CS 331: Algorithms and Complexity (Spring 2024)

Unique numbers: 50930/50935/50940/50945

Discussion Section 7 - Solution

Carpool Fairness.

Description: In this scenario, n people are sharing a carpool for m days. Each person may choose whether to participate in the carpool on each day.

Example. Figure 1 describes a carpool in which 4 people share a carpool 5 days. X's indicate days when people participate in the carpool.

Our goal is to allocate the daily driving responsibilities 'fairly.' One possible approach is to split the responsibilities based on how many people use the car. So, on a day when k people use the carpool, each person incurs a responsibility of $\frac{1}{k}$. That is, for each person i, we calculate his or her driving obligation O_i as shown below. We can then require that person i drives no more than $\lceil O_i \rceil$ times every m days. Figure 2 shows the calculation of these O_i and their ceilings.

Person	**Days:** 1	2	3	4	5
1	X	X	X		
2	X		X		
3	X	X	X	X	X
4		X	X	X	X

Figure 1: Example of a carpool

Person	**Days:** 1	2	3	4	5	O_i	$\lceil O_i \rceil$
1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$			1	1
2	$\frac{1}{3}$		$\frac{1}{4}$			$\frac{7}{12}$	1
3	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{7}{4}$	2
4		$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{19}{12}$	2
$\sum$	1	1	1	1	1	-	-

Figure 2: Driver Responsibilities.

Prove that there always exists a fair solution.

Solution:

To determine whether such an assignment is possible, we formulate the problem as a network, as shown in Figure 3.

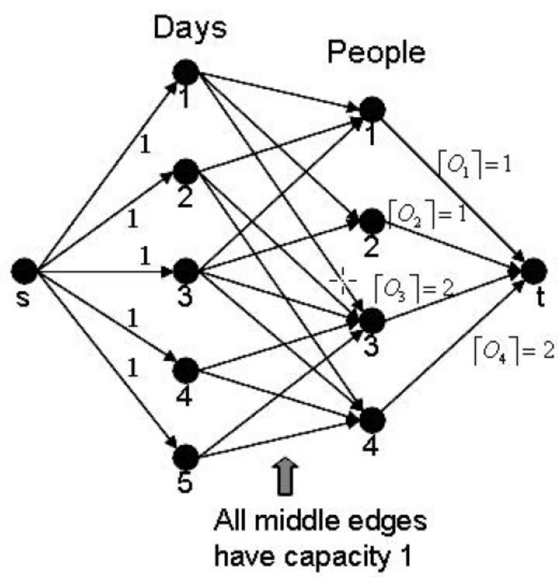

Figure 3: Flow instance for determining a fair carpool.

We use this network to prove a claim for an m day carpool.

Claim 1 If flow of value m exists, then a fair driving schedule exists.
Proof: Note that all capacities are integers and if a flow of value m exists, then an integral flow of value m also exists. So, for each day, exactly one arc pointing outward has a flow of 1. This arc points to some person, and this is the person who should drive for the day. By flow conservation and the capacity of the arcs into t, no one will have to drive more than their obligation.
Note that we do not have to compute the maximum flow to conclude that there always exists a fair driving schedule.

Claim 2 A flow of value m always exists.

Proof: We can always give a fractional flow of value m, where each person present on a given day drives $\frac{1}{k}$ on a day when k people participate in the carpool.