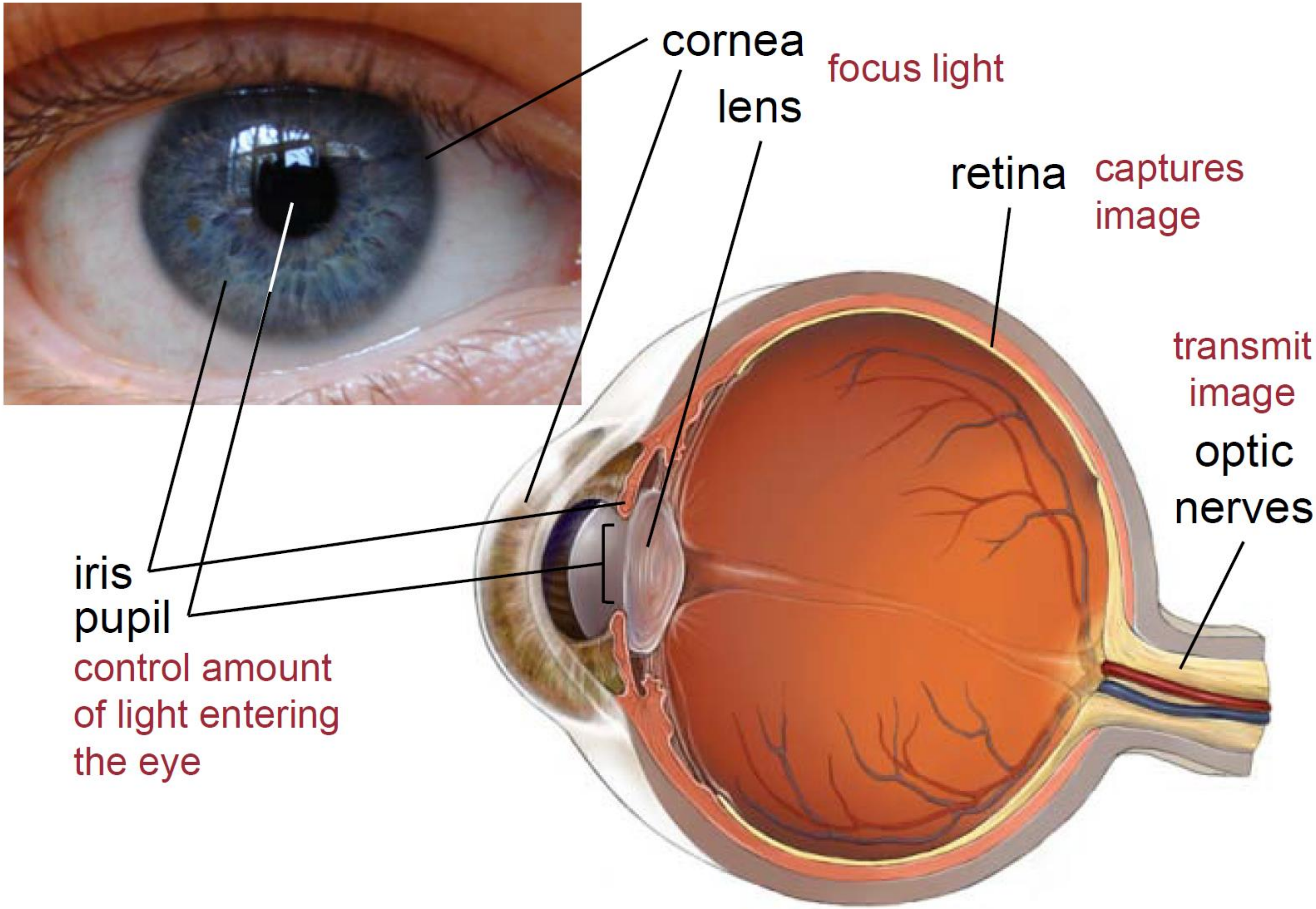
AI6121 Computer Vision

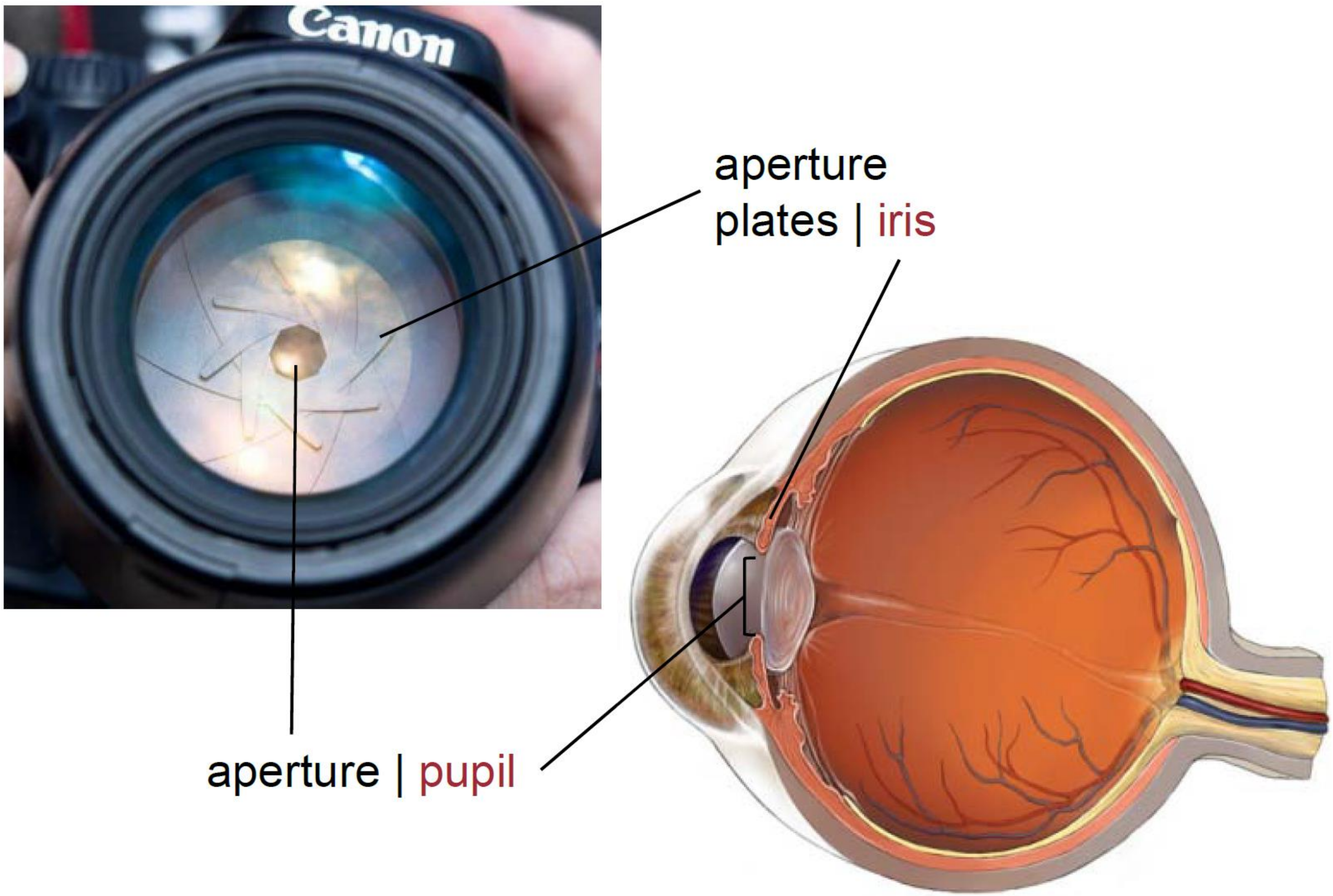
Imaging Geometry

Contents

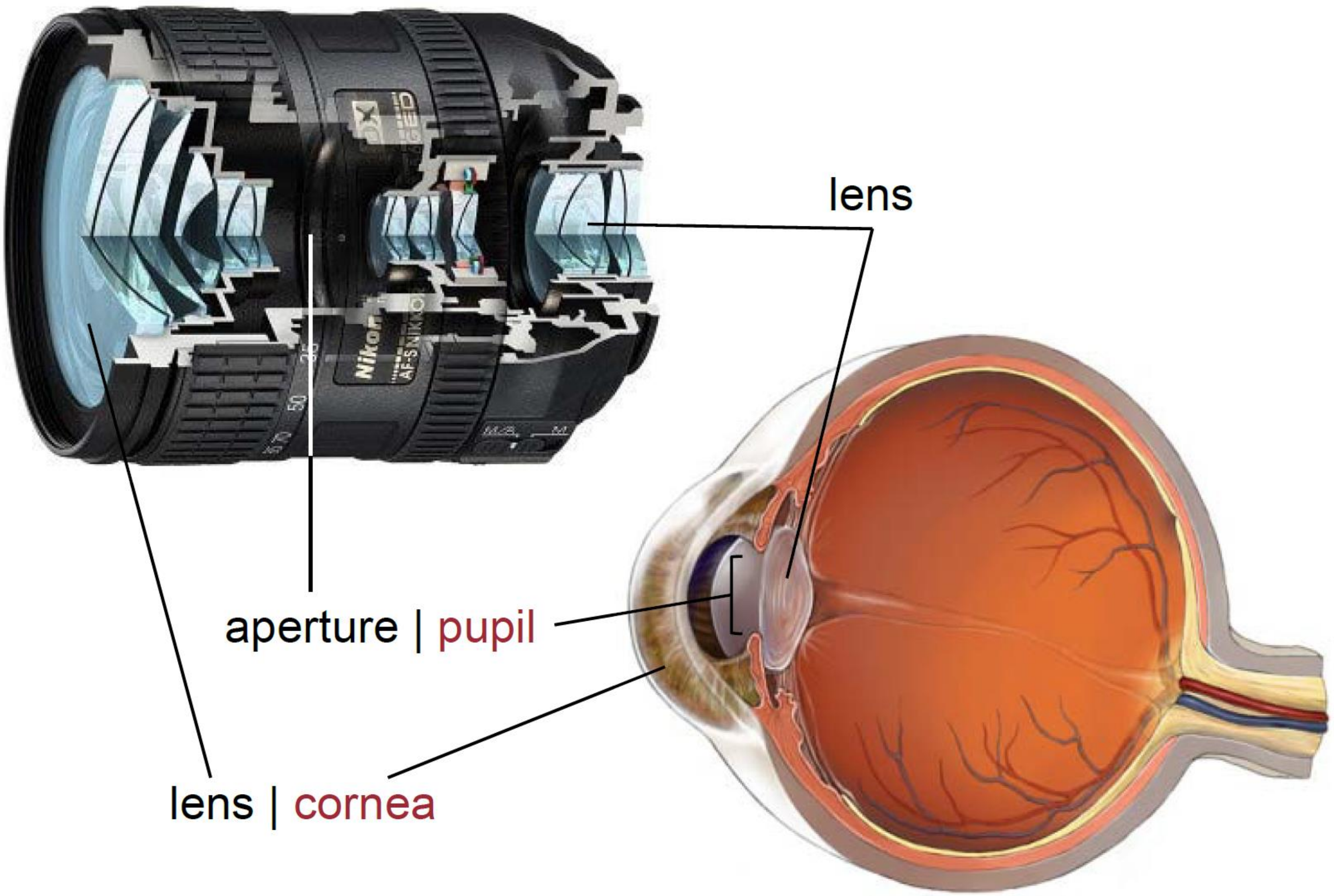
1. Introduction



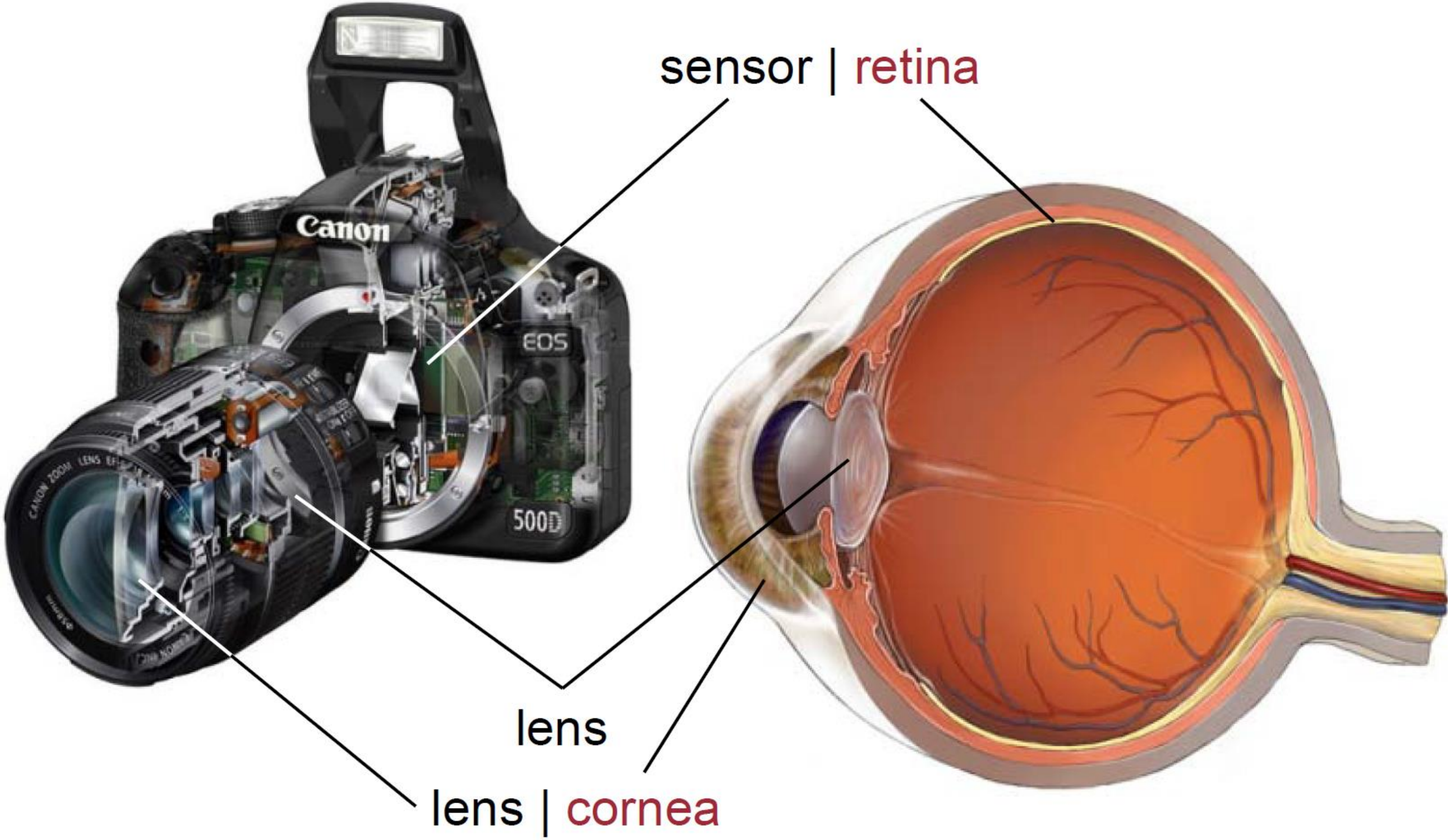
1. Introduction



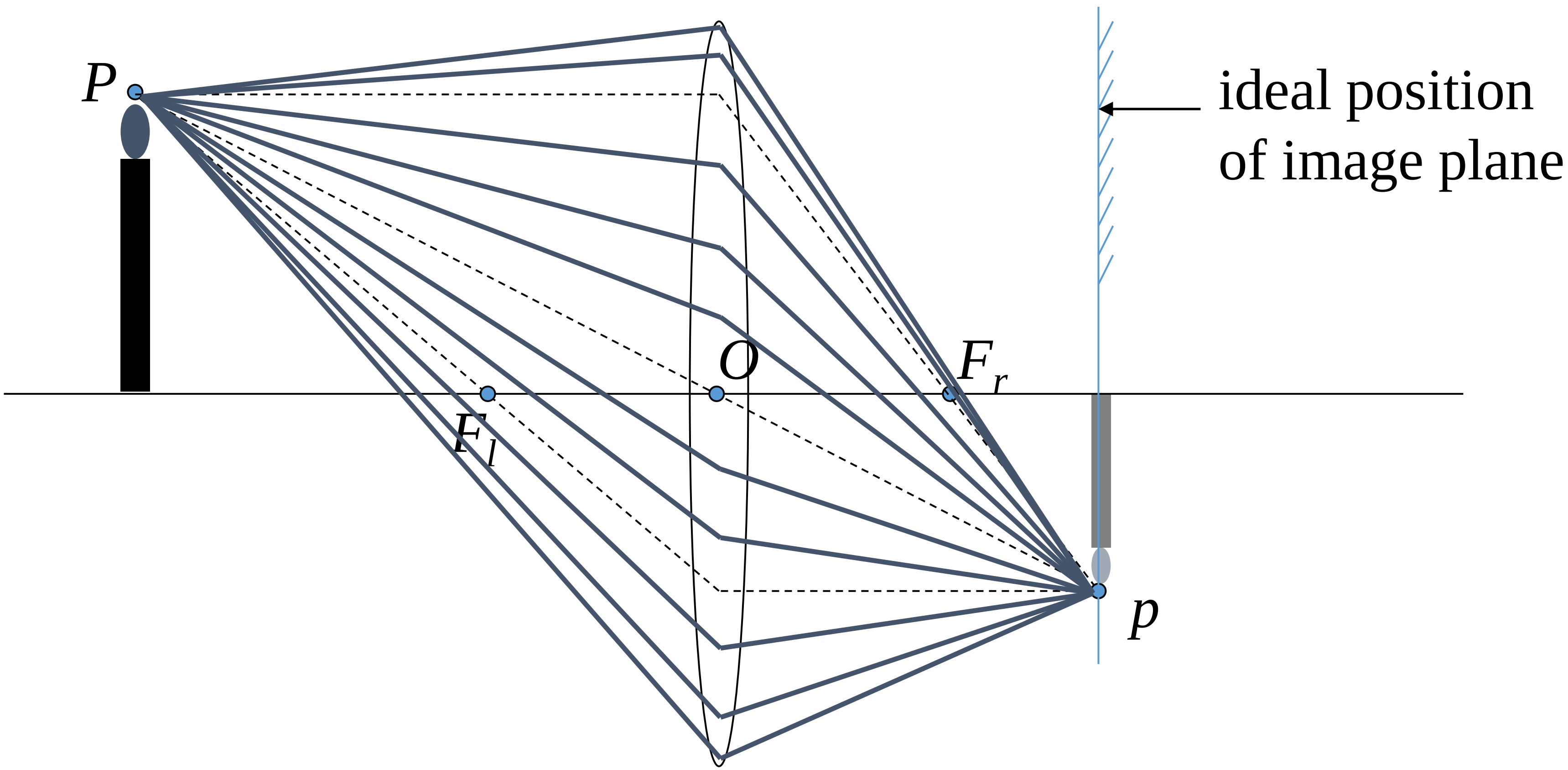
1. Introduction



1. Introduction



