TIME SERIES ANALYSIS

Chapter 9
ARCH and GARCH models

Big pictures:

(a) We use ARMA model for the conditional mean.

(b) We use ARCH model for the conditional variance.

(c) ARMA and ARCH model can be used together to describe both conditional mean and conditional variance

1 Typical financial time series

The models discussed so far concern the conditional mean structure of time series data. However, the linear structural (and time series) models cannot explain a number of important features common to much financial data. More recently, there has been much work on modeling the conditional variance structure of time series data-mainly motivated by the needs for financial modeling.

In finance, the conditional variance of the return of a financial asset is often adopted as a measure of the risk of the asset. This is a key component in the mathematical theory of pricing a financial asset and the VaR (Value at Risk) calculations. See, for example, Tsay (2005).

Let $P_t, t = 0, ..., n$, be a time series of prices of a financial asset, e.g. daily quotes on a share, stock index, currency exchange rate or a commodity. Then the relative return of "buying yesterday and selling today" (assuming no dividend) is

$$r_t = \frac{p_t - p_{t-1}}{p_{t-1}}.$$

Instead of the relative return we often analyze log-returns on P_t, i.e. the series

$$y_t = \log p_t - \log p_{t-1}.$$

By Taylor's expansion, log return y_t is close to the relative return r_t when r_t is small. This is because

$$\log p_t - \log p_{t-1} = \log \frac{p_t}{p_{t-1}} = \log \left(1 + \frac{p_t - p_{t-1}}{p_{t-1}}\right).$$

As an example of financial time series, we consider the daily values of a unit of the CREF stock fund over the period from August 26, 2004 to August 15, 2006.

The CREF stock fund is a fund of several thousand stocks and is not openly traded in the stock market. Fig 1 shows the time series plot of the CREF data. It shows a generally increasing trend with a hint of higher variability with higher level of the stock value.

```
CREF=read.table("C:/Users/gmpan/Desktop/teaching material/Probability
/Time series analysis/CREF.dat");
plot.ts(CREF)
```

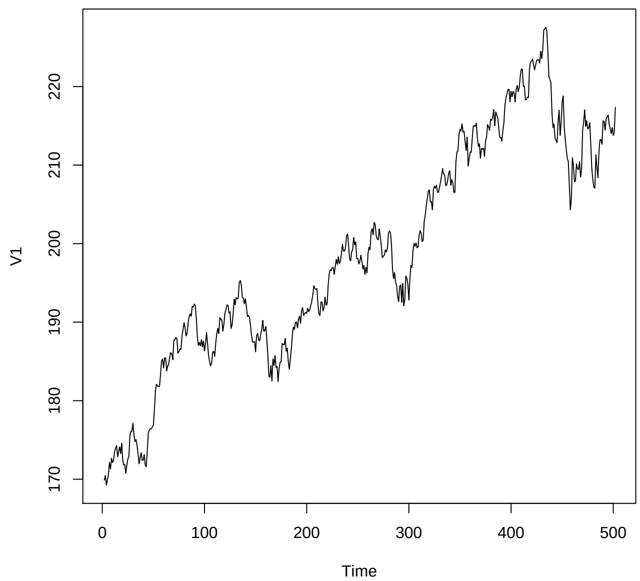

Figure 1:

Fig 2 plots the CREF return series (sample size = 500). The plot shows that the returns were more volatile over some time periods and became very volatile toward the end of the study period. This observation may be more clearly seen by plotting the time sequence plot of the absolute or squared returns. These results might be triggered by the instability in the Middle East due to a war in southern Lebanon from July 12 to August 14, 2006. This pattern of alternating quiet and volatile periods of substantial duration is referred to as volatility clustering in the literature. Volatility in a time series refers to the phenomenon where the conditional variance of the time series varies over time.

```
r.cref=diff(log(CREF))*100
plot(r.cref); abline(h=0)
```

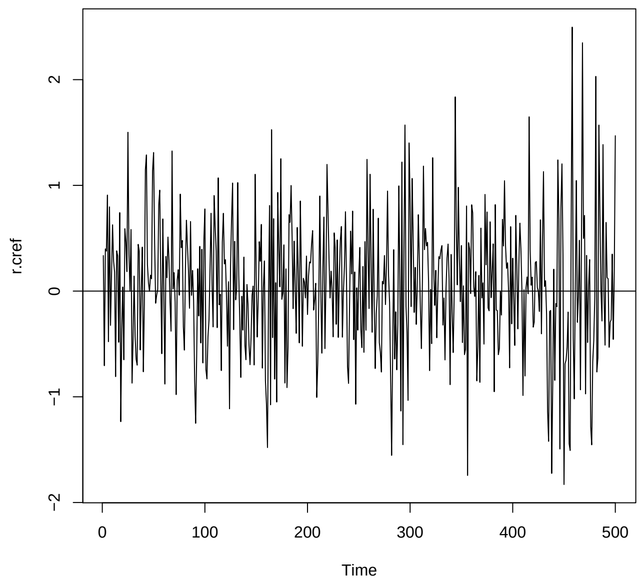

Figure 2: Fig 2

Here the returns are then multiplied by 100 so that they can be interpreted as percentage changes in the price. The multiplication may also reduce numerical errors as the raw returns could be very small numbers and render large rounding errors in some calculations.

The sample ACF and PACF of the daily CREF returns (multiplied by 100), shown in Figs 3, suggest that the returns have little serial correlation at all. The sample EACF (not shown) also suggests that a white noise model is appropriate for these data. The average CREF return equals 0.0493 with a standard error of 0.02885. Thus the mean of the return process is not statistically significantly different from zero.

```
par(mfrow=c(1,2))
acf(r.cref)
pacf(r.cref)
```

However, the volatility clustering observed in the CREF return data gives us a hint that they may not be independently and identically distributed-otherwise the variance would be constant over time. This is the first occasion in our study of time series models where we need to distinguish between series values being uncorrelated and series values being independent. If the absolute or squared returns admit some significant autocorrelations, then these autocorrelations furnish some evidence against the hypothesis that the returns are independently and

