

# AI6104 - MATHEMATICS FOR AI

## TUTORIAL 1 - VECTORS, DOT PRODUCTS, PLANES

### Problem 1

Note that in lecture we introduced Euclidean distance between two points  $x = (x_1, x_2, x_3)$  and  $y = (y_1, y_2, y_3)$ , i.e.,  $d(x, y) = \sqrt{\sum_{i=1}^n |x_i - y_i|^2}$ .

A general definition of distance, Minkowski distance of order  $p$ , also known as  $p$ -norm distance, is defined as

$$D_p(x, y) = \left( \sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}}$$

In the limit case when  $p$  tends to infinity, we have the  $\infty$ -norm distance,

$$D_\infty(x, y) = \lim_{p \rightarrow \infty} \left( \sum_{i=1}^n |x_i - y_i|^p \right)^{\frac{1}{p}} = \max_i (|x_i - y_i|)$$

Calculate the following distance between  $P_1 = (3, -1, 5)$  and  $P_2 = (2, 1, -1)$

- (a) 1-norm distance (Manhattan distance)
- (b) 2-norm distance (Euclidean distance)
- (c)  $\infty$ -norm distance (Chebyshev distance)

*Solution:*

(a)  $D_1(P_1, P_2) = |3 - 2| + |-1 - 1| + |5 - (-1)| = 9$

(b)  $D_2(P_1, P_2) = \sqrt{|3 - 2|^2 + |-1 - 1|^2 + |5 - (-1)|^2} = \sqrt{41}$

(c)  $D_\infty(P_1, P_2) = \max(|3 - 2|, |-1 - 1|, |5 - (-1)|) = 6$

### Problem 2

Let  $\mathbf{u} = \langle 3, 0, 4 \rangle$ ,  $\mathbf{v} = \langle 0, 5, 12 \rangle$  be vectors. Calculate  $\|\mathbf{u}\| + \|\mathbf{v}\|$  and  $\|\mathbf{u} + \mathbf{v}\|$ .

*Solution:*

$$\|\mathbf{u}\| = \sqrt{3^2 + 0^2 + 4^2} = 5$$

$$\|\mathbf{v}\| = \sqrt{0^2 + 5^2 + 12^2} = 13$$

$$\|\mathbf{u} + \mathbf{v}\| = \|\langle 3, 5, 16 \rangle\| = \sqrt{290} \approx 17.03$$

We note that  $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$ , which is well-known as triangle inequality.

**Problem 3**

Let  $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$  and  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$  be vectors. Prove the following properties of the dot product.

(a)  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

*Solution:*

$$\begin{aligned}\mathbf{u} \cdot \mathbf{v} &= \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= u_1v_1 + u_2v_2 + u_3v_3 \\ &= v_1u_1 + v_2u_2 + v_3u_3 \\ &= \langle v_1, v_2, v_3 \rangle \cdot \langle u_1, u_2, u_3 \rangle \\ &= \mathbf{v} \cdot \mathbf{u}\end{aligned}$$

(b)  $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v}$

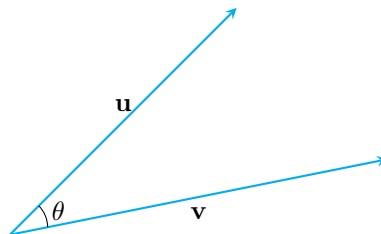
*Solution:*

$$\begin{aligned}c(\mathbf{u} \cdot \mathbf{v}) &= c(u_1v_1 + u_2v_2 + u_3v_3) \\ &= (cu_1)v_1 + c(u_2)v_2 + c(u_3)v_3 \\ &= \langle cu_1, cu_2, cu_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= (c\mathbf{u}) \cdot \mathbf{v}\end{aligned}$$

**Problem 4**

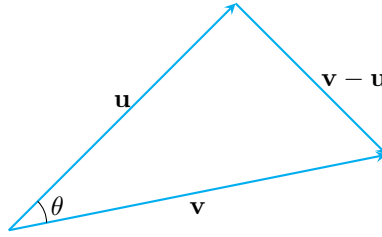
Prove that the dot product of two vectors is the product of the magnitude of each vector and the cosine of the angle  $\theta$  between them, *i.e.*,

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$



*Solution:*

If we connect the endpoints of  $\mathbf{u}$  and  $\mathbf{v}$ , we have a triangle with the sides  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{v} - \mathbf{u}$ .



By the law of cosines,

$$\|\mathbf{v} - \mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

Expanding the left-hand side gives

$$\|\mathbf{v} - \mathbf{u}\|^2 = \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\|^2$$

This gives us the result that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

### Problem 5

Prove that two vectors  $\mathbf{u}$  and  $\mathbf{v}$  are orthogonal if and only if  $\mathbf{u} \cdot \mathbf{v} = 0$ .

