AI6104 - Mathematics for AI

Tutorial 4 - Optimization

Problem 1

Find the critical points of each of the following functions

(a) $f(x,y) = \sqrt{4y^2 - 9x^2 + 24y + 36x + 36}$

(b) $g(x,y) = x^2 + 2xy - 4y^2 + 4x - 6y + 4$

(c) $h(x,y) = x^3 + 2xy - 2x - 4y$

(d) $f(x,y,z) = x + \dfrac{y^2}{4x} + \dfrac{z^2}{y} + \dfrac{2}{z}$ where $x, y, z > 0$

Solution:

(a) First we solve $f_x(x,y) = 0$ and $f_y(x,y) = 0$.

$$f_x(x,y) = \frac{-9x + 18}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} = 0$$
$$f_y(x,y) = \frac{4y + 12}{\sqrt{4y^2 - 9x^2 + 24y + 36x + 36}} = 0$$

We have $x = 2$ and $y = -3$. So $(2,-3)$ is a critical point of f.
We then check if any of the partial derivatives does not exist, *i.e.*, the denominator equals to zero.

$$4y^2 - 9x^2 + 24y + 36x + 36 = 4(y+3)^2 - 9(x-2)^2 + 36 = 0$$

This equation defines a hyperbola, and all points on this hyperbola would cause the partial derivatives not exist. Therefore, the critical points of the function f are $(2,-3)$ and all points on the hyperbola, $4(y+3)^2 - 9(x-2)^2 + 36 = 0$.

(b) We first solve $f_x(x,y) = 0$ and $f_y(x,y) = 0$.

$$g_x(x,y) = 2x + 2y + 4 = 0$$
$$g_y(x,y) = 2x - 8y - 6 = 0$$

Thus, $(-1,-1)$ is a critical point. Since there are no points in $\mathbb{R}^2$ that make the partial derivatives not exist, $(-1,-1)$ is the only critical point.

(c) We first solve $f_x(x,y) = 0$ and $f_y(x,y) = 0$.

$$h_x(x,y) = 3x^2 + 2y - 2 = 0$$
$$h_y(x,y) = 2x - 4 = 0$$

We have $x = 2$ and $y = -5$. So $(2,-5)$ is a critical point. Since there are no points in $\mathbb{R}^2$ that make the partial derivatives not exist, $(2,-5)$ is the only critical point.

(d) Note that we have

$$f_x = 1 - \frac{y^2}{4x^2} = 0, \quad f_y = \frac{y}{2x} - \frac{z^2}{y^2} = 0, \quad f_z = \frac{2z}{y} - \frac{2}{z^2} = 0$$

From $f_x = 0$, we have $y^2 = 4x^2$. Since $x, y > 0$, we conclude that $y = 2x$. Substituting $y = 2x$ into the $f_y = 0$, we get $z = y = 2x$. From the third equation, we have $z = 1$. Therefore, the critical point is $(1/2, 1, 1)$.

Problem 2

Find and classify the critical points of the function

$$f(x, y) = 10x^2y - 5x^2 - 4y^2 - x^4 - 2y^4$$

Solution:

The first partial derivatives are

$$f_x = 20xy - 10x - 4x^3 \qquad f_y = 10x^2 - 8y - 8y^3$$

We need to solve the following equation system to find the critical points

$$\begin{cases} 2x(10y - 5 - 2x^2) = 0 \\ 5x^2 - 4y - 4y^3 = 0 \end{cases}$$

From the first equation, we know that either $x = 0$ or $10y - 5 - 2x^2 = 0$.

(i) If $x = 0$, then the second equation becomes $-4y(1 + y^2) = 0$. So $y = 0$ and we have a critical point at $a_1 = (0, 0)$.

(ii) If $10y - 5 - 2x^2 = 0$, we have $x^2 = 5y - 2.5$, or $x = \pm\sqrt{5y - 2.5}$, and hence $4y^3 - 21y + 12.5 = 0$, which have three real roots,

$$y \approx -2.5452 \quad y \approx 0.6468 \quad y \approx 1.8984$$

This gives us the critical points $a_2 = (0.8567, 0.6468)$, $a_3 = (-0.8567, 0.6468)$, $a_4 = (2.6442, 1.8984)$ and $a_5 = (-2.6442, 1.8984)$.

