Icosahedral Quantum Codes from Twisted Unitary t-groups

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Based on the following papers

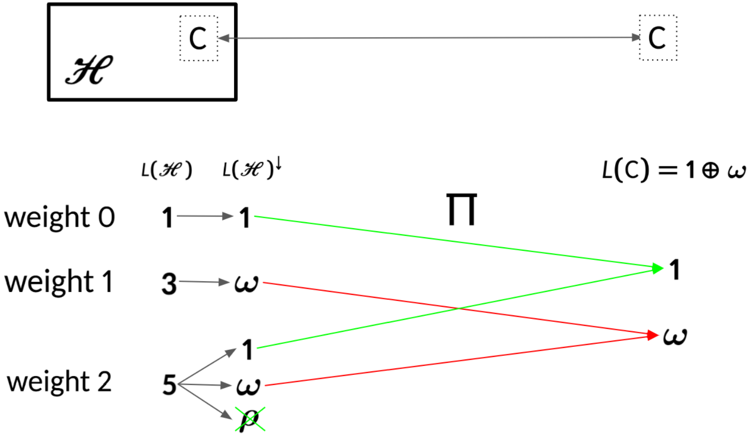
Family of Quantum Codes with Exotic Transversal Gates. *Physical Review Letters, Dec 2023*

Quantum Codes from Twisted Unitary t-groups. *Physical Review Letters, July 2024*

Setup

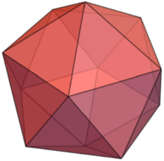
- Let C be code subspace of physical Hilbert space $\mathscr{H}$
- Let P_C be projector onto C
- Define logical error map $\Pi : L(\mathscr{H}) \to L(C)$
  - $\Pi(E) = P_C E P_C$
  - maps physical errors to logical errors
- Let G be transversal gate group for C
  - Π is G-linear
  - $\implies$ We can use Schur's Lemma

Mapping Physical Errors to Logical Errors



Icosahedral Quantum Codes

Binary Icosahedral group



Icosahedral group
$I \simeq A_5 \subset \mathrm{SO}(3)$

$\downarrow$

$\boxed{\text{Binary Icosahedral group } 2I \simeq 2.A_5 \simeq SL(2,5) \subset \mathrm{SU}(2)}$

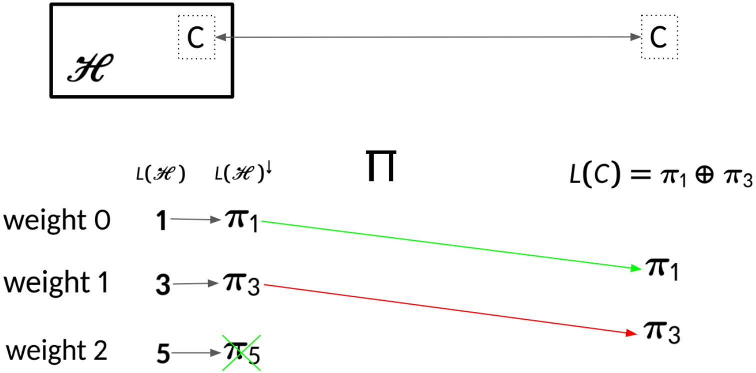
Size	1	1	30	20	20	12	12	12	12
π_1	1	1	1	1	1	1	1	1	1
π_2	2	-2	0	1	-1	φ	φ^{-1}	$-\varphi^{-1}$	$-\varphi$
$\overline{\pi_2}$	2	-2	0	1	-1	$-\varphi^{-1}$	$-\varphi$	φ	φ^{-1}
π_3	3	3	-1	0	0	φ	$-\varphi^{-1}$	$-\varphi^{-1}$	φ
$\overline{\pi_3}$	3	3	-1	0	0	$-\varphi^{-1}$	φ	φ	$-\varphi^{-1}$
π_4	4	-4	0	-1	1	1	-1	1	-1
$\pi_{4'}$	4	4	0	1	1	-1	-1	-1	-1
π_5	5	5	1	-1	-1	0	0	0	0
π_6	6	-6	0	0	0	-1	1	-1	1

$\varphi = (1+\sqrt{5})/2$ is the golden ratio

$C: \quad \lambda = \pi_2$

$L(C): \quad \pi_1 \oplus \pi_3$

Logical Error Map for $\lambda = \pi_2$



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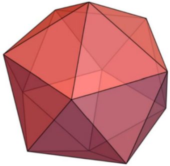

