# Mathematics for AI

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# The Inverse of a Matrix

# The Inverse of a Matrix (1 of 3)

**Definition**

If *A* is an $n \times n$ matrix, an ***inverse*** of *A* is an $n \times n$ matrix *A′* with the property that

$$AA' = I \text{ and } A'A = I$$

where $I = I_n$ is the $n \times n$ identity matrix. If such an *A′* exists, then *A* is called ***invertible*****.**

# The Inverse of a Matrix (2 of 3)

**Theorem 3.6**

If $A$ is an invertible matrix, then its inverse is unique.

**Theorem 3.7**

If $A$ is an invertible $n \times n$ matrix, then the system of linear equations given by $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$ for any **b** in $\mathbb{R}^n$.

# The Inverse of a Matrix (3 of 3)

**Theorem 3.8**

If *A* $= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then *A* is invertible if $ad - bc \neq 0$, in which case

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then *A* is not invertible.

The expression *ad* − *bc* is called the ***determinant*** of *A*, denoted det *A*.

# Example 3.24

Find the inverses of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 12 & -15 \\ 4 & -5 \end{bmatrix}$, if they exist.

Solution:

We have $\det A = 1(4) - 2(3) = -2 \neq 0$, so $A$ is invertible, with

$$A^{-1} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

On the other hand, $\det B = 12(-5) - (-15)(4) = 0$, so $B$ is not invertible.

# Properties of Invertible Matrices

# Properties of Invertible Matrices (1 of 3)

**Theorem 3.9**

a. If *A* is an invertible matrix, then $A^{-1}$ is invertible and

$$(A^{-1})^{-1} = A$$

b. If *A* is an invertible matrix and *c* is a nonzero scalar, then *cA* is an invertible matrix and

$$(cA)^{-1} = \frac{1}{c}A^{-1}$$

c. If *A* and *B* are invertible matrices of the same size, then *AB* is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

# Properties of Invertible Matrices (2 of 3)

d. If *A* is an invertible matrix, then $A^T$ is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

e. If *A* is an invertible matrix, then $A^n$ is invertible for all nonnegative integers *n* and

$$(A^n)^{-1} = (A^{-1})^n$$

# Properties of Invertible Matrices (3 of 3)

The inverse of a product of invertible matrices is the product of their inverses in the reverse order.

**Definition**

If *A* is an invertible matrix and *n* is a positive integer, then $A^{-n}$ is defined by

$$A^{-n} = (A^{-1})^n = (A^n)^{-1}$$

# Example 3.26

Solve the following matrix equation for *X* (assuming that the matrices involved are such that all of the indicated operations are defined):

$$A^{-1}(BX)^{-1} = (A^{-1}B^3)^2$$

# Example 3.26 – Solution

There are many ways to proceed here. One solution is

$$
\begin{aligned}
A^{-1}(BX)^{-1} = (A^{-1}B^3)^2 &\Rightarrow ((BX)A)^{-1} = (A^{-1}B^3)^2 \\
&\Rightarrow [((BX)A)^{-1}]^{-1} = [(A^{-1}B^3)^2]^{-1} \\
&\Rightarrow (BX)A = [(A^{-1}B^3)(A^{-1}B^3)]^{-1} \\
&\Rightarrow (BX)A = B^{-3}(A^{-1})^{-1}B^{-3}(A^{-1})^{-1} \\
&\Rightarrow BXA = B^{-3}AB^{-3}A \\
&\Rightarrow B^{-1}BXAA^{-1} = B^{-1}B^{-3}AB^{-3}AA^{-1} \\
&\Rightarrow IXI = B^{-4}AB^{-3}I \\
&\Rightarrow X = B^{-4}AB^{-3}
\end{aligned}
$$

# Elementary Matrices

# Elementary Matrices (1 of 3)

**Definition**

An ***elementary matrix*** is any matrix that can be obtained by performing an elementary row operation on an identity matrix.

# Elementary Matrices (2 of 3)

**Theorem 3.10**

Let $E$ be the elementary matrix obtained by performing an elementary row operation on $I_n$. If the same elementary row operation is performed on an $n \times r$ matrix $A$, the result is the same as the matrix $EA$.

# Example 3.28 (1 of 2)

Let

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \; E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

Then $E_1$ corresponds to $R_2 \leftrightarrow R_3$, which is undone by doing $R_2 \leftrightarrow R_3$, again. Thus, $E_1^{-1} = E_1$. (Check by showing that $E_1^2 = E_1E_1 = I$.) The matrix $E_2$ comes from $4R_2$, which is undone by performing $\frac{1}{4}R_2$.