To classify the critical points, we consider the Hessian matrix

$$H = \begin{bmatrix} 20y - 10 - 12x^2 & 20x \\ 20x & -8 - 24y^2 \end{bmatrix}$$

We can verify that the determinants of all 5 critical points are all non-zeros, which shows they are all non-degenerate. Furthermore, we find that critical points a_1, a_4 and a_5 are local maximums, since their Hessian matrices are negative definite; while a_2 and a_3 are saddle points since their Hessian matrices are neither positive nor negative definite.

Problem 3

Find the absolute maximum and minimum values of the following functions in the specified domains

(a) $f(x, y) = x^2 - 2xy + 4y^2 - 4x - 2y + 24$ on the domain defined by $0 \le x \le 4$ and $0 \le y \le 2$

(b) $g(x, y) = x^2 + y^2 + 4x - 6y$ on the domain defined by $x^2 + y^2 \le 16$

Solution:

(a) We first find the critical points for f. $(3, 1)$ is the only critical point, where $f(3, 1) = 17$. We then consider the boundary of its domain. Similar to the example in lecture notes, we consider the rectangle enclosed by $x = 4$, $y = 2$ and $xy - axis$.

(i) When $y = 0$, $f(x, 0) = x^2 - 4x + 24$. The minimum is 20, which is taken at $(2, 0)$, and the maximum value 24 is taken when $(0, 0)$ and $(4, 0)$.

(ii) When $x = 0$, $f(0, y) = 4y^2 - 2y + 24$. The minimum value $95/4$ is taken at $(0, 1/4)$, and the maximum value 36 is taken at $(0, 2)$.

(iii) When $x = 4$, $f(4, y) = 4y^2 - 10y + 24$. The minimum value $71/4$ is taken at $(4, 5/4)$, and the maximum value 24 is taken at $(4, 0)$.

(iv) When $y = 2$, $f(x, 2) = x^2 - 8x + 36$. The minimum value 20 is taken at $(4, 2)$, and the maximum value 36 is taken at $(0, 2)$.

Therefore, The absolute maximum of f is 36, which occurs at $(0, 2)$, and the absolute minimum is 17, which occurs at $(3, 1)$

(b) The only critical point of g is $(-2, 3)$, and $g(-2, 3) = -13$. We then consider the boundary of its domain, which is a circle of radius 4 centered at the origin. Consider the parametric function of the boundary,

$$\begin{cases} x(t) = 4\cos t \\ y(t) = 4\sin t \end{cases}$$

where $0 \le t \le 2\pi$. Then $g(x, y)$ can be rewritten as

$$h(t) = g(x(t), y(t)) = 16 + 16\cos t - 24\sin t$$

To find the extreme values on the boundary, we need to consider the critical points of $h(t)$, which occurs when

$$t = \pi - \arctan\frac{3}{2} \qquad \text{and} \qquad t = 2\pi - \arctan\frac{3}{2}$$

and the boundary points, which occurs when $t = 0$ and $t = 2\pi$.

(i) When $t = \pi - \arctan\frac{3}{2}$,

$$\begin{cases} \sin t = \frac{3\sqrt{13}}{13} \\ \cos t = -\frac{2\sqrt{13}}{13} \end{cases}$$

Thus, $(-\frac{8\sqrt{13}}{13}, \frac{12\sqrt{13}}{13})$ is a critical point, and

$$g(-\frac{8\sqrt{13}}{13}, \frac{12\sqrt{13}}{13}) = \frac{208 - 104\sqrt{13}}{13} \approx -12.84$$

(ii) When $t = 2\pi - \arctan\frac{3}{2}$,

$$\begin{cases} \sin t = -\frac{3\sqrt{13}}{13} \\ \cos t = \frac{2\sqrt{13}}{13} \end{cases}$$

Thus, $(\frac{8\sqrt{13}}{13}, -\frac{12\sqrt{13}}{13})$ is a critical point, and

$$h(2\pi - \arctan\frac{3}{2}) = g(\frac{8\sqrt{13}}{13}, -\frac{12\sqrt{13}}{13}) = \frac{208 + 104\sqrt{13}}{13} \approx 44.84$$

(iii) When $t = 0$ or $t = 2\pi$, $h(0) = h(2\pi) = g(1, 0) = 32$.

In conclusion, the maximum value of g is $\frac{208+104\sqrt{13}}{13}$, which is taken at boundary point $(\frac{8\sqrt{13}}{13}, -\frac{12\sqrt{13}}{13})$. The minimum value of g is -13, which is taken at critical point $(-2, 3)$.