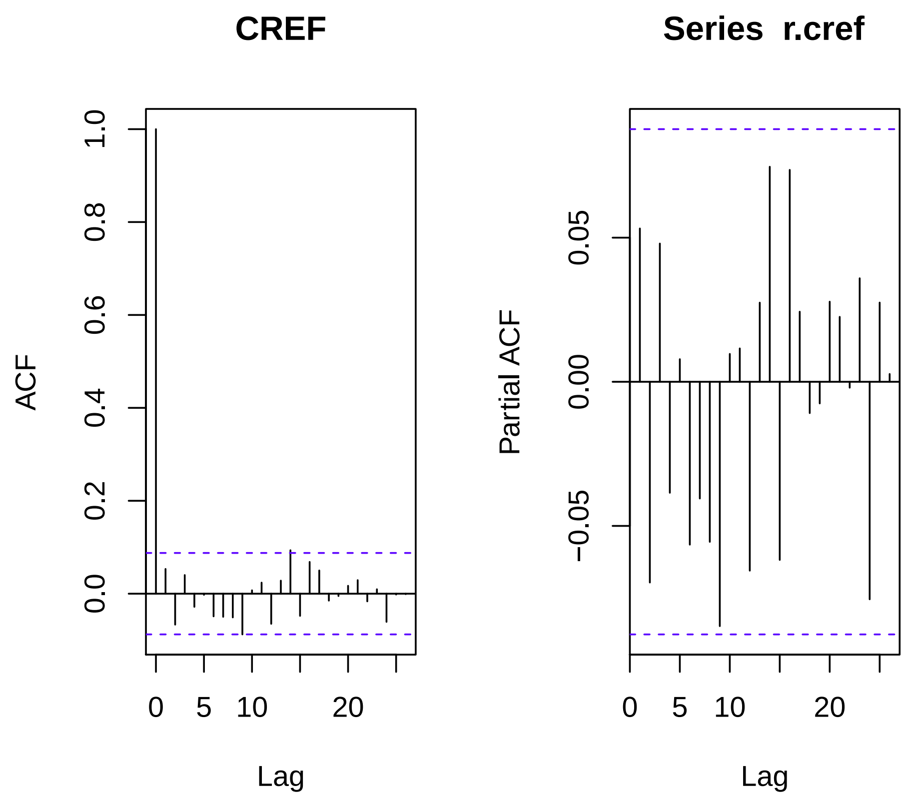

Figure 3: Fig 3

identically distributed. Indeed, the sample ACF and PACF of the absolute returns and those of the squared returns in Exhibits 4 and 5 display some significant autocorrelations and hence provide some evidence that the daily CREF returns are not independently and identically distributed.

```
par(mfrow=c(1,2))
acf(abs(r.cref))
pacf(abs(r.cref))
acf(r.cref^2)
pacf(r.cref^2)
```

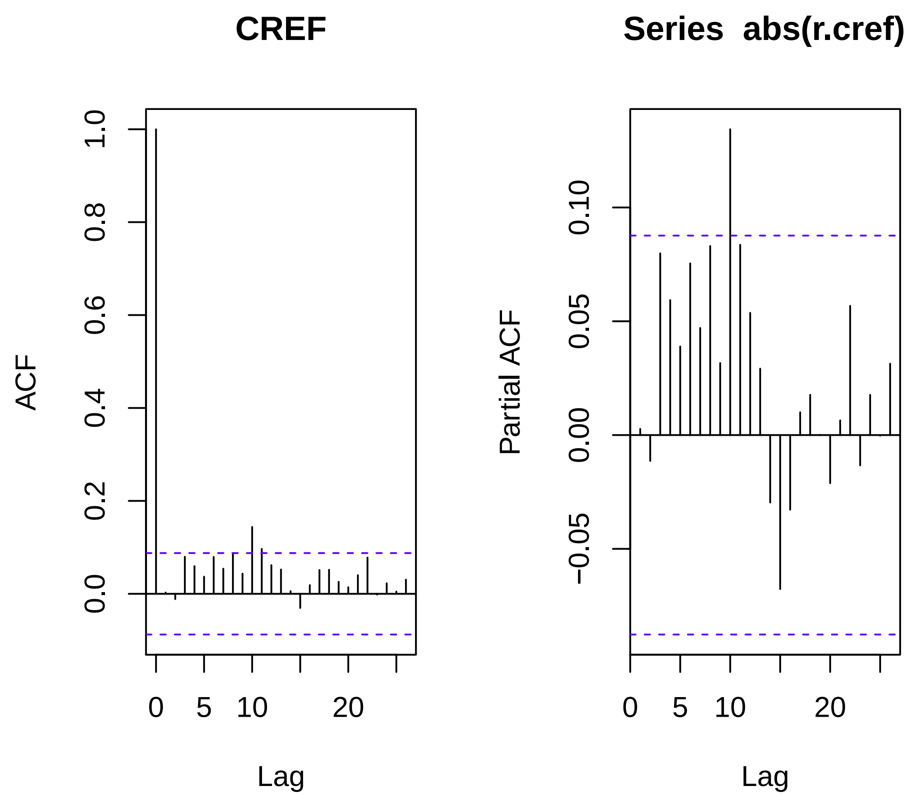

Figure 4: Fig 4

The distributional shape of the CREF returns can be explored by constructing a QQ normal scores plot-see Fig 6. The QQ plot suggests that the distribution of returns may have a tail thicker than that of a normal distribution and may be somewhat skewed to the right. This is referred to be **heavy-tailed** distribution. The thickness of the tail of a distribution relative to that of a normal distribution