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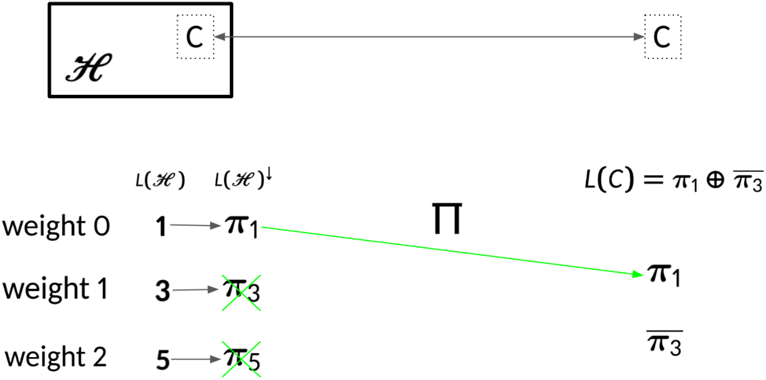
C: $\lambda = \overline{\pi_2}$

$L(\mathrm{C})$: $\pi_1 \oplus \overline{\pi_3}$

Logical Error Map for $\lambda = \overline{\pi_2}$



$\Longrightarrow$ All $\lambda = \overline{\pi_2}$ codes have $d = 3$ automatically!

The smallest $2\mathrm{I}$ code

- Smallest $2\mathrm{I}$ transversal code is in 7 qubits
- Codewords:

$$\left|\overline{0}\right\rangle = \frac{\sqrt{15}}{8}\left|D^7_0\right\rangle + \frac{\sqrt{7}}{8}\left|D^7_2\right\rangle + \frac{\sqrt{21}}{8}\left|D^7_4\right\rangle - \frac{\sqrt{21}}{8}\left|D^7_6\right\rangle$$
$$\left|\overline{1}\right\rangle = -\frac{\sqrt{21}}{8}\left|D^7_1\right\rangle + \frac{\sqrt{21}}{8}\left|D^7_3\right\rangle + \frac{\sqrt{7}}{8}\left|D^7_5\right\rangle + \frac{\sqrt{15}}{8}\left|D^7_7\right\rangle$$

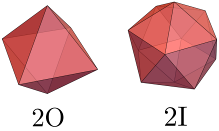
- $\left|D^n_w\right\rangle$ is a Dicke state

$$\left|D^n_w\right\rangle = \frac{1}{\sqrt{\binom{n}{w}}}\sum_{wt(s)=w}\left|s\right\rangle$$

  - sum is over all length n bit strings of Hamming weight w

Why is $2\mathrm{I}$ important?

- **Super golden gate sets** *
  - Single-qubit ($\mathrm{SU}(2)$) universal gate sets $\mathscr{U} = \mathrm{G} + \tau$ with optimal navigation properties, minimizing expensive τ gates



  - $\mathscr{U}_1 = 2\mathrm{O} + T$
    - Most standard universal gate set (Clifford)
    - Many stabilizer codes implement $2\mathrm{O}$ transversally
  - $\mathscr{U}_2 = 2\mathrm{I} + \tau_{60}$
    - Icosahedral gate set
    - Most efficient (single qubit) universal gate set
    - Need codes that implement $2\mathrm{I}$ transversally

* O. Parzanchevski and P. Sarnak, Super-golden-gates for $PU(2)$, Advances in Mathematics, (2018)

Unitary t-Groups and Representation Theory

The following are equivalent:

(1) $\mathsf{G} \subset \mathrm{U}(q)$ is a unitary t-group

(2) $\frac{1}{|\mathsf{G}|}\sum_{g\in\mathsf{G}}\left(\mathbf{F}^{\downarrow}\otimes\mathbf{F}^{\downarrow *}\right)^{\otimes t}(g) = \int_{\mathrm{U}(q)}(\mathbf{F}\otimes\mathbf{F}^{*})^{\otimes t}(g)\,dg$

(3) $\frac{1}{|\mathsf{G}|}\sum_{g\in\mathsf{G}}\left(\mathbf{1}\oplus\mathbf{Ad}^{\downarrow}\right)^{\otimes t}(g) = \int_{\mathrm{U}(q)}(\mathbf{1}\oplus\mathbf{Ad})^{\otimes t}(g)\,dg$