*Solution:*

Let  $\mathbf{u}$  and  $\mathbf{v}$  be nonzero vectors, and let  $\theta$  be the angle between them.

If  $\mathbf{u} \cdot \mathbf{v} = 0$ , then

$$\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta = 0$$

Since both  $\|\mathbf{u}\|$  and  $\|\mathbf{v}\|$  are non-zeros, we must have  $\cos\theta = 0$ . Hence  $\theta = \pi/2$  and the two vectors are orthogonal.

Now assume that  $\mathbf{u}$  and  $\mathbf{v}$  are orthogonal. Then  $\theta = \pi/2$ , and we have

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\frac{\pi}{2} = 0$$

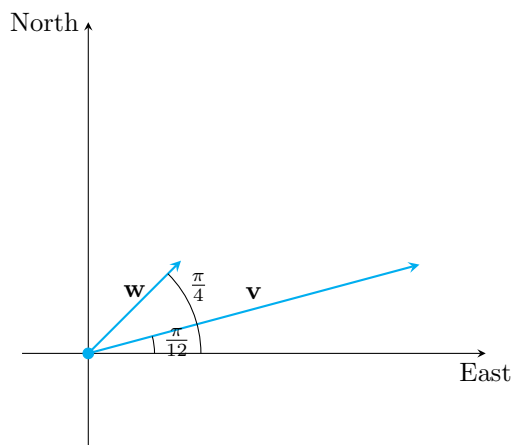
### Problem 6

A container ship leaves port traveling 15 degrees north of east. Its engine generates a speed of 20 knots along that path (see the following figure). In addition, the ocean current moves the ship northeast at a speed of 2 knots. Considering both the engine and the current, how fast is the ship moving in the direction 15 degrees north of east?

*Solution:*

Let  $\mathbf{v}$  be the velocity vector generated by the engine, and let  $\mathbf{w}$  the velocity vector of the current. We already know  $\|\mathbf{v}\| = 20$  along the desired route. We only need to add in the scalar projection of  $\mathbf{w}$  onto  $\mathbf{v}$ . Thus,

$$\text{comp}_{\mathbf{v}}\mathbf{w} = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|} = \frac{\|\mathbf{v}\|\|\mathbf{w}\|\cos(30)}{\|\mathbf{v}\|} = \|\mathbf{w}\|\cos(30) = \sqrt{3}$$



Therefore, the ship moves in the direction 15 degrees north of east at the speed of  $20 + \sqrt{3} \approx 21.73$  knots.

### Problem 7

Let  $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$  and  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$  be vectors, and let  $\theta$  be the angle between them. Prove that

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$$

*Solution:*

By the definition of cross product, we have

$$\begin{aligned} \|\mathbf{u} \times \mathbf{v}\|^2 &= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\ &= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \\ &= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 \\ &= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \cos^2 \theta \\ &= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 (1 - \cos^2 \theta) \\ &= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \sin^2 \theta \end{aligned}$$

Note that  $0 \leq \theta \leq 2\pi$ , we have  $\sin \theta \geq 0$ . Thus,

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$$

### Problem 8

Determine whether each pair of planes is parallel, orthogonal. If neither, calculate the angle between two planes.

- (a)  $x + 3y - 2z = 8$  and  $2x + 6y - 4z = 5$
- (b)  $2x - 3y + 2z = 3$  and  $4x + 2y - z = 6$
- (c)  $x + 2y + z = 4$  and  $x - 3y + 2z = 1$

*Solution:*

- (a) The normal vectors are  $\mathbf{n}_1 = \langle 1, 3, -2 \rangle$  and  $\mathbf{n}_2 = \langle 2, 6, -4 \rangle$ . They are parallel to each other, so are the planes.
- (b) The normal vectors are  $\mathbf{n}_1 = \langle 2, -3, 2 \rangle$  and  $\mathbf{n}_2 = \langle 4, 2, -1 \rangle$ . Note that  $\mathbf{n}_1 \cdot \mathbf{n}_2 = 2 \cdot 4 - 3 \cdot 2 - 2 \cdot 1 = 0$ . The normal vectors are orthogonal, and the corresponding planes are orthogonal.
- (c) The normal vectors are  $\mathbf{n}_1 = \langle 1, 2, 1 \rangle$  and  $\mathbf{n}_2 = \langle 1, -3, 2 \rangle$ .

$$\begin{aligned}\cos \theta &= \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \\ &= \frac{|\langle 1, 2, 1 \rangle \cdot \langle 1, -3, 2 \rangle|}{\sqrt{1^2 + 2^2 + 1^2} \sqrt{1^2 + (-3)^2 + 2^2}} \\ &= \frac{-\sqrt{3}}{2\sqrt{7}}\end{aligned}$$

Thus, the angle between them is  $\arccos(-\sqrt{3}/(2\sqrt{7}))$ .