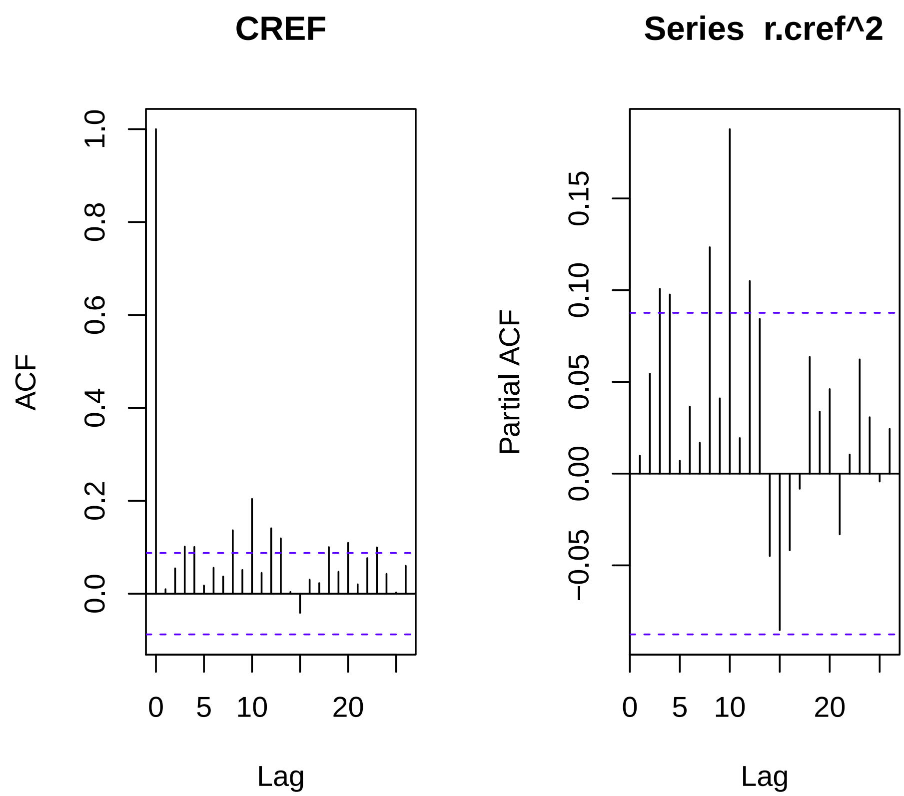

Figure 5: Fig 5

is often measured by the (excess) kurtosis, defined as $E(x - EX)^4/\sigma^4 - 3$. A distribution with positive kurtosis is called a heavy-tailed distribution, whereas it is called light-tailed if its kurtosis is negative. The kurtosis can be estimated by the sample kurtosis

$$\frac{\sum_{i=1}^{n}(x_i - \bar{x})^4}{n\hat{\sigma}^4} - 3.$$

The sample kurtosis of the CREF returns equals 0.6274

```
dev.off()
qqnorm(r.cref)
qqline(r.cref)
kurtosis(r.cref)
```

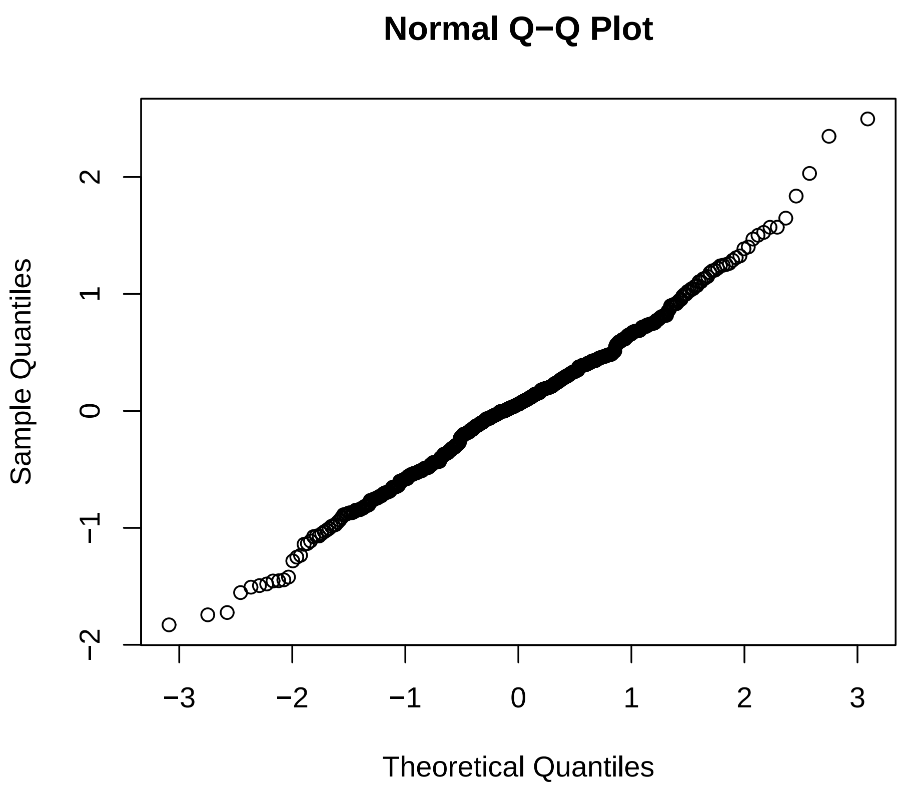

Figure 6: Fig 6

In summary, the CREF return data are found to be serially uncorrelated but admit a higher-order dependence structure, namely volatility clustering, and a heavy-tailed distribution. It is commonly observed that such characteristics are rather prevalent among financial time series data. The GARCH models introduced in the next sections attempt to provide a framework for modeling and analyzing time series that display some of these characteristics.

Volatility clustering: "Large changes tend to follow by large changes, ..., and small changes tend to follow by small changes...".

2 Motivation of using ARCH models

- ▷ A stylized fact about financial market is "volatility clustering". That is, a volatile period tends to be followed by another volatile period, or volatile periods are usually clustered.
- ▷ Intuitively, the market becomes volatile whenever big news comes, and it may take several periods for the market to fully digest the news
- ▷ Statistically, volatility clustering implies time-varying conditional variance: big volatility (variance) today may lead to big volatility tomorrow.
- ▷ The ARCH process has the property of time-varying conditional variance, and therefore can capture the volatility clustering.

Standard estimation of volatility gives equal weight to each observation:

$$\sigma^2 = \frac{1}{n-1}\sum_{t=1}^{n}(r_t - \bar{r})^2.$$

However, it makes more sense to give more weight to the recent data. For example we may use a simple formula for updating volatility estimates:

$$\sigma_t^2 = \lambda\sigma_{t-1}^2 + (1-\lambda)r_{t-1}^2,$$

where λ is a constant between zero and one. To understand this formula, we can use backward substitution to find out that

$$\sigma_t^2 = (1-\lambda)\sum_{i=1}^{m}\lambda^i r_{t-i}^2 + \lambda^m\sigma_{t-m}^2.$$

The last term is small enough to be ignored if m is large.

3 ARCH models

Noting the above difficulties, Engle (1982) was the first to propose a stationary non-linear model for y_t, which he termed ARCH (Auto-Regressive Conditionally Heteroscedastic; it means that the conditional variance of y_t evolves according to an autoregressive-type pro- cess).

As discussed in the previous section, the return series of a financial asset, say r_t, is often a serially uncorrelated sequence with zero mean, even as it exhibits volatility clustering. This suggests that the conditional variance of rt given past returns is not constant.

The ARCH model is formally a regression model with the conditional volatility as the response variable and the past lags of the squared return as the covariates. The conditional variance is denoted by σ_t^2. When r_t is available, the squared return provides an unbiased estimator of σ_t^2. For example, the ARCH(1) model assumes that the return series r_t is generated as follows:

$$r_t = \sigma_t \varepsilon_t \tag{3.1}$$

$$\sigma_t^2 = \omega + \alpha r_{t-1}^2. \tag{3.2}$$

```
> set.seed(1235678); library(TSA)
> garch01.sim=garch.sim(alpha=c(.01,.9),n=500)
plot(garch01.sim,type='l',ylab=expression(r[t]), xlab='t')
```

Fig 7 shows the time series plot of a simulated series of size 500 from an ARCH(1) model with $\omega = 0.01$ and $\alpha = 0.9$. Volatility clustering is evident in the data as larger fluctuations cluster together, although the series is able to recover from large fluctuations quickly because of the very short memory in the conditional variance process.

