Quantum Codes, Transversal Gates, and Representation Theory

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Overview

I. Background

II. A Difficult Problem

III. Getting It All For Free

IV. The View From The Mountaintop

Part I

Background

Error Correction: Especially Important for Quantum!

Qubits/QuDits are inherently noisy

$\Downarrow$

Quantum advantage is lost

$\Downarrow$

Need quantum error correction

A Quantum Code is a Protected Subspace

- Qubit:

$$\alpha |0\rangle + \beta |1\rangle \in \mathbb{C}^2$$

- Quantum code: Pick 2-dimensional subspace of n-qubits

$$\alpha \left|\overline{0}\right\rangle + \beta \left|\overline{1}\right\rangle \in \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{n}$$

- A code has **distance** d if for each error E with weight $< d$

$$\left\langle\overline{0}\right| E \left|\overline{1}\right\rangle = 0 = \left\langle\overline{1}\right| E \left|\overline{0}\right\rangle$$
$$\left\langle\overline{0}\right| E \left|\overline{0}\right\rangle = \left\langle\overline{1}\right| E \left|\overline{1}\right\rangle$$

Stabilizer Codes

- $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- Pauli Group: $\mathrm{P} = \left\langle \{\mathbb{1}, X, Y, Z\}^{\otimes n} \right\rangle$
- Stabilizer Codes:
  - Let $S \subset \mathrm{P}$ be abelian with $-\mathbb{1} \notin S$
  - Code is simultaneous $+1$ eigenspace of S
  - $|S| = 2^{n-k} \implies K = 2^k$
  - Specially denoted $[[n, k, d]]$ (instead of $((n, K, d))$)
- **Almost all known quantum codes are stabilizer**
  - Simple structure and many fault-tolerant techniques exist

Part II

A Difficult Problem

E. Kubischta and I. Teixeira *, **Family of Quantum Codes with Exotic Transversal Gates**, *Physical Review Letters*, December 2023

* Equal Contribution

Transversal Gates

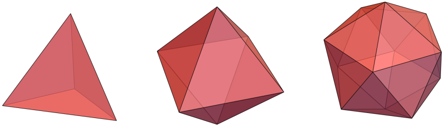
October 2021

- Single qubit gate is a matrix in $\mathrm{U}(2)$
- Called **transversal** if $h^{\otimes n}$ preserves the code space and enacts the logical gate g
- No logical gates $\implies$ quantum storage not quantum computing
- Transversal gates are naturally fault tolerant - they don't spread errors
- Eastin-Knill*: transversal gates cannot be universal, transversal gate group G at most a finite subgroup of $\mathrm{SU}(2)$

* B. Eastin and E. Knill, Restrictions on Transversal Encoded Quantum Gate Sets, Physical Review Letters, (2009)

Finite Subgroups of $\mathrm{SU}(2)$

October 2021



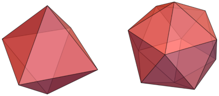
Clifford Gates: $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ $F = HS^\dagger$

$G \subset SU(2)$	Order	Generators
C_m	m	$\langle \mathrm{Ph}(2\pi/m) \rangle$
Dic_m	$4m$	$\langle X, \mathrm{Ph}(\pi/m) \rangle$
$2\mathbb{T}$	24	$\langle X, Z, F \rangle$
$2\mathbb{O}$ = Clifford	48	$\langle X, Z, F, H, S \rangle$
$2\mathbb{I}$	120	$\langle X, Z, F, \Phi \rangle$

Why is $2\mathrm{I}$ important?

May 2022

- **Super golden gate sets** [*]
  - Single-qubit ($\mathrm{SU}(2)$) universal gate sets $\mathscr{U} = \mathrm{G} + \tau$ with optimal navigation properties, minimizing expensive τ gates



$2\mathrm{O}$ $2\mathrm{I}$

  - $\mathscr{U}_1 = 2\mathrm{O} + T$
    - Most standard universal gate set (Clifford)
    - Many stabilizer codes implement $2\mathrm{O}$ transversally
  - $\mathscr{U}_2 = 2\mathrm{I} + \tau_{60}$
    - Icosahedral gate set
    - Most efficient (single qubit) universal gate set
    - Need codes that implement $2\mathrm{I}$ transversally

[*] O. Parzanchevski and P. Sarnak, Super-golden-gates for $PU(2)$, Advances in Mathematics, (2018)

Transversal Gates and Representation Theory

Summer 2022

Goal: Find code with transversal gate group $2\mathrm{I}$. How?

