AI6104 - Mathematics for AI

Systems of Equations

Problem 1: Solve the following linear systems of equations

(i) $$\begin{bmatrix} 1 & 2 & -3 \\ 2 & -1 & 1 \\ 4 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 0 \\ 4 \end{bmatrix}$$

First row-reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 2 & -1 & 1 & 0 \\ 4 & -1 & 1 & 4 \end{array}\right] \xrightarrow[R_3-4R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & -5 & 7 & -18 \\ 0 & -9 & 13 & -32 \end{array}\right] \xrightarrow{-\frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & 1 & -\frac{7}{5} & \frac{18}{5} \\ 0 & -9 & 13 & -32 \end{array}\right]$$

$$\xrightarrow{R_3+9R_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & 1 & -\frac{7}{5} & \frac{18}{5} \\ 0 & 0 & \frac{2}{5} & \frac{2}{5} \end{array}\right] \xrightarrow{\frac{5}{2}R_3} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 9 \\ 0 & 1 & -\frac{7}{5} & \frac{18}{5} \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_2+\frac{7}{5}R_3]{R_1+3R_3} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 12 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1-2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Thus the solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$.

(ii) $$\begin{bmatrix} 1 & -3 & -2 \\ -1 & 2 & 1 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

First row-reduce the augmented matrix of the system:

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 4 & 6 & 0 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2+R_1} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 10 & 10 & 0 \end{array}\right] \xrightarrow{-R_2}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 10 & 10 & 0 \end{array}\right] \xrightarrow{R_3-10R_2} \left[\begin{array}{ccc|c} 1 & -3 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1+3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since there is a row of zeros, and no leading 1 in the third column, we know that x_3 is a free variable, say $x_3 = t$. Back substituting gives $x_2 + t = 0$ and $x_1 + t = 0$, so that $x_1 = x_2 = -t$. So the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

(iii) $\begin{bmatrix} 2 & 1 \\ 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ -1 \end{bmatrix}$

First row-reduce the augmented matrix of the system:

$$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 4 & 1 & 7 \\ 2 & 5 & -1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{3}{2} \\ 4 & 1 & 7 \\ 2 & 5 & -1 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-4R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & -1 & 1 \\ 0 & 4 & -4 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{3}{2} \\ 0 & 1 & -1 \\ 0 & 4 & -4 \end{array}\right] \xrightarrow[R_3-4R_2]{R_1-\frac{1}{2}R_2} \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

The solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Problem 2: Determine if the vector **v** is a linear combination of the remaining vectors

(i) $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, u_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, u_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Yes, it is. We want to write

$$x \begin{bmatrix} 1 \\ -1 \end{bmatrix} + y \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

so we want to solve the system of linear equations

$$\begin{aligned} x + 2y &= 1 \\ -x - y &= 2. \end{aligned}$$

Row reducing the augmented matrix for the system gives

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ -1 & -1 & 2 \end{array}\right] \xrightarrow{R_2+R_1} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 1 & 3 \end{array}\right] \xrightarrow{R_1-2R_2} \left[\begin{array}{cc|c} 1 & 0 & -5 \\ 0 & 1 & 3 \end{array}\right].$$

So the solution is $x = -5$ and $y = 3$, and the linear combination is $\mathbf{v} = -5\mathbf{u}_1 + 3\mathbf{u}_2$.

(ii) $v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Yes, it is. We want to write

$$x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

so we want to solve the system of linear equations

$$\begin{aligned} x + z &= 1 \\ x + y &= 2 \\ y + z &= 3. \end{aligned}$$

Row-reducing the associated augmented matrix gives

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 2 \end{array}\right] \xrightarrow{\frac{1}{2}R_3} \cdots$$

$$\cdots \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R_2+R_3]{R_1-R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

This gives the solution $x = 0$, $y = 2$, and $z = 1$, so that $\mathbf{v} = 2\mathbf{u}_2 + \mathbf{u}_3$.

Problem 3:

(a) Suppose that vector **w** is a linear combination of vector $u_1, \cdots u_k$ and that each u_i is a linear combination of vectors $v_1, \cdots v_m$. Prove that **w** is a linear combination of $v_1, \cdots v_m$ and therefore $span(u_1, \cdots u_k) \subseteq span(v_1, \cdots v_m)$.

(b) In part (a), suppose in additional that each v_i is also linear combination of $u_1, \cdots u_k$. Prove that $span(u_1, \cdots u_k) = span(v_1, \cdots v_m)$.

(c) Use the results of part (b) to prove that

$$\Re^3 = span\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right)$$

(a) Suppose that

$$\mathbf{u}_i = u_{i1}\mathbf{v}_1 + u_{i2}\mathbf{v}_2 + \cdots + u_{im}\mathbf{v}_m, \quad i = 1,\ 2,\ \ldots,\ k.$$

