{P(\boldsymbol{x}^*|y = c)P(y = c)}{P(\boldsymbol{x}^*)}$$

$P(\boldsymbol{x}^*)$ is a constant w.r.t. different c

$$= \arg\max_c \boxed{P(\boldsymbol{x}^*|y = c)P(y = c)}$$

Estimate these two types of probabilities from training data

Bayesian Classifiers (cont.)

$$P(y|\boldsymbol{x}) \propto \boxed{P(\boldsymbol{x}|y)}\boxed{P(y)}$$

Class probabilities from training data, easy to estimate

In general, difficult to estimate as all the possible combinations need to be considered in training

- Consider the risk estimation task

$$P(\text{Assets=Yes,Occ.=Manager,}\boxed{\text{Income=125K}}\mid \text{Rapy=Yes})$$

$$P(\text{Assets=No,Occ.=Manager,Income=125K}\mid \text{Rapy=Yes})$$

$$P(\text{Assets=Yes,Occ.=Engineer,Income=125K}\mid \text{Rapy=Yes})$$

$$P(\text{Assets=Yes,Occ.=Lawyer,Income=125K}\mid \text{Rapy=Yes})$$

…

Bayesian Classifiers (cont.)

- In theory, the estimation of $P(\boldsymbol{x}|y)$ is computationally expensive
  - Need to consider all possible value combination of $\boldsymbol{x}$ and y
  - How to make the estimation of $P(\boldsymbol{x}|y)$ computationally tractable?
- Two implementations of Bayesian classification methods
  - Naïve Bayes classifier
    - Based on a strong conditional independence assumption
  - Bayesian belief network
    - Based on a graph of dependence among variables

Not covered

Naïve Bayes Classifier

- Assume that the features are <u>conditionally independent</u> given the class label:

$$P(\boldsymbol{x}|y=c) = \prod_{i=1}^{d} P(x_i|y=c) \quad \text{where } \boldsymbol{x} = [x_1, x_2, \ldots, x_d]$$

$$P(x_1, x_2, .., x_d|y=c) = \prod_{i=1}^{d} \boxed{P(x_i|y=c)}$$

Only need to different combinations of x_i and y, no need to consider all combinations of $[x_1, \ldots, x_d]$ and y

For example:

$P(\text{Assets=Yes,Occ.=Manager,Income=125K} \mid \text{Rapy=Yes})$

$= P(\text{Assets=Yes} \mid \text{Rapy=Yes}) P(\text{Occ.=Manager} \mid \text{Rapy=Yes}) P(\text{Income=125K} \mid \text{Rapy=Yes})$

Independence

- Let A and B be two random variables
- A is said to be independent of B, if the following condition holds:

$$P(A|B) = P(A)$$

$$P(A,B) = P(A|B) \times P(B) = P(A) \times P(B)$$

- This can be generalized to the setting where **A** and **B** are two sets of random variables
- The variables in **A** are said to be independent of **B**, if the following condition holds:

$$P(\mathbf{A},\mathbf{B}) = P(\mathbf{A}|\mathbf{B}) \times P(\mathbf{B}) = P(\mathbf{A}) \times P(\mathbf{B})$$

Conditional Independence

- Let **A**, **B**, and **C** be three sets of random variables
- The variables in **A** are said to be conditionally independent of **B**, given **C**, if the following condition holds:

$$P(\mathbf{A}|\mathbf{B},\mathbf{C}) = P(\mathbf{A}|\mathbf{C})$$

$$\Downarrow$$

$$P(\boldsymbol{x}|y=c) = \prod_{i=1}^{d} P(x_i|y=c)$$

Conditional Independence (cont.)

- The conditional independence between **A** and **B** given **C** can also be written as follows

$$P(\mathbf{A},\mathbf{B}|\mathbf{C}) = \frac{P(\mathbf{A},\mathbf{B},\mathbf{C})}{P(\mathbf{C})}$$

Product rule: $P(\mathbf{A},\mathbf{B}|\mathbf{C})P(\mathbf{C}) = P(\mathbf{A},\mathbf{B},\mathbf{C})$

$$= \frac{P(\mathbf{A},\mathbf{B},\mathbf{C})}{\boxed{P(\mathbf{B},\mathbf{C})}} \times \frac{\boxed{P(\mathbf{B},\mathbf{C})}}{P(\mathbf{C})}$$

Product rule:
$P(\mathbf{A}|\mathbf{B},\mathbf{C})P(\mathbf{B},\mathbf{C}) = P(\mathbf{A},\mathbf{B},\mathbf{C})$
$P(\mathbf{B}|\mathbf{C})P(\mathbf{C}) = P(\mathbf{B},\mathbf{C})$

$$= P(\mathbf{A}|\mathbf{B},\mathbf{C}) \times P(\mathbf{B}|\mathbf{C})$$

$$= P(\mathbf{A}|\mathbf{C}) \times P(\mathbf{B}|\mathbf{C})$$

Conditional independence:
$P(\mathbf{A}|\mathbf{B},\mathbf{C}) = P(\mathbf{A}|\mathbf{C})$

Naïve Bayes Classifier (cont.)