Novel Idea: Code must transform in faithful 2-dim irrep of $2\mathrm{I}$.

- Representation theory primer:
  - A **representation** of G is a homomorphism: $\lambda : \mathrm{G} \to \mathrm{U}(V)$
  - Called **irrep** if no subspace is invariant under G action
  - λ-Isotypic projector: $P_\lambda = \frac{\dim\lambda}{|\mathrm{G}|} \sum_{g \in \mathrm{G}} \mathrm{Tr}(\lambda^*(g)) g^{\otimes n}$

Character Table 2I

Summer 2022

Size	1	1	30	20	20	12	12	12	12
π_1	1	1	1	1	1	1	1	1	1
π_2	2	-2	0	1	-1	φ	φ^{-1}	$-\varphi^{-1}$	$-\varphi$
$\overline{\pi_2}$	2	-2	0	1	-1	$-\varphi^{-1}$	$-\varphi$	φ	φ^{-1}
π_3	3	3	-1	0	0	φ	$-\varphi^{-1}$	$-\varphi^{-1}$	φ
$\overline{\pi_3}$	3	3	-1	0	0	$-\varphi^{-1}$	φ	φ	$-\varphi^{-1}$
π_4	4	-4	0	-1	1	1	-1	1	-1
$\pi_{4'}$	4	4	0	1	1	-1	-1	-1	-1
π_5	5	5	1	-1	-1	0	0	0	0
π_6	6	-6	0	0	0	-1	1	-1	1

$\varphi = (1+\sqrt{5})/2$ is the golden ratio

Code Finding Procedure

Summer 2022

- Looking for codespace $\mathrm{C} = \{ \left|\bar{0}\right\rangle, \left|\bar{1}\right\rangle \}$ transforming in π_2
  - Write $\left|\bar{0}\right\rangle = \sum_{i=1}^{\mu} a_i \left|b_i\right\rangle$
    - $\mu = \mathrm{rank}\left(P_{\pi_2}\right)$
    - $\left|b_i\right\rangle$ is eigenvector of P_{π_2}
    - $a_i \in \mathbb{C}$
  - $\mathrm{C} = \mathbb{C}[\mathrm{G}] \cdot \left|\bar{0}\right\rangle$
- Arbitrary choice of $a_i \implies$ C has distance $d = 1 \implies$ useless
- Knill-Laflamme conditions:
  - $\left\langle\bar{0}\right| E \left|\bar{0}\right\rangle = \left\langle\bar{1}\right| E \left|\bar{1}\right\rangle$ and $\left\langle\bar{0}\right| E \left|\bar{1}\right\rangle = 0$
  - Number of equations is $N_d = 2\sum_{i=1}^{d-1} 3^i \binom{n}{i}$
- **Goal: Find solution of N_d equations in μ variables**

Lemma. Any code that implements $2\mathrm{I}$ transversally must be non-stabilizer.

Proof sketch.

- Stabilizer codes:
  - Every $g \in G$ defined over field $\mathbb{Q}(e^{i\pi/2^r})$ for some r
- $2\mathrm{I}$ contains gate $\Phi = \frac{1}{2}\begin{pmatrix} \varphi + i\varphi^{-1} & 1 \\ -1 & \varphi - i\varphi^{-1} \end{pmatrix}$ where $\varphi = \frac{1+\sqrt{5}}{2}$
- $\sqrt{5}$ is not in this field
  - $\Longrightarrow$ $2\mathrm{I}$ not in Clifford hierarchy (we dubbed this exotic)
  - $\Longrightarrow$ no stabilizer code can implement $2\mathrm{I}$ transversally

