CS 331: Algorithms and Complexity (Spring 2024)

Unique numbers: 50930/50935/50940/50945

Discussion Section 3 - Solution

Road Trip!

You and some friends have decided to take a road trip across the country. It has fallen to you to plan the route. Due to an unfortunate incident during your last road trip, you all agree that you should not travel at night.

Your friends have already identified stopping points along the route (usually hotels in larger cities). All you need to do is plan the route so that you spend the fewest number of nights on the road as possible.

As you sit down to start planning, one of your friends messages you with an idea. She says that in order to minimize the number of nights you spend on the road, you should use the following strategy: whenever you arrive at a stopping point, figure out whether you can make it to the next point before sundown. If so, continue on. Otherwise, stop for the night. She claims that this will minimize the number of nights you spend away from home, and is easy-to-calculate to boot.

Let's suppose that your estimates for how far you can travel in a given period of time are always accurate. Does your friend's strategy always guarantee that you'll spend the least number of nights on the road, regardless of how your stopping points are arranged along the route? If not, give a counterexample. If so, prove that this strategy is optimal.

Solution

Let $G = \{p_1, p_2, ..., p_k\}$ denote the set of stopping points chosen by greedy solution and let $O = \{q_1, q_2, ..., q_m\}$ denote the set of stopping points chosen by the optimal solution.

Let $x(p_j)$ denote the distance of stopping point p_j from the origin along the route. For example, if stopping point p_1 comes before stopping point p_2 on the route, then $x(p_2) > x(p_1)$.

To prove that the greedy solution is the optimal solution, we have to prove the following:

- The stopping point reached by the greedy solution on each day i is at least as far as the stopping point reached by the optimal solution on day i. Therefore, we have to show $x(p_j) \geq x(q_j)$, for $j = 1, 2, ..., m$.
- We have to also show that $k = m$.

Proof of first part: We prove by induction on the number j of night stay.

Base case: $j = 1$ is the case of first night stay. The stopping point reached by greedy solution is p_1 and the stopping point reached by the optimal solution is q_1. Note that both of these points are reached from the origin before sundown. Because greedy solution chooses the farthest stopping point among the stopping points those can be reached before sundown, q_1 is not farther than p_1. Therefore, $x(p_1) \geq x(q_1)$.

Induction hypothesis: Assume that upto j-th night stay $x(p_j) \geq x(q_j)$ holds.

Inductive case: At $(j+1)-th$ night the stopping point selected by greedy solution is p_{j+1} and the stopping point selected by optimal solution is q_{j+1}.

Let d denote the maximum distance that can be traveled in a day before sundown along the route.

We now observe the following facts.

- $x(q_{j+1}) - x(q_j) \leq d$
- $x(q_{j+1}) - x(p_j) \leq x(q_{j+1}) - x(q_j)$ because $x(p_j) \geq x(q_j)$ from the induction hypothesis.

Combining the above two facts, we obtain $x(q_{j+1}) - x(p_j) \leq d$. Therefore, starting from point p_j the greedy solution can reach the stopping point q_{j+1} before sundown. But instead of choosing stopping point q_{j+1}, greedy chooses stopping point p_{j+1} which implies that $x(p_{j+1}) \geq x(q_{j+1})$.

Proof of second part: We have to show that $k = m$. By contradiction we assume that $m < k$. Suppose greedy solution chooses one extra stopping point p_{m+1}. Greedy solution chose this stopping point because the distance between p_m and the destination is greater than d.

Now we have already proved in the first part that $x(p_m) \geq x(q_m)$. That means, p_m is farther that q_m along that route. So the the distance between q_m and the destination is also greater than d. Thus, if optimal solution has q_m as its last stopping point, then optimal solution did not reach destination. This contradicts the validity of the optimal solution. Therefore, k cannot be greater than m, $k = m$ which proves that greedy solution is the optimal solution under this scenario.