- The set of varilables **A** and **B** are said to be independent given **C** if $P(\mathbf{A}, \mathbf{B}|\mathbf{C}) = P(\mathbf{A}|\mathbf{C}) \times P(\mathbf{B}|\mathbf{C})$
- Recall that naïve Bayes classifier assumes that the features are conditionally independent given the class label

$$\mathbf{A} = \{x_1, \ldots, x_{d-1}\}, \mathbf{B} = \{x_d\}, \mathbf{C} = \{y = c\}$$

$$P(x_1, x_2, .., x_d|y = c) = \boxed{P(x_1, \ldots, x_{d-1}|y = c)} P(x_d|y = c)$$

$$\Downarrow$$

$$P(x_1, .., x_{d-1}|y = c) = \boxed{P(x_1, \ldots, x_{d-2}|y = c)} P(x_{d-1}|y = c)$$

$$\Downarrow$$

$$\cdots$$

$$P(x_1, x_2, .., x_d|y = c) = P(x_1|y = c) P(x_2|y = c) \ldots P(x_d|y = c) = \prod_{i=1}^{d} P(x_i|y = c)$$

Naïve Bayes Classifier (cont.)

- For any test data instance $\boldsymbol{x}^*$

$$c^* = \arg\max_c P(y = c|\boldsymbol{x}^*)$$

$$= \arg\max_c \frac{P(\boldsymbol{x}^*|y = c)P(y = c)}{P(\boldsymbol{x}^*)}$$

$$= \arg\max_c P(\boldsymbol{x}^*|y = c)P(y = c)$$

$$= \arg\max_c P(y = c) \prod_{i=1}^{d} P(x_i^*|y = c)$$

- In training, we need to estimate $P(y)$ for different classes, and for each class c and feature x_i, $P(x_i|y = c)$ for different possible values of x_i

Credit Risk Estimation

ID	Assets	Occupation	Income	Repay
1	Yes	Manager	125K	Yes
2	No	Engineer	100K	Yes
3	No	Manager	70K	Yes
4	Yes	Engineer	120K	Yes
5	No	Lawyer	95K	No
6	No	Engineer	60K	Yes
7	Yes	Lawyer	220K	Yes
8	No	Manager	85K	No
9	No	Engineer	75K	Yes
10	No	Manager	90K	No

Training

$P(x_i|y = c)$

P(Assets=Yes | Repay=No)
P(Assets=No | Repay=No)
P(Assets=Yes | Repay=Yes)
P(Assets=No | Repay=Yes)

P(Occ.=Manager | Repay=No)
P(Occ.=Engineer | Repay=No)
P(Occ.=Lawyer | Repay=No)
P(Occ.=Manager | Repay=Yes)
P(Occ.=Engineer | Repay=Yes)
P(Occ.=Lawyer | Repay=Yes)

P(Income=v | Repay=Yes)
P(Income=v | Repay=No)
where $v \geq 0$

$P(y = c)$

P(Repay=Yes)
P(Repay=No)

Testing

ID	Assets	Occupation	Income	Repay
11	No	Engineer	85K	?

P(No)P(Assets=No | No)P(Occu.=Engineer | No)P(Income=85K | No)

v.s.

P(Yes)P(Assets=No | Yes)P(Occu.=Engineer | Yes)P(Income=85K | Yes)

Margin Probability of Class

ID	Assets	Occupation	Income	Repay
1	Yes	Manager	125K	Yes
2	No	Engineer	100K	Yes
3	No	Manager	70K	Yes
4	Yes	Engineer	120K	Yes
5	No	Lawyer	95K	No
6	No	Engineer	60K	Yes
7	Yes	Lawyer	220K	Yes
8	No	Manager	85K	No
9	No	Engineer	75K	Yes
10	No	Manager	90K	No

Number of data instances of class *c*

$$P(y=c) = \frac{\boxed{|y=c|}}{\boxed{N}}$$

Total number of training data instances

$$P(\text{Repay=Yes}) = \frac{7}{10}$$

$$P(\text{Repay=No}) = \frac{3}{10}$$

Conditional Probability on Discrete Features

ID	Assets	Occupation	Income	Repay
1	Yes	Manager	125K	Yes
2	No	Engineer	100K	Yes
3	No	Manager	70K	Yes
4	Yes	Engineer	120K	Yes
5	No	Lawyer	95K	No
6	No	Engineer	60K	Yes
7	Yes	Lawyer	220K	Yes
8	No	Manager	85K	No
9	No	Engineer	75K	Yes
10	No	Manager	90K	No

$$P(\text{Assets} = \text{Yes} \mid \text{Repay} = \text{Yes})$$

$$= \frac{\#(\text{Assets} = \text{Yes} \wedge \text{Repay} = \text{Yes})}{\#(\text{Repay} = \text{Yes})} = \frac{3}{7}$$

$$P(\text{Occ.} = \text{Manager} \mid \text{Repay} = \text{No})$$

$$= \frac{\#(\text{Occ.} = \text{Manager} \wedge \text{Repay} = \text{No})}{\#(\text{Repay} = \text{No})} = \frac{2}{3}$$

Number of data instances of class c, whose values of the i-th feature are z

$$P(x_i = z | y = c) = \frac{\boxed{|(x_i = z) \wedge (y = c)|}}{\boxed{|y = c|}}$$

Value of the i-th feature equals to z

Number of data instances of class c

Conditional Probability on Continuous Features

- Assume the values of a specific feature x_i given a specific class c follow a Guassian distribution, i.e., $P(x_i|y=c)$ is a Guassian distribution

$$\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$$

  - Use training data of class c to estimate parameters of the Gaussian distribution, i.e., mean μ and variance σ^2
  - Once the parameters are estimated, the Guassian distribution is known, and we can use it to compute conditional probability
- Note: more methods will be introduced when introducing density estimation

Conditional Probability on Continuous Features (cont.)

