

## Proof of Correctness

Total Correctness: Termination and Partial Correctness

Partial Correctness: Loop invariants and induction

Loop Invariant: A property that holds before and after each iteration of a loop

Initialization: The loop invariant holds before the first iteration

Maintenance: If the loop invariant holds before an iteration, it holds after the iteration

Termination: When loop terminates, invariant gives useful property to show the algorithm is correct

Iterative: Usually loop invariants

Recursive: Usually induction

## Stable Marriage

Perfect Matching with Stability

Self-enforcing: Self-interest itself prevents offers from being retracted and redirected

## Gale-Shapley Algorithm

Men get the best women while women get the worst men

Greedy algorithm that picks the best available women for each man

Time:  $O(n^2)$

## Complexity

$$a^{\log_b x} = x^{\log_b a}$$

$$\log_b x = \frac{\log_c x}{\log_c b}$$

$$\log_b M \cdot N = \log_b M + \log_b N$$

$$\log_b \frac{M}{N} = \log_b M - \log_b N$$

$$\log_b M^k = k \log_b M$$

Big Oh:  $f(n)$  is  $O(g(n))$  if  $f(n) \leq cg(n)$  for  $n \geq n_0$  :  $c, n_0 > 0$

Big Omega:  $f(n)$  is  $\Omega(g(n))$  if  $f(n) \geq cg(n)$  for  $n \geq n_0$  :  $c, n_0 > 0$

Big Theta:  $f(n)$  is  $\Theta(g(n))$  if  $f(n)$  is  $O(g(n))$  and  $f(n)$  is  $\Omega(g(n))$

Little Oh: Strict Big Oh

Little Omega: Strict Big Omega

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)}:$$

0 if  $f(n)$  is  $o(g(n))$ ,  $\infty$  if  $f(n)$  is  $\omega(g(n))$

$< \infty$  if  $f(n)$  is  $O(g(n))$ ,  $> 0$  if  $f(n)$  is  $\Omega(g(n))$

$0 < \infty$  if  $f(n)$  is  $\Theta(g(n))$

Growth Rates:  $1 < \log(n) < \sqrt{n} < n < n \log(n) < n^2 < n^c < 2^n < c^n < n! < n^n$

$$\text{Harmonic: } \sum_{k=1}^n \frac{1}{k} \sim \ln n$$

$$\text{Triangular: } \sum_{k=1}^n k = \frac{n(n+1)}{2} \sim \frac{n^2}{2}$$

$$\text{Squares: } \sum_{k=1}^n k^2 \sim \frac{n^3}{3}$$

$$\text{Geometric: } \sum_{k=0}^n ar^k = \frac{a(r^{n+1}-1)}{r-1}$$

Stirling's Approximation:  $\log_2(n!) \sim n \log_2 n$

Master's Theorem:  $T(n) = aT(\frac{n}{b}) + f(n)$ ,  $\epsilon > 0$

$$f(n) = O(n^{\log_b(a)-\epsilon}) \rightarrow T(n) = \Theta(n^{\log_b a})$$

$$f(n) = \Theta(n^{\log_b a}) \rightarrow T(n) = \Theta(n^{\log_b a} \log n)$$

$$f(n) = \Omega(n^{\log_b(a)+\epsilon}) \wedge af(\frac{n}{b}) \leq cf(n) \text{ for some } c < 1 \rightarrow T(n) = \Theta(f(n))$$

## Graphs

$$n = |V|, m = |E|, e = (u, v)$$

$$\text{Complete: } |E| = \frac{n(n-1)}{2}$$

## Adjacency Matrix

Space:  $n^2$ , Check if  $(u, v)$  is an edge:  $\Theta(1)$ , Check all edges:  $\Theta(n^2)$

## Adjacency List

Space:  $n + m$ , Check if  $(u, v)$  is an edge:  $O(\deg(u))$  (actually  $O(1 + \deg(u))$ ), Check all edges:  $\Theta(n + m)$  (actually  $\Theta(n + 2m)$ )

## Paths

Sequence of vertices  $v_1, v_2, \dots, v_k$  such that  $(v_i, v_{i+1})$  is an edge

