AI6104 - Mathematics for AI

Tutorial 2 - Limits, Partial Derivatives

Problem 1

Evaluate the following limits

(a) $\lim_{(x,y)\to(2,-1)} (x^2 - 2xy + 3y^2 - 4x + 3y - 6)$

(b) $\lim_{(x,y)\to(2,-1)} \dfrac{2x+3y}{4x-3y}$

(c) $\lim_{(x,y)\to(5,-2)} \sqrt[3]{\dfrac{x^2-y}{y^2+x-1}}$

(d) $\lim_{(x,y)\to(0,0)} \dfrac{2xy}{3x^2+y^2}$

(e) $\lim_{(x,y)\to(0,0)} \dfrac{4xy^2}{x^2+3y^4}$

(f) $\lim_{(x,y,z)\to(4,1,-3)} \dfrac{x^2y-3z}{2x+5y-z}$

Solution:

(a) The function can be decomposed into different parts

$$
\begin{aligned}
&\lim_{(x,y)\to(2,-1)} (x^2 - 2xy + 3y - 4x + 3y - 6) \\
=&\left(\lim_{(x,y)\to(2,-1)} x\right)^2 - 2\left(\lim_{(x,y)\to(2,-1)} x\right)\left(\lim_{(x,y)\to(2,-1)} y\right) + 3\left(\lim_{(x,y)\to(2,-1)} y\right)^2 \\
&- 4\left(\lim_{(x,y)\to(2,-1)} x\right) + 3\left(\lim_{(x,y)\to(2,-1)} y\right) - \lim_{(x,y)\to(2,-1)} 6 \\
=&2^2 - 2(2)(-1) + 3(-1)^2 - 4(2) + 3(-1) - 6 \\
=&-6
\end{aligned}
$$

(b) We need to verify that the denominator is nonzero first.

$$
\begin{aligned}
\lim_{(x,y)\to(2,-1)} (4x-3y) &= \lim_{(x,y)\to(2,-1)} 4x - \lim_{(x,y)\to(2,-1)} 3y \\
&= 4(2) - 3(-1) = 11
\end{aligned}
$$

Then we can apply the quotient law

$$
\begin{aligned}
\lim_{(x,y)\to(2,-1)} \frac{2x+3y}{4x-3y} &= \frac{\lim_{(x,y)\to(2,-1)}(2x+3y)}{\lim_{(x,y)\to(2,-1)}(4x-3y)} \\
&= \frac{1}{11}
\end{aligned}
$$

(c) We first verify that the denominator is nonzero

$$\lim_{(x,y)\to(5,-2)} (y^2 + x - 1) = (-2)^2 + 5 - 1 = 8$$

Then we decompose the function as

$$\begin{aligned}\lim_{(x,y)\to(5,-2)} \sqrt[3]{\frac{x^2 - y}{y^2 + x - 1}} &= \sqrt[3]{\frac{\lim_{(x,y)\to(5,-2)}(x^2 - y)}{\lim_{(x,y)\to(5,-2)}(y^2 + x - 1)}} \\ &= \sqrt[3]{\frac{27}{8}} = \frac{3}{2}\end{aligned}$$

(d) Consider approaching $(0,0)$ along the line $y = 0$ in the xy-plane. By substituting $y = 0$ into $f(x,y)$, we have

$$f(x,0) = \frac{2x(0)}{3x^2 + 0^0} = 0$$

for all values of x. Thus, $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along y-axis.

If we consider approaching along the line $y = x$, we have

$$f(x,x) = \frac{2x(x)}{3x^2 + x^2} = \frac{1}{2}$$

for all points on the line $y = x$, which means $f(x,y) \to \frac{1}{2}$ as $(x,y) \to (0,0)$ along the line $y = x$.

Therefore, the limit does not exist.

(e) If we approach $(0,0)$ along x-axis or y-axis, the function $f(x,y)$ remains fixed at zero. If we approach the origin along a straight line of slope k, *i.e.*, $y = kx$, the limit becomes

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2 + 3y^4} &= \lim_{(x,y)\to(0,0)} \frac{4x(kx)^2}{x^2 + 3(kx)^4} \\ &= \lim_{(x,y)\to(0,0)} \frac{4k^2x^3}{x^2 + 3k^4x^4} \\ &= \frac{\lim_{(x,y)\to(0,0)}(4k^2x)}{\lim_{(x,y)\to(0,0)}(1 + 3k^4x^2)} \\ &= 0\end{aligned}$$

It seems that the limit equals to zero from every directions. However, if we consider the parabola $x = y^2$, the limit becomes

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2 + 3y^4} &= \lim_{(x,y)\to(0,0)} \frac{4y^2y^2}{(y^2)^2 + 3y^4} \\ &= 1\end{aligned}$$

The gives us the conclusion that the limit does not exist.

(f) We first verify that denominator is nonzero

$$\begin{aligned}\lim_{(x,y,z)\to(4,1,-3)}(2x+5y-z) =& 2\left(\lim_{(x,y,z)\to(4,1,-3)} x\right)+5\left(\lim_{(x,y,z)\to(4,1,-3)} y\right)\\ &-\left(\lim_{(x,y,z)\to(4,1,-3)} z\right)\\ =& 2(4)+5(1)-(-3)=16\end{aligned}$$

We then apply the quotient law that

$$\begin{aligned}\lim_{(x,y,z)\to(4,1,-3)}\frac{x^2y-3z}{2x+5y-z} &= \frac{\lim_{(x,y,z)\to(4,1,-3)}(x^2y-3z)}{\lim_{(x,y,z)\to(4,1,-3)}(2x+5y-z)}\\ &=\frac{25}{16}\end{aligned}$$

