CS 331: Algorithms and Complexity (Spring 2022)

Unique numbers: 51310/51315

Discussion Section 1

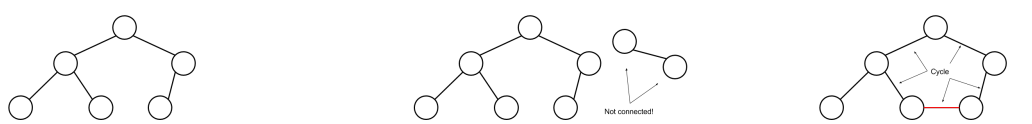
Problem 1

Recall that:

- A *graph* is a data structure with nodes and edges. Each edge connects two nodes.
- A *tree* is a connected acyclic graph.

In a connected graph, you can reach every node from every other node by following edges. In Figure 1b, the two nodes on the right are not reachable from the nodes on the left. Therefore, this is not a tree. Acyclic means that there are no cycles. In Figure 1c, a cycle is formed by the addition of the red edge.

We'll say that n is the number of nodes in a tree and m is the number of edges.



(a) This is a tree, with $n = 6$ and $m = 5$.

(b) The two nodes on the right hand side are disconnected. Not a tree.

(c) The addition of the red edge forms a cycle. This is not a valid tree.

Figure 1: Examples of trees and non-trees.

Prove that for any tree, the number of edges is one less than the number of nodes, i.e. $n - m = 1$.

Problem 2

Compression algorithms take in data and attempt to store it in a form that takes less space. These are sometimes separated into *lossy* algorithms, which can only return an approximation of the data, and *lossless* algorithms, which always return a perfect copy of the data.

Every once in a while, someone claims to have invented a "perfect" compression algorithm: one that is both lossless and guarantees compression (i.e. if you give it n bits, it will always give you a compressed form that is $n-1$ bits or smaller).

Prove that such an algorithm cannot possibly exist.

Hint: Remember that the algorithm is guaranteed to be both compressing *and* lossless. Start by assuming that the algorithm is always compressing, and show that there it cannot possibly be lossless (thus creating a contradiction).

Problem 3

Consider an $n \times n$ chess-board ($n \geq 1$). Let the bottom-left square be colored black. Prove that a bishop placed on that square can go to any black colored square on the chess-board.

Problem 4

Suppose that you have a chessboard and a set of dominoes, where each domino covers two squares on the chessboard (one black and one white). A *tiling* of the chessboard is an arrangement of dominoes such that every square on the chessboard is covered, and every domino covers two adjacent squares: no dominoes are "hanging off the edge," and no dominoes are placed diagonally.

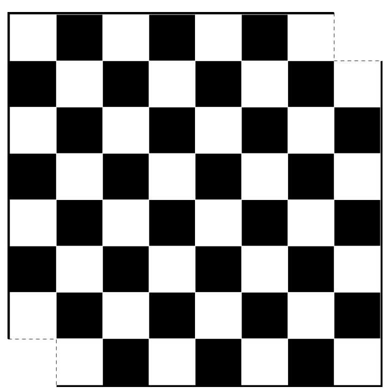


Figure 2: A chessboard with opposite corners removed.

Now take the chessboard and remove two corner squares of the same color, as seen in Figure 2. Prove that this new board cannot be tiled with dominoes—that is, any attempt to cover the chessboard with dominoes must always have either an uncovered square or a domino hanging off the edge.