AI6104 - Mathematics for AI

Tutorial 3 - Chain Rule, Directional Derivatives

Problem 1

Use the Chain Rule to calculate dz/dt for the following functions

(a) $z = f(x, y) = 4x^2 + 3y^2$, $x = x(t) = \sin t$, $y = y(t) = \cos t$.

(b) $z = f(x, y) = \sqrt{x^2 - y^2}$, $x = x(t) = e^{2t}$, $y = y(t) = e^{-t}$.

(c) $z = f(x, y) = x^2 - 3xy + 2y^2$, $x = x(t) = 3\sin 2t$, $y = y(t) = 4\cos 2t$.

Problem 2

Use the Chain Rule to calculate $\partial z/\partial u$ and $\partial z/\partial v$ for the following functions

(a) $z = f(x, y) = 3x^2 - 2xy + y^2$, $x = x(u, v) = 3u + 2v$, $y = y(u, v) = 4u - v$.

(b) $z = f(x, y) = \dfrac{x}{y}$, $x = x(u, v) = 2\cos u$, $y = y(u, v) = 3\sin v$.

Problem 3

Write down the equation of tangent plane to each of the following surfaces at given points

(a) $z = x^2 + y^2$ at point $(1, 2, 5)$

(b) $x^2 + y^2 + z^2 = 169$ at point $(3, 4, 12)$

(c) $z = y + \ln \dfrac{x}{z}$ at point $(1, 1, 1)$

Problem 4

Find the gradient $\nabla f(x, y)$ of each of the following functions

(a) $f(x, y) = x^2 - xy + 3y^2$

(b) $f(x, y) = \sin 3x \cos 3y$

(c) $f(x, y) = \dfrac{x^2 - 3y^2}{2x + y}$

Problem 5

Suppose the function $z = f(x, y)$ is differentiable at $\mathbf{x}$, and $\mathbf{u}$ is a unit vector. Show that $D_{\mathbf{u}}f(\mathbf{x})$ is maximized when $\mathbf{u}$ has the same direction as $\nabla f(\mathbf{x})$ and minimized when $\mathbf{u}$ has the opposite direction as $\nabla f(\mathbf{x})$. Calculate the maximum and minimum value of $D_{\mathbf{u}}f(\mathbf{x})$.

Problem 6

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function differentiable at $\mathbf{a} \in \mathbb{R}^n$. Let k and l be constants. $\mathbf{u}$ and $\mathbf{v}$ are unit vectors. Prove that $D_{k\mathbf{u}+l\mathbf{v}}f(\mathbf{a}) = kD_{\mathbf{u}}f(\mathbf{a}) + lD_{\mathbf{v}}f(\mathbf{a})$.