AI6104 - Mathematics for AI

Tutorial 5 - Lagrange Multipliers, Jacobians

Problem 1

Use the method of Lagrange multipliers to find the minimum value of $f(x, y) = x^2 + 4y^2 - 2x + 8y$, subject to the constraint $x + 2y = 7$.

Solution:

Let $g(x, y) = x+2y$. Consider that $\nabla f(x, y) = \langle 2x-2, 8y+7\rangle$ and $nablag(x, y) = \langle 1, 2\rangle$, and let $\nabla f(x, y) = \lambda \nabla g(x, y)$, we have

$$\langle 2x - 2, 8y + 8\rangle = \lambda\langle 1, 2\rangle$$

Since the constraint is $x + 2y = 7$, we have the following equation system

$$\begin{cases} 2x - 2 = \lambda \\ 8y + 8 = 2\lambda \\ x + 2y = 7 \end{cases}$$

which can be solved by

$$\begin{cases} x = 5 \\ y = 1 \\ \lambda = 8 \end{cases}$$

The corresponding minimum value is $f(5, 1) = 27$.

Problem 2

Find the shortest distance from the origin to the curve $x^2y = 16$.

Solution:

Suppose the closest point on the curve $x^2y = 16$ from the origin is (x_0, y_0), which satisfies the curve equation $x_0^2 y = 16$. The distance is $x_0^2 + y_0^2$. The problem can be converted to a constraint minimization problem,

$$\begin{cases} \min x_0^2 + y_0^2 \\ s.t.\ x_0^2 y - 16 = 0 \end{cases}$$

Hence the Lagrange multipliers method gives us

$$\begin{cases} 2x_0 = 2\lambda x_0 y_0 \\ 2y_0 = \lambda x_0^2 \\ x_0^2 y_0 = 16 \end{cases}$$

Note that the first equation implies that either $x_0 = 0$ or $\lambda y_0 = 1$. Since $x_0 = 0$ is inconsistent with the third equation, we have $\lambda y_0 = 1$.
The second equation gives us $x_0^2 = 2y_0^2$. Hence, $x_0 = \pm\sqrt{2}y_0$. Substituting in $x_0^2 y_0 = 16$ we get $2y_0^3 = 16$, *i.e.*, $y_0 = 2$.
Therefore, $(\pm 2\sqrt{2}, 2)$ are two points with the minimum distance from the origin, and the distance is

$$\sqrt{\left(\pm 2\sqrt{2}\right)^2 + 2^2} = 2\sqrt{3}$$

Problem 3

Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

Solution:

Take $g(x, y, z) = x - y + z - 1$ and $h(x, y, z) = x^2 + y^2 - 1$. Hence, the Lagrange condition is $\nabla f = \lambda \nabla g + \mu \nabla h$, which gives us the equation system

$$\begin{cases} 1 = \lambda + 2\mu x \\ 2 = -\lambda + 2\mu y \\ 3 = \lambda \\ x - y + z - 1 = 0 \\ x^2 + y^2 - 1 \end{cases}$$

We substitute $\lambda = 3$ into the first two equations, hence

$$x = -\frac{1}{\mu}, y = \frac{5}{2\mu}$$

Substituting into the last equation gives

$$\frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1$$

so $\mu = \pm\sqrt{29}/2$. Therefore, $x = \pm 2/\sqrt{29}$ and $y = \mp 5/\sqrt{29}$. From the forth equation, we get

$$z = 1 - x + y = 1 \mp 7/\sqrt{29}$$

The corresponding values of f are

$$\pm\frac{2}{\sqrt{29}} + 2\left(\mp\frac{5}{\sqrt{29}}\right) + 3\left(1 \mp \frac{7}{\sqrt{29}}\right) = 3 \pm \sqrt{29}$$

Therefore, the maximum value of f is $3 + \sqrt{29}$.

Problem 4

Find the maximum and minimum values of $f(x, y, z) = xy + 2z$ on the intersection of the plane $x + y + z = 0$ and the sphere $x^2 + y^2 + z^2 = 24$.

Solution:

We first write $g(x, y, z) = x + y + z$ and $h(x, y, z) = x^2 + y^2 + z^2 - 24$. Hence, the Lagrange condition $\nabla f = \lambda \nabla g + \mu \nabla h$ gives us the equation system

$$\begin{cases} y = \lambda + 2\mu x \\ x = \lambda + 2\mu y \\ 2 = \lambda + 2\mu z \\ x + y + z = 0 \\ x^2 + y^2 + z^2 - 24 = 0 \end{cases}$$

Subtracting the first two equation gives us $(x - y)(1 + 2\mu) = 0$. So either $\mu = -1/2$ or $x = y$.

(i) If $\mu = -1/2$, we have

$$x = \lambda - y, 2 = \lambda - z$$

Thus, $x + y = 2 + z$. Combining with $x + y + z = 0$, we get $z = -1$ and $x + y = 1$. Now $x^2 + y^2 = 23$ and since $x^2 + y^2 + 2xy = 1$, we have $xy = -11$ and $(x - y)^2 = x^2 + y^2 - 2xy = 45$. So $x - y = \pm 3\sqrt{5}$. Combining this with $x + y = 1$, we obtain two solutions from $\mu = -1/2$, namely

$$\left(\left(1 + 3\sqrt{5}\right)/2, \left(1 - 3\sqrt{5}\right)/2, -1\right)$$

and

$$\left(\left(1 - 3\sqrt{5}\right)/2, \left(1 + 3\sqrt{5}\right)/2, -1\right)$$

At both points, we find that $f(x, y, z) = xy + 2z = -13$.

(ii) If $x = y$, from the last two equations, we have $z = -2x$, and $6x^2 = 24$. So $x = \pm 2$. Therefore we have two solutions $(2, 2, -4)$ and $(-2, -2, 4)$, which yields $f(2, 2, -4) = -4$ and $f(-2, -2, 4) = 12$.

We conclude that the maximum value of f is 12 and the minimum value is -13.

Problem 5

Calculate the Jacobian Matrix of the following functions

(a) $f(x, y) = (x \sin(xy), \arctan(x + y))$

(b) $g(x, y) = \left(xy^2, \sin(x^2 - y), e^{x-2y}\right)$

Solution:

(a) The Jacobian matrix of f is a 2×2 matrix

$$J = \begin{bmatrix} \frac{\partial}{\partial x} f_1 & \frac{\partial}{\partial y} f_1 \\ \frac{\partial}{\partial x} f_2 & \frac{\partial}{\partial y} f_2 \end{bmatrix} = \begin{bmatrix} \sin(xy) + xy\cos(xy) & x^2\cos(xy) \\ \frac{1}{1+(x+y)^2} & \frac{1}{1+(x+y)^2} \end{bmatrix}$$

(b) The Jacobian of g is a 3×2 matrix

$$J = \begin{bmatrix} \frac{\partial}{\partial x} g_1 & \frac{\partial}{\partial y} g_1 \\ \frac{\partial}{\partial x} g_2 & \frac{\partial}{\partial y} g_2 \\ \frac{\partial}{\partial x} g_3 & \frac{\partial}{\partial y} g_3 \end{bmatrix} = \begin{bmatrix} y^2 & 2xy \\ 2x\cos(x^2 - y) & -\cos(x^2 - y) \\ e^{x-2y} & -2e^{x-2y} \end{bmatrix}$$