CS 331: Algorithms and Complexity (Spring 2024)
Unique Number: 50930, 50935 50940, 50945

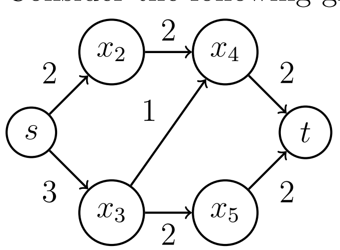
Assignment 6 - Solution

Due on Tuesday, 19 March, by 11.59pm

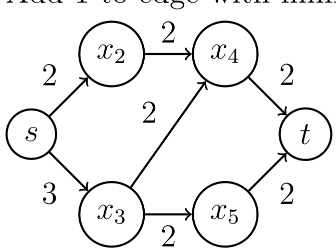
Problem 1

(10 points) State whether the following are true or false. Justify if true. If false, give a counter-example.

a) **(1 point)**
False, it can stay the same
Consider the following graph:



Max flow: 4
Add 1 to edge with minimum capacity

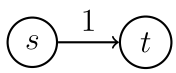


However, the max flow remains the same since $x_4 \rightarrow t$ and $s \rightarrow x_3$ can't pass more flow

b) **(2 point)**
True, we can describe the max flow as the min-cut where $s \in A$ and $t \in B$
Since the sum of integers divisible by 3 is divisible by 3, we can say that every cut is divisible by 3, and therefore the min-cut is also divisible by 3.

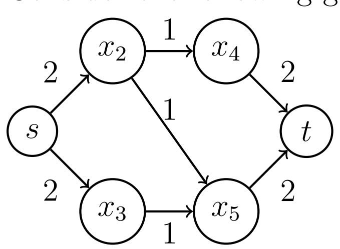
Therefore, the max flow is divisible by 3.

c) **(1 point)**
False, it can still be unique
Consider this graph:

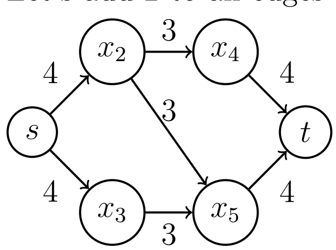


There is one edge, so all edges have the same capacity
There is only one possible min-cut, so it's unique

d) **(2 points)**
False, the min-cuts may change
Consider the following graph:



The min-cuts are $A = \{s, x_3\}$ and $B = \{x_2, x_4, x_5, t\}$ with a capacity of 3 and $A = \{s, x_2, x_3\}$ and $B = \{x_4, x_5, t\}$ with a capacity of 3 and $A = \{s, x_2, x_3, x_5\}$ and $B = \{x_4, t\}$ with a capacity of 3
Let's add 2 to all edges

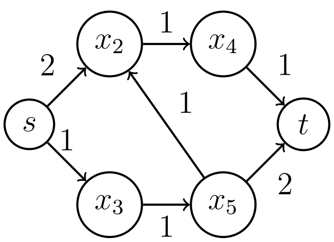


However, the min-cuts are now $A = \{s, x_3\}$ and $B = \{x_2, x_4, x_5, t\}$ with a capacity of 7 and $A = \{s, x_2, x_3, x_5\}$ and $B = \{x_4, t\}$ with a capacity of 7

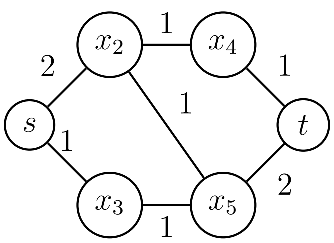
e) **(2 points)**
True, the min-cuts stay the same
Assume this is false
Now we have 2 cases: A min-cut previously is no longer a min-cut, or a new min-cut is added to the sets
Assume a previous min-cut is no longer a min-cut, lets call it α
Then, some other cut, lets call it β has a smaller capacity than α
$\therefore$, $cap(\beta) < cap(\alpha)$, and since every edge was multiplied by a constant factor c, then this implies $\frac{cap(\beta)}{c} < \frac{cap(\alpha)}{c}$

However, α was a min-cut before, so $\frac{cap(\beta)}{c} \geq \frac{cap(\alpha)}{c}$
This is a contradiction, so a previous min-cut can't be no longer a min-cut
Now for the second case, assume a new min-cut is added to the sets, lets call it γ
Therefore, for every cut ξ in the transformed graph, $cap(\gamma) \leq cap(\xi)$
Divide both sides by c, which yields $\frac{cap(\gamma)}{c} \leq \frac{cap(\xi)}{c}$
This means that γ is a min-cut in the original graph, which is a contradiction
Therefore, the min-cuts stay the same

f) **(2 points)**
False, the max flow may change
Consider the following graph:



The max-flow is 2
Now we transform this directed graph to an undirected graph



The max flow is now 3

Problem 2

(10 points)
We know that the profit is the revenue of the stages being performed minus the cost of inviting those artists.
I would create two new nodes being the source and target in the network flow.
The first layer is just the source, and it connects to each artist with the weights being the cost of inviting the artist.
The second layer is the artists, which connect to each stage they're needed in with the weights being infinite.
The third layer is the stages to the target with the weights being the revenue of performing on that stage.
The final layer is the target.

Problem 3

(10 points)
Structure: Farm → Produce → Stores → Target.
Layer 1- The farm connects to each produce with the weights being the amount of each it can supply.
Layer 2- The produce connects to each store they're needed in with the weights being the amount its willing to buy.
Layer 3- The stores connect to the target with the weights being the budget.
Layer 4- The target.
This works since the max-flow will push the maximum amount of produce through the network, which can be constricted based on a store's budget or their list of needed items.
The algorithm is Ford-Fulkerson, which is $O(mC)$.