AI6104 - Mathematics for AI

Extra Practice Questions - Limits & Derivatives

Problem 1

Evaluate the following limits

(a) $\displaystyle\lim_{(x,y)\to(0,0)} x^4 \sin\left(\frac{1}{x^2+|y|}\right)$

(b) $\displaystyle\lim_{(x,y)\to(0,0)} e^{x+y^2}$

(c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2}$

(d) $\displaystyle\lim_{(x,y)\to(-1,0)} \frac{x^2+xy+3}{x^2y-5xy+y^2+1}$

(e) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^4}{x^2+y^4}$

(f) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{e^x e^y}{x+y+2}$

(g) $\displaystyle\lim_{(x,y)\to(2,0)} \frac{x^2-y^2-4x+4}{x^2+y^2-4x+4}$

(h) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$

(i) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2+y^2}{x^2+y^2}$

(j) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+2xy+y^2}{x+y}$

(k) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^2+y^2}$

(l) $\displaystyle\lim_{(x,y)\to(0,0),x\neq y} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

(m) $\displaystyle\lim_{(x,y,z)\to(0,\sqrt{\pi},1)} e^{xz}\cos y^2 - x$

(n) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy-xz+yz}{x^2+y^2+z^2}$

(o) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} x^2+2xy+yz+z^3+2$

Answers:

(a) 0

(b) 1

(c) does not exist

(d) 4

(e) does not exist

(f) 1/2

(g) does not exist

(h) does not exist

(i) does not exist

(j) 0

(k) 0

(l) 0

(m) -1

(n) does not exist

(o) 2

Problem 2

Calculate the partial derivatives with respect to all variables in following functions

(a) $f(x, y) = xy^2 + x^2y$

(b) $f(x, y) = \sin xy + \cos xy$

(c) $f(x, y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

(d) $f(x, y) = \cos x^3y$

(e) $f(x, y) = xe^y + y\sin(x^2 + y)$

(f) $f(x, y) = e^{x^2+y^2}$

(g) $f(x, y) = \dfrac{x^3 - y^2}{1 + x^2 + 3y^4}$

(h) $f(x, y) = \ln(x^2 + y^2)$

(i) $f(x, y) = \ln\left(\dfrac{x}{y}\right)$

(j) $f(x, y, z) = \dfrac{x - y}{y + z}$

(k) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

(l) $f(x, y, z) = \dfrac{x + y + z}{(1 + x^2 + y^2 + z^2)^{3/2}}$

(m) $f(x, y, z) = \dfrac{x^3 + yz}{x^2 + z^2 + 1}$

Answers:

(a) $f_x = y^2 + 2xy,\ f_y(x, y) = 2xy + x^2$

(b) $f_x = y \cos xy - y \sin xy,\ f_y = x \cos xy - x \sin xy$

(c) $f_x = 4xy^2/(x^2 + y^2)^2,\ f_y = -4x^2y/(x^2 + y^2)^2$

(d) $f_x = -3x^2y \sin x^3y,\ f_y = -x^3 \sin x^3y$

(e) $f_x = e^y + 2xy \cos(x^2 + y),\ f_y = xe^y + \sin(x^2 + y) + y \cos(x^2 + y)$

(f) $f_x = 2xe^{x^2+y^2},\ f_y = 2ye^{x^2+y^2}$

(g) $$f_x = \frac{x\left(x^3 + x\left(9y^4 + 3\right) + 2y^2\right)}{\left(x^2 + 3y^4 + 1\right)^2},$$
$$f_y = -\frac{2\left(6x^3y^3 + x^2y - 3y^5 + y\right)}{\left(x^2 + 3y^4 + 1\right)^2}$$

(h) $f_x = \dfrac{2x}{x^2 + y^2}, f_y = \dfrac{2y}{x^2 + y^2}$

(i) $f_x = 1/x, f_y = -1/y$

(j) $f_x = 1/(y + z),\ f_y = -(x + z)/(y + z)^2,\ f_z = (y - x)/(y + z)^2$

(k) $f_x = x/\sqrt{x^2 + y^2 + z^2},\ f_y = y/\sqrt{x^2 + y^2 + z^2},\ f_z = z/\sqrt{x^2 + y^2 + z^2}$