# Example 3.28 (2 of 2)

Thus,

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which can be easily checked. Finally, $E_3$ corresponds to the elementary row operation $R_3 - 2R_1$, which can be undone by the elementary row operation $R_3 + 2R_1$. So, in this case,

$$E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

# Elementary Matrices (3 of 3)

**Theorem 3.11**

Each elementary matrix is invertible, and its inverse is an elementary matrix of the same type.

# The Fundamental Theorem of Invertible Matrices

**Theorem 3.12**

**The Fundamental Theorem of Invertible Matrices: Version 1**

Let $A$ be an $n \times n$ matrix. The following statements are equivalent:

a. $A$ is invertible.

b. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.

c. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

d. The reduced row echelon form of $A$ is $I_n$.

e. $A$ is a product of elementary matrices.

# Example 3.29

If possible, express $A = \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}$ as a product of elementary matrices.

Solution:

We row reduce *A* as follows:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 1 & 3 \\ 0 & -3 \end{bmatrix}$$

$$\xrightarrow{R_1 + R_2} \begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

# Example 3.29 – Solution (1 of 2)

Thus, the reduced row echelon form of *A* is the identity matrix, so the Fundamental Theorem assures us that *A* is invertible and can be written as a product of elementary matrices. We have $E_4E_3E_2E_1A = I$, where

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \quad E_3 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad E_4 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{3} \end{bmatrix}$$

are the elementary matrices corresponding to the four elementary row operations used to reduce *A* to *I*.

# Example 3.29 – Solution (2 of 2)

As in the proof of the theorem, we have

$$A = (E_4E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix}$$

as required.

**Theorem 3.13**

Let $A$ be a square matrix. If $B$ is a square matrix such that either $AB = I$ or $BA = I$, then $A$ is invertible and $B = A^{-1}$.

**Theorem 3.14**

Let $A$ be a square matrix. If a sequence of elementary row operations reduces $A$ to $I$, then the same sequence of elementary row operations transforms $I$ into $A^{-1}$.

# The Gauss-Jordan Method for Computing the Inverse

# Example 3.30

Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$$

if it exists.

# Example 3.30 – Solution (1 of 3)

Gauss-Jordan elimination produces

$$[A \mid I] = \left[\begin{array}{rrr|rrr} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{rrr|rrr} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 6 & -2 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right]$$

# Example 3.30 – Solution (2 of 3)

$$\xrightarrow{(-\frac{1}{2})R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \end{array}\right]$$

$$\xrightarrow[\longrightarrow]{\substack{R_1 + R_3 \\ R_2 + 3R_3}} \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -1 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & -5 & 1 & 3 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \end{array}\right]$$

# Example 3.30 – Solution (3 of 3)

$$\xrightarrow{R_1 - 2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -\frac{3}{2} & -5 \\ 0 & 1 & 0 & -5 & 1 & 3 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \end{array}\right]$$

Therefore,

$$A^{-1} = \begin{bmatrix} 9 & -\frac{3}{2} & -5 \\ -5 & 1 & 3 \\ -2 & \frac{1}{2} & 1 \end{bmatrix}$$

# Subspaces, Basis, Dimension, and Rank

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# Subspaces, Basis, Dimension, and Rank (1 of 2)

## Definition

A ***subspace*** of $\mathbb{R}^n$ is any collection $S$ of vectors in $\mathbb{R}^n$ such that:

1. The zero vector $\mathbf{0}$ is in $S$.

2. If $\mathbf{u}$ and $\mathbf{v}$ are in $S$, then $\mathbf{u} + \mathbf{v}$ is in $S$. ($S$ is ***closed under addition***.)

3. If $\mathbf{u}$ is in $S$ and $c$ is a scalar, then $c\mathbf{u}$ is in $S$. ($S$ is ***closed under scalar multiplication***.)

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# Example 3.37 (1 of 2)

Every line and plane through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$. It should be clear geometrically that properties (1) through (3) are satisfied. Here is an algebraic proof in the case of a plane through the origin.