ID	Assets	Occupation	Income	Repay
1	Yes	Manager	125K	Yes
2	No	Engineer	100K	Yes
3	No	Manager	70K	Yes
4	Yes	Engineer	120K	Yes
5	No	Lawyer	95K	No
6	No	Engineer	60K	Yes
7	Yes	Lawyer	220K	Yes
8	No	Manager	85K	No
9	No	Engineer	75K	Yes
10	No	Manager	90K	No

{Income, Repay=Yes}, Gaussian distribution:

$$P(\text{Inc.}|\text{Yes}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\text{Inc.}-\mu)^2}{2\sigma^2}}$$

μ and σ^2 are the mean and variance of the income of the data instances whose labels are Yes (Repay=Yes)

$$\mu_x = \frac{1}{N}\sum_{k=1}^{N} x_k \quad \sigma_x = \sqrt{\frac{1}{N-1}\sum_{k=1}^{N}(x_i - \mu_x)^2}$$

$$\mu_{\{\text{inc., Yes}\}} = 110$$

$$\sigma^2_{\{\text{Inc., Yes}\}} = 2975$$

$$\sigma_{\{\text{Inc., Yes}\}} = 54.54$$

$$P(\text{Inc.}|\text{Yes}) = \frac{1}{\sqrt{2\pi} \times 54.54} e^{-\frac{(\text{Inc.}-110)^2}{2\times 2975}}$$

Conditional Probability on Continuous Features (cont.)

ID	Assets	Occupation	Income	Repay
1	Yes	Manager	125K	Yes
2	No	Engineer	100K	Yes
3	No	Manager	70K	Yes
4	Yes	Engineer	120K	Yes
5	No	Lawyer	95K	No
6	No	Engineer	60K	Yes
7	Yes	Lawyer	220K	Yes
8	No	Manager	85K	No
9	No	Engineer	75K	Yes
10	No	Manager	90K	No

{Income, Repay=No}, Gaussian distribution:

$$P(\text{Inc.}|\text{No}) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\text{Inc.}-\mu)^2}{2\sigma^2}}$$

μ and σ^2 are the mean and variance of the income of the data instances whose labels are No (Repay=No)

$$\mu_{\{\text{inc., No}\}} = 90$$

$$\sigma^2_{\{\text{Inc.,No}\}} = 25$$

$$\sigma_{\{\text{Inc.,No}\}} = 5$$

$$P(\text{Inc.}|\text{No}) = \frac{1}{\sqrt{2\pi} \times 5} e^{-\frac{(\text{Inc.}-90)^2}{2\times 25}}$$

Conditional Probability on Continuous Features (cont.)

$$P(\text{Inc.}|\text{No}) = \frac{1}{\sqrt{2\pi} \times 5} e^{-\frac{(\text{Inc.}-90)^2}{2\times 25}}$$

$$P(\text{Inc.}|\text{Yes}) = \frac{1}{\sqrt{2\pi} \times 54.54} e^{-\frac{(\text{Inc.}-110)^2}{2\times 2975}}$$

ID	Fixed Assets	Occupation	Income	Repay
11	No	Engineer	85k	?

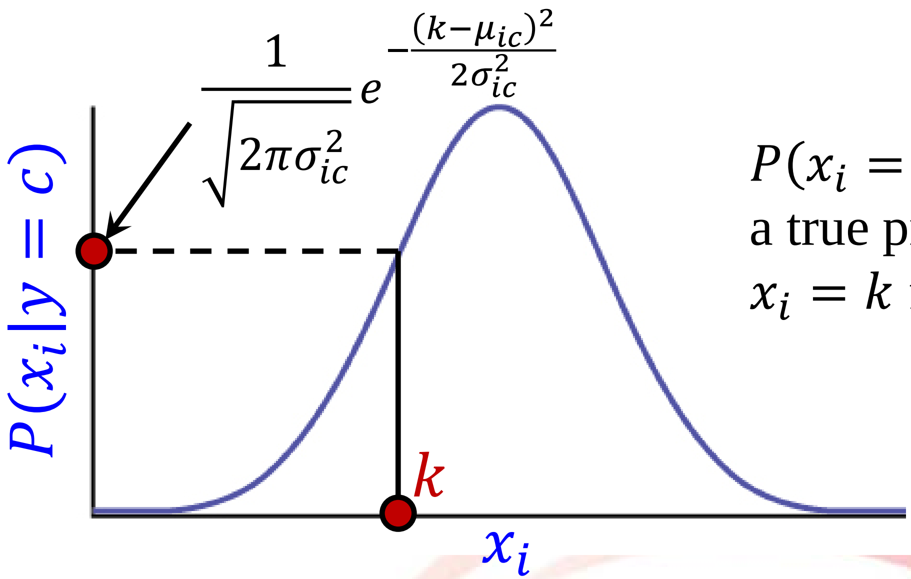
$$P(\text{Income}{=}85|\text{No}) = \frac{1}{\sqrt{2\pi} \times 5} e^{-\frac{(85-90)^2}{2\times 25}} = 0.048$$