1. Revisiting Imaging Systems



1. Introduction

The 3D-to-2D imaging is sufficient and useful to many real-world applications.

https://clouard.users.greyc.fr/Pantheon/experiments/licenseplate-detection/index-en.html

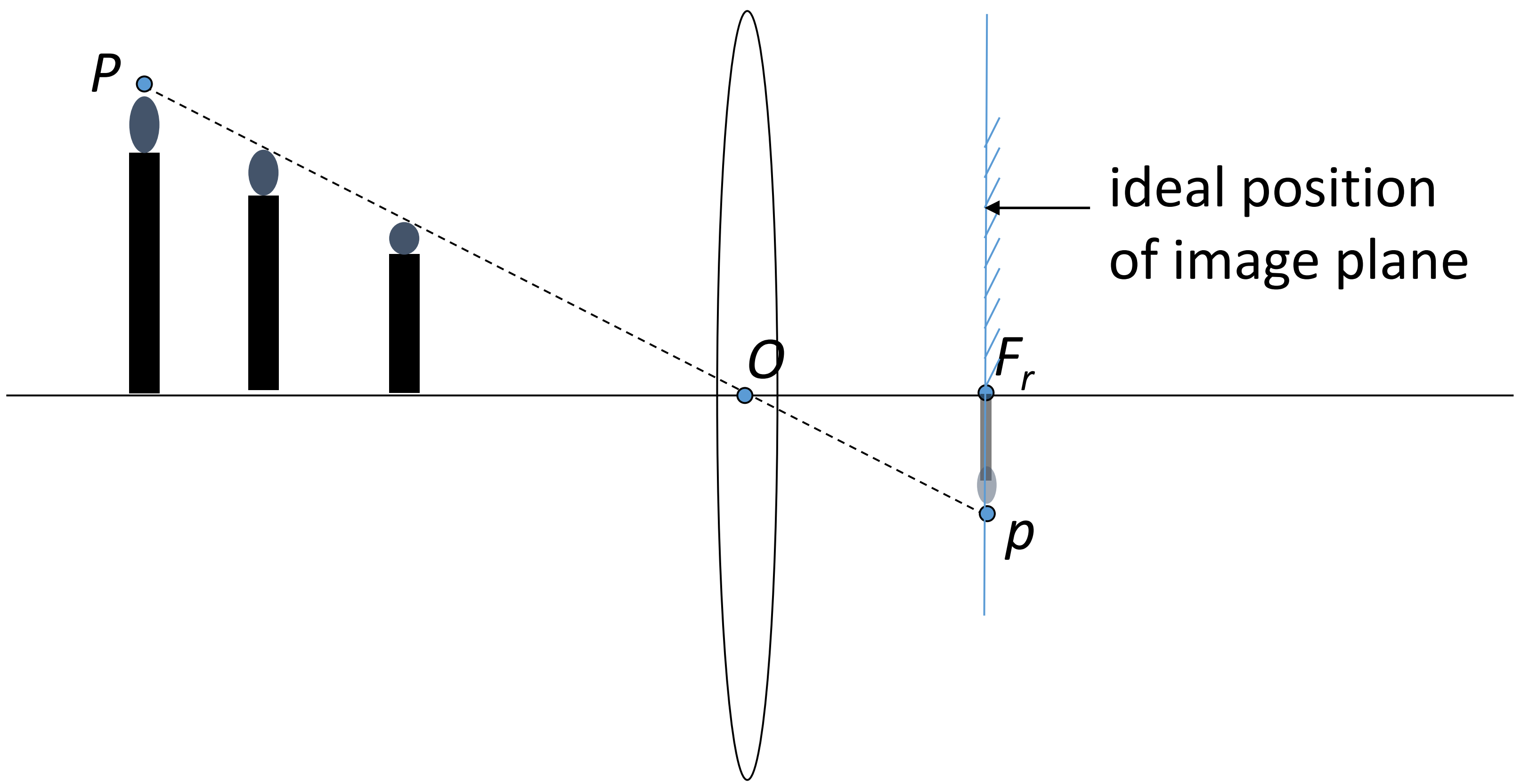
1. Introduction

But it might not be sufficient for many applications that requires to get 3D information from 2D images.

2 Imaging Geometry

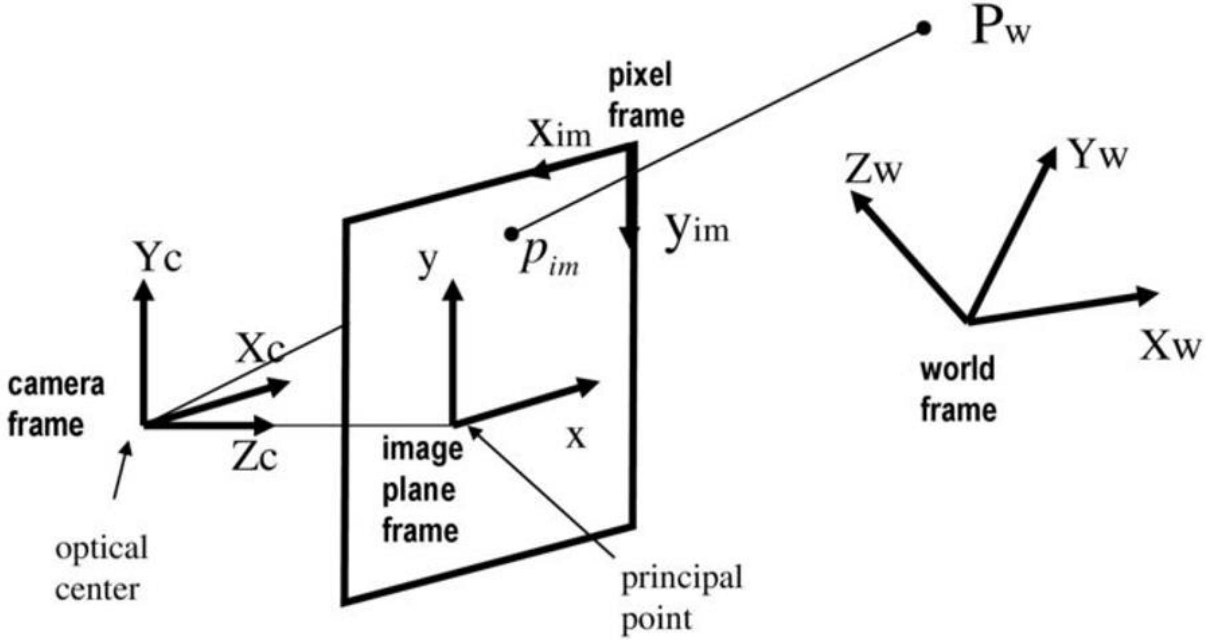
2 Imaging Geometry: the Pinhole Model

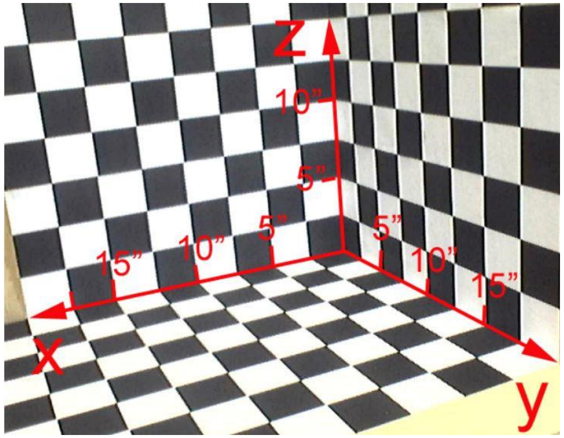
We start to build up the imaging geometry by using a simplified pinhole camera



2 Imaging Geometry: the Pinhole Model

There are four coordinate systems in the pinhole model.



