AI6104 - Mathematics for AI

Tutorial 3 - Chain Rule, Directional Derivatives

Problem 1

Use the Chain Rule to calculate dz/dt for the following functions

(a) $z = f(x,y) = 4x^2 + 3y^2$, $x = x(t) = \sin t$, $y = y(t) = \cos t$.

(b) $z = f(x,y) = \sqrt{x^2 - y^2}$, $x = x(t) = e^{2t}$, $y = y(t) = e^{-t}$.

(c) $z = f(x,y) = x^2 - 3xy + 2y^2$, $x = x(t) = 3\sin 2t$, $y = y(t) = 4\cos 2t$.

Solution:

(a) The Chain Rule gives

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}\\ &= 8x\cos t - 6y\sin t\\ &= 8\sin t\cos t - 6\cos t\sin t\\ &= 2\sin t\cos t\end{aligned}$$

We may also substitute $x(t)$ and $y(t)$ into $f(x,y)$, then differentiate from this form

$$\frac{dz}{dt} = \frac{d}{dt}\left(4\sin^2 t + 3\cos^2 t\right) = 8\sin t\cos t - 6\cos t\sin t = 2\sin t\cos t$$

(b) The Chain Rule gives

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}\\ &= \frac{x}{\sqrt{x^2-y^2}}\cdot 2e^{2t} + \frac{-y}{\sqrt{x^2-y^2}}\cdot(-e^{-t})\\ &= \frac{2xe^{2t}+ye^{-t}}{\sqrt{x^2-y^2}}\\ &= \frac{2e^{4t}+e^{-2t}}{\sqrt{e^{4t}-e^{-2t}}}\end{aligned}$$

(c) The Chain Rule gives

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}\\ &= (2x-3y)(6\cos 2t) + (-3x+4y)(-8\sin 2t)\\ &= (6\sin 2t - 12\cos 2t)(6\cos 2t) + (9\sin 2t - 16\cos 2t)(8\sin 2t)\\ &= 72\sin^2 2t - 72\cos^2 2t - 92\sin 2t\cos 2t\end{aligned}$$

Problem 2

Use the Chain Rule to calculate $\partial z/\partial u$ and $\partial z/\partial v$ for the following functions

(a) $z = f(x, y) = 3x^2 - 2xy + y^2$, $x = x(u, v) = 3u + 2v$, $y = y(u, v) = 4u - v$.

(b) $z = f(x, y) = \dfrac{x}{y}$, $x = x(u, v) = 2\cos u$, $y = y(u, v) = 3\sin v$.

Solution:

(a) We use the Chain Rule for each partial derivative

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\\ &= 3(6x - 2y) + 4(-2x + 2y)\\ &= 10x + 2y\\ &= 38u + 18v\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}\\ &= 14x - 6y\\ &= 18u + 34v\end{aligned}$$

(b) Similarly,

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\\ &= \frac{1}{y}(-2\sin u)\\ &= -\frac{2\sin u}{3\sin v}\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}\\ &= -\frac{x}{y^2}(3\cos v)\\ &= -\frac{2\cos u\cos v}{3\sin^2 v}\end{aligned}$$

Problem 3

Write down the equation of tangent plane to each of the following surfaces at given points

(a) $z = x^2 + y^2$ at point $(1, 2, 5)$

(b) $x^2 + y^2 + z^2 = 169$ at point $(3, 4, 12)$

(c) $z = y + \ln\dfrac{x}{z}$ at point $(1, 1, 1)$

Solution:

(a) Note that $z_x = 2x$ and $z_y = 2y$, thus $z_x(1,2) = 2$ and $z_y(1,2) = 4$. Therefore, the tangent plane is

$$z = 5 + 2(x-1) + 4(y-2) \Leftrightarrow 2x + 4y - z = 5$$

(b) We notice that $z = \pm\sqrt{169 - x^2 - y^2}$. Since at the given point $(3,4,12)$ we have $z = 12 > 0$, so we use $z = \sqrt{169 - x^2 - y^2}$. The partial derivatives are

$$z_x = -\frac{x}{\sqrt{169 - x^2 - y^2}} = -\frac{x}{z}$$

and

$$z_y = -\frac{y}{\sqrt{169 - x^2 - y^2}} = -\frac{y}{z}$$

Thus, $z_x(3,4,12) = -1/4$ and $z_y(3,4,12) = -1/3$. Therefore the tangent plane is

$$z = 12 - \frac{x-3}{4} - \frac{y-4}{3} \Longleftrightarrow 3x + 4y + 12z = 169$$

(c) We notice here it is hard to write the function in the form of $z = f(x,y)$. Alternatively, we write in the form of $y = f(x,z) = z - \ln x + \ln z$. So the partial derivatives are $y_x = -1/x$ and $y_z = 1 + 1/z$. Further, $y_x(1,1) = -1$ and $y_z(1,1) = 2$. Therefore, the tangent plane is

$$y = 1 - (x-1) + 2(z-1) \Longleftrightarrow x + y - 2z = 0$$

Problem 4

Find the gradient $\nabla f(x,y)$ of each of the following functions

(a) $f(x,y) = x^2 - xy + 3y^2$

(b) $f(x,y) = \sin 3x \cos 3y$

(c) $f(x,y) = \dfrac{x^2 - 3y^2}{2x + y}$

Solution:

(a) Since $f_x(x,y) = 2x - y$ and $f_y(x,y) = -x + 6y$,

$$\nabla f(x,y) = \langle 2x - y, -x + 6y \rangle$$

(b) Since $f_x(x,y) = 3\cos 3x \cos 3y$ and $f_y(x,y) = -3\sin 3x \sin 3y$,

$$\nabla f(x,y) = \langle 3\cos 3x \cos 3y, -3\sin 3x \sin 3y \rangle$$

(c) Since $f_x(x,y) = \dfrac{2x(2x+y) - 2(x^2 - 3y^2)}{(2x+y)^2}$, $f_y(x,y) = \dfrac{-6y(2x+y) - (x^2 - 3y^2)}{(2x+y)^2}$, we have

$$\nabla f(x,y) = \left\langle \frac{2(x^2 + xy + 3y^2)}{(2x+y)^2}, -\frac{x^2 + 12xy + 3y^2}{(2x+y)^2} \right\rangle$$

Problem 5

Suppose the function $z = f(x, y)$ is differentiable at $\mathbf{x}$, and $\mathbf{u}$ is a unit vector. Show that $D_{\mathbf{u}}f(\mathbf{x})$ is maximized when $\mathbf{u}$ has the same direction as $\nabla f(\mathbf{x})$ and minimized when $\mathbf{u}$ has the opposite direction as $\nabla f(\mathbf{x})$. Calculate the maximum and minimum value of $D_{\mathbf{u}}f(\mathbf{x})$.

Solution:

Denote the angle between $\nabla f(\mathbf{x})$ and $\mathbf{u}$ by ψ. Recall that the dot product between two vectors $\mathbf{u}$ and $\mathbf{v}$ can be represented as $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Since $\mathbf{u}$ is a unit vector, $\|\mathbf{u}\| = 1$, and we can rewrite the directional derivative as

$$D_{\mathbf{u}}f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u} = \|\nabla f(\mathbf{x})\|\|\mathbf{u}\| \cos\psi = \|\nabla f(\mathbf{x})\| \cos\psi$$

Therefore, the directional derivative is maximized when $\psi = 0$, where its maximum value is $\|\nabla f(\mathbf{x})\|$, and minimized when $\psi = \pi$, where its minimum value is $-\|\nabla f(\mathbf{x})\|$.

Problem 6

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function differentiable at $\mathbf{a} \in \mathbb{R}^n$. Let k and l be constants. $\mathbf{u}$ and $\mathbf{v}$ are unit vectors. Prove that $D_{k\mathbf{u}+l\mathbf{v}}f(\mathbf{a}) = kD_{\mathbf{u}}f(\mathbf{a}) + lD_{\mathbf{v}}f(\mathbf{a})$.

Solution:

Since f is differentiable, we have

$$\begin{aligned} D_{k\mathbf{u}+l\mathbf{v}}f(\mathbf{a}) &= (k\mathbf{u} + l\mathbf{v}) \cdot \nabla f(\mathbf{a}) \\ &= k\mathbf{u} \cdot \nabla f(\mathbf{a}) + l\mathbf{v} \cdot \nabla f(\mathbf{a}) \\ &= kD_{\mathbf{u}}f(\mathbf{a}) + lD_{\mathbf{v}}f(\mathbf{a}) \end{aligned}$$