AI6104 - Mathematics for AI

Tutorial 1 - Vectors, Dot Products, Planes

Problem 1

Note that in lecture we introduced Euclidean distance between two points $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, *i.e.*, $d(x, y) = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2}$.

A general definition of distance, Minkowski distance of order p, also known as p-norm distance, is defined as

$$D_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}}$$

In the limit case when p tends to infinity, we have the ∞-norm distance,

$$D_\infty(x, y) = \lim_{p \to \infty} \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{\frac{1}{p}} = \max_i (|x_i - y_i|)$$

Calculate the following distance between $P_1 = (3, -1, 5)$ and $P_2 = (2, 1, -1)$

(a) 1-norm distance (Manhattan distance)

(b) 2-norm distance (Euclidean distance)

(c) ∞-norm distance (Chebyshev distance)

Solution:

(a) $D_1(P_1, P_2) = |3 - 2| + |-1 - 1| + |5 - (-1)| = 9$

(b) $D_2(P_1, P_2) = \sqrt{|3 - 2|^2 + |-1 - 1|^2 + |5 - (-1)|^2} = \sqrt{41}$

(c) $D_\infty(P_1, P_2) = \max(|3 - 2|, |-1 - 1|, |5 - (-1)|) = 6$

Problem 2

Let $\mathbf{u} = \langle 3, 0, 4 \rangle$, $\mathbf{v} = \langle 0, 5, 12 \rangle$ be vectors. Calculate $\|\mathbf{u}\| + \|\mathbf{v}\|$ and $\|\mathbf{u} + \mathbf{v}\|$.

Solution:

$$\|\mathbf{u}\| = \sqrt{3^2 + 0^2 + 4^2} = 5$$
$$\|\mathbf{v}\| = \sqrt{0^2 + 5^2 + 12^2} = 13$$
$$\|\mathbf{u} + \mathbf{v}\| = \|\langle 3, 5, 16 \rangle\| = \sqrt{290} \approx 17.03$$

We note that $\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$, which is well-known as triangle inequality.

Problem 3

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors. Prove the following properties of the dot product.

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

Solution:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= u_1 v_1 + u_2 v_2 + u_3 v_3 \\ &= v_1 u_1 + v_2 u_2 + v_3 u_3 \\ &= \langle v_1, v_2, v_3 \rangle \cdot \langle u_1, u_2, u_3 \rangle \\ &= \mathbf{v} \cdot \mathbf{u} \end{aligned}$$

(b) $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v}$

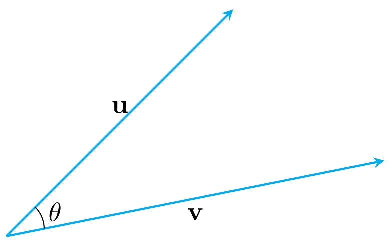
Solution:

$$\begin{aligned} c(\mathbf{u} \cdot \mathbf{v}) &= c(u_1 v_1 + u_2 v_2 + u_3 v_3) \\ &= (cu_1)v_1 + c(u_2)v_2 + c(u_3)v_3 \\ &= \langle cu_1, cu_2, cu_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle \\ &= (c\mathbf{u}) \cdot \mathbf{v} \end{aligned}$$

Problem 4

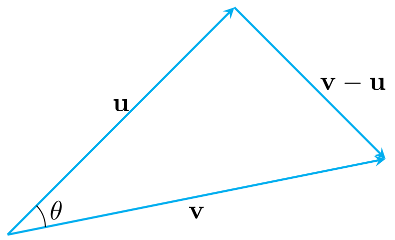
Prove that the dot product of two vectors is the product of the magnitude of each vector and the cosine of the angle θ between them, *i.e.*,

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \|\mathbf{v}\| \cos \theta$$



Solution:

If we connect the endpoints of $\mathbf{u}$ and $\mathbf{v}$, we have a triangle with the sides $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{v} - \mathbf{u}$.



By the law of cosines,

$$\|\mathbf{v}-\mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

Expanding the left-hand side gives

$$\|\mathbf{v}-\mathbf{u}\|^2 = \|\mathbf{v}\|^2 - 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{u}\|^2$$

This gives us the result that

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$$

Problem 5

Prove that two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\mathbf{u}\cdot\mathbf{v} = 0$.

Solution:

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors, and let θ be the angle between them.
If $\mathbf{u}\cdot\mathbf{v} = 0$, then

$$\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta = 0$$

Since both $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ are non-zeros, we must have $\cos\theta = 0$. Hence $\theta = \pi/2$ and the two vectors are orthogonal.