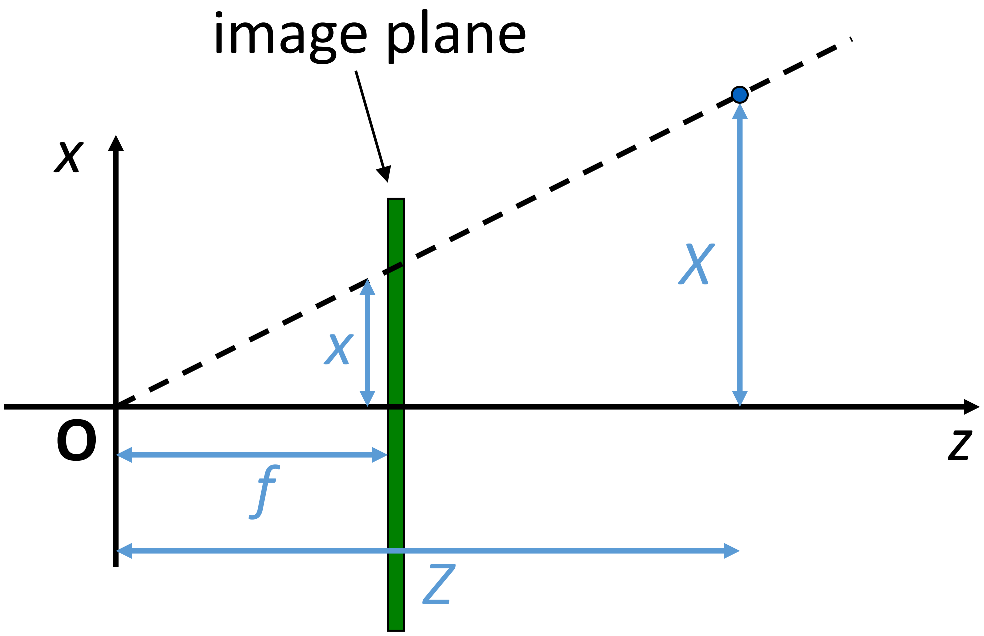


Camera model:

$$p_{im} = \begin{bmatrix} transformation \\ matrix \end{bmatrix} P_w$$

World Coords		Camera Coords		Image Coords		Pixel Coords
X_W, Y_W, Z_W	→	X_C, Y_C, Z_C	→	x, y	→	x_{im}, y_{im}

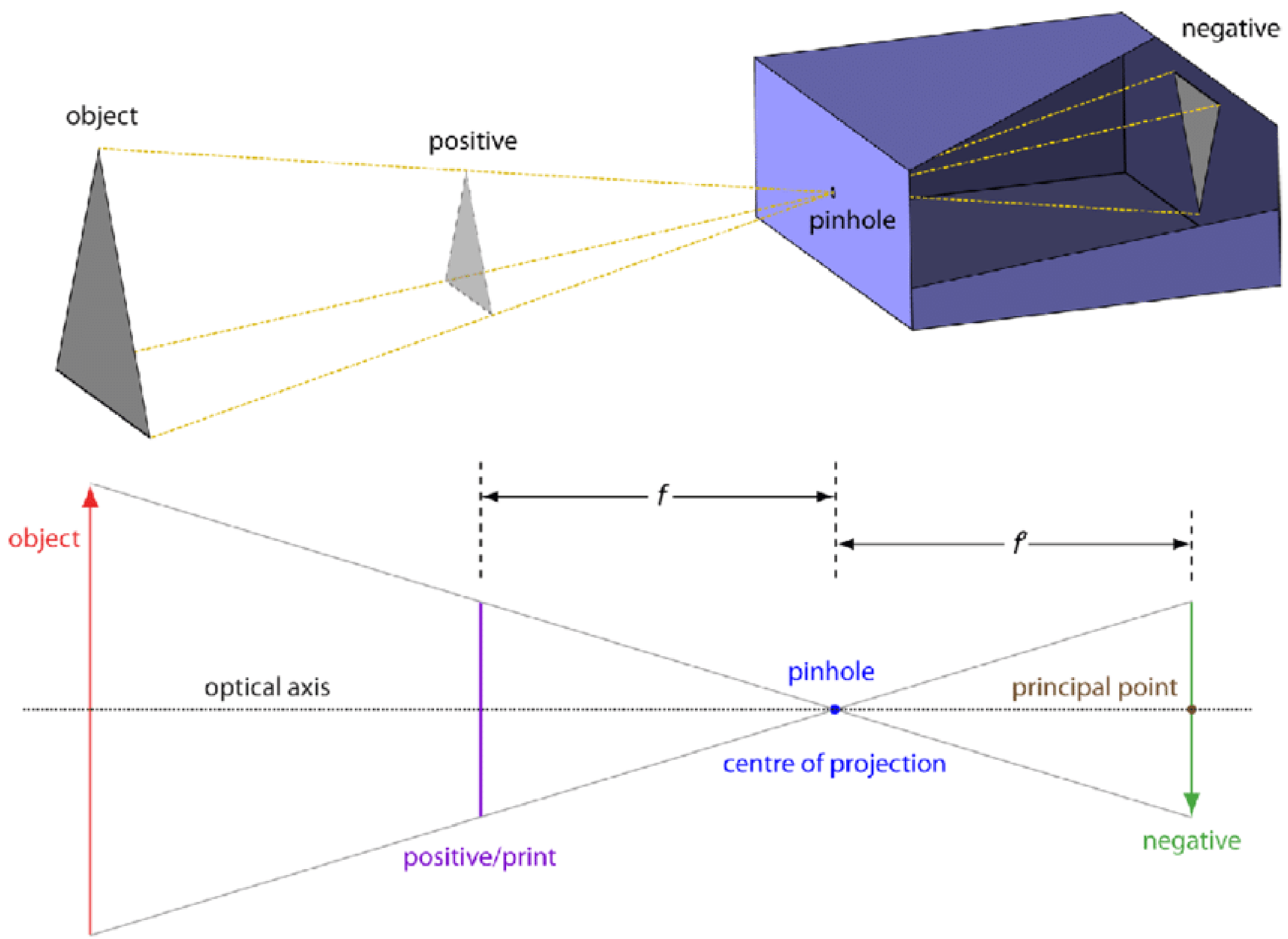
2 Imaging Geometry: Camera vs Image Frames



Based on similar triangles, in the camera and image frames:

$$x = f\frac{X}{Z}, \qquad y = f\frac{Y}{Z}$$

2 Imaging Geometry: Camera vs Image Frames



2 Camera Geometry: Camera vs Image Frames

$$\begin{bmatrix} kx \\ ky \\ k \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

