Q1 a) $\vec{r} = \vec{r_0} + t\vec{v}$ $\vec{r_0} = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

$\vec{r} = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

$x = 5 + (-t)$ $y = 1 + t$ $z = 3 + 2t$.

$x^2 + y^2 + z^2 = 25.$

$(5-t)^2 + (1+t)^2 + (3+2t)^2 = 25.$

$25 - 10t + t^2 + 1 + 2t + t^2 + 9 + 6t + 4t^2 = 25.$

$10 - 2t + 6t^2 = 0.$

$(-2)^2 - 4(6)(10) < 0$

$\therefore$ ~~No~~ The line doesn't pass the sphere

(5 marks)

Q1 b) $ax + by + cz + d = 0.$

$5a + b + 3c + d = 0$ — (1)

$a + b + c + d = 0$ — (2)

$5a + 5b + 9c + d = 0$ — (3)

(3) − (1) $4b + 6c = 0 \Rightarrow b = \frac{-3}{2}c$ — (5)

(1) − (2) $4a + 2c = 0 \Rightarrow a = \frac{-1}{2}c$ — (6)

$\therefore$ (2) becomes: $\frac{-1}{2}c + \frac{-3}{2}c + c + d = 0.$

$\Rightarrow -2c + c + d = 0. \Rightarrow d = c$ — (7)

sub (5) - (7) into original equation

$-\frac{1}{2}cx + \frac{-3}{2}cy + cz + c = 0.$

$\Rightarrow x + 3y + (-2z) - 2 = 0.$

$x + 3y - 2z - 2 = 0.$

(5 marks)

r=(2,1,0)+t(1,3,-2)
=(2+t)+3(1+3t)-2(-2t)-2=0
closest point (2+t)+3(1+3t)-2(-2t)-2=0
(25/14, 5/14,3/7)

Q1 c) i) $\lim_{(x,y)\to(1,1)} \frac{y}{x^2+y^2} = \frac{1}{1+1} = \frac{1}{2}$. (2 marks)

ii) $\lim_{(x,\theta)\to(0,\infty)} x\sin(\theta) = 0$ (2 marks)

iii) $\lim_{(x,y)\to(0,0)} \frac{2x^2+y^2}{x^2-y^2}$

Consider the path ~~(x,y)→x~~ along y-axis.

$\lim_{(x\to y)\to(0,0)} \frac{2x^2}{x^2} = 2$.

Consider the path along x-axis.

$\lim_{(x,y)\to(0,0)} \frac{+y^2}{-y^2} = -1$

$\therefore$ limit doesn't exist (2 marks)

d) i) $f(x,y) = \left(\frac{x^2}{y^2}\right) + xy$

$\frac{\partial f(x,y)}{\partial x} = \frac{2x}{y^2} + y$ $\qquad \frac{\partial f(x,y)}{\partial y} = \frac{-2x^2}{y^3} + x$ (3 marks)

ii)

$$f(x_1 \cdots x_n) = \sum_{i=1}^{n-1} x_i^2 + x_i x_{i+1}$$

$\frac{\partial f}{\partial x_1} = 2x_1 + x_2$, $\frac{\partial f}{\partial x_n} = x_{n-1}$, $\frac{\partial f}{\partial x_i} = 2x_i + x_{i-1} + x_{i+1}$, where $1 < i < n$

1e) $Z = 4x^2 + 2y^2 + xy + 4y + 1$

$\frac{\partial z}{\partial x} = 8x + \cancel{4y} + y \Big|_{1}^{1} = 9$

$\frac{\partial z}{\partial y} = 4y + x + 4 \Big|_{1}^{1} = 9.$

$Z(1,1) = 4 + 2 + 1 + 4 + 1 = 12$

Tangent plane.

$$z = 12 + 9(x-1) + 9(y-1)$$
$$= 9x + 9y - 6$$

(3 marks)

2a) $z = x^2 + xy^2 + 1$ $\quad \frac{\partial z}{\partial x} = 2x + y^2$ $\quad \frac{\partial z}{\partial y} = 2xy$

$\frac{\partial x}{\partial u} = \frac{\partial (uv^2 + w^2)}{\partial u} = v^2$ $\qquad \frac{\partial y}{\partial u} = \frac{\partial (u + ve^w)}{\partial u} = 1$

$\frac{\partial x}{\partial v} = 2uv$ $\qquad \frac{\partial y}{\partial v} = e^w$

$\frac{\partial x}{\partial w} = 2w$ $\qquad \frac{\partial y}{\partial w} = ve^w$

when $u = 2$, $v = 1$, $w = 0$

$\frac{\partial x}{\partial u} = 1$, $\frac{\partial y}{\partial u} = 1$, $\frac{\partial x}{\partial v} = 4$, $\frac{\partial y}{\partial v} = 1$, $\frac{\partial x}{\partial w} = 0$, $\frac{\partial y}{\partial w} = 1.$

$x = uv^2 + w^2 = 2$ $\qquad y = u + ve^w = 3$

$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}$

$\frac{\partial z}{\partial x} = 2(2) + 3^2 = 13$ $\qquad \frac{\partial z}{\partial y} = 2(2)(3) = 12.$