$$P(\text{Income}{=}85|\text{Yes}) = \frac{1}{\sqrt{2\pi} \times 54.54} e^{-\frac{(85-110)^2}{2\times 2975}} = 0.007$$

Additional Notes

Probability density function $P(x_i|y=c) = \frac{1}{\sqrt{2\pi\sigma_{ic}^2}} e^{-\frac{(x_i-\mu_{ic})^2}{2\sigma_{ic}^2}}$

- The probability density function is continuous, the probability is defined as the area under the curve of the probability density function



$P(x_i = k|y=c)$ is not a true probability that $x_i = k$ for class c

Additional Notes (cont.)

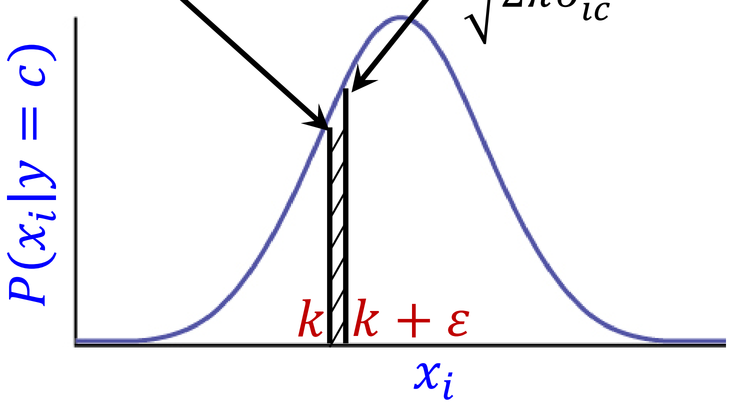
- Instead, we should compute

Small positive constant

$$P(k \le x_i \le k + \varepsilon | y = c) = \int_{k}^{k+\varepsilon} \frac{1}{\sqrt{2\pi\sigma_{ic}^2}} e^{-\frac{(x_i-\mu_{ic})^2}{2\sigma_{ic}^2}} dx_i$$

$$\frac{1}{\sqrt{2\pi\sigma_{ic}^2}} e^{-\frac{(k-\mu_{ic})^2}{2\sigma_{ic}^2}}$$

$$\frac{1}{\sqrt{2\pi\sigma_{ic}^2}} e^{-\frac{(k+\varepsilon-\mu_{ic})^2}{2\sigma_{ic}^2}}$$



Assume $\frac{1}{\sqrt{2\pi\sigma_{ic}^2}} e^{-\frac{(k-\mu_{ic})^2}{2\sigma_{ic}^2}} \approx \frac{1}{\sqrt{2\pi\sigma_{ic}^2}} e^{-\frac{(k+\varepsilon-\mu_{ic})^2}{2\sigma_{ic}^2}}$

$$\approx \frac{1}{\sqrt{2\pi\sigma_{ic}^2}} e^{-\frac{(k-\mu_{ic})^2}{2\sigma_{ic}^2}} \times \varepsilon$$

Additional Notes (cont.)

- Since ε appears as a constant multiplicative factor for each class, it cancels out when comparing posterior probabilities $P(y=c|\boldsymbol{x})$ for each class
- E.g., consider binary classification and instance is represented by a single feature of continues values

$$P(y=0|x=k) \quad \textit{vs.} \quad P(y=1|x=k)$$

$$\Downarrow$$

$$P(x=k|y=0)P(y=0) \quad \textit{vs.} \quad P(x=k|y=1)P(y=1)$$

$$\Downarrow$$

$$\frac{1}{\sqrt{2\pi\sigma_0^2}}e^{-\frac{(k-\mu_0)^2}{2\sigma_0^2}} \times \cancel{\varepsilon} \times P(y=0) \quad \textit{vs.} \quad \frac{1}{\sqrt{2\pi\sigma_1^2}}e^{-\frac{(k-\mu_1)^2}{2\sigma_1^2}} \times \cancel{\varepsilon} \times P(y=1)$$

Additional Notes (cont.)