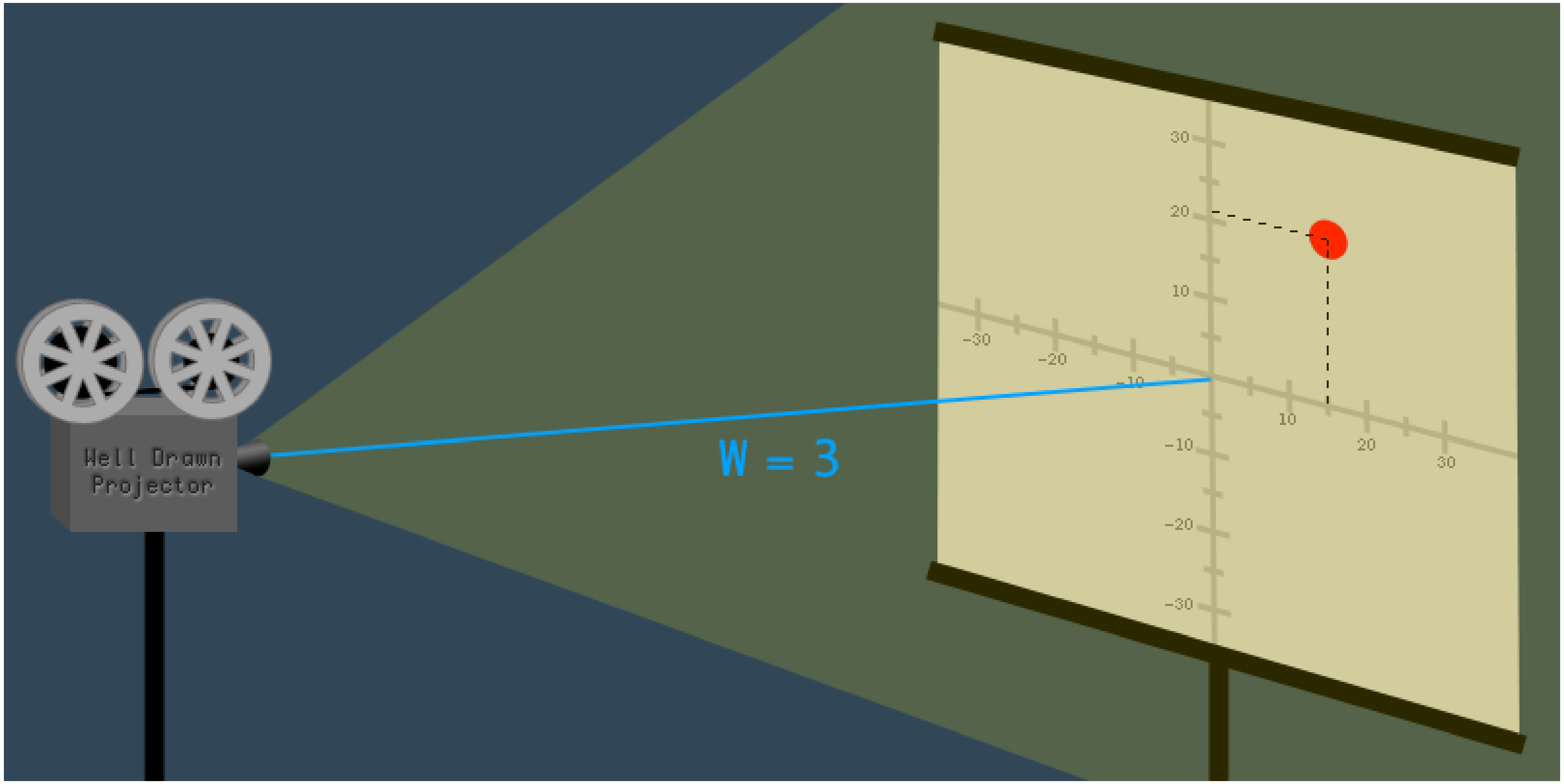
2D point in the camera frame

Perspective projection

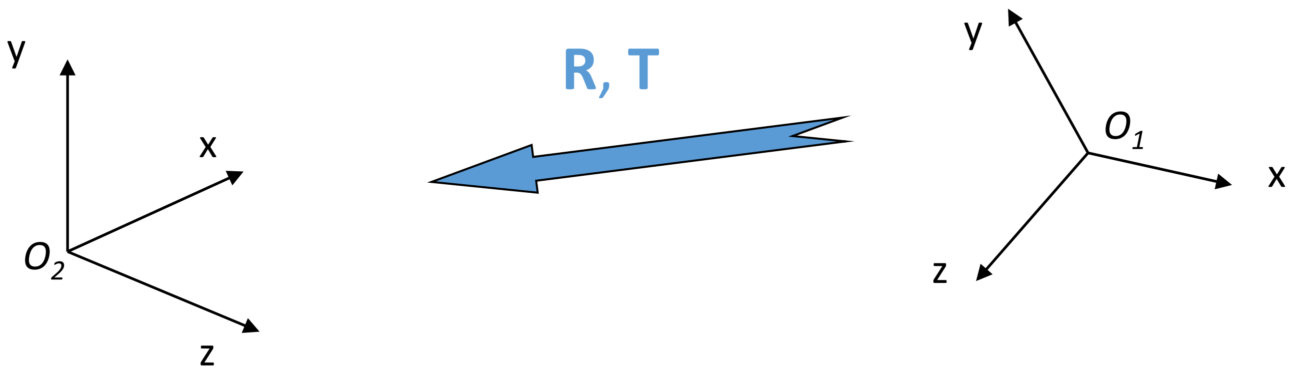
3D point in the camera frame

2 Camera Geometry: Camera vs Image Frames

Homogenous coordinate is widely used in 3D vision. It includes one more dimension beyond the Euclidean coordinate, acting as a scaling factor. A point (5, 7) in Euclidean coordinate can thus be represented by (5, 7, 1) with w=1 or (15, 21, 3) with w=3 in the homogenous coordinate.



2 Camera Geometry: Camera vs World Frames



Defined by 3D rotation and translation

- **R** matrix - Rotation about some 3D axis through O_1
- **T** vector – Translation of origin from O_1 to O_2

2 Camera Geometry: Camera vs World Frames

Translation

$$\mathbf{P}' = \mathbf{P} + \mathbf{T} \qquad \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

Rotation

$$\mathbf{P}' = \mathbf{R}\mathbf{P} \qquad \begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

2 Camera Geometry: Camera vs World Frames

Combined: rotation about origin of first coordinate frame, followed by translation

$$\mathbf{P} = \mathbf{R}\mathbf{P}_w + \mathbf{T}$$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \end{bmatrix} + \begin{bmatrix} T_X \\ T_Y \\ T_Z \end{bmatrix}$$

2 Camera Geometry: Camera vs World Frames

A neater form

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & T_X \\ r_{21} & r_{22} & r_{23} & T_Y \\ r_{31} & r_{32} & r_{33} & T_Z \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

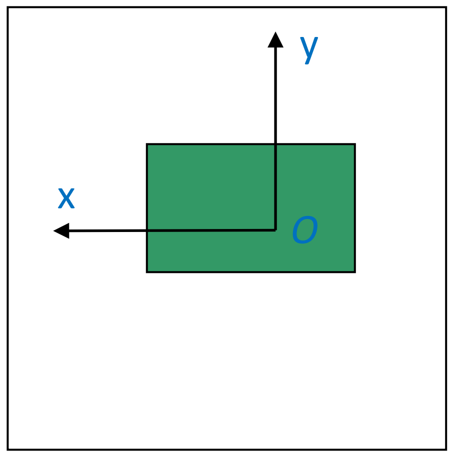
3D point in the camera frame

Rigid transform

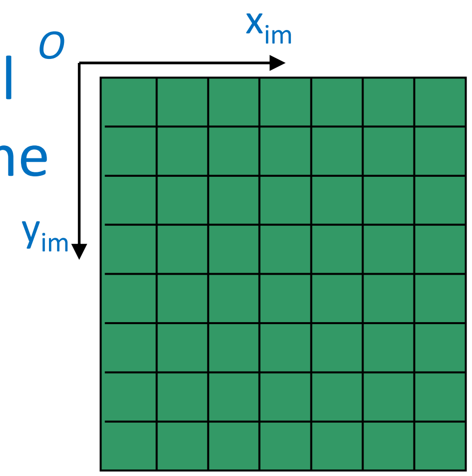
3D point in the world frame

2 Camera Geometry: Image vs Pixel Frames

Image frame



Pixel frame



$$\begin{cases} x = -(x_{im} - o_x) s_x \\ y = -(y_{im} - o_y) s_y \end{cases}$$

(o_x,o_y): the coordinates in pixel of the image center

(s_x,s_y): the effective physical size of the pixel

$$\begin{cases} x_{im} = -x / s_x + o_x \\ y_{im} = -y / s_y + o_y \end{cases}$$

An equivalent form

2 Camera Geometry: Image vs Pixel Frames

A neater form

$$\begin{bmatrix} x_{im} \\ y_{im} \\ 1 \end{bmatrix} = \begin{bmatrix} -1/s_x & 0 & o_x \\ 0 & -1/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

2D point in the pixel frame

discretization

2D point in the image frame

2 Camera Geometry: World vs Pixel Frames

$$\begin{bmatrix} kx \\ ky \\ k \end{bmatrix} = \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

$$\begin{bmatrix} x_{im} \\ y_{im} \\ 1 \end{bmatrix} = \begin{bmatrix} -1/s_x & 0 & o_x \\ 0 & -1/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} kx_{im} \\ ky_{im} \\ k \end{bmatrix} = \begin{bmatrix} -1/s_x & 0 & o_x \\ 0 & -1/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} kx \\ ky \\ k \end{bmatrix}$$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & T_X \\ r_{21} & r_{22} & r_{23} & T_Y \\ r_{31} & r_{32} & r_{33} & T_Z \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

2 Camera Geometry: World vs Pixel Frames

$$\begin{bmatrix} kx_{im} \\ ky_{im} \\ k \end{bmatrix} = \begin{bmatrix} -1/s_x & 0 & o_x \\ 0 & -1/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & T_X \\ r_{21} & r_{22} & r_{23} & T_Y \\ r_{31} & r_{32} & r_{33} & T_Z \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

image point in the pixel frame

From the image frame to the pixel frame

From the camera frame to the image frame (perspective projection)