But The Problem Is Hard

Winter 2023

- Not feasible (even for $d = 3$)

n	3	5	7	9	11
$N_{d=3}$	72	210	420	702	1056
μ	4	10	28	84	266

- Simplifying assumptions
  -
    - » $\lvert\overline{0}\rangle$ is a $+1$ eigenvector of $Z^{\otimes n}$
    - » $\lvert\overline{1}\rangle = X^{\otimes n}\lvert\overline{0}\rangle$
    - » $a_i \in \mathbb{R}$
  - $\implies$ Even weight errors automatically satisfy Knill-Laflamme conditions*

* E. Kubischta, I. Teixeira, and J. M. Silvester, Quantum weight enumerators for real codes with X and Z exactly transversal, (2023)

Finding a Needle in a Much Smaller Haystack

Spring 2023

- With these simplifying assumptions:

n	3	5	7	9	11
$N_{d=3}$	18	30	42	54	66
μ	4	10	28	84	266

- We finally find a code!
  - $((11, 2, 3))$ code with transversal gate group $2\mathrm{I}$
  - Novel non-stabilizer code
  - Found using UMD Zaratan
  - Numerical precision coefficients

Can We Do Better?

Spring 2023

- For π_2 irrep: no $\implies$ try $\overline{\pi_2}$ instead
- We find a smaller code!
  - $((7,2,3))$ with transversal gate group $2\mathrm{I}$
  - Exact coefficients

$$|\overline{0}\rangle = \frac{\sqrt{15}}{8}|D^7_0\rangle + \frac{\sqrt{7}}{8}|D^7_2\rangle + \frac{\sqrt{21}}{8}|D^7_4\rangle - \frac{\sqrt{21}}{8}|D^7_6\rangle$$

$$|\overline{1}\rangle = -\frac{\sqrt{21}}{8}|D^7_1\rangle + \frac{\sqrt{21}}{8}|D^7_3\rangle + \frac{\sqrt{7}}{8}|D^7_5\rangle + \frac{\sqrt{15}}{8}|D^7_7\rangle$$

  - $|D^n_w\rangle$ is a Dicke state

$$|D^n_w\rangle = \frac{1}{\sqrt{\binom{n}{w}}}\sum_{wt(s)=w}|s\rangle$$

    - » sum is over all length n bit strings of Hamming weight w

A Hint of Something Deeper

June 2023

First part of implementation of the Parzanchevski/Sarnak super efficient universal gate set is accomplished

- Finding higher distance codes intractable but we can go in another direction
- For $\overline{\pi_2}$ we notice that $P_{\overline{\pi_2}} E P_{\overline{\pi_2}} = 0$ for all weight-1 errors E
- $\implies$ weight 1 errors automatically satisfy Knill-Laflamme condition
- Is there a more general theory here?

Part III

Free Codes as in Free Beer and Free Speech

E. Kubischta and I. Teixeira*, **Quantum Codes from Twisted Unitary t-Groups**. *Physical Review Letters*, July 2024

*Equal Contribution

Unitary t-Groups and Representation Theory

The following are equivalent:

(1) $\mathsf{G} \subset \mathrm{U}(q)$ is a unitary t-group

(2) $\frac{1}{|\mathsf{G}|} \sum_{g \in \mathsf{G}} \left(\mathbf{F}^{\downarrow} \otimes \mathbf{F}^{\downarrow *}\right)^{\otimes t}(g) = \int_{\mathrm{U}(q)} (\mathbf{F} \otimes \mathbf{F}^{*})^{\otimes t}(g)\, dg$

(3) $\frac{1}{|\mathsf{G}|} \sum_{g \in \mathsf{G}} \left(\mathbf{1} \oplus \mathbf{Ad}^{\downarrow}\right)^{\otimes t}(g) = \int_{\mathrm{U}(q)} (\mathbf{1} \oplus \mathbf{Ad})^{\otimes t}(g)\, dg$

