TIME SERIES ANALYSIS

Chapter 8: Unit root test

A shock is usually used to describe an unexpected change in a variable or in the value of the error terms at a particular time period. We have two types of non-stationarity. In an AR(1) model

$$y_t = \phi_1 y_{t-1} + \varepsilon_t.$$

we have:

- ▷ Stationary: $|\phi_1| < 1$;
- ▷ Unit root: $|\phi_1| = 1$: homogeneous non-stationarity;
- ▷ Explosive root: $|\phi_1| > 1$: explosive non-stationarity.

In the last case, a shock to the system become more influential as time goes on. It can never be seen in real life. We will not consider them.

1 Augmented Dickey Fuller (ADF)

The Augmented Dickey Fuller test is as follows. Consider the model

$$y_t = \sum_{j=1}^{p} a_j y_{t-j} + \varepsilon_t.$$

Its characteristic function is

$$\phi(z) = 1 - \sum_{j=1}^{p} a_j z^j.$$

When $z = 1$ is a root to $\phi(z) = 0$ we have

$$1 = \sum_{j=1}^{p} a_j.$$

If $z = 1$ is a root to $\phi(z) = 0$ then we may factor $\phi(z)$ as

$$\phi(z) = (1 - z)\varphi(z),$$

where

$$\varphi(z) = 1 - \sum_{j=1}^{p-1} \beta_j z^j.$$

Hence we can write the model as

$$\Delta y_t = \sum_{j=1}^{p-1} \Delta\beta_j y_{t-j} + \varepsilon_t.$$

This suggests estimating:

$$y_t = \rho y_{t-1} + \sum_{j=1}^{p-1} \Delta\beta_j y_{t-j} + \varepsilon_t$$

and testing whether $\rho = 1$ to see if we have a unit root.

Let

$$x_t = (y_{t-1}, \Delta y_{t-1}, \cdots, \Delta y_{t-p}), \quad \theta = (\rho, \beta_1, \cdots, \beta_{p-1})'.$$

It follows from the least square estimate that

$$\hat{\theta} = (X'X)^{-1}X'Y$$

where $Y = (y_1, \cdots, y_p)$. It can be proved that

$$\frac{T(\hat{\rho} - 1)}{1 - \sum \hat{\beta}_j} \to \frac{\int W(t)dW(t)}{\int (W(t))^2 dt},$$

where W is a Brownian motion.

EXAMPLE 1 Consider

$$y_t = \phi_1 y_{t-1} + \varepsilon_t.$$

- H_0 (y_t nonstationary): $\phi_1 = 1$
- H_1 (y_t stationary):$\phi_1 < 1$

```
adf.test(x, alternative = c("stationary", "explosive"),
k = trunc((length(x)-1)^(1/3)))


Examples
x <- rnorm(1000)  # no unit-root
adf.test(x)

y <- diffinv(x)   # contains a unit-root
plot(y,type="l")
adf.test(y)
```

Main criticism to ADF.test: Power of tests is low if the process is stationary but with a root close to the non-stationary boundary. For example, the tests are poor at distinguishing between $\varphi = 1$ or $\varphi = 0.976$, especially with small sample sizes. Suppose the true model is

$$y_t = 0.976 y_{t-1} + e_t$$

=> $H_0 : \rho = 1$ should be rejected.

One way to get around this is to use a stationarity test (like KPSS test) as well as the unit root ADF tests.

2 KPSS (Kwiatkowski, Phillips, Schmidt and Shin) Test (1992)

In the statistical analysis of time series, a stochastic process is trend stationary if an underlying trend (function solely of time) can be removed, leaving a stationary process. The trend does not have to be linear. Conversely, if the process requires differencing to be made stationary, then it is called difference stationary and possesses one or more unit roots.

A different test is the KPSS (Kwiatkowski, Phillips, Schmidt and Shin) Test (1992). It can be used to test whether we have a deterministic trend vs. stochastic trend:

- ▹ $H_0 : Y_t \sim I(0) \longrightarrow$ level (trend) stationary
- ▹ $H_1 : Y_t \sim I(1) \longrightarrow$ difference stationary.

Consider the models

$$\begin{aligned} y_t &= \mu + \delta t + r_t + u_t && (2.1) \\ r_t &= r_{t-1} + e_t && (2.2) \end{aligned}$$

where e_t is white noise with variance σ^2 uncorrelated with the white noise u_t. Then,

- ▹ H_0 : trend stationary: $\sigma^2 = 0$
- ▹ H_0 : level stationary:$\sigma^2 = 0$ and $\delta = 0$.

Under H_1, $\sigma^2 \neq 0$ there is a random walk in y_t.

The KPSS test is constructed as follows

$$KPSS = T^{-2} \sum_{t=1}^{T} \frac{S_t}{s_u^2}$$

where S_t is the partial sum of the residuals $\hat{u}_t$ (OLS residuals), i.e., $S_t = \sum_{i=1}^{t} \hat{u}_i$, and where s_u^2 is the estimate of the long-run variance of the residuals. Here

$$s_u^2 = \frac{1}{T}\sum_{t=1}^{T} \hat{u}_t^2 + \frac{2}{T}\sum_{s=1}^{T} w(s,l) \sum_{t=s+l}^{T} \hat{u}_t \hat{u}_{t-l}.$$

where $w(s,l)$ is a kernel function, for example, the Bartlett kernel. We also need to specify the number of lags, which should grow with T.

Reject H_0 when KPSS is large. It is a very powerful unit root test, but if there is a volatility shift it cannot catch this type non-stationarity.

```
kpss.test(x, null = c("Level", "Trend"), lshort = TRUE)

Examples
x <- rnorm(1000)  # is level stationary
kpss.test(x)

y <- cumsum(x)  # has unit root
plot(y,type="l")
kpss.test(y)

x <- 0.3*(1:1000)+rnorm(1000)  # is trend stationary
kpss.test(x, null = "Trend")
```