Let $\wp$ be a plane through the origin with direction vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Hence, $\wp = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$. The zero vector $\mathbf{0}$ is in $\wp$, since $\mathbf{0} = 0\mathbf{v}_1 + 0\mathbf{v}_2$. Now let

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 \text{ and } \mathbf{v} = d_1\mathbf{v}_1 + d_2\mathbf{v}_2$$

be two vectors in $\wp$.

# Example 3.37 (2 of 2)

Then

$$\mathbf{u} + \mathbf{v} = (c_1\mathbf{v}_1 + c_2\mathbf{v}_2) + (d_1\mathbf{v}_1 + d_2\mathbf{v}_2) = (c_1 + d_1)\mathbf{v}_1 + (c_2 + d_2)\mathbf{v}_2$$

Thus, $\mathbf{u} + \mathbf{v}$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ and so is in $\wp$.

Now let $c$ be a scalar. Then

$$c\mathbf{u} = c(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = (cc_1)\mathbf{v}_1 + (cc_2)\mathbf{v}_2$$

which shows that $c\mathbf{u}$ is also a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ and is therefore in $\wp$. We have shown that $\wp$ satisfies properties (1) through (3) and hence is a subspace of $\mathbb{R}^3$.

**Theorem 3.19**

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. Then span $(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is a subspace of $\mathbb{R}^n$.

# Example 3.38

Show that the set of all vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ that satisfy the conditions $x = 3y$ and $z = -2y$ forms a subspace of $\mathbb{R}^3$.

## Solution:

Substituting the two conditions into $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ yields

$$\begin{bmatrix} 3y \\ y \\ -2y \end{bmatrix} = y\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

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# Example 3.38 – Solution

Since *y* is arbitrary, the given set of vectors is span $\left(\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}\right)$ and is thus a subspace of $\mathbb{R}^3$, by Theorem 3.19.

# Subspaces Associated with Matrices

# Subspaces Associated with Matrices (1 of 3)

**Definition**

Let *A* be an $m \times n$ matrix.

1. The ***row space*** of *A* is the subspace row(*A*) of $\mathbb{R}^n$ spanned by the rows of *A*.

2. The ***column space*** of *A* is the subspace col(*A*) of $\mathbb{R}^m$ spanned by the columns of *A*.

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# Example 3.41

Consider the matrix

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 3 & -3 \end{bmatrix}$$

(a) Determine whether $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in the column space of $A$.

(b) Determine whether $\mathbf{w} = \begin{bmatrix} 4 & 5 \end{bmatrix}$ is in the row space of $A$.

(c) Describe row($A$) and col($A$).

# Example 3.41 – Solution (1 of 4)

(a) We know that **b** is a linear combination of the columns of *A* if and only if the linear system $A\mathbf{x} = \mathbf{b}$ is consistent. We row reduce the augmented matrix as follows:

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 3 & -3 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

Thus, the system is consistent (and, in fact, has a unique solution). Therefore, **b** is in col(*A*).

# Example 3.41 – Solution (2 of 4)

(b) Elementary row operations simply create linear combinations of the rows of a matrix. That is, they produce vectors only in the row space of the matrix. If the vector **w** is in row($A$), then **w** is a linear combination of the rows of $A$, so if we augment $A$ by **w** as $\left[\frac{A}{\mathbf{w}}\right]$, it will be possible to apply elementary row operations to this augmented matrix to reduce it to form $\left[\frac{A'}{\mathbf{0}}\right]$ using only elementary row operations of the form $R_i + kR_j$, where $i > j$—in other words, *working from top to bottom in each column*.

# Example 3.41 – Solution (3 of 4)

We have

$$\left[\begin{array}{c} A \\ \hline \mathbf{w} \end{array}\right] = \left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \\ 3 & -3 \\ \hline 4 & 5 \end{array}\right] \xrightarrow[R_4 - 4R_1]{R_3 - 3R_1} \left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \\ 0 & 0 \\ \hline 0 & 9 \end{array}\right] \xrightarrow{R_4 - 9R_2} \left[\begin{array}{cc} 1 & -1 \\ 0 & 1 \\ 0 & 0 \\ \hline 0 & 0 \end{array}\right]$$

Therefore, **w** is a linear combination of the rows of *A* and thus **w** is in row(*A*).

# Example 3.41 – Solution (4 of 4)

(c) It is easy to check that, for any vector $\mathbf{w} = [x \quad y]$, the augmented matrix $\left[\frac{A}{\mathbf{w}}\right]$ reduces to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \hline 0 & 0 \end{bmatrix}$$

in a similar fashion. Therefore, every vector in $\mathbb{R}^2$ is in row(*A*), and so $\text{row}(A) = \mathbb{R}^2$.