From the world frame to the camera frame

3D point in world frame

2 Camera Geometry: World vs Pixel Frames

$$\begin{bmatrix} kx_{im} \\ ky_{im} \\ k \end{bmatrix} = \underbrace{\begin{bmatrix} -f/s_x & 0 & o_x \\ 0 & -f/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix}}_{M_{\text{int}}} \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & T_X \\ r_{21} & r_{22} & r_{23} & T_Y \\ r_{31} & r_{32} & r_{33} & T_Z \end{bmatrix}}_{M_{ext}} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

Involves only intrinsic parameters

Involves only extrinsic parameters

2 Camera Geometry: World vs Pixel Frames

$$\begin{bmatrix} kx_{im} \\ ky_{im} \\ k \end{bmatrix} = \begin{bmatrix} -fr_{11}/s_x + o_x r_{31} & -fr_{12}/s_x + o_x r_{32} & -fr_{13}/s_x + o_x r_{33} & -fT_X/s_x + o_x T_Z \\ -fr_{21}/s_y + o_y r_{31} & -fr_{22}/s_y + o_y r_{32} & -fr_{23}/s_y + o_y r_{33} & -fT_Y/s_x + o_y T_Z \\ r_{31} & r_{32} & r_{33} & T_Z \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix}}_{M} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

Projection matrix

2 Camera Geometry: World vs Pixel Frames

$$\begin{bmatrix} kx_{im} \\ ky_{im} \\ k \end{bmatrix} = \underbrace{\begin{bmatrix} -f/s_x & 0 & o_x \\ 0 & -f/s_y & o_y \\ 0 & 0 & 1 \end{bmatrix}}_{M_{\text{int}}} \underbrace{\begin{bmatrix} r_{11} & r_{12} & r_{13} & T_X \\ r_{21} & r_{22} & r_{23} & T_Y \\ r_{31} & r_{32} & r_{33} & T_Z \end{bmatrix}}_{M_{ext}} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

2 Imaging Geometry: Summary

- The methodology
  - —Build the basic model that link the camera frame and the image frame
  - —Link the camera and world frames
  - —Link the image and pixel frames
  - —Build the final model between the world and pixel frames
- A scene point and its image point can be linked by a 3*4 projection matrix *M*.
- This is useful and important for 3D reconstruction.
- But *M* is unknown yet. To make it known, we need camera calibration.

3. Camera Calibration

The model we have obtained

$$\begin{bmatrix} kx_{im} \\ ky_{im} \\ k \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

In the vision problem, what we know is

$$\begin{bmatrix} x_{im} \\ y_{im} \end{bmatrix}$$

What we want to know is

$$\begin{bmatrix} X_w \\ Y_w \\ Z_w \end{bmatrix}$$

The difficulty is that we do not know

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & 1 \end{bmatrix}$$

The good news is, if the two vectors are known, *M* can be made known. This is called **camera calibration**.

3. Camera Calibration

The above relationship can be written as

$$\begin{cases} kx_{im} = m_{11}X_w + m_{12}Y_w + m_{13}Z_w + m_{14} \\ ky_{im} = m_{21}X_w + m_{22}Y_w + m_{23}Z_w + m_{24} \\ k = m_{31}X_w + m_{32}Y_w + m_{33}Z_w + 1 \end{cases}$$

And rearranged as

$$\begin{cases} X_w m_{11} + Y_w m_{12} + Z_w m_{13} + m_{14} - x_{im}X_w m_{31} - x_{im}Y_w m_{32} - x_{im}Z_w m_{33} = x_{im} \\ X_w m_{21} + Y_w m_{22} + Z_w m_{23} + m_{24} - y_{im}X_w m_{31} - y_{im}Y_w m_{32} - y_{im}Z_w m_{33} = y_{im} \end{cases}$$

These are two equations but 11 unknowns m_{ij}

3. Camera Calibration

If we have *n* point correspondences, we have 2*n* equations

$$X_w^1 m_{11} + Y_w^1 m_{12} + Z_w^1 m_{13} + m_{14} - x_{im}^1 X_w^1 m_{31} - x_{im}^1 Y_w^1 m_{32} - x_{im}^1 Z_w^1 m_{33} = x_{im}^1$$

$$X_w^1 m_{21} + Y_w^1 m_{22} + Z_w^1 m_{23} + m_{24} - y_{im}^1 X_w^1 m_{31} - y_{im}^1 Y_w^1 m_{32} - y_{im}^1 Z_w^1 m_{33} = y_{im}^1$$

$$X_w^2 m_{11} + Y_w^2 m_{12} + Z_w^2 m_{13} + m_{14} - x_{im}^2 X_w^2 m_{31} - x_{im}^2 Y_w^2 m_{32} - x_{im}^2 Z_w^2 m_{33} = x_{im}^2$$

$$X_w^2 m_{21} + Y_w^2 m_{22} + Z_w^2 m_{23} + m_{24} - y_{im}^2 X_w^2 m_{31} - y_{im}^2 Y_w^2 m_{32} - y_{im}^2 Z_w^2 m_{33} = y_{im}^2$$

$$\cdots$$

$$X_w^n m_{11} + Y_w^n m_{12} + Z_w^n m_{13} + m_{14} - x_{im}^n X_w^n m_{31} - x_{im}^n Y_w^n m_{32} - x_{im} Z_w^n m_{33} = x_{im}^n$$

$$X_w^n m_{21} + Y_w^n m_{22} + Z_w^n m_{23} + m_{24} - y_{im}^n X_w^n m_{31} - y_{im}^n Y_w^n m_{32} - y_{im}^n Z_w^n m_{33} = y_{im}^n$$

3. Camera Calibration

We form a linear equation system Au = v

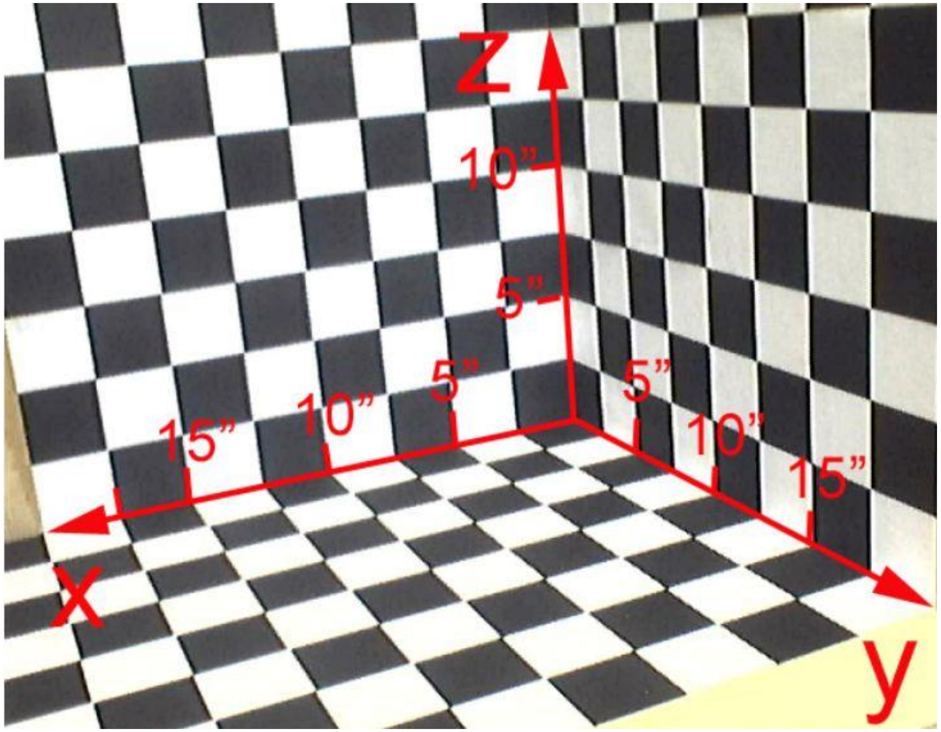
$$
\begin{bmatrix}
X_w^1 & Y_w^1 & Z_w^1 & 1 & 0 & 0 & 0 & 0 & -x_{im}^1 X_w^1 & -x_{im}^1 Y_w^1 & -x_{im}^1 Z_w^1 \\
0 & 0 & 0 & 0 & X_w^1 & Y_w^1 & Z_w^1 & 1 & -y_{im}^1 X_w^1 & -y_{im}^1 Y_w^1 & -y_{im}^1 Z_w^1 \\
X_w^2 & Y_w^2 & Z_w^2 & 1 & 0 & 0 & 0 & 0 & -x_{im}^2 X_w^2 & -x_{im}^2 Y_w^2 & -x_{im}^2 Z_w^2 \\
0 & 0 & 0 & 0 & X_w^2 & Y_w^2 & Z_w^2 & 1 & -y_{im}^2 X_w^2 & -x_{im}^2 Y_w^2 & -y_{im}^2 Z_w^2 \\
. & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . \\
X_w^n & Y_w^n & Z_w^n & 1 & 0 & 0 & 0 & 0 & -x_{im}^n X_w^n & -x_{im}^n Y_w^n & -x_{im}^n Z_w^n \\
0 & 0 & 0 & 0 & X_w^n & Y_w^n & Z_w^n & 1 & -y_{im}^n X_w^n & -x_{im}^n Y_w^n & -y_{im}^n Z_w^n
\end{bmatrix}
\begin{bmatrix}
m_{11} \\ m_{12} \\ m_{13} \\ m_{14} \\ m_{21} \\ m_{22} \\ m_{23} \\ m_{24} \\ m_{31} \\ m_{32} \\ m_{33}
\end{bmatrix}
=
\begin{bmatrix}
x_{im}^1 \\ y_{im}^1 \\ x_{im}^2 \\ y_{im}^2 \\ . \\ . \\ x_{im}^n \\ y_{im}^n
\end{bmatrix}
$$