(4) $\frac{1}{|\mathsf{G}|}\sum_{g\in\mathsf{G}}\mathbf{R}^{\downarrow}(g) = \int_{\mathrm{U}(q)}\mathbf{R}(g)\,dg, \qquad \forall\mathbf{R}\in\mathscr{E}_t$

(5) $\frac{1}{|\mathsf{G}|}\sum_{g\in\mathsf{G}}\mathbf{R}^{\downarrow}(g) = \mathbf{0}, \quad \forall\mathbf{R}\in\mathscr{E}_t, \mathbf{R}\neq\mathbf{1}$

(6) $\left\langle 1, R^{\downarrow}\right\rangle = 0, \quad \forall\mathbf{R}\in\mathscr{E}_t, \mathbf{R}\neq\mathbf{1}$

$\boldsymbol{\lambda}$-Twisted Unitary t-Groups

The following are equivalent:

(1) $\mathrm{G} \subset \mathrm{U}(q)$ is a $\boldsymbol{\lambda}$-twisted unitary t-group

(2) $\frac{1}{|\mathrm{G}|} \sum_{g \in \mathrm{G}} |\lambda(g)|^2 \left(\mathbf{F}^{\downarrow} \otimes \mathbf{F}^{\downarrow *}\right)^{\otimes t}(g) = \int_{\mathrm{U}(q)} (\mathbf{F} \otimes \mathbf{F}^*)^{\otimes t}(g)\, dg$

(3) $\frac{1}{|\mathrm{G}|} \sum_{g \in \mathrm{G}} |\lambda(g)|^2 \left(\mathbf{1} \oplus \mathbf{Ad}^{\downarrow}\right)^{\otimes t}(g) = \int_{\mathrm{U}(q)} (\mathbf{1} \oplus \mathbf{Ad})^{\otimes t}(g)\, dg$

(4) $\frac{1}{|\mathrm{G}|} \sum_{g \in \mathrm{G}} |\lambda(g)|^2 \, \mathbf{R}^{\downarrow}(g) = \int_{\mathrm{U}(q)} \mathbf{R}(g)\, dg, \quad \forall \mathbf{R} \in \mathscr{E}_t$

(5) $\frac{1}{|\mathrm{G}|} \sum_{g \in \mathrm{G}} |\lambda(g)|^2 \, \mathbf{R}^{\downarrow}(g) = \mathbf{0}, \quad \forall \mathbf{R} \in \mathscr{E}_t, \mathbf{R} \neq \mathbf{1}$

(6) $\left\langle \lambda^* \lambda, R^{\downarrow} \right\rangle = \left\langle 1 + \omega, R^{\downarrow} \right\rangle = 0, \quad \forall \mathbf{R} \in \mathscr{E}_t, \mathbf{R} \neq \mathbf{1}$

Main Results

Theorem

If G *is a* $\boldsymbol{\lambda}$*-twisted unitary t-group then every subspace of* $(\mathbf{F}^{\downarrow})^{\otimes n}$ *that transforms in* $\boldsymbol{\lambda}$ *is a* $|\boldsymbol{\lambda}|$*-dimensional quantum code with distance* $d \geq t+1$ *and transversal gate group* $\boldsymbol{\lambda}(\mathrm{G})$*.*

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Corollary

$2\mathrm{I} \subset \mathrm{SU}(2)$ *is a* $\overline{\boldsymbol{\pi}}_2$*-twisted unitary 2-group thus every* $\overline{\boldsymbol{\pi}}_2$ *subspace of* $\mathsf{F}^{\downarrow\otimes n}$ *is a 2-dimensional quantum code with* $d \geq 3$ *and transversal gate group* $2\mathrm{I}$*.*

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Corollary

$\boldsymbol{\Sigma}(360\phi) \subset \mathrm{SU}(3)$ *is a* $\overline{\boldsymbol{\chi}}_4$*-twisted unitary 1-group thus every* $\overline{\boldsymbol{\chi}}_4$ *subspace of* $\mathbf{F}^{\downarrow\otimes n}$ *is a 3-dimensional quantum code with* $d \geq 2$ *and transversal gate group* $\boldsymbol{\Sigma}(360\phi)$.

Thank you

Questions?