# Subspaces Associated with Matrices (2 of 3)

**Theorem 3.20**

Let $B$ be any matrix that is row equivalent to a matrix $A$. Then $\text{row}(B) = \text{row}(A)$.

**Theorem 3.21**

Let $A$ be an $m \times n$ matrix and let $N$ be the set of solutions of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. Then $N$ is a subspace of $\mathbb{R}^n$.

# Subspaces Associated with Matrices (3 of 3)

**Definition**

Let $A$ be an $m \times n$ matrix. The ***null space*** of $A$ is the subspace of $\mathbb{R}^n$ consisting of solutions of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. It is denoted by null($A$).

**Theorem 3.22**

Let $A$ be a matrix whose entries are real numbers. For any system of linear equations $A\mathbf{x} = \mathbf{b}$, exactly one of the following is true:

a. There is no solution.

b. There is a unique solution.

c. There are infinitely many solutions.

# Basis

# Basis (1 of 3)

**Definition**

A ***basis*** for a subspace *S* of $\mathbb{R}^n$ is a set of vectors in *S* that

1. spans *S* and

2. is linearly independent.

# Example 3.42

The standard unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, ... $\mathbf{e}_n$ in $\mathbb{R}^n$ are linearly independent and span $\mathbb{R}^n$. Therefore, they form a basis for $\mathbb{R}^n$, called the ***standard basis***.

# Example 3.44

Find a basis for $S = \mathrm{span}(\mathbf{u}, \mathbf{v}, \mathbf{w})$, where

$$\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 0 \\ -5 \\ 1 \end{bmatrix}$$

Solution:

The vectors **u**, **v**, and **w** already span *S*, so they will be a basis for *S* if they are also linearly independent. It is easy to determine that they are not; indeed, $\mathbf{w} = 2\mathbf{u} - 3\mathbf{v}$.

# Example 3.44 – Solution

Therefore, we can ignore **w**, since any linear combinations involving **u**, **v**, and **w** can be rewritten to involve **u** and **v** alone. This implies that $S = \text{span}\,(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \text{span}\,(\mathbf{u}, \mathbf{v})$, and since **u** and **v** are certainly linearly independent, they form a basis for *S*. (Geometrically, this means that **u**, **v**, and **w** all lie in the same plane and **u** and **v** can serve as a set of direction vectors for this plane.)

# Basis (2 of 3)

Following is a summary of the most effective procedure to use to find bases for the row space, the column space, and the null space of a matrix $A$.

1. Find the reduced row echelon form $R$ of $A$.
2. Use the nonzero row vectors of $R$ (containing the leading 1s) to form a basis for row($A$).
3. Use the column vectors of $A$ that correspond to the columns of $R$ containing the leading 1s (the pivot columns) to form a basis for col($A$).

# Basis (3 of 3)

4. Solve for the leading variables of $R\mathbf{x} = \mathbf{0}$ in terms of the free variables, set the free variables equal to parameters, substitute back into **x**, and write the result as a linear combination of *f* vectors (where *f* is the number of free variables). These *f* vectors form a basis for null(*A*).

# Example

- A= $\begin{bmatrix} 1 & 0 & 1|0 \\ 0 & 2 & 3|0 \\ 0 & 4 & 6|0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1\ |0 \\ 0 & 1 & 3/2|0 \\ 0 & 0 & 0\ |0 \end{bmatrix}$
- $row(A) = \left( \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3/2 \end{bmatrix} \right), col(A) = \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 4 \end{bmatrix} \right)$
- Solve $\begin{bmatrix} 1 & 0 & 1\ |0 \\ 0 & 1 & 3/2|0 \\ 0 & 0 & 0\ |0 \end{bmatrix}$. We have $x = -z, y = -3/2z, null(A) = \left( \begin{bmatrix} -1 \\ -3/2 \\ 1 \end{bmatrix} \right)$

# Dimension and Rank

# Dimension and Rank (1 of 8)

**Theorem 3.23**

**The Basis Theorem**

Let *S* be a subspace of $\mathbb{R}^n$. Then any two bases for *S* have the same number of vectors.

**Definition**

If *S* is a subspace of $\mathbb{R}^n$, then the number of vectors in a basis for *S* is called the ***dimension*** of *S*, denoted dim *S*.