As the forecasting of the future conditional variances in an ARCH(1) model only involves the most recent squared return. In practice, one may expect that the accuracy of forecasting may improve by including all past squared returns with lesser weight for more distant volatilities. One approach is to include further lagged squared returns in the model

Definition 1 The ARCH(q) model is defined by

$$y_k = \sigma_k \varepsilon_k \tag{3.3}$$

$$\sigma_k^2 = \omega + \sum_{j=1}^{q} \beta_j y_{k-j}^2, \tag{3.4}$$

where $\omega > 0$, $\beta_j \geq 0$ and ε_k's are independent and identically distributed with mean zero and variance one. Here, q is referred to as the ARCH order.

Another approach, proposed by Bollerslev (1986) and Taylor (1986), introduces p lags of the conditional variance in the model, where p is referred to as the GARCH order. The combined model is called the generalized autoregressive conditional heteroscedasticity.

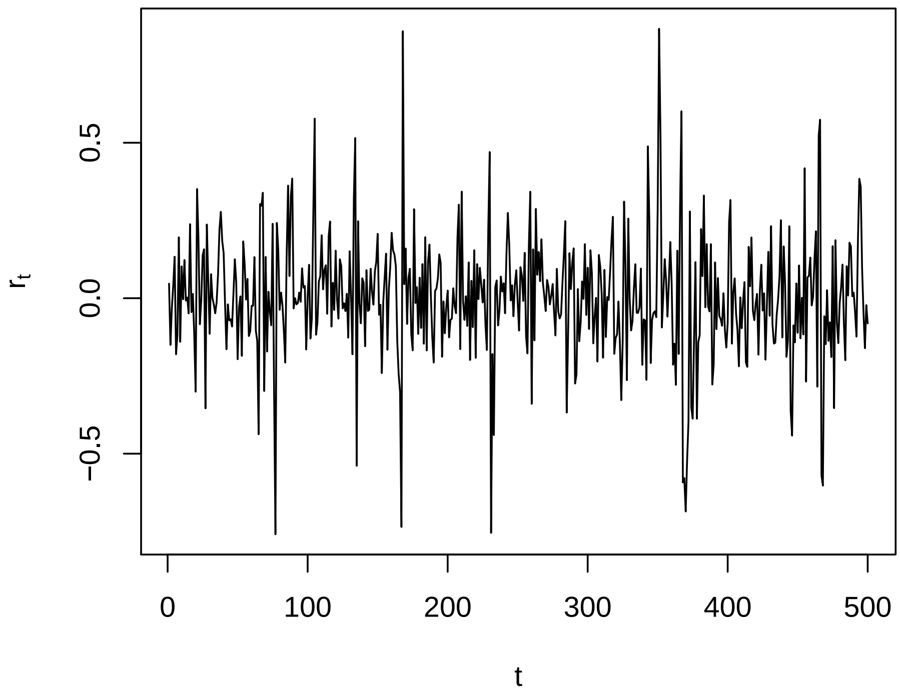

Figure 7: Fig 7

Definition 2 The GARCH(p, q) model is defined by

$$y_k = \sigma_k \varepsilon_k \tag{3.5}$$

$$\sigma_k^2 = \omega + \sum_{i=1}^{p} \alpha_i y_{k-1}^2 + \sum_{j=1}^{q} \beta_j \sigma_{k-j}^2, \tag{3.6}$$

where $\omega > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$ and ε_k's are independent and identically distributed with mean zero and variance one.

The main idea is that σ_k^2 , the conditional variance of y_k given information available up to time k - 1, has an autoregressive structure and is positively correlated to its own recent past and to recent values of the squared returns y^2. This captures the idea of volatility (conditional variance) being persistent: large (small) values of y_k^2 are likely to be followed by large (small) values. Because conditional variances must be nonnegative, the coefficients in a GARCH model are often constrained to be nonnegative

Arising from the use of conditional versus unconditional mean, the key insight offered by the ARCH model lies in the distinction between conditional and the unconditional second moments.

Fig 8 shows the time series plot of a time series, of size 500, simulated from a GARCH(1,1) model with standard normal innovations and parameter values ù = 0.02, á = 0.05, and â = 0.9. Volatility clustering is evident in the plot, as large (small) fluctuations are usually succeeded by large (small) fluctuations. Moreover, the inclusion of the lag 1 of the conditional variance in the model successfully enhances the memory in the volatility.

```
>set.seed(1234567)
>library(TSA)
> garch11.sim=garch.sim(alpha=c(0.02,0.05),beta=.9,n=500)
> plot(garch11.sim,type='l',ylab=expression(r[t]), xlab='t')
```

Except for lags 3 and 20, which are mildly significant, the sample ACF and PACF of the simulated data, shown in Figs 9 and 10, do not show significant correlations. Hence, the simulated process seems to be basically serially uncorrelated as it is.

```
>acf(garch11.sim)
>pacf(garch11.sim)

>acf(garch11.sim^2)
>pacf(garch11.sim^2)
```

Fig 17 indicate the existence of significant autocorrelation patterns in the squared data and indicate that the simulated process is in fact serially dependent.