3. Camera Calibration

- The projection matrix can be determined by matrix inverse $\mathbf{u} = \mathbf{A}^{-1}\mathbf{v}$ (when the matrix is square). The MATLAB function is inv(A).
- Or by matrix psudo-inverse $\mathbf{u} = \mathbf{A}^{+}\mathbf{v}$ (when the matrix is non-square). The MATLAB function is pinv(A).
- 6 point pairs are sufficient, but we often use more points to make the solution more robust.
- From the projection matrix *M*, we can further get all intrinsic and extrinsic parameters.
- Now we need to find scene and image point pairs (why?)

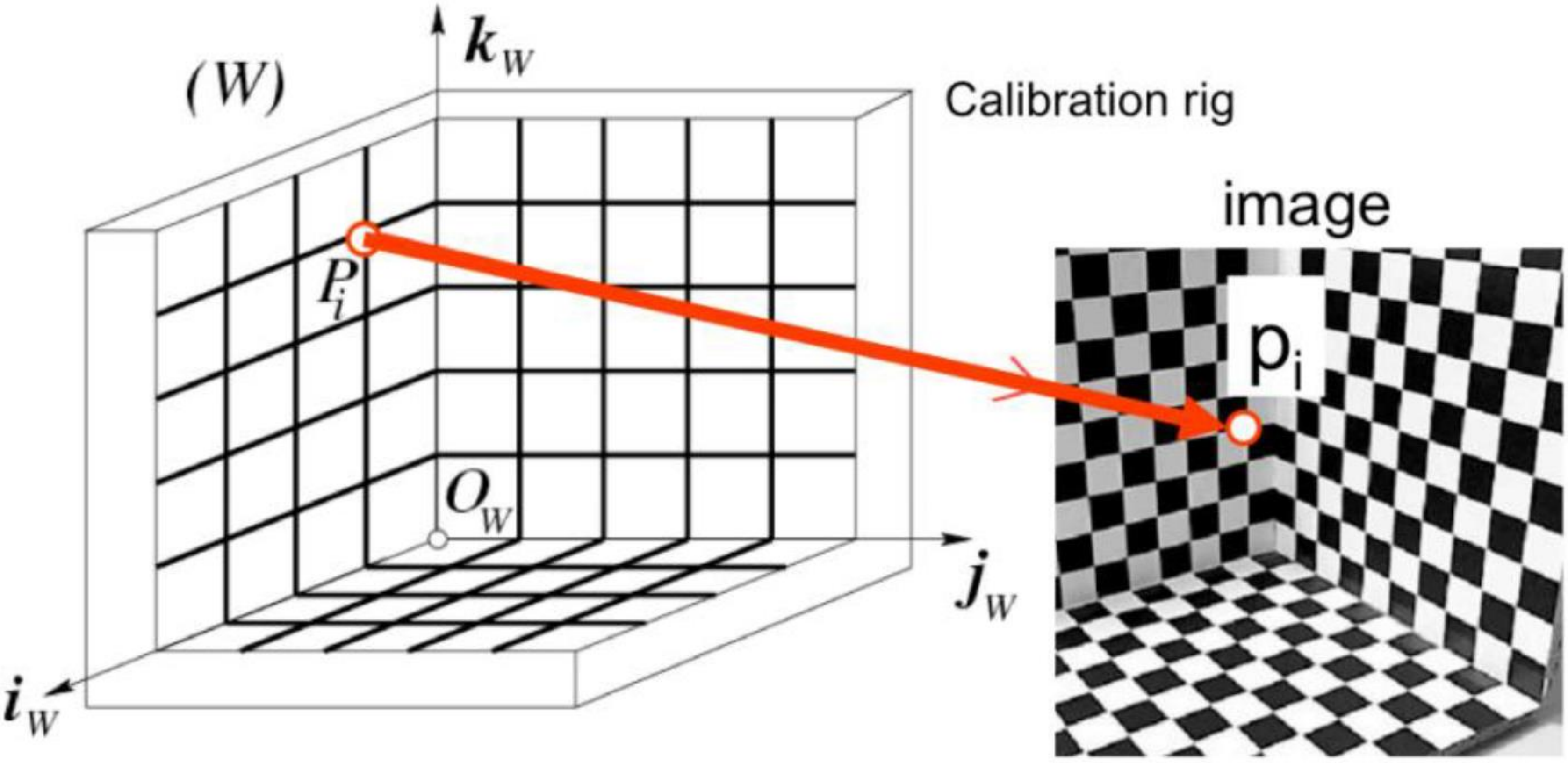
3. Camera Calibration

Calibration Chart or cube consists of a known structure fabricated with high accuracy, so the coordinates of corners in the world frame are known.



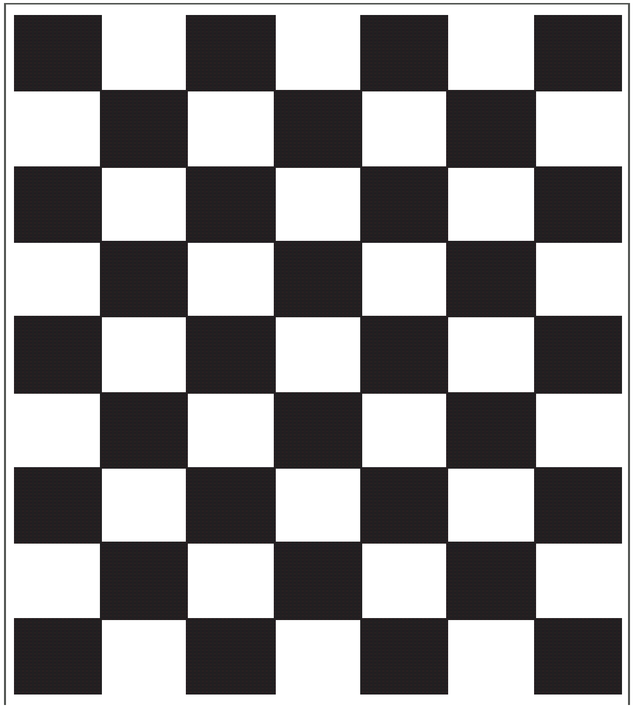
3. Camera Calibration

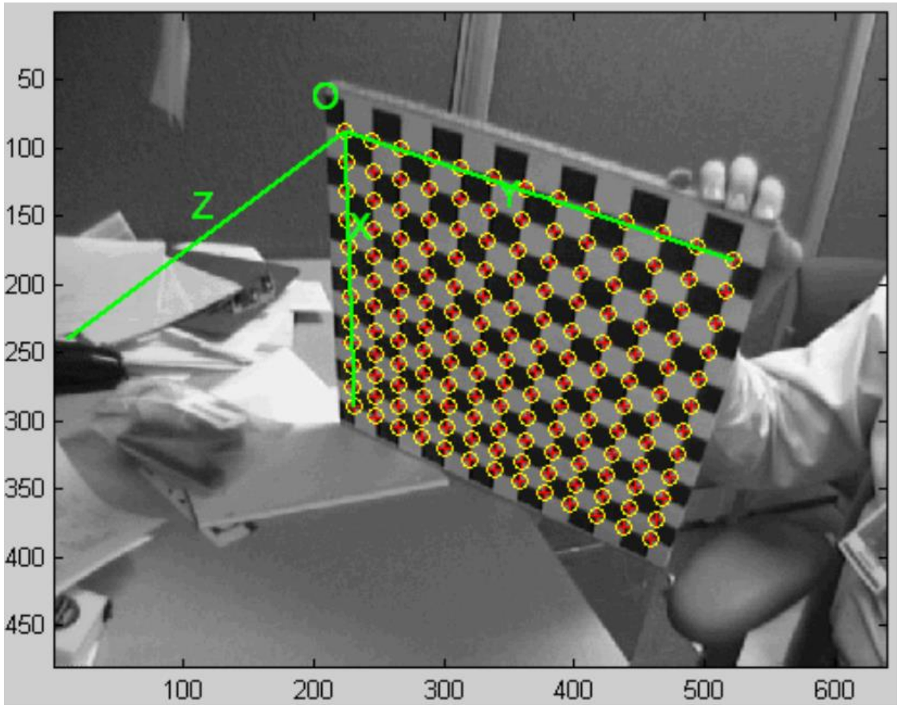
The corners in the image are detected by edge intersection or corner detection and thus also known.