Problem 2

Calculate $\partial f/\partial x$ and $\partial f/\partial y$ for the following functions

(a) $f(x,y)=x^2-3xy+2y^2-4x+5y-12$

(b) $f(x,y)=\sin(x^2y-2x+4)$

(c) $f(x,y)=\tan(x^3-3x^2y^2+2y^4)$

Solution:

(a) Taking y as a constant, then the partial derivative becomes

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}x^2-3xy+2y^2-4x+5y-12\\ &=\frac{\partial}{\partial x}x^2-\frac{\partial}{\partial x}3xy+\frac{\partial}{\partial x}2y^2-\frac{\partial}{\partial x}4x+\frac{\partial}{\partial x}5y-\frac{\partial}{\partial x}12\\ &=2x-3y-4\end{aligned}$$

Similarly, taking x as constant, we have

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}x^2-3xy+2y^2-4x+5y-12\\ &=\frac{\partial}{\partial y}x^2-\frac{\partial}{\partial y}3xy+\frac{\partial}{\partial y}2y^2-\frac{\partial}{\partial y}4x+\frac{\partial}{\partial y}5y-\frac{\partial}{\partial y}12\\ &=-3x+4y+5\end{aligned}$$

(b) By fixing y and x, respectively, we have the partial derivatives

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\sin(x^2y-2x+4) \\
&= \cos(x^2y-2x+4)\frac{\partial}{\partial x}(x^2y-2x+4) \\
&= (2xy-2)\cos(x^2y-2x+4) \\
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\sin(x^2y-2x+4) \\
&= \cos(x^2y-2x+4)\frac{\partial}{\partial y}(x^2y-2x+4) \\
&= x^2\cos(x^2y-2x+4)
\end{aligned}$$

(c) By fixing y and x, respectively, we have the partial derivatives

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\tan(x^3-3x^2y^2+2y^4) \\
&= \sec(x^3-3x^2y^2+2y^4)\frac{\partial}{\partial x}(x^3-3x^2y^2+2y^4) \\
&= (3x^2-6xy^2)\sec(x^3-3x^2y^2+2y^4) \\
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\tan(x^3-3x^2y^2+2y^4) \\
&= \sec(x^3-3x^2y^2+2y^4)\frac{\partial}{\partial y}(x^3-3x^2y^2+2y^4) \\
&= (-6x^2y+8y^3)\sec(x^3-3x^2y^2+2y^4)
\end{aligned}$$

Problem 3

Calculate $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial y^2}$, $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ for

$$f(x,y) = xe^{-3y} + \sin(2x-5y)$$

Solution:

We first calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$:

$$\begin{aligned}
\frac{\partial f}{\partial x} &= e^{-3y} + 2\cos(2x-5y) \\
\frac{\partial f}{\partial y} &= -3xe^{-3y} - 5\cos(2x-5y)
\end{aligned}$$

Thus, by taking the derivative of $\partial f/\partial x$ with respect to x and y, we have

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= -4\sin(2x-5y) \\
\frac{\partial^2 f}{\partial x \partial y} &= -3e^{-3y} + 10\sin(2x-5y)
\end{aligned}$$

Then, using $\partial f/\partial y$ with respect to x and y, we have

$$\frac{\partial^2 f}{\partial y \partial x} = -3e^{-3y} + 10\sin(2x - 5y)$$
$$\frac{\partial^2 f}{\partial y^2} = 9xe^{-3y} - 25\sin(2x - 5y)$$

Note that we have verified that $\partial^2 f/\partial x\partial y = \partial^2 f/\partial y\partial x$.

Problem 4

Find the equation of the tangent plane to the surface defined by the function $f(x, y) = \sin(2x)\cos(3y)$ at the point$(\pi/3, \pi/4)$.

Solution:

We first calculate the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ at point$(\pi/3, \pi/4)$.

$$f_x(x,y) = 2\cos(2x)\cos(3y) \qquad \Rightarrow f_x(\frac{\pi}{3}, \frac{\pi}{4}) = \frac{\sqrt{2}}{2}$$
$$f_y(x,y) = -3\sin(2x)\sin(3y) \qquad \Rightarrow f_y(\frac{\pi}{3}, \frac{\pi}{4}) = -\frac{3\sqrt{6}}{4}$$

Also note that

$$f(\frac{\pi}{3}, \frac{\pi}{4}) = -\frac{\sqrt{6}}{4}$$

Then we use Equation 2 from the lecture and get

$$z + \frac{\sqrt{6}}{4} = \frac{\sqrt{2}}{2}(x - \frac{\pi}{3}) - \frac{3\sqrt{6}}{4}(y - \frac{\pi}{4})$$

or

$$z = \frac{\sqrt{2}}{2}x - \frac{3\sqrt{6}}{4}y - \frac{\sqrt{6}}{4} - \frac{\pi\sqrt{2}}{6} + \frac{3\pi\sqrt{6}}{16}$$

Problem 5

Show that $f(x, y) = x^2 + 3y$ is differentiable at every point. *hint: use the definition.*

Solution:

Note that the increment of $f(x, y)$ at an arbitrary point (x, y) can be written as

$$\begin{aligned}\Delta z &= f(x + \Delta x, y + \Delta y) - f(x, y)\\ &= (x + \Delta x)^2 + 3(y + \Delta y) - x^2 - 3y\\ &= 2x\Delta x + 3\Delta y + (\Delta x)^2\\ &= f_x\Delta x + f_y\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y\end{aligned}$$

where $\varepsilon_1 = \Delta x$ and $\varepsilon_2 = 0$. Since $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$, it follows that $f(x, y)$ is differentiable at every point in the plane.