- Therefore, we can still apply the following equation to approximate the probability of $x_i = k$ for class c

$$P(x_i = k | y = c) = \frac{1}{\sqrt{2\pi\sigma_{ic}^2}} e^{-\frac{(k-\mu_{ic})^2}{2\sigma_{ic}^2}}$$

Naïve Bayes Classifier: An Example

ID	Assets	Occupation	Income	Repay
1	Yes	Manager	125K	Yes
2	No	Engineer	100K	Yes
3	No	Manager	70K	Yes
4	Yes	Engineer	120K	Yes
5	No	Lawyer	95K	No
6	No	Engineer	60K	Yes
7	Yes	Lawyer	220K	Yes
8	No	Manager	85K	No
9	No	Engineer	75K	Yes
10	No	Manager	90K	No

$$\arg\max_c P(y = c) \prod_{i=1}^{d} P(x_i^* | y = c)$$

$P(\text{Assets=Yes} \mid \text{Repay=No}) = 0/3$
$P(\text{Assets=No} \mid \text{Repay=No}) = 3/3$
$P(\text{Assets=Yes} \mid \text{Repay=Yes}) = 3/7$
$P(\text{Assets=No} \mid \text{Repay=Yes}) = 4/7$

$P(\text{Occ.=Manager} \mid \text{Repay=No}) = 2/3$
$P(\text{Occ.=Engineer} \mid \text{Repay=No}) = 0/3$
$P(\text{Occ.=Lawyer} \mid \text{Repay=No}) = 1/3$
$P(\text{Occ.=Manager} \mid \text{Repay=Yes}) = 2/7$
$P(\text{Occ.=Engineer} \mid \text{Repay=Yes}) = 4/7$
$P(\text{Occ.=Lawyer} \mid \text{Repay=Yes}) = 1/7$

$P(\text{Income} \mid \text{Repay=Yes})$
$\mu_{\{\text{inc., Yes}\}} = 110,\ \sigma^2_{\{\text{Inc.,Yes}\}} = 2975$
$P(\text{Income} \mid \text{Repay=No})$
$\mu_{\{\text{inc., No}\}} = 90,\ \sigma^2_{\{\text{Inc.,No}\}} = 25$

$P(\text{Repay=Yes}) = 7/10$
$P(\text{Repay=No}) = 3/10$

ID	Fixed Assets	Occupation	Income	Repay
11	No	Engineer	85k	?

$P(\text{Assets=Yes} \mid \text{Repay=No}) = 0/3$
$P(\text{Assets=No} \mid \text{Repay=No}) = 3/3$
$P(\text{Assets=Yes} \mid \text{Repay=Yes}) = 3/7$
$P(\text{Assets=No} \mid \text{Repay=Yes}) = 4/7$

$P(\text{Occ.=Manager} \mid \text{Repay=No}) = 2/3$
$\boxed{P(\text{Occ.=Engineer} \mid \text{Repay=No}) = 0/3}$
$P(\text{Occ.=Lawyer} \mid \text{Repay=No}) = 1/3$
$P(\text{Occ.=Manager} \mid \text{Repay=Yes}) = 2/7$
$P(\text{Occ.=Engineer} \mid \text{Repay=Yes}) = 4/7$
$P(\text{Occ.=Lawyer} \mid \text{Repay=Yes}) = 1/7$

$P(\text{Income} \mid \text{Repay=Yes})$
$\mu_{\{\text{inc., Yes}\}} = 110,\ \sigma^2_{\{\text{Inc.,Yes}\}} = 2975$
$P(\text{Income} \mid \text{Repay=No})$
$\mu_{\{\text{inc., No}\}} = 90,\ \sigma^2_{\{\text{Inc.,No}\}} = 25$

$P(\text{Repay=Yes}) = 7/10$
$P(\text{Repay=No}) = 3/10$

$$P(\boldsymbol{x}^* \mid \text{No}) = P(\text{Assets=No} \mid \text{No}) \times P(\text{Occ.=Engineer} \mid \text{No}) \times P(\text{Income=85} \mid \text{No})$$
$$\boxed{= 1 \times 0 \times 0.048 = 0}$$

one of the conditional probabilities is 0, the entire expression is 0

$$P(\boldsymbol{x}^* \mid \text{Yes}) = P(\text{Assets=No} \mid \text{Yes}) \times P(\text{Occ.=Engineer} \mid \text{Yes}) \times P(\text{Income=85} \mid \text{Yes})$$
$$= 4/7 \times 4/7 \times 0.007 = 0.0023$$

$$P(\boldsymbol{x}^* \mid \text{No}) \times P(\text{No}) = 0 \times 0.3 = 0$$

$$P(\boldsymbol{x}^* \mid \text{Yes}) \times P(\text{Yes}) = 0.0023 \times 0.7 = 0.0016$$

predict Repay=Yes

Laplace Estimate or Smoothing

Original: $P(x_i = z|y = c) = \frac{|(x_i = z) \wedge (y = c)|}{|y = c|}$

Laplace: $P(x_i = z|y = c) = \frac{|(x_i = z) \wedge (y = c)| + 1}{|y = c| + \boxed{n_i}}$

Number of possible values of x_i

$$P(\text{Engineer}|\text{No}) = \frac{\#(\text{Engineer} \wedge \text{No})}{\#(\text{No})} = \frac{0}{3}$$

$$P(\text{Engineer}|\text{No}) = \frac{\#(\text{Engineer} \wedge \text{No}) + 1}{\#(\text{No}) + 3} = \frac{1}{6}$$

