AI6104 - Mathematics for AI

Tutorial 1 - Vectors, Dot Products, Planes

Problem 1

Note that in lecture we introduced Euclidean distance between two points $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, *i.e.*, $d(x, y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$.

A general definition of distance, Minkowski distance of order p, also known as p-norm distance, is defined as

$$D_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

In the limit case when p tends to infinity, we have the ∞-norm distance,

$$D_\infty(x, y) = \lim_{p \to \infty} \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}} = \max_i(|x_i - y_i|)$$

Calculate the following distance between $P_1 = (3, -1, 5)$ and $P_2 = (2, 1, -1)$

(a) 1-norm distance (Manhattan distance)

(b) 2-norm distance (Euclidean distance)

(c) ∞-norm distance (Chebyshev distance)

Problem 2

Let $\mathbf{u} = \langle 3, 0, 4 \rangle$, $\mathbf{v} = \langle 0, 5, 12 \rangle$ be vectors. Calculate $\|\mathbf{u}\| + \|\mathbf{v}\|$ and $\|\mathbf{u} + \mathbf{v}\|$.

Problem 3

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors. Prove the following properties of the dot product.

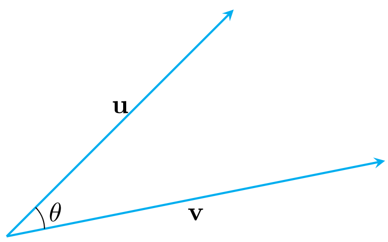
(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

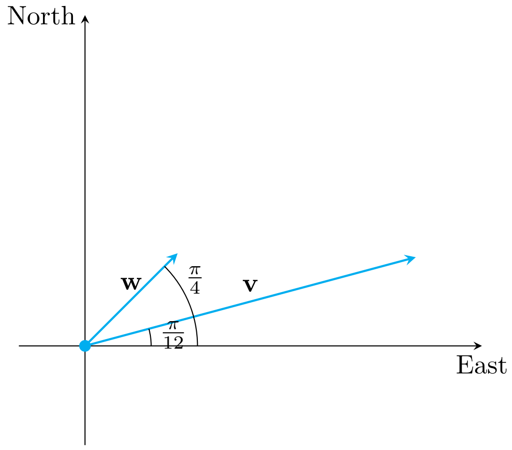
(b) $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v}$

Problem 4

Prove that the dot product of two vectors is the product of the magnitude of each vector and the cosine of the angle θ between them, *i.e.*,

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos\theta$$





Problem 5

Prove that two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

Problem 6

A container ship leaves port traveling 15 degrees north of east. Its engine generates a speed of 20 knots along that path (see the following figure). In addition, the ocean current moves the ship northeast at a speed of 2 knots. Considering both the engine and the current, how fast is the ship moving in the direction 15 degrees north of east?

Problem 7

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors, and let θ be the angle between them. Prove that

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$$

Problem 8

Determine whether each pair of planes is parallel, orthogonal. If neither, calculate the angle between two planes.

(a) $x + 3y - 2z = 8$ and $2x + 6y - 4z = 5$

(b) $2x - 3y + 2z = 3$ and $4x + 2y - z = 6$

(c) $x + 2y + z = 4$ and $x - 3y + 2z = 1$