(4) $\frac{1}{|\mathsf{G}|} \sum_{g \in \mathsf{G}} \mathbf{R}^{\downarrow}(g) = \int_{\mathrm{U}(q)} \mathbf{R}(g)\, dg, \qquad \forall \mathbf{R} \in \mathscr{E}_t$

(5) $\frac{1}{|\mathsf{G}|} \sum_{g \in \mathsf{G}} \mathbf{R}^{\downarrow}(g) = \mathbf{0}, \quad \forall \mathbf{R} \in \mathscr{E}_t, \mathbf{R} \neq \mathbf{1}$

(6) $\left\langle 1, R^{\downarrow} \right\rangle = 0, \quad \forall \mathbf{R} \in \mathscr{E}_t, \mathbf{R} \neq \mathbf{1}$

$\boldsymbol{\lambda}$-Twisted Unitary t-Groups

The following are equivalent:

(1) $\mathrm{G} \subset \mathrm{U}(q)$ is a $\boldsymbol{\lambda}$-twisted unitary t-group

(2) $$\frac{1}{|\mathrm{G}|}\sum_{g\in\mathrm{G}} |\lambda(g)|^2 \left(\mathbf{F}^{\downarrow} \otimes \mathbf{F}^{\downarrow\,*}\right)^{\otimes t}(g) = \int_{\mathrm{U}(q)} (\mathbf{F} \otimes \mathbf{F}^*)^{\otimes t}(g)\, dg$$

(3) $$\frac{1}{|\mathrm{G}|}\sum_{g\in\mathrm{G}} |\lambda(g)|^2 \left(\mathbf{1} \oplus \mathbf{Ad}^{\downarrow}\right)^{\otimes t}(g) = \int_{\mathrm{U}(q)} (\mathbf{1} \oplus \mathbf{Ad})^{\otimes t}(g)\, dg$$

(4) $\frac{1}{|\mathrm{G}|}\sum_{g\in\mathrm{G}} |\lambda(g)|^2\, \mathbf{R}^{\downarrow}(g) = \int_{\mathrm{U}(q)} \mathbf{R}(g)\, dg, \quad \forall \mathbf{R} \in \mathscr{E}_t$

(5) $\frac{1}{|\mathrm{G}|}\sum_{g\in\mathrm{G}} |\lambda(g)|^2\, \mathbf{R}^{\downarrow}(g) = \mathbf{0}, \quad \forall \mathbf{R} \in \mathscr{E}_t, \mathbf{R} \neq \mathbf{1}$

(6) $\left\langle \lambda^* \lambda, R^{\downarrow} \right\rangle = 0, \quad \forall \mathbf{R} \in \mathscr{E}_t, \mathbf{R} \neq \mathbf{1}$

Key Lemma

Lemma

Suppose a code transforms in an irrep $\boldsymbol{\lambda}$ *of* G *and an error* E *transforms in an irrep* $\mathbf{R}$ *of* $\mathrm{U}(q)$*. If* $\langle 1, \lambda^* R^{\downarrow} \lambda \rangle = 0$ *then the KL conditions are automatically satisfied for the error* E*.*

Main Result

Theorem

If G is a $\boldsymbol{\lambda}$-twisted unitary t-group then every subspace of $(\mathsf{F}^{\downarrow})^{\otimes n}$ that transforms in $\boldsymbol{\lambda}$ is a $|\boldsymbol{\lambda}|$-dimensional quantum code with distance $d \geq t+1$ and transversal gate group $\boldsymbol{\lambda}(\mathrm{G})$.

Corollary

$2\mathrm{I} \subset \mathrm{SU}(2)$ is a $\overline{\boldsymbol{\pi}}_2$-twisted unitary 2-group thus every $\overline{\boldsymbol{\pi}}_2$ subspace of ${\boldsymbol{\pi}_2}^{\otimes n}$ is a 2-dimensional quantum code with $d \geq 3$ and transversal gate group $2\mathrm{I}$.