3. Camera Calibration

- Download and try the codes available from the internet.
- http://www.vision.caltech.edu/bouguetj/calib_doc/





4 Applications

When the scene points are in a plane

$$\begin{bmatrix} kx_{im} \\ ky_{im} \\ k \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ \alpha X_w + \beta Y_w + \gamma \\ 1 \end{bmatrix}$$

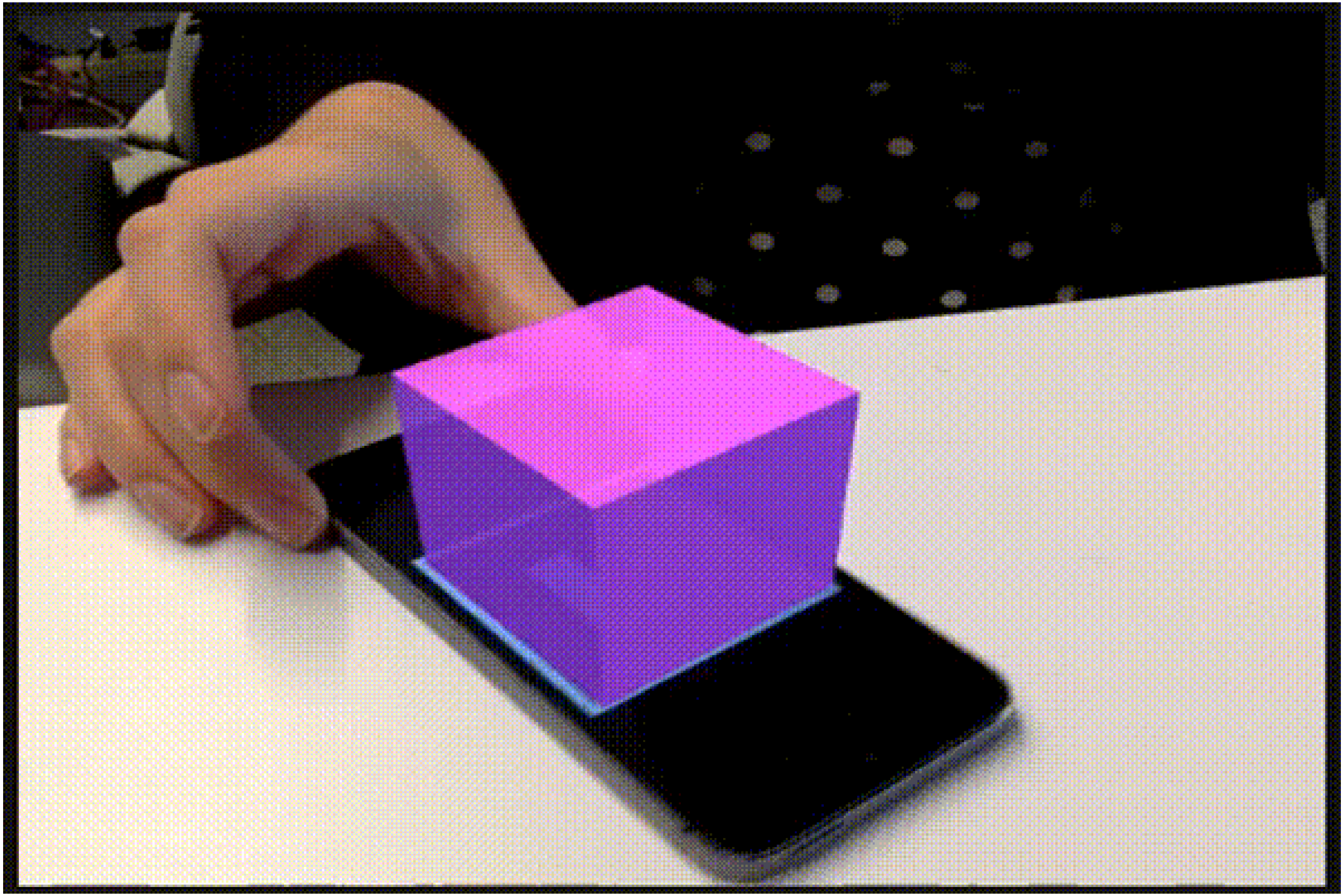
By reducing the redundancy, we get

$$\begin{bmatrix} kx_{im} \\ ky_{im} \\ k \end{bmatrix} = \begin{bmatrix} m'_{11} & m'_{12} & m'_{13} \\ m'_{21} & m'_{22} & m'_{23} \\ m'_{31} & m'_{32} & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ 1 \end{bmatrix}$$

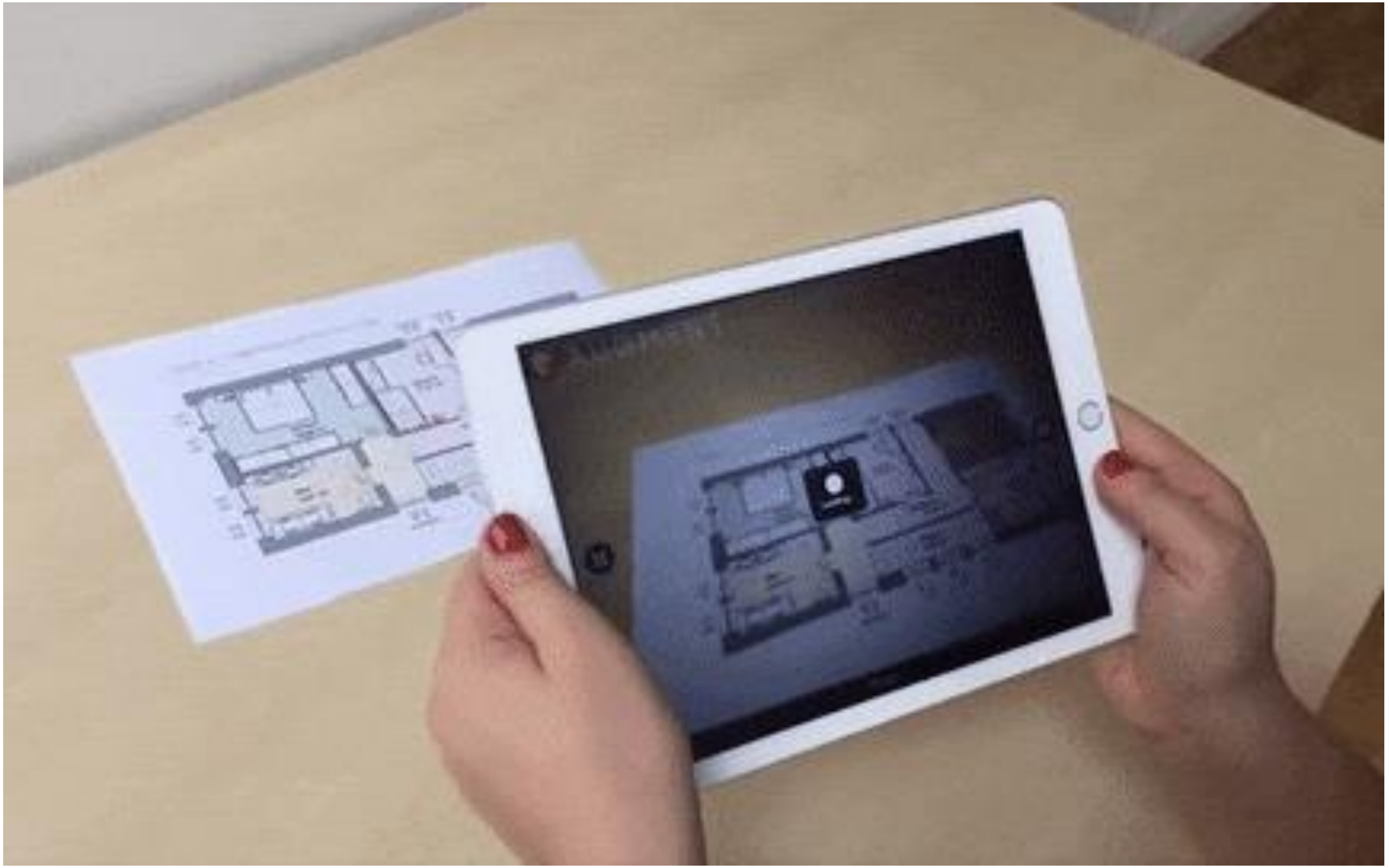
The 3*4 projective transformation becomes a 3*3 ‘homography’ matrix which can be applied to many different tasks.

4 Applications: Image Stitching

4 Applications: Mixed Reality



4 Applications: Mixed Reality



Summary

1. How 2D and 3D points correlate
2. How to calibrate cameras
3. Applications