Suppose that $\mathbf{w}$ is a linear combination of the $\mathbf{u}_i$. Then

$$\begin{aligned}
\mathbf{w} &= w_1\mathbf{u}_1 + w_2\mathbf{u}_2 + \cdots + w_k\mathbf{u}_k \\
&= w_1(u_{11}\mathbf{v}_1 + u_{12}\mathbf{v}_2 + \cdots + u_{1m}\mathbf{v}_m) + w_2(u_{21}\mathbf{v}_1 + u_{22}\mathbf{v}_2 + \cdots + u_{2m}\mathbf{v}_m) + \cdots \\
&\qquad w_k(u_{k1}\mathbf{v}_1 + u_{k2}\mathbf{v}_2 + \cdots + u_{km}\mathbf{v}_m) \\
&= (w_1u_{11} + w_2u_{21} + \cdots + w_ku_{k1})\mathbf{v}_1 + (w_1u_{12} + w_2u_{22} + \cdots + w_ku_{k2})\mathbf{v}_2 + \cdots \\
&\qquad (w_1u_{1m} + w_2u_{2m} + \cdots + w_ku_{km})\mathbf{v}_m \\
&= w_1'\mathbf{v}_1 + w_2'\mathbf{v}_2 + \cdots + w_m'\mathbf{v}_m.
\end{aligned}$$

This shows that $\mathbf{w}$ is a linear combination of the $\mathbf{v}_j$. Thus $\mathrm{span}(\mathbf{u}_1, \ldots, \mathbf{u}_k) \subseteq \mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$.

(b) Reversing the roles of the $\mathbf{u}$'s and the $\mathbf{v}$'s in part (a) gives $\mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) \subseteq \mathrm{span}(\mathbf{u}_1, \ldots, \mathbf{u}_k)$. Combining that with part (a) shows that each of the spans is contained in the other, so they are equal.

(c) Let

$$\mathbf{u}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix}1\\1\\1\end{bmatrix}.$$

We want to show that each $\mathbf{u}_i$ is a linear combination of $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, and the reverse. The first of these is obvious: since $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ span $\mathbb{R}^3$, any vector is $\mathbb{R}^3$ is a linear combination of those, so in particular $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are. Next, note that

$$\begin{aligned}
\mathbf{e}_1 &= \mathbf{u}_1 \\
\mathbf{e}_2 &= \mathbf{u}_2 - \mathbf{u}_1 \\
\mathbf{e}_3 &= \mathbf{u}_3 - \mathbf{u}_2,
\end{aligned}$$

so that the $\mathbf{e}_j$ are linear combinations of the $\mathbf{u}_i$. Part (b) then implies that $\mathrm{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \mathrm{span}(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = \mathbb{R}^3$.

Problem 4:

(a) If the columns of an $n \times n$ matrix **A** are linearly independent as vectors in $\Re^n$, what is the rank of **A**? Explain.

(b) If the row

(a) In this case, $\text{rank}(A) = n$. To see this, recall that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are linearly independent if and only if the only solution of $[A \,|\, 0] = [\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n \,|\, 0]$ is the trivial solution. Since the trivial solution is always a solution, it is the only one if and only if the number of free variables in the associated system is zero. Then the Rank Theorem (Theorem 2.2 in Section 2.2) says that the number of free variables is $n - \text{rank}(A)$, so that $n = \text{rank}(A)$.

(b) Let

$$A = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{bmatrix}.$$

Then by Theorem 2.7, the vectors, which are the rows of A, are linearly independent if and only if the rank of A is greater than or equal to n. But A has n rows, so its rank is at most n. Thus, the rows of A are linearly independent if and only if the rank of A is exactly n.

s of an $n \times n$ matrix **A** are linearly independent as vectors in $\Re^n$, what is the rank of **A**? Explain.

Problem 5: Show that $\Re^3 = span\left(\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\1\\-1\end{bmatrix}\right)$

We want to show that any vector can be written as a linear combination of the three given vectors, i.e. that

$$\begin{bmatrix}a\\b\\c\end{bmatrix} = x\begin{bmatrix}1\\1\\0\end{bmatrix} + y\begin{bmatrix}1\\2\\3\end{bmatrix} + z\begin{bmatrix}2\\1\\-1\end{bmatrix}$$

for some x, y, z. Row-reduce the associated augmented matrix:

$$\left[\begin{array}{ccc|c}1&1&2&a\\1&2&1&b\\0&3&-1&c\end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{ccc|c}1&1&2&a\\0&1&-1&b-a\\0&3&-1&c\end{array}\right] \xrightarrow{R_3-3R_2} \left[\begin{array}{ccc|c}1&1&2&a\\0&1&-1&b-a\\0&0&2&c+3a-3b\end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|c}1&1&2&a\\0&1&-1&b-a\\0&0&1&\frac{1}{2}(c+3a-3b)\end{array}\right] \xrightarrow[R_2+R_3]{R_1-2R_3} \left[\begin{array}{ccc|c}1&1&0&3b-2a-c\\0&1&0&\frac{1}{2}(c+a-b)\\0&0&1&\frac{1}{2}(c+3a-3b)\end{array}\right]$$

$$\xrightarrow{R_1-R_2} \left[\begin{array}{ccc|c}1&0&0&\frac{1}{2}(7b-5a-3c)\\0&1&0&\frac{1}{2}(c+a-b)\\0&0&1&\frac{1}{2}(c+3a-3b)\end{array}\right]$$

Thus

$$\begin{bmatrix}a\\b\\c\end{bmatrix} = \frac{1}{2}(7b-5a-3c)\begin{bmatrix}1\\1\\0\end{bmatrix} + \frac{1}{2}(c+a-b)\begin{bmatrix}1\\2\\3\end{bmatrix} + \frac{1}{2}(c+3a-3b)\begin{bmatrix}2\\1\\-1\end{bmatrix}.$$