(l) $$f_x = \frac{1 - 2x^2 - 3xy - 2y^2 - 3xz + z^2}{(1 + x^2 + y^2 + z^2)^{5/2}},$$
$$f_z = \frac{1 + x^2 - 3xy - 2y^2 - 3yz + z^2}{(1 + x^2 + y^2 + z^2)^{5/2}},$$
$$f_z = \frac{1 + x^2 + y^2 - 3xz - 3yz - 2z^2}{(1 + x^2 + y^2 + z^2)^{5/2}}$$

(m) $$f_x = \frac{x^4 - 2xyz + 3x^2z^2 + 3x^2}{(x^2 + z^2 + 1)^2},$$
$$f_y = \frac{z}{x^2 + z^2 + 1},$$
$$f_z = \frac{x^2y - 2x^3z0yz^2 + y}{x^2 + z^2 + 1}$$

Problem 3

Find the second-order partial derivatives for the following functions

(a) $f(x, y) = x^3y^7 + 3xy^2 - 7xy$

(b) $f(x, y) = e^{y/x} - ye^{-x}$

(c) $f(x, y) = 1/(\sin^2 x + 2e^y)$

(d) $f(x, y) = y \sin x - x \cos y$

(e) $f(x, y) = x^2e^y + e^{2z}$

(f) $f(x, y, z) = x^2yz + xy^2z + xyz^2$

(g) $f(x, y, z) = e^{ax} \sin y + e^{bx} \cos z$

Answers:

(a) $f_{xx} = 6xy^7, f_{yy} = 42x^3y^5 + 6x, f_{xy} = f_{yx} = 21x^2y^6 + 6y - 7$

(b) $f_{xx} = -ye^{-x} + 2yx^{-3}e^{y/x} + y^2x^{-4}e^{y/x}$,
$f_{yy} = x^{-2}e^{y/x}$,
$f_{xy} = f_{yx} = e^{-x} - x^{-2}e^{y/x} - yx^{-3}e^{y/x}$

(c) $f_{xx} = \dfrac{2\left[\sin^2 2x - \cos 2x(\sin^2 x + 2e^y)\right]}{(\sin^2 x + 2e^y)^3}$,
$f_{yy} = \dfrac{2e^y(2e^y - \sin^2 x)}{(\sin^2 x + 2e^y)^3}$,
$f_{xy} = f_{yx} = \dfrac{4e^y \sin 2x}{(\sin^2 x + 2e^y)^3}$

(d) $f_{xx} = -y\sin x, f_{yy} = x\cos y, f_{xy} = f_{yx} = \cos x + \sin y$

(e) $f_{xx} = 2e^y, f_{yy} = x^2e^y, f_{zz} = 4e^{2z}, f_{xy} = f_{yx} = 2xe^y, f_{xz} = f_{zx} = 0, f_{yz} = f_{zy} = 0$

(f) $f_{xx} = 2yz, f_{yy} = 2xz, f_{zz} = 2xy$,
$f_{xy} = f_{yx} = 2xz + 2yz + z^2$,
$f_{xz} = f_{zx} = 2xy + y^2 + 2yz$,
$f_{yz} = f_{zy} = x^2 + 2xy + 2xz$

(g) $f_{xx} = b^2e^{bx}\cos z + a^2e^{ax}\sin y$,
$f_{yy} = -e^{ax}\sin y$,
$f_{zz} = -e^{bx}\cos z$,
$f_{xy} = f_{yx} = ae^{ax}\cos y$,
$f_{xz} = f_{zx} = -be^{bx}\sin z$,
$f_{yz} = f_{zy} = 0$

Problem 4

Find the gradient ∇f of the following functions

(a) $f(x,y) = x^2y + e^{y/x}$

(b) $f(x,y) = \frac{x-y}{x^2+y^2+1}$

(c) $f(x,y) = e^{xy} + \ln(x-y)$

(d) $f(x,y,z) = \sin xyz$

(e) $f(x,y,z) = xy + y\cos z - x\sin yz$

(f) $f(x,y,z) = \frac{x+y}{e^z}$

(g) $f(x,y,z) = \cos z \ln(x+y^2)$

(h) $f(x,y,z) = \frac{xy^2 - x^2z}{y^2+z^2+1}$

Answers:

(a) $\left[2xy - \dfrac{ye^{y/x}}{x^2}, \dfrac{x^3 + e^{y/x}}{x}\right]$

(b) $\left[\dfrac{-x^2+2xy+y^2+1}{(x^2+y^2+1)^2}, \dfrac{-x^2-2xy+y^2-1}{(x^2+y^2+1)^2}\right]$

(c) $\left[ye^{xy}+\dfrac{1}{x-y}, xe^{xy}+\dfrac{1}{y-x}\right]$

(d) $[yz\cos(xyz), xz\cos(xyz), xy\cos(xyz)]$

(e) $[y-\sin(yz), x(-z)\cos(yz)+x+\cos(z), -y(x\cos(yz)+\sin(z))]$

(f) $\left[e^{-z}, e^{-z}, -e^{-z}(x+y)\right]$

(g) $\left[\dfrac{\cos(z)}{x+y^2}, \dfrac{2y\cos(z)}{x+y^2}, \sin(z)\left(-\log\left(x+y^2\right)\right)\right]$

(h) $\left[\dfrac{y^2-2xz}{y^2+z^2+1}, \dfrac{2xy\left(xz+z^2+1\right)}{\left(y^2+z^2+1\right)^2}, -\dfrac{x\left(x\left(y^2-z^2+1\right)+2y^2z\right)}{\left(y^2+z^2+1\right)^2}\right]$

Problem 5

Find the matrix Df of partial derivatives for the following functions

(a) $f(x,y)=\frac{x}{y}$

(b) $f(x,y,z)=x^2+x\ln(yz)$

(c) $f(x,y,z)=(2x-3y+5z, x^2+y, \ln(yz))$

(d) $f(x,y,z)=\left(xyz, \sqrt{x^2+y^2+z^2}\right)$

(e) $f(t)=(t, \cos 2t, \sin 5t)$

(f) $f(x,y,z,w)=(3x-7y+z, 5x+2z-8w, y-17z+3w)$

(g) $f(x,y)=(x^2y, x+y^2, \cos\pi xy)$

(h) $f(s,t)=(s^2, st, t^2)$

Answers:

(a) $\left[\frac{1}{y}, -\frac{x}{y^2}\right]$

(b) $[2x+\ln(yz), x/y, x/z]$

(c) $\begin{bmatrix} 2 & -3 & 5 \\ 2x & 1 & 0 \\ 0 & 1/y & 1/z \end{bmatrix}$

(d) $\begin{bmatrix} yz & xz & xy \\ \frac{x}{\sqrt{x^2+y^2+z^2}} & \frac{y}{\sqrt{x^2+y^2+z^2}} & \frac{z}{\sqrt{x^2+y^2+z^2}} \end{bmatrix}$

(e) $\begin{bmatrix} 1 \\ -2\sin 2t \\ 5\cos 5t \end{bmatrix}$

(f) $\begin{bmatrix} 3 & -7 & 1 & 0 \\ 5 & 0 & 2 & -8 \\ 0 & 1 & -17 & 3 \end{bmatrix}$

(g) $\begin{bmatrix} 2xy & x^2 \\ 1 & 2y \\ -y & -x \end{bmatrix}$

(h) $\begin{bmatrix} 2s & 0 \\ t & s \\ 0 & 2t \end{bmatrix}$

Problem 6

(a) Let $f(x, y) = ye^{3x}$. Give general formulas for $\partial^n f/\partial x^n$ and $\partial^n f/\partial y^n$, where $n \geq 2$.

(b) Let $f(x, y, z) = \ln\left(\frac{xy}{z}\right)$. Give general formulas for $\partial^n f/\partial x^n$, $\partial^n f/\partial y^n$ and $\partial^n f/\partial z^n$, where $n \geq 1$.

Answers:

(a) $\partial^n f/\partial x^n = 3^n ye^{3x}$,
$\partial^n f/\partial y^n = 0$

(b) $\partial^n f/\partial x^n = (-1)^{n-1}(n-1)!/x^n$,
$\partial^n f/\partial y^n = (-1)^{n-1}(n-1)!/y^n$,
$\partial^n f/\partial z^n = (-1)^n(n-1)!/z^n$