The same to $P(\text{Manager}|\text{No})$ and $P(\text{Lawyer}|\text{No})$

ID	Assets	Occupation	Income	Repay
1	Yes	Manager	125K	Yes
2	No	Engineer	100K	Yes
3	No	Manager	70K	Yes
4	Yes	Engineer	120K	Yes
5	No	Lawyer	95K	No
6	No	Engineer	60K	Yes
7	Yes	Lawyer	220K	Yes
8	No	Manager	85K	No
9	No	Engineer	75K	Yes
10	No	Manager	90K	No

Extreme case - no training data:

$$P(\text{Manager}|\text{No}) = P(\text{Engineer}|\text{No}) = P(\text{Lawyer}|\text{No}) = \frac{1}{3}$$

More General Form

Laplace: $P(x_i = z|y = c) = \dfrac{|(x_i = z) \wedge (y = c)| + \boxed{\alpha}}{|y = c| + \alpha n_i}$

$\alpha > 0$ is a smoothing parameter

For example, $\alpha = 0.1$

$$P(\text{Engineer}|\text{No}) = \frac{\#(\text{Engineer} \wedge \text{No}) + 0.1}{\#(\text{No}) + 0.3} = \frac{1}{33}$$

For example, $\alpha = 10$

$$P(\text{Engineer}|\text{No}) = \frac{\#(\text{Engineer} \wedge \text{No}) + 10}{\#(\text{No}) + 30} = \frac{10}{33}$$

ID	Assets	Occupation	Income	Repay
1	Yes	Manager	125K	Yes
2	No	Engineer	100K	Yes
3	No	Manager	70K	Yes
4	Yes	Engineer	120K	Yes
5	No	Lawyer	95K	No
6	No	Engineer	60K	Yes
7	Yes	Lawyer	220K	Yes
8	No	Manager	85K	No
9	No	Engineer	75K	Yes
10	No	Manager	90K	No

Practice

Use Laplace smoothing with $\alpha = 1$ to re-estimate $P(\text{Assets}|\text{Repay})$ and $P(\text{Occ.}|\text{Repay})$

$P(\text{Assets=Yes} \mid \text{Repay=No}) = 0/3$
$P(\text{Assets=No} \mid \text{Repay=No}) = 3/3$
$P(\text{Assets=Yes} \mid \text{Repay=Yes}) = 3/7$
$P(\text{Assets=No} \mid \text{Repay=Yes}) = 4/7$

$P(\text{Occ.=Manager} \mid \text{Repay=No}) = 2/3$
$P(\text{Occ.=Engineer} \mid \text{Repay=No}) = 0/3$
$P(\text{Occ.=Lawyer} \mid \text{Repay=No}) = 1/3$
$P(\text{Occ.=Manager} \mid \text{Repay=Yes}) = 2/7$
$P(\text{Occ.=Engineer} \mid \text{Repay=Yes}) = 4/7$
$P(\text{Occ.=Lawyer} \mid \text{Repay=Yes}) = 1/7$

$P(\text{Income} \mid \text{Repay=Yes})$
$\mu_{\{\text{inc., Yes}\}} = 110,\ \sigma^2_{\{\text{Inc.,Yes}\}} = 2975$
$P(\text{Income} \mid \text{Repay=No})$
$\mu_{\{\text{inc., No}\}} = 90,\ \sigma^2_{\{\text{Inc.,No}\}} = 25$

$P(\text{Repay=Yes}) = 7/10$
$P(\text{Repay=No}) = 3/10$

$P(\text{Assets=Yes} \mid \text{Repay=No}) = ?$
$P(\text{Assets=No} \mid \text{Repay=No}) = ?$
$P(\text{Assets=Yes} \mid \text{Repay=Yes}) = ?$
$P(\text{Assets=No} \mid \text{Repay=Yes}) = ?$

$P(\text{Occ.=Manager} \mid \text{Repay=No}) = ?$
$P(\text{Occ.=Engineer} \mid \text{Repay=No}) = ?$
$P(\text{Occ.=Lawyer} \mid \text{Repay=No}) = ?$
$P(\text{Occ.=Manager} \mid \text{Repay=Yes}) = ?$
$P(\text{Occ.=Engineer} \mid \text{Repay=Yes}) = ?$
$P(\text{Occ.=Lawyer} \mid \text{Repay=Yes}) = ?$

$P(\text{Income} \mid \text{Repay=Yes})$
$\mu_{\{\text{inc., Yes}\}} = 110,\ \sigma^2_{\{\text{Inc.,Yes}\}} = 2975$
$P(\text{Income} \mid \text{Repay=No})$
$\mu_{\{\text{inc., No}\}} = 90,\ \sigma^2_{\{\text{Inc.,No}\}} = 25$

$P(\text{Repay=Yes}) = 7/10$
$P(\text{Repay=No}) = 3/10$

$$P(x_i = z | y = c) = \frac{|(x_i = z) \wedge (y = c)| + \alpha}{|y = c| + \alpha n_i} \qquad \alpha = 1$$

$P(\text{Assets=Yes} \mid \text{Repay=No}) = 0/3$
$P(\text{Assets=No} \mid \text{Repay=No}) = 3/3$
$P(\text{Assets=Yes} \mid \text{Repay=Yes}) = 3/7$
$P(\text{Assets=No} \mid \text{Repay=Yes}) = 4/7$