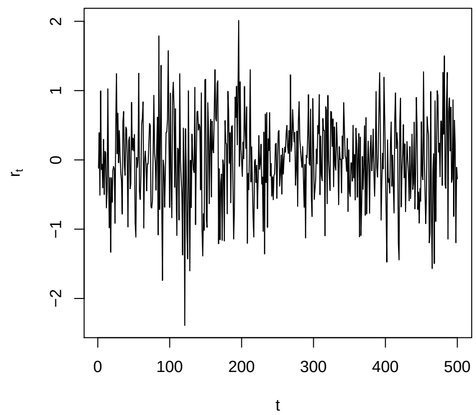

Figure 8: Simulated GARCH(1,1) Process

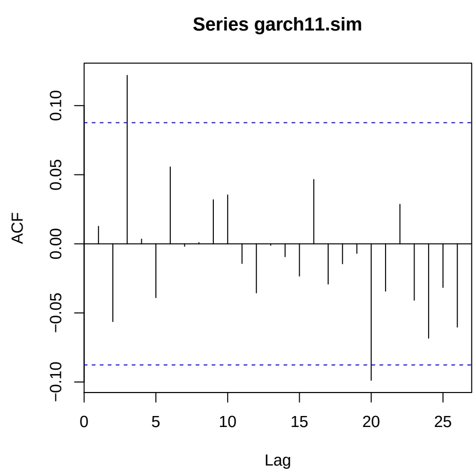

Figure 9: Sample ACF of Simulated GARCH(1,1) Process

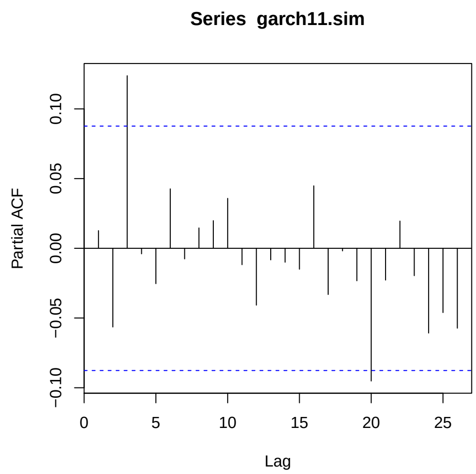

Figure 10: Sample PACF of Simulated GARCH(1,1) Process

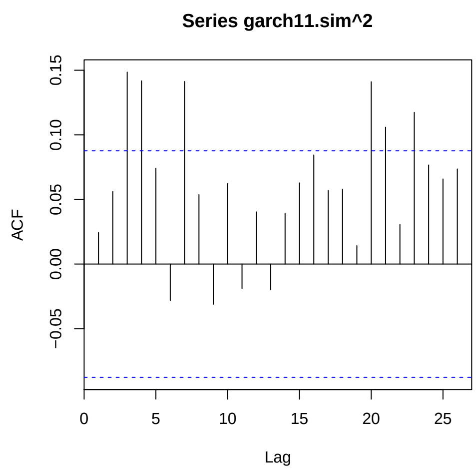

Figure 11: Sample ACF of the Squared Values of the Simulated GARCH(1,1) Process

4 Model identification

For model identification of the GARCH orders, it is again advantageous to express the model for the conditional variances in terms of the squared returns.

Define

$$\eta_t = y_t^2 - \sigma_t^2.$$

It turns out that η_t is a serially uncorrelated sequence and uncorrelated with past squared returns. Substituting the expression $\sigma_t^2 = y_t^2 - \eta_t$ into the above GARCH model yields

$$y_t^2 = \omega + (\alpha_1 + \beta_1) y_{t-1}^2 + \cdots + (\alpha_{max(p,q)} + \beta_{max(p,q)}) y_{t-max(p,q)}^2 + \eta_t - \beta_1 \eta_{t-1} + \cdots + \beta_p \eta_{t-p} \tag{4.1}$$

where $\beta_k = 0$ for all integers $k > p$ and $\alpha_k = 0$ for $k > q$. This shows that the GARCH(p,q) model for the return series implies that the model for the squared returns is an ARMA(max(p, q),p) model. Thus, we can apply the model identification techniques for ARMA models to the squared return series to identify p and max(p,q). Notice that if q is smaller than p, it will be masked in the model identification. In such cases, we can first fit a GARCH(p,p) model and then estimate q by examining the significance of the resulting ARCH coefficient estimates.

4.1 Extended Autocorrelation Function(EACF)

The sample ACF and PACF provide effective tools for identifying pure AR(p) or MA(q) models. However, for a mixed ARMA model, its theoretical ACF and PACF have infinitely many nonzero values, making it difficult to identify mixed models from the sample ACF and PACF. The extended autocorrelation function (EACF) is one method proposed to assess the orders of a ARMA(p, q) model. Other methods for specifying ARMA(p, q) models include the corner method and the smallest canonical correlation (SCAN) method, which we will not discuss here.

In an ARMA(p, q) model, if we "filter out" (i.e., subtract off) the autoregressive component(s), we are left with a pure MA(q) process that can be specified using the ACF approach. For example, consider

$$Y_t = \phi Y_{t-1} + Z_t - \theta Z_{t-1}.$$

If we regress Y_t on Y_{t-1}, we get an inconsistent estimator of ϕ, but this regression's residuals do tell us about the behavior of the error process $\{Z_t\}$. So then let's regress Y_t on Y_{t-1} and the lag 1 of the first regression's residuals, which stand in for Z_{t-1}. In this second regression, the estimated coefficient of Y_{t-1} (denoted by $\tilde{\phi}$) is a consistent estimator of ϕ. Then $W_t = Y_t - \tilde{\phi} Y_{t-1}$ is approximately an MA(1) process. For higher-order ARMA processes, we would need more of these sequential regressions to consistently estimate the AR coefficients (we'd need q extra regressions for an AR(p, q) model).

As the AR and MA orders are unknown, an iterative procedure is required. Let

$$W_{t,k,j} = Y_t - \tilde{\phi}_1 T_{t-1} - \cdots - \tilde{\phi}_k T_{t-k}$$

be the autoregressive residuals defined with the AR coefficients estimated iteratively assuming the AR order is k and the MA order is j. The sample autocorrelations of $W_{t,k,j}$ are referred to as the extended sample autocorrelations (EACF). For $k = p$ and $j > q$, $\{W_{t,k,j}\}$ is approximately an MA(q) model, so that its theoretical autocorrelations of lag q + 1 or higher are equal to zero. We can summarize the EACFs Tsay and Tiao (1984) suggested summarizing the information in the sample EACF by a table with the element in the kth row and jth column equal to the symbol X if the lag $j+1$ sample correlation of $\{W_{t,k,j}\}$ is significantly different from zero (that is, if its magnitude is greater than since the sample autocorrelation is asymptotically $N(0, 1/(n-k-j))$ if the W's are approximately an MA(j) process) and 0 otherwise. In such a table, an MA(p,q) process will have a theoretical pattern of a triangle of zeroes, with the upper left-hand vertex corresponding to the ARMA orders. Exhibit 6.4 displays the schematic pattern for an $ARMA(1, 1)$ model. The upper left-hand vertex of the triangle of zeros is marked with the symbol 0 and is located in the $p = 1$ row and $q = 1$ columnŮan indication of an $ARMA(1, 1)$ model.

Exhibit 6.4 Theoretical Extended ACF (EACF) for an ARMA(1,1) Model

AR/MA	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	x	x	x	x	x	x	x	x	x	x	x	x	x	x
1	x	0*	0	0	0	0	0	0	0	0	0	0	0	0
2	x	x	0	0	0	0	0	0	0	0	0	0	0	0
3	x	x	x	0	0	0	0	0	0	0	0	0	0	0
4	x	x	x	x	0	0	0	0	0	0	0	0	0	0
5	x	x	x	x	x	0	0	0	0	0	0	0	0	0
6	x	x	x	x	x	x	0	0	0	0	0	0	0	0
7	x	x	x	x	x	x	x	0	0	0	0	0	0	0

Figure 12: Exhibit 6.4

Of course, the sample EACF will never be this clear-cut. Displays like Exhibit 6.4 will contain 8Œ14 = 112 different estimated correlations, and some will be statistically significantly different from zero by chance.

