CS 331: Algorithms and Complexity (Spring 2024)
Unique Number: 50930, 50935 50940, 50945

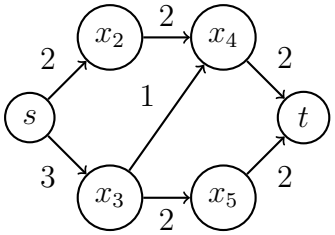
Assignment 6 - Solution

Due on Tuesday, 19 March, by 11.59pm

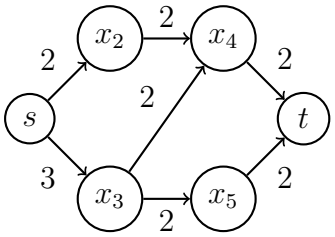
Problem 1

(10 points) State whether the following are true or false. Justify if true. If false, give a counter-example.

a) **(1 point)**
False, it can stay the same
Consider the following graph:



Max flow: 4
Add 1 to edge with minimum capacity

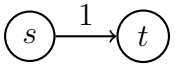


However, the max flow remains the same since $x_4 \to t$ and $s \to x_3$ can't pass more flow

b) **(2 point)**
True, we can describe the max flow as the min-cut where $s \in A$ and $t \in B$
Since the sum of integers divisible by 3 is divisible by 3, we can say that every cut is divisible by 3, and therefore the min-cut is also divisible by 3.

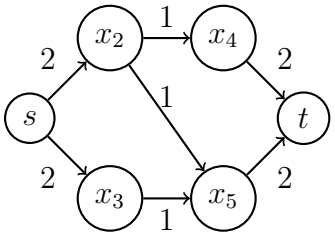
Therefore, the max flow is divisible by 3.

c) **(1 point)**
False, it can still be unique
Consider this graph:

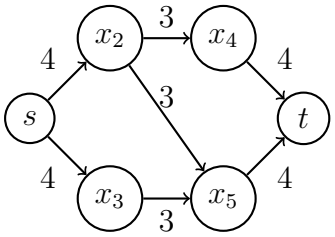


There is one edge, so all edges have the same capacity
There is only one possible min-cut, so it's unique

d) **(2 points)**
False, the min-cuts may change
Consider the following graph:



The min-cuts are $A = \{s, x_3\}$ and $B = \{x_2, x_4, x_5, t\}$ with a capacity of 3 and $A = \{s, x_2, x_3\}$ and $B = \{x_4, x_5, t\}$ with a capacity of 3 and $A = \{s, x_2, x_3, x_5\}$ and $B = \{x_4, t\}$ with a capacity of 3
Let's add 2 to all edges

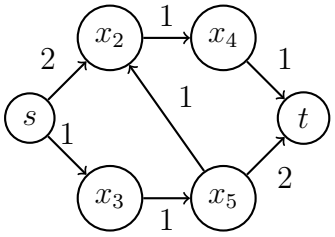


However, the min-cuts are now $A = \{s, x_3\}$ and $B = \{x_2, x_4, x_5, t\}$ with a capacity of 7 and $A = \{s, x_2, x_3, x_5\}$ and $B = \{x_4, t\}$ with a capacity of 7

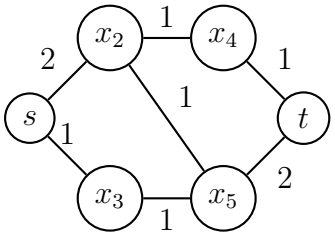
e) **(2 points)**
True, the min-cuts stay the same
Assume this is false
Now we have 2 cases: A min-cut previously is no longer a min-cut, or a new min-cut is added to the sets
Assume a previous min-cut is no longer a min-cut, lets call it α
Then, some other cut, lets call it β has a smaller capacity than α
$\therefore$, $cap(\beta) < cap(\alpha)$, and since every edge was multiplied by a constant factor c, then this implies $\frac{cap(\beta)}{c} < \frac{cap(\alpha)}{c}$

However, α was a min-cut before, so $\frac{cap(\beta)}{c} \geq \frac{cap(\alpha)}{c}$
This is a contradiction, so a previous min-cut can't be no longer a min-cut
Now for the second case, assume a new min-cut is added to the sets, lets call it γ
Therefore, for every cut ξ in the transformed graph, $cap(\gamma) \leq cap(\xi)$
Divide both sides by c, which yields $\frac{cap(\gamma)}{c} \leq \frac{cap(\xi)}{c}$
This means that γ is a min-cut in the original graph, which is a contradiction
Therefore, the min-cuts stay the same

f) **(2 points)**
False, the max flow may change
Consider the following graph:



The max-flow is 2
Now we transform this directed graph to an undirected graph



The max flow is now 3

Problem 2

(10 points)
Structure: Source → Artists → Stages → Target.
Layer 1- The source connects to each artist with the weights being the cost of inviting that artist.
Layer 2- The artists connect to each stage they're needed in with the weights being at least the cost of inviting that artist.
Layer 3- The stages connect to the target with the weights being the revenue generated by performing on that stage.
Layer 4- The target.
We know that the profit is the revenue of the stages being performed minus the cost of inviting those artists.
When we compute the max-flow through this network, we will have the flows from the source

to each artist.
Take the artists which has flow = capacity, as it means the cost of inviting that artist is fulfilled and from the stages to the target, the entire cost is consumed, meaning we will at least break even.
We then take all stages that have their artist requirements satisfied by our chosen subset of artists.

Problem 3

(10 points)
Structure: Farm → Produce → Stores → Target.
Layer 1- The farm connects to each produce with the weights being the amount of each it can supply.
Layer 2- The produce connects to each store they're needed in with the weights being the amount its willing to buy.
Layer 3- The stores connect to the target with the weights being the budget.
Layer 4- The target.
This works since the max-flow will push the maximum amount of produce through the network, which is constrained based on a store's budget and list of needed items.
The algorithm is Ford-Fulkerson, which is $O(mC)$, where m = number of edges and C = produce supplied by the farm.
The example from the question is shown below: