AI6104 - Mathematics for AI

Tutorial 8 - Determinants, Eigenvalues and Eigenvectors

Problem 1

Compute the determinant of following matrices using the Laplace expansion along the first row.

(a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ (c) $C = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

Problem 2

Find the eigenvalues and a basis for each eigenspace of the following matrices

(a) $A = \begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ (c) $A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$

Problem 3

Compute the indicated power of matrix

$$A^{2020} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{2020}$$

Problem 4

Principal Component Analysis, or simply PCA, is a statistical procedure that allows us to identify the principal directions in which the data varies. Suppose we have input data $x \in \mathbb{R}^p$, by PCA we may construct another random vector y which has lower dimension $m \ll p$ but has similar statistical properties. This often makes it easier to interpret the data and to identify interesting features.

The idea behind PCA is to ignore the redundant variables, *i.e.*, variables that are highly correlated with others. Thinking of x as attributes of automobiles, and two different attributes, some x_i and x_j, respectively give a car's maximum speed measured in miles per hour, and kilometers per hour. These two attributes are therefore almost linearly dependent; thus, the data really lies approximately on an $p-1$ dimensional subspace. The measure of such correlation is the covariance matrix, Σ.

(a) One important step in PCA is to find a linear function of x, $\alpha^\top x$, which has the maximum variance, *i.e.*, $\mathrm{Var}(\alpha^\top x) = \alpha^\top \Sigma \alpha$, where $\alpha^\top \alpha = 1$.

Show that the eigenvector α_1 corresponding to the largest eigenvalue λ_1 is the solution to the maximization problem.

(b) In general, α_k will give the k-th largest variance. Consider $k = 2$, we wish to find the the largest linear function of x, $\alpha^\top x$, which is uncorrelated with $\alpha_1 x$, and has the largest variance. The constraint of uncorrelation can be expressed as $\mathrm{Cov}(\alpha^\top x, \alpha_1^\top x) =$

$\lambda_1 \alpha^\top \alpha_1 = 0$. Prove that eigenvector α_2 corresponding to the second largest eigenvalue λ_2 is the solution to the maximization problem

$$\begin{aligned} \max_{\alpha} \quad & \alpha^\top \Sigma \alpha \\ \text{subject to} \quad & \alpha^\top \alpha = 1 \\ & \alpha^\top \alpha_1 = 0 \end{aligned}$$

Hint: you may use Lagrange multipliers to solve the maximization problem.