Here is one more example. The triangular region of zeros shown in the sample EACF in Exhibit 6.17 indicates quite clearly that a mixed model with q = 1 and with p = 1 or 2 would be more appropriate.

4.2 GARCH models

As an illustration, Exhibit 12.19 shows the sample EACF of the squared values from the simulated GARCH(1,1) series.

Exhibit 6.17 Sample EACF for Simulated ARMA(1,1) Series

AR / MA	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	x	x	x	x	o	o	o	o	o	o	o	o	o	o
1	x	o	o	o	o	o	o	o	o	o	o	o	o	o
2	x	o	o	o	o	o	o	o	o	o	o	o	o	o
3	x	x	o	o	o	o	o	o	o	o	o	o	o	o
4	x	o	x	o	o	o	o	o	o	o	o	o	o	o
5	x	o	o	o	o	o	o	o	o	o	o	o	o	o
6	x	o	o	o	x	o	o	o	o	o	o	o	o	o
7	x	o	o	o	x	o	o	o	o	o	o	o	o	o

```
> eacf(arma11.s)
```

Figure 13: Exhibit 6.17

Exhibit 12.19 Sample EACF for the Squared Simulated GARCH(1,1) Series

AR/MA	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	o	o	x	x	o	o	x	o	o	o	o	o	o	o
1	x	o	o	o	x	o	x	x	o	o	o	o	o	o
2	x	o	o	o	o	o	x	o	o	o	o	o	o	o
3	x	x	x	o	o	x	o	o	o	o	o	o	o	o
4	x	x	o	x	x	o	o	o	o	o	o	o	o	o
5	x	o	x	x	o	o	o	o	o	o	o	o	o	o
6	x	o	x	x	o	x	o	o	o	o	o	o	o	o
7	x	x	x	x	x	x	o	o	o	o	o	o	o	o

```
> eacf((garch11.sim)^2)
```

Figure 14: Exhibit 12.19

The pattern in the EACF table is not very clear, although an ARMA(2,2) model seems to be suggested. The fuzziness of the signal in the EACF table is likely caused by the larger sampling variability when we deal with higher moments. Shin and Kang (2001) argued that, to a first-order approximation, a power transformation preserves the theoretical autocorrelation function and hence the order of a stationary ARMA process. Their result suggests that the GARCH order may also be identified by studying the absolute returns. Indeed, the sample EACF table for the absolute returns, shown in Exhibit 12.20, more convincingly suggests an ARMA(1,1) model, and therefore a GARCH(1,1) model for the original data, although there is also a hint of a GARCH(2,2) model.

Exhibit 12.20 Sample EACF for Absolute Simulated GARCH(1,1) Series

AR/MA	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	o	o	x	x	o	o	x	o	o	o	o	o	o	o
1	x	o	o	o	x	o	o	o	o	o	o	o	o	o
2	x	x	o	o	o	o	o	o	o	o	o	o	o	o
3	x	x	o	o	o	x	o	o	o	o	o	o	o	o
4	x	x	o	x	o	x	o	o	o	o	o	o	o	o
5	x	o	x	x	x	o	o	o	o	o	o	o	o	o
6	x	o	x	x	x	x	o	o	o	o	o	o	o	o
7	x	x	x	x	x	o	x	o	o	o	o	o	o	o

```
> eacf(abs(garch11.sim))
```

Figure 15: Exhibit 12.20

For the absolute CREF daily return data, the sample EACF table is reported in Exhibit 12.21, which suggests a GARCH(1,1) model. The corresponding EACF table for the squared CREF returns (not shown) is, however, less clear and may suggest a GARCH(2,2) model.

Furthermore, the parameter estimates of the fitted ARMA model for the absolute data may yield initial estimates for maximum likelihood estimation of the GARCH model. For example, Exhibit 12.22 reports the estimated parameters of the fitted ARMA(1,1) model for the absolute simulated GARCH(1,1) process. It can be seen that β is estimated by 0.9445, using (4.1) α is estimated by $(0.9821 - 0.9445) = 0.03763$, and ω can be estimated as the variance of the original data times the estimate of $1 - \alpha - \beta$, which equals 0.0073.

We now derive the condition for a GARCH model to be weakly stationary. Assume for the moment that the return process is weakly stationary. Taking expectations on both sides of Equation (4.1) gives an equation for the unconditional

Exhibit 12.21 Sample EACF for the Absolute Daily CREF Returns

AR/MA	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	o	o	o	o	o	o	o	o	o	x	x	o	o	o
1	x	o	o	o	o	o	o	o	o	o	o	o	o	o
2	x	o	o	o	o	o	o	o	o	o	o	o	o	o
3	x	o	x	o	o	o	o	o	o	o	o	o	o	o
4	x	o	x	o	o	o	o	o	o	o	o	o	o	o
5	x	x	x	x	o	o	o	o	o	o	o	o	o	o
6	x	x	x	x	o	o	o	o	o	o	o	o	o	o
7	x	x	x	x	o	o	o	o	o	o	o	o	o	o

```
> eacf(abs(r.cref))
```

Figure 16: Exhibit 12.21

Exhibit 12.22 Parameter Estimates with ARMA(1,1) Model for the Absolute Simulated GARCH(1,1) Series

Coefficient	ar1	ma1	Intercept
Estimate	0.9821	−0.9445	0.5077
s.e.	0.0134	0.0220	0.0499

```
> arima(abs(garch11.sim),order=c(1,0,1))
```

Figure 17: Exhibit 12.22

variance σ^2:

$$\sigma^2 = \frac{\omega}{1 - \sum_{i=1}^{\max(p,q)} (\alpha_i + \beta_i)}$$

which is positive and finite if

$$\sum_{i=1}^{\max(p,q)} (\alpha_i + \beta_i) < 1.$$

This condition can be shown to be necessary and sufficient for the weak stationarity of a GARCH(p,q) model.