Now assume that $\mathbf{u}$ and $\mathbf{v}$ are orthogonal. Then $\theta = \pi/2$, and we have

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\|\cos\frac{\pi}{2} = 0$$

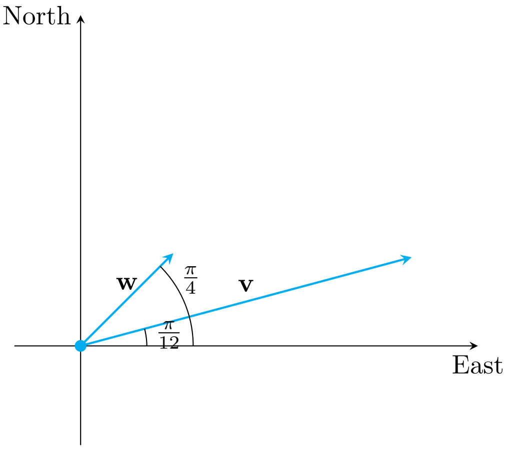
Problem 6

A container ship leaves port traveling 15 degrees north of east. Its engine generates a speed of 20 knots along that path (see the following figure). In addition, the ocean current moves the ship northeast at a speed of 2 knots. Considering both the engine and the current, how fast is the ship moving in the direction 15 degrees north of east?

Solution:

Let $\mathbf{v}$ be the velocity vector generated by the engine, and let $\mathbf{w}$ the velocity vector of the current. We already know $\|\mathbf{v}\| = 20$ along the desired route. We only need to add in the scalar projection of $\mathbf{w}$ onto $\mathbf{v}$. Thus,

$$\text{comp}_{\mathbf{v}}\mathbf{w} = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|} = \frac{\|\mathbf{v}\|\|\mathbf{w}\|\cos(30)}{\|\mathbf{v}\|} = \|\mathbf{w}\|\cos(30) = \sqrt{3}$$



Therefore, the ship moves in the direction 15 degrees north of east at the speed of $20 + \sqrt{3} \approx 21.73$ knots.

Problem 7

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors, and let θ be the angle between them. Prove that

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$$

Solution:

By the definition of cross product, we have

$$\begin{aligned}
\|\mathbf{u} \times \mathbf{v}\|^2 &= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\
&= (u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2 \\
&= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2 \\
&= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \cos^2 \theta \\
&= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 (1 - \cos^2 \theta) \\
&= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \sin^2 \theta
\end{aligned}$$

Note that $0 \leq \theta \leq 2\pi$, we have $\sin \theta \geq 0$. Thus,

$$\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \|\mathbf{v}\| \sin \theta$$

Problem 8

Determine whether each pair of planes is parallel, orthogonal. If neither, calculate the angle between two planes.

(a) $x + 3y - 2z = 8$ and $2x + 6y - 4z = 5$

(b) $2x - 3y + 2z = 3$ and $4x + 2y - z = 6$

(c) $x + 2y + z = 4$ and $x - 3y + 2z = 1$

Solution:

(a) The normal vectors are $\mathbf{n}_1 = \langle 1, 3, -2\rangle$ and $\mathbf{n}_2 = \langle 2, 6, -4\rangle$. They are parallel to each other, so are the planes.

(b) The normal vectors are $\mathbf{n}_1 = \langle 2, -3, 2\rangle$ and $\mathbf{n}_2 = \langle 4, 2, -1\rangle$. Note that $\mathbf{n}_1 \cdot \mathbf{n}_2 = 2 \cdot 4 - 3 \cdot 2 - 2 \cdot 1 = 0$. The normal vectors are orthogonal, and the corresponding planes are orthogonal.

(c) The normal vectors are $\mathbf{n}_1 = \langle 1, 2, 1\rangle$ and $\mathbf{n}_2 = \langle 1, -3, 2\rangle$.

$$\begin{aligned}\cos\theta &= \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\|\mathbf{n}2\|} \\ &= \frac{|\langle 1, 2, 1\rangle \cdot \langle 1, -3, 2\rangle|}{\sqrt{1^2 + 2^2 + 1^2}\sqrt{1^2 + (-3)^2 + 2^2}} \\ &= \frac{\sqrt{3}}{2\sqrt{7}}\end{aligned}$$

Thus, the angle between them is $\arccos(\sqrt{3}/(2\sqrt{7}))$.