# Dimension and Rank (2 of 8)

**Theorem 3.24**

The row and column spaces of a matrix $A$ have the same dimension.

**Definition**

The ***rank*** of a matrix $A$ is the dimension of its row and column spaces and is denoted by rank($A$).

# Dimension and Rank (3 of 8)

**Theorem 3.25**

For any matrix *A*,

$$\operatorname{rank}(A^T) = \operatorname{rank}(A)$$

**Definition**

The ***nullity*** of a matrix *A* is the dimension of its null space and is denoted by nullity(*A*).

# Dimension and Rank (4 of 8)

**Theorem 3.26**

**The Rank Theorem**

If $A$ is an $m \times n$ matrix, then

$$\text{rank}(A) + \text{nullity}(A) = n$$

# Example 3.51

Find the nullity of each of the following matrices:

$$M = \begin{bmatrix} 2 & 3 \\ 1 & 5 \\ 4 & 7 \\ 3 & 6 \end{bmatrix} \quad \text{and}$$

$$N = \begin{bmatrix} 2 & 1 & -2 & -1 \\ 4 & 4 & -3 & 1 \\ 2 & 7 & 1 & 8 \end{bmatrix}$$

# Example 3.51 – Solution (1 of 2)

Since the two columns of *M* are clearly linearly independent, $\text{rank}(M) = 2$. Thus, by the Rank Theorem, $\text{nullity}(M) = 2 - \text{rank}(M) = 2 - 2 = 0$.

There is no obvious dependence among the rows or columns of *N*, so we apply row operations to reduce it to

$$\begin{bmatrix} 2 & 1 & -2 & -1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

# Example 3.51 – Solution (2 of 2)

We have reduced the matrix far enough (we do not need *reduced* row echelon form here, since we are not looking for a basis for the null space). We see that there are only two nonzero rows, so $\operatorname{rank}(N) = 2$.

Hence, $\operatorname{nullity}(N) = 4 - \operatorname{rank}(N) = 4 - 2 = 2$.

**Theorem 3.27**

**The Fundamental Theorem of Invertible Matrices: Version 2**

Let $A$ be an $n \times n$ matrix. The following statements are equivalent:

a. $A$ is invertible.

b. $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.

c. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

d. The reduced row echelon form of $A$ is $I_n$.

# Dimension and Rank (6 of 8)

e. $A$ is a product of elementary matrices.

f. $\operatorname{rank}(A) = n$

g. $\operatorname{nullity}(A) = 0$

h. The column vectors of $A$ are linearly independent.

i. The column vectors of $A$ span $\mathbb{R}^n$.

j. The column vectors of $A$ form a basis for $\mathbb{R}^n$.

k. The row vectors of $A$ are linearly independent.

# Dimension and Rank (7 of 8)

l. The row vectors of $A$ span $\mathbb{R}^n$.

m. The row vectors of $A$ form a basis for $\mathbb{R}^n$.

# Dimension and Rank (8 of 8)

**Theorem 3.28**

Let *A* be an $m \times n$ matrix. Then:

a. $\operatorname{rank}(A^T A) = \operatorname{rank}(A)$

b. The $n \times n$ matrix $A^T A$ is invertible if and only if $\operatorname{rank}(A) = n$.

# Coordinates

# Coordinates (1 of 2)

**Theorem 3.29**

Let *S* be a subspace of $\mathbb{R}^n$ and let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a basis for *S*. For every vector **v** in *S*, there is exactly one way to write **v** as a linear combination of the basis vectors in $\mathcal{B}$:

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$$

# Coordinates (2 of 2)

## Definition

Let $S$ be a subspace of $\mathbb{R}^n$ and let $\mathscr{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a basis for $S$. Let $\mathbf{v}$ be a vector in $S$, and write $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$. Then $c_1, c_2, \ldots, c_k$ are called the ***coordinates of* v *with respect to*** $\mathcal{B}$, and the column vector

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

is called the ***coordinate vector of* v *with respect to*** $\mathcal{B}$**.**

# Example 3.53

Let $\mathscr{E} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$. Find the coordinate vector of

$$\mathbf{v} = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$$

with respect to $\mathcal{E}$.

# Example 3.53 – Solution

Since $\mathbf{v} = 2\mathbf{e}_1 + 7\mathbf{e}_2 + 4\mathbf{e}_3$,

$$[\mathbf{v}]_{\mathcal{E}} = \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix}$$