The model fit of the GARCH(1,1) model is reported in the following.

```
>g2=garch(garch11.sim,order=c(1,1))
> summary(g2)

Call:
garch(x = garch11.sim, order = c(1, 1))

Model:
GARCH(1,1)

Residuals:
      Min        1Q    Median        3Q       Max
-3.307030 -0.637977  0.009156  0.741977  3.019441

Coefficient(s):
    Estimate  Std. Error  t value Pr(>|t|)
a0  0.007575    0.007590    0.998   0.3183
a1  0.047184    0.022308    2.115   0.0344 *
b1  0.935377    0.035839   26.100   <2e-16 ***
---
Signif. codes:  0 Ś***Š 0.001 Ś**Š 0.01 Ś*Š 0.05 Ś.Š 0.1 Ś Š 1

Diagnostic Tests:
        Jarque Bera Test

data:  Residuals
X-squared = 0.82911, df = 2, p-value = 0.6606


        Box-Ljung test

data:  Squared.Residuals
```

```
X-squared = 0.53659, df = 1, p-value = 0.4638
```

For the CREF return data, we earlier identified either a GARCH(1,1) or GARCH(2,2) model. The AIC of the fitted GARCH(1,1) model is 969.6, whereas that of the GARCH(2,2) model is 970.3. Hence the GARCH(1,1) model provides a marginally better fit to the data. Maximum likelihood estimates of the fitted GARCH(1,1) model are reported below.

```
>library(tseries)
>g2=garch(r.cref,order=c(1,1))
> summary(g2)
>AIC(g2)

Call:
garch(x = r.cref, order = c(1, 1))

Model:
GARCH(1,1)

Residuals:
     Min       1Q   Median       3Q      Max
-2.78577 -0.61949  0.08695  0.67933  3.30810

Coefficient(s):
    Estimate  Std. Error  t value Pr(>|t|)
a0   0.01633     0.01237    1.320   0.1869
a1   0.04414     0.02097    2.105   0.0353 *
b1   0.91704     0.04570   20.066   <2e-16 ***
---
Signif. codes:  0 Ś***Š 0.001 Ś**Š 0.01 Ś*Š 0.05 Ś.Š 0.1 Ś Š 1

Diagnostic Tests:
Jarque Bera Test

data:  Residuals
X-squared = 1.0875, df = 2, p-value = 0.5806


Box-Ljung test

data:  Squared.Residuals
X-squared = 0.77654, df = 1, p-value = 0.3782
```

How do we know if ARCH/GARCH is necessary for the times series in concern? Firstly, check if residual plot displays any cluster of volatility. Next, observe the squared residual plot. If there are clusters of volatility, ARCH/GARCH should be used to model the volatility of the series to reflect more recent changes and fluctuations in the series. Finally, ACF & PACF of squared residuals will help confirm if the residuals (noise term) are not independent and can be predicted.

5 Model diagnostic

Before we accept a fitted model and interpret its findings, it is essential to check whether the model is correctly specified, that is, whether the model assumptions are supported by the data. If some key model assumptions seem to be violated, then a new model should be specified; fitted, and checked again until a model is found that provides an adequate fit to the data. Recall that the standardized residuals are defined as

$$\hat{\varepsilon}_t = \frac{r_t}{\sigma_t},$$

which are approximately independently and identically distributed if the model is correctly specified. As in the case of model diagnostics for ARIMA models, the standardized residuals are very useful for checking the model specification. The normality assumption of the innovations can be explored by plotting the QQ normal scores plot.

For the GARCH(1,1) model fitted to the CREF return data, the standardized residuals are plotted in Fig 18. There is some tendency for the residuals to be larger in magnitude towards the end of the study period, perhaps suggesting that there is some residual pattern in the volatility. The QQ plot of the standardized residuals is shown in Fig 19. The QQ plot shows a largely straight-line pattern.

```
>plot(residuals(g2),type='h',ylab='Standardized Residuals')
>win.graph(width=2.5,height=2.5,pointsize=8)
> qqnorm(residuals(g2)); qqline(residuals(g2))
>acf(residuals(m1)^2,na.action=na.omit)
>gBox(g3,method='squared')
```

The p-value of the Jarque-Bera test equals 0.58 and that of the Shapiro-Wilk test is 0.34. Hence, the normality assumption cannot be rejected.

The general impression from the figure is that the squared residuals are serially uncorrelated (see Fig 20) . Fig 21 displays the p-values of the generalized portmanteau tests with the squared standardized residuals from the fitted GARCH(1,1) model of the CREF data for m = 1 to 20. All p-values are higher than 5%, suggesting that the squared residuals are uncorrelated over time, and hence the standardized residuals may be independent.

Below are R codes to perform ARCH/GARCH model for the residuals of some financial data (denoted by arima1):

```
res.arima1=arima1$res
arch03=garch(res.arima1,order=c(0,3),trace=F)
```