$P(\text{Occ.=Manager} \mid \text{Repay=No}) = 2/3$
$P(\text{Occ.=Engineer} \mid \text{Repay=No}) = 0/3$
$P(\text{Occ.=Lawyer} \mid \text{Repay=No}) = 1/3$
$P(\text{Occ.=Manager} \mid \text{Repay=Yes}) = 2/7$
$P(\text{Occ.=Engineer} \mid \text{Repay=Yes}) = 4/7$
$P(\text{Occ.=Lawyer} \mid \text{Repay=Yes}) = 1/7$

$$P(\text{Assets=Yes} \mid \text{Repay=No}) = \frac{0+1}{3+2} = \frac{1}{5}$$
$$P(\text{Assets=No} \mid \text{Repay=No}) = \frac{3+1}{3+2} = \frac{4}{5}$$
$$P(\text{Assets=Yes} \mid \text{Repay=Yes}) = \frac{3+1}{7+2} = \frac{4}{9}$$
$$P(\text{Assets=No} \mid \text{Repay=Yes}) = \frac{4+1}{7+2} = \frac{5}{9}$$

$$P(\text{Occ.=Manager} \mid \text{Repay=No}) = \frac{2+1}{3+3} = \frac{1}{2}$$
$$P(\text{Occ.=Engineer} \mid \text{Repay=No}) = \frac{0+1}{3+3} = \frac{1}{6}$$
$$P(\text{Occ.=Lawyer} \mid \text{Repay=No}) = \frac{1+1}{3+3} = \frac{1}{3}$$
$$P(\text{Occ.=Manager} \mid \text{Repay=Yes}) = \frac{2+1}{7+3} = \frac{3}{10}$$
$$P(\text{Occ.=Engineer} \mid \text{Repay=Yes}) = \frac{4+1}{7+3} = \frac{1}{2}$$
$$P(\text{Occ.=Lawyer} \mid \text{Repay=Yes}) = \frac{1+1}{7+3} = \frac{1}{5}$$

Naïve Bayes vs. Logistic Regression

- Both are probabilistic models for classification
- Use different ways to estimate $P(y|\boldsymbol{x})$
  - Naïve Bayes:

Generative model

$$P(y|\boldsymbol{x}) = \frac{P(\boldsymbol{x}, y)}{P(\boldsymbol{x})} = \frac{P(\boldsymbol{x}|y)P(y)}{P(\boldsymbol{x})}$$

  - Logistic Regression:

Discriminative model

$$P(y=1|\boldsymbol{x}) = \frac{1}{1+\exp(-\boldsymbol{w}^T\boldsymbol{x})}$$

Binary classification

$$P(y=0|\boldsymbol{x}) = \frac{\exp\left(-\boldsymbol{w}^T\boldsymbol{x}\right)}{1+\exp(-\boldsymbol{w}^T\boldsymbol{x})}$$

On Discriminative vs. Generative Classifiers: A Comparison of Logistic Regression and Naïve Bayes, Andrew Ng and Michael Jordon, NIPS 2001

Deal with Missing Values

- In training, we only need to compute $P(x_i = z|y = c)$ for each feature independently
  - Ignore the missing value, e.g., when we compute $P(\text{Occ.} = z|\text{Repay} = \text{Yes})$ and $P(\text{Occ.} = z|\text{Repay} = \text{No})$, where $z \in \{\text{Manager,Engineer, Laywer}\}$, we only consider the data instances without missing values of Occ.
  - No need to remove whole data instances or features from the training dataset

ID	Assets	Occupation	Income	Repay
1	Yes	Manager	125K	Yes
2	No	?	100K	Yes
3	No	Manager	70K	Yes
4	Yes	Engineer	120K	Yes
5	?	Lawyer	95K	No
6	No	Engineer	60K	Yes
7	Yes	Lawyer	220K	Yes
8	No	Manager	85K	No
9	No	Engineer	75K	Yes
10	No	Manager	90K	No

- In testing,

ID	Assets	Occupation	Income	Repay
11	?	Engineer	85k	?

v.s. $\begin{cases} \boxed{P(\text{No} \mid \text{Occ.=Engineer,Income=85})} \\ P(\text{Yes} \mid \text{Occ.=Engineer,Income=85}) \end{cases}$

$$P(\text{No} \mid \text{Occ.=Engineer,Income=85}) \propto P(\text{Occ.=Engineer,Income=85} \mid \text{No})P(\text{No})$$

$$= P(\text{Occ.=Engineer,Income=85,No})$$

By using the sum rule $\sum_B P(\boldsymbol{A}, B) = P(\boldsymbol{A})$

$$= P(\text{Assets=No,Occ.=Eng.,Income=85,No}) + P(\text{Assets=Yes,Occ.=Eng.,Income=85,No})$$

$$= P(\text{Assets=No} \mid \text{No})\times P(\text{Occ.=Engineer} \mid \text{No})\times P(\text{Income=85} \mid \text{No})\times P(\text{No})$$
$$+P(\text{Assets=Yes} \mid \text{No})\times P(\text{Occ.=Engineer} \mid \text{No})\times P(\text{Income=85} \mid \text{No})\times P(\text{No})$$

