CS 331: Algorithms and Complexity (Spring 2024)
Unique Number: 50930, 50935 50940, 50945

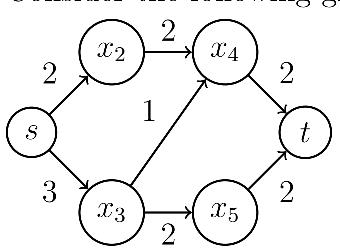
Assignment 6 - Solution

Due on Tuesday, 19 March, by 11.59pm

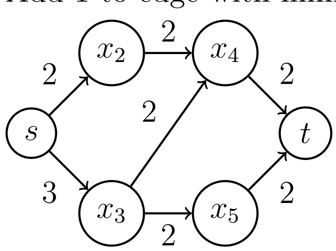
Problem 1

(10 points) State whether the following are true or false. Justify if true. If false, give a counter-example.

a) **(1 point)**
False, it can stay the same
Consider the following graph:



Max flow: 4
Add 1 to edge with minimum capacity

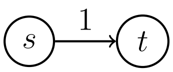


However, the max flow remains the same since $x_4 \to t$ and $s \to x_3$ can't pass more flow

b) **(2 point)**
True, we can describe the max flow as the min-cut where $s \in A$ and $t \in B$
Since the sum of integers divisible by 3 is divisible by 3, we can say that every cut is divisible by 3, and therefore the min-cut is also divisible by 3.

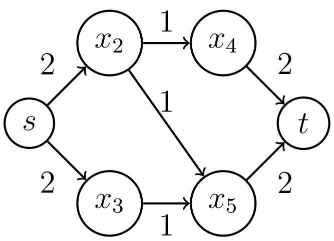
Therefore, the max flow is divisible by 3.

c) **(1 point)**
False, it can still be unique
Consider this graph:

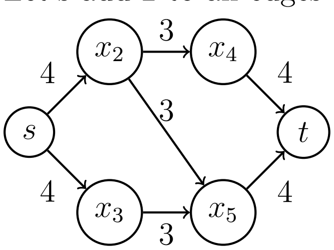


There is one edge, so all edges have the same capacity
There is only one possible min-cut, so it's unique

d) **(2 points)**
False, the min-cuts may change
Consider the following graph:



The min-cuts are $\{s, x_3\}$ and $\{x_2, x_4, x_5, t\}$ with a capacity of 3 and $\{s, x_2, x_3\}$ and $\{x_4, x_5, t\}$ with a capacity of 3 and $\{s, x_2, x_3, x_5\}$ and $\{x_4, t\}$ with a capacity of 3
Let's add 2 to all edges

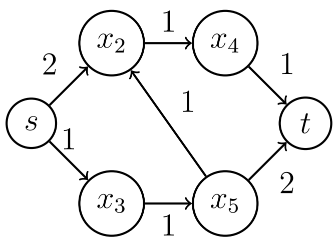


However, the min-cuts are now $\{s, x_3\}$ and $\{x_2, x_4, x_5, t\}$ with a capacity of 7 and $\{s, x_2, x_3, x_5\}$ and $\{x_4, t\}$ with a capacity of 7

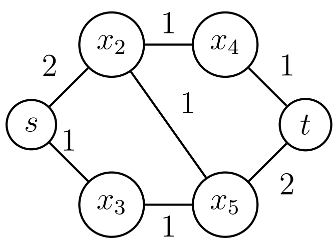
e) **(2 points)**
True, the min-cuts stay the same
Assume this is false
Now we have 2 cases: A min-cut previously is no longer a min-cut, or a new min-cut is added to the sets
Assume a previous min-cut is no longer a min-cut, lets call it α
Then, some other cut, lets call it β has a smaller capacity than α
$\therefore$, $cap(\beta) < cap(\alpha)$, and since every edge was multiplied by a constant factor c, then this implies $\frac{cap(\beta)}{c} < \frac{cap(\alpha)}{c}$
However, α was a min-cut before, so $\frac{cap(\beta)}{c} \geq \frac{cap(\alpha)}{c}$

This is a contradiction, so a previous min-cut can't be no longer a min-cut
Now for the second case, assume a new min-cut is added to the sets, lets call it γ
Therefore, for every cut ξ in the transformed graph, $cap(\gamma) \leq cap(\xi)$
Divide both sides by c, which yields $\frac{cap(\gamma)}{c} \leq \frac{cap(\xi)}{c}$
This means that γ is a min-cut in the original graph, which is a contradiction
Therefore, the min-cuts stay the same

f) **(2 points)**
False, the max flow may change
Consider the following graph:



The max-flow is 2
Now we transform this directed graph to an undirected graph



The max flow is now 3

Problem 2

(10 points)

Problem 3

(10 points)