$\frac{\partial z}{\partial u} = (13)(1) + (12)(1) = 25$

$\frac{\partial z}{\partial v} = (13)(4) + (12)(1) = 64$

$\frac{\partial z}{\partial w} = (13)(0) + (12)(1) = 12$

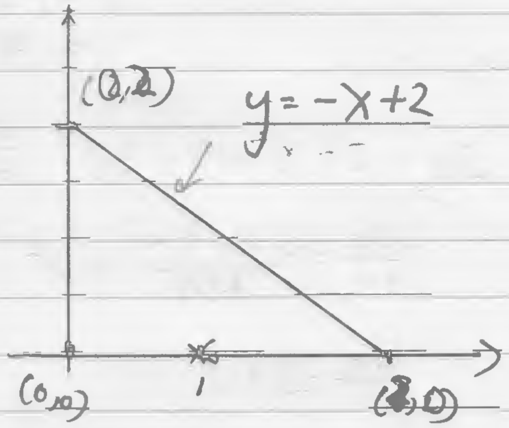
(7 marks)

b)

$$f(x,y) = x^2 \ln y, \frac{\partial f}{\partial x} = 2x\ln y, \frac{\partial f}{\partial y} = \frac{x^2}{y},$$

at P=(3,1) $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 9, Df = \langle 0,9\rangle \cdot \langle 1,1\rangle \frac{1}{\sqrt{2}} = \frac{9}{\sqrt{2}}$

c) $f(x,y,z) = x^2 + y^2 - 2x$



$\frac{\partial f}{\partial x} = 2x - 2 = 0 \quad x = 1$

$\frac{\partial f}{\partial y} = 2y = 0 \quad y = 0$

when $f(1,0) = 1 - 2(1) = -1$

when $y = 0$ $\quad f(x,0) = x^2 - 2x \quad \frac{d}{dx} x^2 - 2x = 2x - 2 = 0.$

$f(0,0) = 0$, $\quad f(2,0) = 0$

when $x = 0$ $\quad f(0,y) = y^2 \quad \frac{dy}{dy} = y = 0$

$f(0,0) = 0$ $\quad f(0,2) = 4$.

when $y = -x+2$.

$f(x,y) = x^2 + (-x+2)^2 - 2x$

$= x^2 + x^2 - 4x + 4 - 2x = 2x^2 - 6x + 4 = x^2 - 3x + 2.$

$= \cancel{4x} \quad \cancel{(x-1)+x}$

$f_x(x,y) = 2x - 3 = 0 \quad x = \frac{3}{2}$

$f(x,y) = (\frac{3}{2})^2 + (\frac{-3}{2} + 2)^2 - 2(\frac{3}{2}) = -\frac{1}{2}$

min $= -1$ at $(1,0)$ ~~max at~~

max $= 4$ at $(0,2)$

(7 marks)

2d) ~~$\partial ||x|| - a$~~

$$\frac{\partial ||y-Ax||}{\partial x} = \frac{1}{2||y-Ax||} \cdot \frac{\partial}{\partial x}(y-Ax)^T(y-Ax)$$

$$= \frac{1}{2||y-Ax||} \cdot 2(y-Ax)^T(-A)$$

$$= \frac{(y-Ax)^T(-A)}{||y-Ax||}$$

(5 marks)

3a) eigenvalue $= \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}$

eigen vector $= \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$

(5 marks)

3b) $BAB^{-1}(Bx) = BAx = B\lambda x = \lambda Bx$

(5 marks)

3c) $X^{-1}(A+5I)X = X^{-1}AX + 5X^{-1}X = \Lambda + 5I$

$\therefore$ ~~eigenvector is X & eigenmatix~~

$\therefore$ eigenvector matrix is X

eigenvalue matrix is $\Lambda + 5I$

(5 marks)

d) $A \sim B$ $P^{-1}AP = B$ $B \sim C$ $Q^{-1}BQ = C$

Consider $Q^{-1}P^{-1}APQ = Q^{-1}BQ = C.$

$\therefore$ $A \sim C$ //

(5 marks)

e) $Ax = \lambda x$ $Bx = \mu x$

$(A+B)x = \lambda x + \mu x = (\lambda + \mu)x$

$(A \cdot B)x = A\mu x = \lambda \cdot \mu x$

(5 marks)

4(a)

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$ Using row operations, we have $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Since rank 3, they are linearly independent.

(b) consider $\begin{bmatrix} 1 & 1 & 1 & a \\ 1 & -1 & 3 & b \\ 0 & 1 & 2 & c \\ -1 & 0 & 1 & d \end{bmatrix}$. Applied R2-R1 and R4-R1, we have $\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & -2 & 2 & b-a \\ 0 & 1 & 2 & c \\ 0 & 1 & 2 & d+a \end{bmatrix}$.

Apply R4-R3 we have $\begin{bmatrix} 1 & 1 & 1 & a \\ 0 & -2 & 2 & b-a \\ 0 & 1 & 2 & c \\ 0 & 0 & 0 & d+a-c \end{bmatrix}$. From 4(a), we know that the rank of this matrix should be at least rank 3. As long as $d+a-c \neq 0$, it is full rank. We can select a=0, b=0, c=0 and d=1 to extend it to a basis of $\Re^4$.

c) rank=2, nullity=1