$$= \boxed{\left(P(\text{Assets=No} \mid \text{No}) + P(\text{Assets=Yes} \mid \text{No})\right)} = 1$$
$$\times P(\text{Occ.=Engineer} \mid \text{No})\times P(\text{Income=85} \mid \text{No})\times P(\text{No})$$

$$= P(\text{Occ.=Engineer} \mid \text{No})\times P(\text{Income=85} \mid \text{No})\times P(\text{No})$$

$$P(\text{Yes} \mid \text{Occ.=Engineer,Income=85}) \propto P(\text{Occ.=Engineer} \mid \text{Yes})\times P(\text{Income=85} \mid \text{Yes})\times P(\text{Yes})$$

Summary

- Computationally efficient
- Computational efficiency is obtained based on a very strong assumption of conditional independence
  - The assumption may not hold in practice (most of the time)
  - That is why we call it “naïve”
  - It was widely used for text classification in the past

Implementation using scikit-learn

- API: sklearn.naive_bayes: Naive Bayes
  https://scikit-learn.org/stable/modules/classes.html#module-sklearn.naive_bayes

sklearn.naive_bayes: Naive Bayes

The sklearn.naive_bayes module implements Naive Bayes algorithms. These are supervised learning methods based on applying Bayes' theorem with strong (naive) feature independence assumptions.

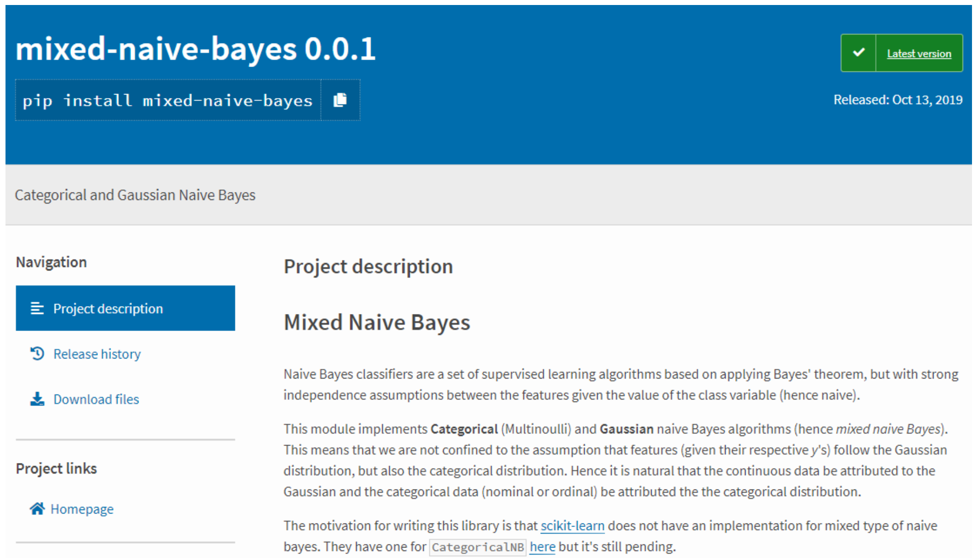
User guide: See the Naive Bayes section for further details.

naive_bayes.BernoulliNB(*[, alpha, ...])	Naive Bayes classifier for multivariate Bernoulli models.
naive_bayes.CategoricalNB(*[, alpha, ...])	Naive Bayes classifier for categorical features
naive_bayes.ComplementNB(*[, alpha, ...])	The Complement Naive Bayes classifier described in Rennie et al.
naive_bayes.GaussianNB(*[, priors, ...])	Gaussian Naive Bayes (GaussianNB)
naive_bayes.MultinomialNB(*[, alpha, ...])	Naive Bayes classifier for multinomial models

Documentation: https://scikit-learn.org/stable/modules/naive_bayes.html

Mixed Naïve Bayes Implementation

https://pypi.org/project/mixed-naive-bayes/#installation



mixed-naive-bayes 0.0.1

pip install mixed-naive-bayes

Latest version

Released: Oct 13, 2019

Categorical and Gaussian Naive Bayes

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Mixed Naive Bayes

Naive Bayes classifiers are a set of supervised learning algorithms based on applying Bayes' theorem, but with strong independence assumptions between the features given the value of the class variable (hence naive).

This module implements **Categorical** (Multinoulli) and **Gaussian** naive Bayes algorithms (hence *mixed naive Bayes*). This means that we are not confined to the assumption that features (given their respective *y*'s) follow the Gaussian distribution, but also the categorical distribution. Hence it is natural that the continuous data be attributed to the Gaussian and the categorical data (nominal or ordinal) be attributed the the categorical distribution.

The motivation for writing this library is that scikit-learn does not have an implementation for mixed type of naive bayes. They have one for `CategoricalNB` here but it's still pending.

```
>>> from mixed_naive_bayes import MixedNB
```

```
>>> nbC = MixedNB(categorical_features=[0,1,3])
>>> nbC.fit(X, y)
>>> nbC.predict(X)
```

Specify which columns are categorical features

Thank you!