Part IV

The View From the Mountaintop

E. Kubischta and I. Teixeira.* **Quantum Codes from Irreducible Products of Characters**. *Designs, Codes and Cryptography*, Accepted February 2025

*Equal Contribution

Ad irreducible subgroups of SU_2

- Tetrahedral group A_4
- Octahedral group S_4, equivalently qubit $Cl_1(2)$ (3-design)
- Icosahedral group A_5 (5-design)

(maximal) Ad irreducible subgroups of SU_3

- $Cl_1(3)$
- $\Sigma_{168} \cong PSL_2(\mathbb{F}_7)$
- $\Sigma_{360} \cong A_6$, 3-design

Table of free codes with transversal unitary t-groups

K	$\mathsf{G} = \boldsymbol{\lambda}(\mathcal{G})$	$N = q^n$	$\mathcal{G}$	t
2	$2.\mathrm{A}_5$	2^7	$2.\mathrm{A}_5$	5
3	$3.\mathrm{A}_6$	3^5	$3.\mathrm{A}_6$	3
3	$3.\mathrm{A}_6$	3^7	$3.\mathrm{A}_6$	3
3	$\mathrm{SL}(3,2)$	7^4	$2^3 \rtimes \mathrm{SL}(3,2)$	2
4	$2.\mathrm{A}_7$	6^9	$6.\mathrm{A}_7$	3
5	$\mathrm{PSp}(4,3)$	4^4	$\mathrm{Sp}(4,3)$	2
5	$\mathrm{PSp}(4,3)$	4^6	$\mathrm{Sp}(4,3)$	2
5	$\mathrm{SU}(4,2)$	6^6	$\mathrm{SU}(4,2)$	2
8	$4_1.\mathrm{PSL}(3,4).2_3$	8^5	$4_1.\mathrm{PSL}(3,4).2_3$	2
10	$2.M_{12}.2$	12^5	$2.M_{12}.2$	2
10	$2.M_{22}.2$	56^3	$2.M_{22}.2$	2
11	$\mathrm{SU}(5,2)$	10^5	$\mathrm{SU}(5,2)$	2
12	$6.Suz$	11088^5	$6.Suz$	2
13	$\mathrm{PSp}(6,3)$	14^6	$\mathrm{Sp}(6,3)$	2
18	$3.J_3$	18^5	$3.J_3$	3
26	${}^2F_4(2)'$	27^5	${}^2F_4(2)'$	2
28	$2.Ru$	1248^3	$2.Ru$	2
43	$\mathrm{SU}(7,2)$	42^9	$\mathrm{SU}(7,2)$	2
45	M_{23}	22^5	M_{23}	2
45	M_{24}	23^6	M_{24}	2
342	$3.ON$	495^4	$3.ON$	2

Table of small free codes for other notable groups

K	$\mathsf{G} = \boldsymbol{\lambda}(\mathcal{G})$	$N = q^n$	$\mathcal{G}$
3	A_5	2^6	$2.A_5$
4	$SL(2,9) \cong 2.A_6$	15^3	$2^5 \rtimes A_6$
4	$SL(2,9).2 \cong 2.\mathsf{S}_6$	15^3	$2^5 \rtimes \mathsf{S}_6$
5	A_6	3^6	$3.A_6$
6	$3.A_7$	6^4	$6.A_7$
6	A_7	6^6	$6.A_7$
6	$2.J_2$	6^9	$2.J_2$
7	$A_8 \cong SL(4,2)$	15^3	$2^4 \rtimes SL(4,2)$
8	A_9	21^4	A_9
14	$G_2(3)$	27^6	$3.G_2(3)$
22	McL	126^3	$3.McL$
22	$2.HS.2$	56^4	$2.HS.2$
23	$Co2$	9625^3	$Co2$
23	$Co3$	896^4	$Co3$
24	$2.Co1$	9152000^3	$2.Co1$
27	$3.G_2(3)$	27^4	$3.G_2(3)$
78	$Fi22$	352^6	$2.Fi22$
248	Thompson	27000^4	Thompson
4371	Baby Monster	53936390144^4	Baby Monster
8671	$Fi24$	1603525^3	$Fi24$
196,883	Monster Group	8980616927734375^3	Monster Group

Thank you

Questions?