Simple Path: No repeated vertices

Connected: There is a path between every pair of vertices

Cycle: Simple path with  $v_1 = v_k$  and  $k > 2$

Tree: Connected graph with no cycles,  $|E| = n - 1$

## Breadth First Search

$O(n + m)$  if adjacency list,  $O(n^2)$  if adjacency matrix

## Depth First Search

$O(n + m)$  if adjacency list,  $O(n^2)$  if adjacency matrix

## Directed Graphs

Mutually Reachable:  $u \rightarrow v$  and  $v \rightarrow u$

Strongly Connected: Every node mutually reachable

Directed Acyclic Graph: No directed cycles

Topological Ordering: Ordering of vertices  $v_1, v_2, \dots, v_n$  such that if  $(v_i, v_j)$  is an edge,  $i < j$

DAG  $\Leftrightarrow$  Topological Ordering

## Bipartite Graphs

No odd cycles, can be colored with 2 colors

## Connected Components

Strongly Connected Components (Di-graph only)

## Greedy Algorithms

Use local optimization to find a global solution.

## Interval Scheduling

Maximum number of non-overlapping intervals  
 Sort by finish time, take the first interval that doesn't overlap with the previous interval  
 Time:  $O(n \log n)$

## Interval Partitioning/Scheduling all Intervals

Minimum number of rooms to schedule all intervals  
 Sort by start time, if interval overlaps with all preceding intervals, allocate new room  
 Time:  $O(n \log n)$

## Scheduling to Minimize Lateness

Minimizes the maximum lateness  
 Sort by deadline, schedule in order of increasing deadline  
 Time:  $O(n \log n)$

## Dijkstra

Single-source shortest path; weighted BFS  
 Assumes non-negative edge weights, non-deterministic(consistent) otherwise  
 Use Bellman-Ford if negative edge weights, which uses dynamic programming  
 Time:  $O(nm)$  if naive,  $O(m \log n)$  if priority heap

## Minimum Spanning Trees

Assumes connected, undirected, distinct and non-negative edges

### Kruskal

Sort edges by weight, add edge if it doesn't form a cycle  
 Time:  $O(m^2)$  if naive,  $O(m \log n)$  if union-find

### Prim

Start with any vertex, add the minimum weight edge that connects to the tree(one inside tree and one outside)  
 Time:  $O(n^2)$  if naive,  $O(m \log n)$  if priority heap

## Reverse-Delete

Sort edges in decreasing order, remove edge if it doesn't disconnect the graph  
 Time:  $O(e \log e)$  in PPT,  $O(E \log(V)(\log \log V)^3)$  online

## Greedy Stays Ahead

Define: Optimal-  $\mathbb{O}$  and Greedy-  $\mathbb{G}$   
 Measurement: Define criteria to compare each element of  $\mathbb{O}$  and  $\mathbb{G}$   
 Stays Ahead: Show  $\mathbb{G}_i$  is better than  $\mathbb{O}_i$  ( $\forall i$ ), usually through induction

Optimality: Show that  $\mathbb{G}$  is optimal since it is always better than  $\mathbb{O}$ , usually through contradiction by assuming greedy isn't optimal but it stays ahead of optimal

## Exchange Argument

Define: Optimal-  $\mathbb{O}$  and Greedy-  $\mathbb{G}$   
 Compare Solutions: Show differences in order or existence  
 Exchange Pieces: Show that transforming  $\mathbb{O}$  to  $\mathbb{G}$  doesn't reduce optimality  
 Iterate: Repeat until  $\mathbb{O}$  is  $\mathbb{G}$

## Divide and Conquer/Combine

Divide: Break problem into smaller subproblems  
 Conquer/Combine: Solve subproblems recursively and combine

## Recurrence Relation

A function defined in terms of itself  
 Mirrors the recursive algorithm it represents  
 Analysis of the running time of a divide and conquer algorithm generally involves solving a recurrence relation  
 1,2,3 Method Example: (Merge Sort)  
 $T(n) = 2T(\frac{n}{2}) + n - 1$ ,  $T(1) = 0$

Level	Problem Size	Total Time
0	$n$	$n$
1	$\frac{n}{2}$	$2 \frac{n}{2} = n$
2	$\frac{n}{4}$	$4 \frac{n}{4} = n$
$\vdots$	$\vdots$	$\vdots$
$k$	$\frac{n}{2^k} = 1$	$2^k \frac{n}{2^k} = n$
$\rightarrow (\sum_{i=0}^{k-1} n) + 0 \cdot 2^{\log_2 n} \rightarrow \sum_{i=0}^{\log_2 n - 1} n \rightarrow n \log_2 n$		
or $\rightarrow \sum_{i=0}^k n \rightarrow \sum_{i=0}^{\log_2 n} n \rightarrow n \log_2 n$		

## Dynamic Programming