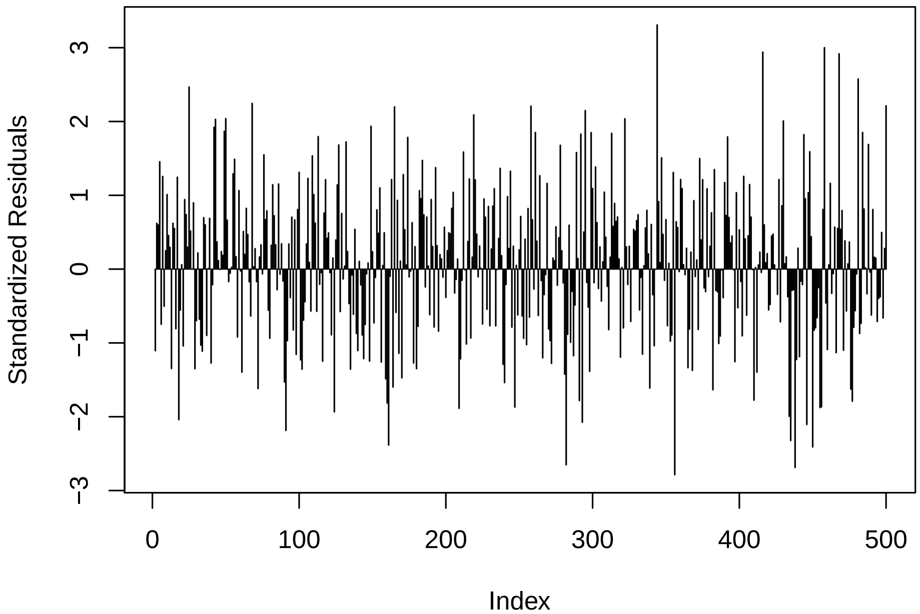

Figure 18: Residuals from the Fitted GARCH(1,1) Model of Daily CREF Returns

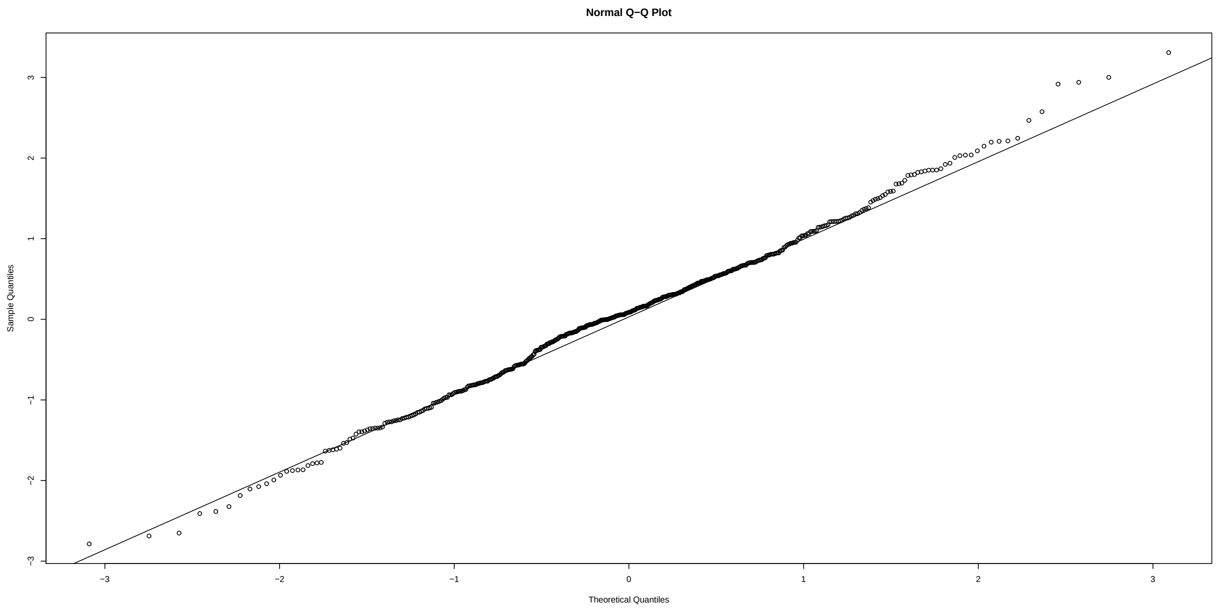

Figure 19: QQ Plot of Standardized Residuals from the Fitted GARCH(1,1) Model of Daily CREF Returns

```
loglik03=logLik(arch03)
summary(arch03)
```

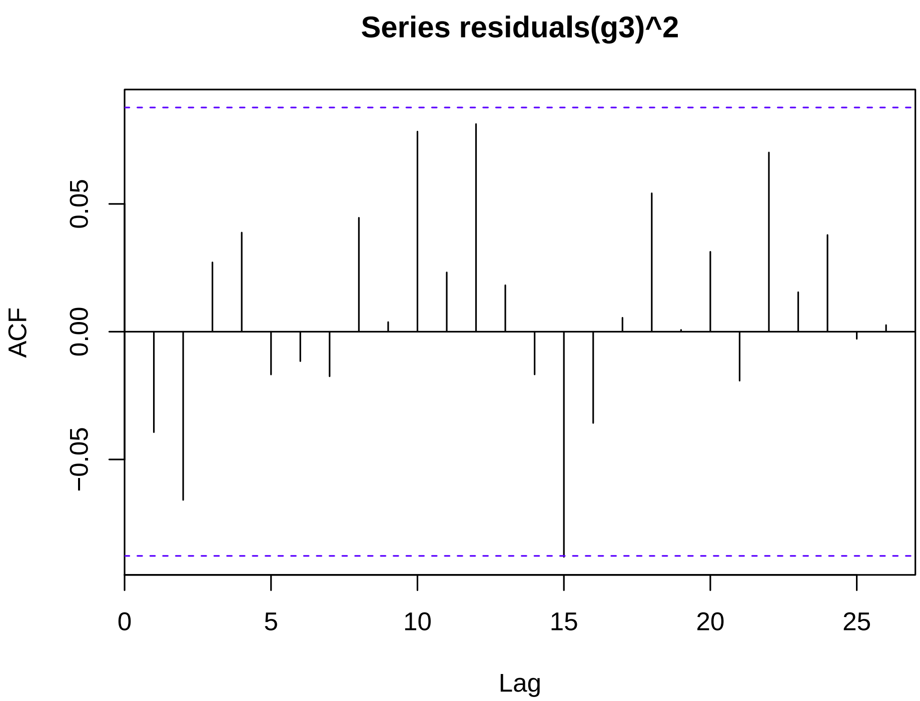

Figure 20: Sample ACF of Squared Residuals from the GARCH(1,1) Model of the Daily CREF Returns

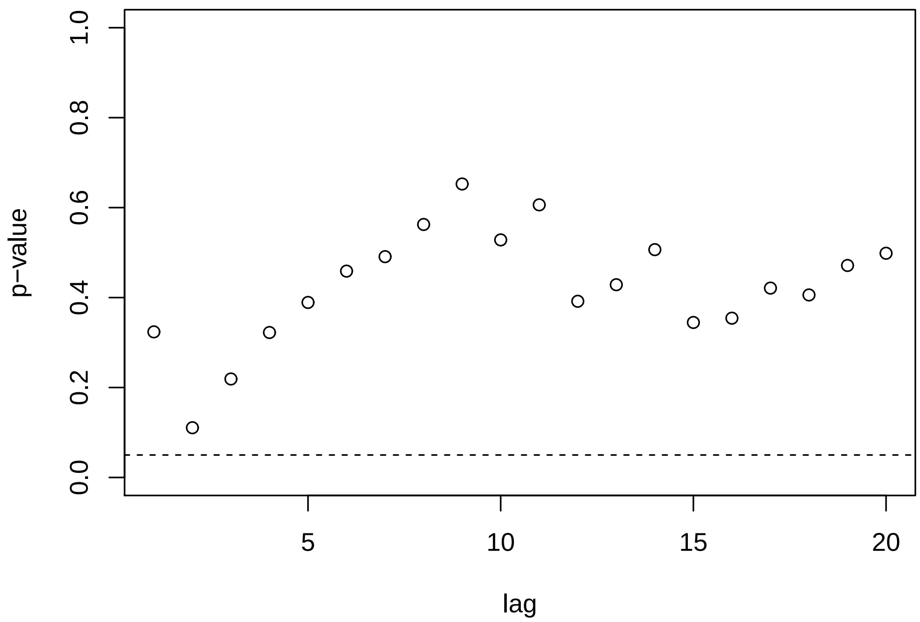

Figure 21: Generalized Portmanteau Test p-Values for the Squared Standardized Residuals for the GARCH(1,1) Model of the Daily CREF Returns